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A DUALITY RELATION FOR CERTAIN TRIPLE PRODUCTS OF AUTOMORPHIC FORMS

BY

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ABSTRACT

We prove a Poisson-type summation formula. The new formula is closely related to automorphic forms, since it contains certain triple products of automorphic forms as weights.

1. Introduction

1.1. In order to be able to describe our formula we first introduce some notation concerning automorphic forms. Then, before actually describing the formula, we will give an interpretation of the classical Poisson formula which will help us show that our formula is analogous to the Poisson formula.

1.2. Notation. We denote by H the open upper half plane. We write

$$\Gamma_0(4) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) : c \equiv 0 \pmod 4 \right\}.$$

Let D_4 be a fundamental domain of $\Gamma_0(4)$ on H, let

$$d\mu_z = \frac{dxdy}{y^2}$$

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(this is the $SL(2,\mathbf{R})$-invariant measure on H), and introduce the notation

$$(f_1,f_2)=\int_{D_4} f_1(z)\overline{f_2(z)}d\mu_z.$$

Introduce the hyperbolic Laplace operator of weight l:

$$\Delta_l := y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)-ily\frac{\partial}{\partial x}.$$

For a complex number $z\neq 0$ we set its argument in $(-\pi,\pi]$, and write $\log z = \log|z|+i\arg z$, where $\log|z|$ is real. We define the power z^s for any $s\in\mathbf{C}$ by $z^s=e^{s\log z}$. We write

$$e(x)=e^{2\pi i x}\quad \text{and}\quad (w)_n=\frac{\Gamma(w+n)}{\Gamma(w)},$$

as usual.

For $z\in H$ we write $\theta(z)=\sum_{m=-\infty}^{\infty} e(m^2 z)$, and we define

$$B_0(z):=(\mathrm{Im} z)^{\frac{1}{4}}\,\theta(z)\,. \tag{1.1}$$

If ν is the well-known multiplier system (see e.g. [D], (2.1) for its explicit form), we have

$$B_0(\gamma z)=\nu(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{1/2}B_0(z)\quad \text{for } \gamma\in\Gamma_0(4),$$

where for

$$\gamma=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbf{R})$$

we write $j_\gamma(z)=cz+d$. Note that $\nu^4=1$.

Let $l=\frac{1}{2}+2n$ or $l=2n$ with some integer n. We say that a function f on H is an automorphic form of weight l for $\Gamma=SL(2,\mathbf{Z})$ or $\Gamma_0(4)$ (but, if $l=\frac{1}{2}+2n$, we can take only $\Gamma=\Gamma_0(4)$), if it satisfies, for every $z\in H$ and $\gamma\in\Gamma$, the transformation formula

$$f(\gamma z)=\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^l f(z)$$

in the case $l=2n$,

$$f(\gamma z)=\nu(\gamma)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^l f(z)$$

in the case $l=\frac{1}{2}+2n$, and f has at most polynomial growth in cusps. The operator Δ_l acts on smooth automorphic forms of weight l. We say that f is a Maass form of weight l for Γ, if f is an automorphic form, it is a smooth function, and it is an eigenfunction on H of the operator Δ_l. If a Maass form f has exponential decay at cusps, it is called a cusp form.

Denote by $L_l^2(D_4)$ the space of automorphic forms of weight l for $\Gamma_0(4)$ for which we have $(f, f) < \infty$.

Take $u_{0,1/2} = c_0 B_0$, where c_0 is chosen such that $(u_{0,1/2}, u_{0,1/2}) = 1$. It is not hard to prove (using [Sa], p. 290) that the only Maass form (up to a constant factor) of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{1/2}$-eigenvalue $-\frac{3}{16}$ is B_0, and the other eigenvalues are smaller. Let $u_{j,1/2}$ $(j \geq 0)$ be a Maass form orthonormal basis of the subspace of $L^2_{1/2}(D_4)$ generated by Maass forms; write

$$\Delta_{1/2} u_{j,1/2} = \Lambda_j u_{j,1/2}, \quad \Lambda_j = S_j(S_j - 1), \quad S_j = \frac{1}{2} + iT_j.$$

Then $\Lambda_0 = -\frac{3}{16}$, $\Lambda_j < -\frac{3}{16}$ for $j \geq 1$, and $\Lambda_j \to -\infty$.

