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An upper estimate in Turán's pure power sum problem

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ABSTRACT

Let $z_1, z_2, \ldots, z_n$ be complex numbers, and write

$$S_j = z_1^j + \ldots + z_n^j$$

for their power sums. Let

$$R_n = \min_{z_1, z_2, \ldots, z_n} \max_{1 \le j \le n} |S_j|,$$

where the minimum is taken under the condition that

$$\max_{1 \le t \le n} |z_t| = 1.$$

In this paper we prove that

$$\limsup_{n \to \infty} R_n < 1.$$

1. INTRODUCTION

The investigation of the above sequence R_n is a classical problem of the power sum theory of Turán (for this theory see [T], and also [M]). The minimum R_n exists by Weierstrass' theorem, and one can easily see that the condition can be replaced by $z_1 = 1$.

The 1942 conjecture of Turán that $R_n > c$ for some $c > 0$ independent of n

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was proved by F.V. Atkinson (see [A]) in 1961, showing that $R_n > \frac{1}{6}$. We proved that $R_n > \frac{1}{2}$ (see [B1]), and in the recent paper [B2] we proved that $R_n > q$ for every n, where q is an absolute constant larger than $\frac{1}{2}$.

Concerning upper bounds, it is trivial that $R_n \leq 1$. Komlós, Sárközy and Szemerédi ([K-S-Sz]) proved that

$$R_n < 1 - \frac{1}{250n}$$

for $n > n_0$, and

$$R_n < 1 - \frac{1}{3}\frac{\log n}{n}$$

for infinitely many n. Recently, in [B3], we improved this result to

$$R_n < 1 - (1-\varepsilon)\frac{\log\log n}{\log n} \qquad (1)$$

for large n. In this paper we prove the following theorem, which can be viewed as an upper bound analogue of Atkinson's theorem from 1961. This theorem solves Problem 15 of the book [T] of Turán.

Theorem. *We have*

$$\limsup_{n\to\infty} R_n < 1.$$

We prove this theorem in Section 2. Our aim in that section is just to prove this theorem quickly, so we use very crude estimates there. Then, in Section 3, we sketch the precise computation, and show what is the exact numerical result (depending on our basic parameter α) obtainable by the present construction (see formulas (22) and (23)). The dependence on α turns out to be somewhat complicated. Using a convenient choice of α, we show in Section 3 that $R_n < \frac{5}{6}$ for large n, hereby obtaining also the 'numerical analogue' of Atkinson's result. However, this explicit value is far from being optimal, even for the present proof: computer work of Gergely Harcos shows that a nearly optimal choice of α in formulas (22) and (23) gives $R_n < 0.694$ for large n (see Addendum at the end of Section 3). This value is very interesting in view of the paper [C-G], where the authors conjecture (based on numerical work) that R_n has a limit about 0.7.

What concerns the proof, we used in [B3] (proving (1)) the formulas

$$S_l + b_1 S_{l-1} + \ldots + b_{l-1} S_1 = 1 + b_1 + \ldots + b_{l-1} - l b_l \quad (l = 1, 2 \ldots, n-1);$$
$$S_n + b_1 S_{n-1} + \ldots + b_{n-1} S_1 = 1 + b_1 + \ldots + b_{n-1},$$

where $z_1 = 1, z_2, \ldots, z_n$ are complex numbers, and

$$(Z - z_2)(Z - z_3)\ldots(Z - z_n) = Z^{n-1} + b_1 Z^{n-2} + \ldots + b_{n-1}.$$

These formulas follow easily by the Newton-Girard formulas for this polynomial (and these formulas were the basic tools in [B1] and are also important in [B2]), and conversely, it is clear that if $z_1 = 1, z_2, \ldots, z_n$ are given, $b_1, b_2, \ldots, b_{n-1}$ are the coefficients of the above polynomial, and some complex numbers S_l satisfy these formulas, then for the system $z_1 = 1, z_2, \ldots, z_n$ we have the numbers $S_1, S_2, \ldots, S_n$ as first n power sums.

Now, in [B3] we put

$$\frac{lb_l}{1 + b_1 + \ldots + b_{l-1}} = \alpha \qquad (1 \le l \le n-1)$$

with some complex parameter α (note that these quotients occurred in our proof in [B1]). This gives $S_l = 1 - \alpha$ for $1 \le l \le n-1$. The novelty of the present proof is to take $S_l = 1 - \alpha$ only for $l \le \frac{n}{2}$, and then choosing optimally the larger power sums.

2. PROOF OF THE THEOREM

Let $2T \le n \le 2T+1$ (i.e. $T = \left[\frac{n}{2}\right]$), and define the numbers S_l in the following way. Let α be a complex number (to be chosen later), set

$$(2) \qquad S_l = 1 - \alpha \qquad (1 \le l \le T),$$

and

$$(3) \qquad S_l = (1-\alpha) + w_l \qquad (T+1 \le l \le n),$$

with some complex numbers w_l. The numbers b_l are defined inductively by $b_0 = 1$, and

$$(4) \qquad S_l + b_1 S_{l-1} + \ldots + b_{l-1} S_1 = 1 + b_1 + \ldots + b_{l-1} - lb_l \qquad (1 \le l \le n).$$

We will choose $S_1, S_2, \ldots, S_n$ in such a way that $b_n = 0$ will hold.

We set $\beta_0 = 1$, and

$$(5) \qquad l\beta_l = \alpha(1 + \beta_1 + \ldots + \beta_{l-1}) \qquad (1 \le l \le n).$$

Then

$$(6) \qquad (1-\alpha)(1 + \beta_1 + \ldots + \beta_{l-1}) = 1 + \beta_1 + \ldots + \beta_{l-1} - l\beta_l \quad (1 \le l \le n).$$

One has

$$(7) \qquad b_l = \beta_l \qquad (0 \le l \le T)$$

by induction, using (2), (4) and (6). We set

$$(8) \qquad b_l = \beta_l + d_l \qquad (T+1 \le l \le n).$$

Taking the difference of (4) and (6) we obtain by (2), (3), (7) and (8) that

$$(9) \qquad w_l + \beta_1 w_{l-1} + \ldots + \beta_{l-T-1} w_{T+1} = \alpha(d_{T+1} + d_{T+2} + \ldots + d_{l-1}) - ld_l$$

for $T+1 \le l \le n$ (we used that $l - T - 1 \le T$). These equations can be expressed in matrix notation as

$$(10)\qquad (I-A)\begin{pmatrix} -(T+1)d_{T+1} \\ \vdots \\ -nd_n \end{pmatrix} = B\begin{pmatrix} w_{T+1} \\ \vdots \\ w_n \end{pmatrix},$$

where

$$I = \{\delta_{k,l}\}_{k,l=T+1}^{n},\ A = \{\alpha_{k,l}\}_{k,l=T+1}^{n},\ B = \{\beta_{k,l}\}_{k,l=T+1}^{n},$$

and $\delta_{k,l}$ is the Kronecker symbol (which means that I is the unit matrix), furthermore

$$\alpha_{k,l} = \begin{cases} 0, & \text{if } k \le l \\ \frac{\alpha}{l}, & \text{if } k > l, \end{cases}$$

and

$$(11)\qquad \beta_{k,l} = \begin{cases} 0, & \text{if } k < l \\ \beta_{k-l}, & \text{if } k \ge l. \end{cases}$$

Since A is a nilpotent matrix, we have

$$(I-A)^{-1} = I + A + A^2 + \dots$$

(this is of course a finite sum), hence

$$(12)\qquad (I-A)^{-1} = \{a_{k,l}\}_{k,l=T+1}^{n}$$

with

$$(13)\qquad a_{k,l} = \begin{cases} 0, & \text{if } k < l \\ 1, & \text{if } k = l \\ \sum\limits_{r \ge 1} \alpha^r \sum\limits_{k=n_1>n_2>\dots>n_{r+1}=l} \frac{1}{n_2 n_3 \dots n_{r+1}}, & \text{if } k > l. \end{cases}$$

Of course the sum over r here is also finite.

By (10) and (12) we obtain

$$(14)\qquad -nd_n = \sum_{l=T+1}^{n} w_l P_l$$

with the notation

$$(15)\qquad P_l = \sum_{m=T+1}^{n} a_{n,m}\beta_{m,l}.$$

Lemma 1. *If $q > 0$, and*

$$\left| n\beta_n + (1-\alpha)\sum_{l+T+1}^{n} P_l \right| \le q \sum_{l=T+1}^{n} |P_l|,$$

then we can choose $S_{T+1}, S_{T+2}, \dots, S_n$ such that

$$|S_{T+1}| \leq q, |S_{T+2}| \leq q, \ldots, |S_n| \leq q$$

and $b_n = 0$.

Proof. By (8), $b_n = 0$ is equivalent to $-nd_n = n\beta_n$, so by (3) and (14) it is enough to choose $S_{T+1}, S_{T+2}, \ldots, S_n$ such that

$$n\beta_n + (1-\alpha) \sum_{l=T+1}^{n} P_l = \sum_{l=T+1}^{n} S_l P_l.$$

The lemma follows by elementary geometry.

We remark here that the 'main term' of P_l will come from $m = n$ in (15), and $a_{n,n}\beta_{n,l} = \beta_{n-l}$ by (11) and (13), so we introduce the notation

(16) $\qquad P_l = \beta_{n-l} + E_l.$

The following statement is straightforward.

Corollary of Lemma 1. *If* $|1-\alpha| < 1$ *and*

$$\left|\sum_{l=0}^{n-T-1} \beta_l\right| \leq q\left(\sum_{l=0}^{n-T-1} |\beta_l|\right) - |n\beta_n| - (1+q) \sum_{l=T+1}^{n} |E_l|,$$

then the condition of Lemma 1 is satisfied.

Lemma 2. *We have*

(i) $$\sum_{l=T+1}^{n} |E_l| \leq \left(\sum_{l=0}^{n-T-1} |\beta_l|\right)\left(|\alpha| e^{|\alpha|}\right)$$

and

(ii) $$|n\beta_n| \leq \left(\sum_{l=0}^{n-T-1} |\beta_l|\right)\left(|\alpha| e^{|\alpha|}\right).$$

So, if $|1-\alpha| < 1$, *and*

$$\left|\sum_{l=0}^{n-T-1} \beta_l\right| \leq \left(\sum_{l=0}^{n-T-1} |\beta_l|\right)\left(q - (2+q)|\alpha| e^{|\alpha|}\right),$$

then the condition of Lemma 1 is satisfied.

Proof. Observe that if $T + 1 \leq m < n$ and $r \geq 1$, then

$$(r-1)! \sum_{n=n_1 > n_2 > \ldots > n_{r+1} = m} \frac{1}{n_2 n_3 \ldots n_{r+1}} \leq \frac{1}{m}\left(\sum_{m<t<n} \frac{1}{t}\right)^{r-1} \leq \frac{1}{T+1},$$

since $m \geq T+1$, $n - m \leq T$. We then obtain by (13) for $T + 1 \leq m < n$ that

$$|a_{n,m}| \leq \frac{1}{T+1} |\alpha| e^{|\alpha|},$$

and (i) follows by (11), (13), (15) and (16), since $n - T \leq T + 1$.

For the proof of (ii) we remark that (by induction, using (5)) one has

$$(17) \qquad 1 + \beta_1 + \ldots + \beta_l = \prod_{r=1}^{l} \left(1 + \frac{\alpha}{r}\right)$$

for $l \geq 0$, and so (using again (5))

$$|n\beta_n| = |\alpha| \left|\sum_{l=0}^{n-T-1} \beta_l\right| \left(\prod_{r=n-T}^{n-1} \left|1 + \frac{\alpha}{r}\right|\right).$$

Since $|1 + \frac{\alpha}{r}| \leq e^{|\alpha|/r}$, (ii) follows, because

$$\prod_{r=n-T}^{n-1} \frac{1}{r} \leq \frac{T}{n-T} \leq 1.$$

The last assertion is then clear, in view of the Corollary of Lemma 1. □

Lemma 3. *If*

$$|1 - \alpha| < 1, \ |\alpha| < \frac{1}{2},$$

and

$$e^{4|\alpha|^2} \leq \frac{|\alpha|}{2\,\mathrm{Re}\,\alpha}\left(\frac{1}{2} - \frac{5}{2}|\alpha|e^{|\alpha|}\right),$$

furthermore

$$\prod_{r=1}^{n-T-1} \left(1 + \frac{\mathrm{Re}\,\alpha}{r}\right) > 2,$$

then the condition of Lemma 1 is satisfied with $q = 1/2$.

Proof. Since by (17) (for l and $l-1$) we have

$$(18) \qquad \beta_l = \frac{\alpha}{l} \prod_{r=1}^{l-1} \left(1 + \frac{\alpha}{r}\right)$$

for $l \geq 1$, we obtain (because $\mathrm{Re}\,\alpha > 0$ by $|1 - \alpha| < 1$)

$$|\beta_l| \geq \frac{|\alpha|}{l} \prod_{r=1}^{l-1} \left(1 + \frac{\mathrm{Re}\,\alpha}{r}\right)$$

and then by induction

$$\frac{|\alpha|}{\mathrm{Re}\,\alpha} + \sum_{r=1}^{l} |\beta_r| \geq \frac{|\alpha|}{\mathrm{Re}\,\alpha} \prod_{r=1}^{l} \left(1 + \frac{\mathrm{Re}\,\alpha}{r}\right)$$

for $l \geq 1$. Hence

$$(19) \qquad \sum_{r=0}^{l} |\beta_r| \geq \frac{|\alpha|}{2\operatorname{Re}\alpha} \prod_{r=1}^{l} \left(1 + \frac{\operatorname{Re}\alpha}{r}\right),$$

if

$$(20) \qquad \prod_{r=1}^{l} \left(1 + \frac{\operatorname{Re}\alpha}{r}\right) > 2.$$

On the other hand, for complex x with $|x| < 1/2$ one has

$$|\log(1+x) - x| \leq \frac{|x|^2}{2(1-|x|)} \leq |x|^2,$$

so for $|x| < 1/2$ we have $1 + x = e^x e^{h(x)}$, where $|h(x)| \leq |x|^2$. Consequently, assuming $|\alpha| < \frac{1}{2}$, we have

$$(21) \qquad \prod_{r=1}^{n-T-1} \left| \left(1 + \frac{\alpha}{r}\right) \Big/ \left(1 + \frac{\operatorname{Re}\alpha}{r}\right) \right| \leq e^{2|\alpha|^2 \sum_{r=1}^{\infty} r^{-2}} \leq e^{4|\alpha|^2}.$$

The lemma is proved, using Lemma 2, (19) and (20) with $l = n - T - 1$, (17) and (21). □

Conclusion of the proof of the Theorem. Using Lemma 1, Lemma 3 and formula (4), we see that if the conditions of Lemma 3 are satisfied, then we can choose complex numbers $S_1, S_2, \ldots, S_n$ and $b_1, b_2, \ldots, b_{n-1}$ in such a way that

$$S_l = S_2 = \ldots = S_T = 1 - \alpha$$

and

$$|S_{T+1}| \leq \frac{1}{2}, |S_{T+2}| \leq \frac{1}{2}, \ldots, |S_n| \leq \frac{1}{2},$$

furthermore,

$$S_l + b_1 S_{l-1} + \ldots + b_{l-1} S_1 = 1 + b_1 + \ldots + b_{l-1} - l b_l \quad (1 \leq l \leq n-1),$$

and

$$S_n + b_1 S_{n-1} + \ldots + b_{n-1} S_1 = 1 + b_1 + \ldots + b_{n-1}.$$

Let $z_2, z_3, \ldots, z_n$ be the roots of the polynomial $Z^{n-1} + b_1 Z^{n-2} + \ldots + b_{n-1}$. Then (as we already remarked in the Introduction) for the system $z_1 = 1, z_2, \ldots, z_n$ we have the numbers $S_1, S_2, \ldots, S_n$ as first n power sums.

Observe that if we first choose $\frac{|\alpha|}{\operatorname{Re}\alpha} = 5$, say (so we first fix $|\arg\alpha|$; of course $\operatorname{Re}\alpha > 0$ by this choice), and then fix α in such a way that $|\alpha|$ is a small enough positive number, then for this fixed α the conditions of Lemma 3 (including $|1-\alpha| < 1$) are satisfied, if n is large enough (since $T = \left[\frac{n}{2}\right]$). This proves our Theorem, because $|1 - \alpha| < 1$ (and of course $\frac{1}{2} < 1$).

In this section we assume that $|1-\alpha|<1$, and we consider α to be a fixed number, so the o- and O-symbols may depend on α.

It is easy to see that instead of the upper bound derived in the proof of Lemma 2, one has in fact the relation

$$a_{n,m}=\frac{1}{m}\sum_{r=1}^{\infty}\frac{\alpha^r}{(r-1)!}\left(\log\frac{n}{m}\right)^{r-1}+o\left(\frac{1}{n}\right),$$

so

$$a_{n,m}=\alpha\frac{n^{\alpha}}{m^{\alpha+1}}+o\left(\frac{1}{n}\right)$$

for $T+1\le m<n$. Then, using that

$$\frac{\beta_{m-l}}{n\beta_n}=\frac{1}{m-l}\left(\frac{m-l}{n}\right)^{\alpha}+O\left(\frac{1}{(m-l)^2}\left(\frac{m-l}{n}\right)^{\operatorname{Re}\alpha}\right)$$

for $T+1\le l<m\le n$ by (18), we get (considering separately the cases $m=l$, $l<m<n$ and $m=n$ in the sum (15) defining P_l, and approximating the sum over $l<m<n$ by an integral) for $T+1\le l<n$ that

$$\frac{P_l}{n\beta_n}=\frac{1}{n-l}\left(\frac{n-l}{n}\right)^{\alpha}+\int_l^n\left(\frac{\alpha n^{\alpha}}{m^{\alpha+1}}\right)\left(\frac{1}{m-l}\left(\frac{m-l}{n}\right)^{\alpha}\right)dm+o\left(\frac{1}{n}\right)$$
$$+O\left(\frac{(n-l)^{\operatorname{Re}\alpha-2}}{n^{\operatorname{Re}\alpha}}\right),$$

so, after explicit computation of the integral,

$$\frac{P_l}{n\beta_n}=\frac{1}{l}\left(\frac{n-l}{n}\right)^{\alpha-1}+o\left(\frac{1}{n}\right)+O\left(\frac{(n-l)^{\operatorname{Re}\alpha-2}}{n^{\operatorname{Re}\alpha}}\right)$$

for $T+1\le l<n$. On the other hand, $P_n=1$, and $|n\beta_n|\to\infty$ as $n\to\infty$ by (18) whence $P_n/n\beta_n=o(1)$. Then, if q is also a fixed number, the condition of Lemma 1 is satisfied for large enough n, if

$$\left|1+(1-\alpha)\int_{1/2}^{1}(1-l)^{\alpha-1}\frac{dl}{l}\right|<q\int_{1/2}^{1}(1-l)^{\operatorname{Re}\alpha-1}\frac{dl}{l}. \tag{22}$$

This means that if (22) is true, then

$$\limsup_{n\to\infty}R_n\le\max(|1-\alpha|,q). \tag{23}$$

Now,

$$\int_{1/2}^{1}(1-l)^{A-1}\frac{dl}{l}=\frac{2^{-A}}{A}+\int_{1/2}^{1}(1-l)^{A}\frac{dl}{l}$$

for $\mathrm{Re}A > 0$. We apply this formula for both sides of (22) (with $A = \alpha$ and $A = \mathrm{Re}\,\alpha$, respectively), then we choose $q = |1-\alpha|$ and estimate trivially, obtaining that if we have

$$(24) \qquad |1-\alpha|\left(\frac{1}{\mathrm{Re}\,\alpha} - \frac{1}{|\alpha|}\right) > 2^{\mathrm{Re}\,\alpha},$$

then

$$(25) \qquad \limsup_{n\to\infty} R_n \le |1-\alpha|.$$

We now choose $\alpha = \frac{1+i}{5}$. Then

$$(26) \qquad |1-\alpha|^2 = \frac{17}{25} < \frac{25}{36} = \left(\frac{5}{6}\right)^2.$$

On the other hand, $|1-\alpha| > 4/5$, and

$$\frac{1}{\mathrm{Re}\,\alpha} - \frac{1}{|\alpha|} = 5\frac{2-\sqrt{2}}{2}.$$

Using $\sqrt{2} < 1.42$, we thus have that the left-hand side of (24) is greater than 1.16, and it is easy to check that $(1.16)^5 > 2$, hence (24) is true. So (25) holds, and in view of (26) this means that $R_n < 5/6$ for large enough n.

Addendum. G. Harcos ([H]) made computer work based on formulas (22) and (23), and he found that the value

$$\alpha = 0.56754 + 0.54237i$$

gives

$$\limsup_{n\to\infty} R_n < 0.69368.$$

He also observed that our basic identity (14) (and even a more compact form of it) can be derived also from the inverse Newton-Girard formulas (these formulas express the coefficients of an n-degree polynomial with the first n power sums of its roots).

I am grateful to him for these remarks.

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