For the cusps $a = 0, \infty$ denote by $E_a\left(z, s, \frac{1}{2}\right)$ the Eisenstein series of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$ at the cusp a (for a precise definition see Section 2). As a function of z, it is an eigenfunction of $\Delta_{1/2}$ of eigenvalue $s(s-1)$. If f is an automorphic form of weight $1/2$ and the following integral is absolutely convergent, introduce the notation

$$\zeta_a(f, r) := \int_{D_4} f(z) \overline{E_a\left(z, \frac{1}{2} + ir, \frac{1}{2}\right)} d\mu_z.$$

If $l \geq 1$ is an integer, let $S_{l+\frac{1}{2}}$ be the space of holomorphic cusp forms of weight $l + \frac{1}{2}$ with the multiplier system ν^{1+2l} for the group $\Gamma_0(4)$. Note that $\nu^{1+2l} = \nu$ if and only if l is even.

We will be mainly concerned with the case when l is even. If $k \geq 1$, let $f_{k,1}, f_{k,2}, \ldots, f_{k,s_k}$ be an orthonormal basis of $S_{2k+\frac{1}{2}}$, and write $g_{k,j}(z) = (\mathrm{Im} z)^{\frac{1}{4}+k} f_{k,j}(z)$. We note that $g_{k,j}$ is a Maass cusp form of weight $2k + \frac{1}{2}$, and $\Delta_{2k+\frac{1}{2}} g_{k,j} = \left(k + \frac{1}{4}\right)\left(k - \frac{3}{4}\right) g_{k,j}$ (see [F], formulas (4) and (7)).

We also introduce the Maass operators

$$K_k := (z - \overline{z})\frac{\partial}{\partial z} + k = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + k,$$

$$L_k := (\overline{z} - z)\frac{\partial}{\partial \overline{z}} - k = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - k.$$

For basic properties of these operators see [F], pp. 145–146. We just mention now that if f is a Maass form of weight k, then $K_{k/2} f$ and $L_{k/2} f$ are Maass forms of weight $k + 2$ and $k - 2$, respectively.

1.3. Poisson's summation and our formula. Now, to state the Poisson formula, consider the space of smooth, 1-periodic functions on the real line $\mathbf{R}$, and let $D = d/dx$ be the derivation operator. Then the eigenfunctions of D in this space are the functions $e^{2\pi inx}$, the eigenvalues are $2\pi in$, and these eigenfunctions form an orthonormal basis of the Hilbert space $L^2(\mathbf{Z} \setminus \mathbf{R})$. We parametrize the eigenvalues with the numbers n, these parameters are contained in the set $\mathbf{R}$, and the Poisson formula states that if F is a "nice" function on $\mathbf{R}$ and we write $w(n) = 1$ for every n, then the expression

$$\sum_{n=-\infty}^{\infty} w(n)F(n)$$

remains unchanged if we replace F by G, where G is the Fourier transform of F. We inserted the notation $w(n)$ for the identically 1 function to emphasize the analogy, since in our case we will indeed have nontrivial weights.

In our case, instead of the smooth, 1-periodic functions on $\mathbf{R}$, consider all the smooth automorphic forms on H of any weight $\frac{1}{2} + 2k$, where $k \geq 0$ is any integer. Instead of the eigenfunctions of D, we will consider the eigenfunctions of the operators $\Delta_{2k+\frac{1}{2}}$, $k \geq 0$. In fact, if $k \geq 0$ is fixed, the eigenfunctions of $\Delta_{2k+\frac{1}{2}}$ are almost in a one-to-one correspondence with the eigenfunctions of $\Delta_{2(k+1)+\frac{1}{2}}$ through the Maass operators, except that the eigenfunctions of weight $2(k+1) + \frac{1}{2}$ corresponding to holomorphic forms are annihilated by $L_{(k+1)+\frac{1}{4}}$. Hence, the essentially different eigenfunctions of the operators $\Delta_{2k+\frac{1}{2}}$ (playing a role in the spectral expansion of functions in the spaces $L^2_{2k+\frac{1}{2}}(D_4)$) are the following:

$$u_{j,1/2}\,(j \geq 0), \quad E_a\left(*, \frac{1}{2} + ir, \frac{1}{2}\right)(a = 0, \infty, r \in \mathbf{R}), \quad g_{k,j}\,(k \geq 1, 1 \leq j \leq s_k).$$

If u is one of these functions, we will parametrize its Laplace eigenvalue by a number T such that

$$\Delta_{2k+\frac{1}{2}} u = \left(\frac{1}{2} + iT\right)\left(-\frac{1}{2} + iT\right) u$$

with the suitable k. In particular, this parameter will be

$$T_j \text{ in case of } u_{j,1/2}, \quad r \text{ in case of } E_a\left(*, \frac{1}{2} + ir, \frac{1}{2}\right), \quad i\left(\frac{1}{4} - k\right) \text{ in case of } g_{k,j}.$$

These numbers correspond to the numbers n in Poisson's formula. In our case these parameters are contained (at least with finitely many possible exceptions:

call j exceptional, if $T_j \notin \mathbf{R}$) in the set $\mathbf{R} \cup D^+$, where

$$D^+ = \left\{ i\left(\frac{1}{4} - k\right) : \ k \geq 1 \text{ is an integer} \right\}. \tag{1.2}$$

Now, in fact we prove not just one summation formula, but many formulas: to every pair u_1,u_2 of Maass cusp forms of weight 0 there will correspond a summation formula. So let us fix two such cusp forms. Our formula states that there are some weights $w_{u_1,u_2}(j)$, $w_{u_1,u_2}(a,r)$ and $w_{u_1,u_2}(k,j)$ such that if F is a "nice" function on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ (note that "nice" will mean, in particular, that the continuous part of F, i.e., the restriction of F to $\mathbf{R}$, extends as a holomorphic function to a relatively large strip containing $\mathbf{R}$, so we can speak about $F(T_j)$ even for the exceptional js), then the expression

$$\sum_{j=0}^{\infty} w_{u_1,u_2}(j) F(T_j) + \sum_{a=0,\infty} \int_{-\infty}^{\infty} w_{u_1,u_2}(a,r) F(r)\, dr + \sum_{k=1}^{\infty} \sum_{j=1}^{s_k} w_{u_1,u_2}(k,j) F\left(i\left(\frac{1}{4} - k\right)\right)$$

remains unchanged if we write $\overline{u_2}$ in place of u_1, $\overline{u_1}$ in place of u_2, and we replace F by G, where G is obtained from F by applying a certain integral transform which maps functions on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ again to such functions: this integral transform is a so-called Wilson function transform of type II, which was introduced quite recently by Groenevelt in [G1]. This integral transform plays the role that the Fourier transform played in the case of Poisson's formula. We will speak in more detail about the Wilson function transform of type II in Section 1.5 below. We just mention here that it shares some nice properties of the Fourier transform: it is an isometry on a suitably defined Hilbert space, and it is its own inverse (this last property is true at least on the even functions in the case of the Fourier transform).

The weights w_{u_1,u_2} in the above formula contain very interesting automorphic quantities. We give now only $w_{u_1,u_2}(j)$, since the other weights will be analogous, and everything will be given precisely in the theorem. So we will have for $j \geq 0$ that $w_{u_1,u_2}(j)$ equals

$$\Gamma\left(\frac{3}{4}+iT_j\right)\Gamma\left(\frac{3}{4}-iT_j\right) \int_{D_4} B_0(z)\, u_1(4z)\, \overline{u_{j,\frac{1}{2}}(z)} d\mu_z \overline{\int_{D_4} B_0(z)\, u_2(4z)\, \overline{u_{j,\frac{1}{2}}(z)} d\mu_z}.$$

1.4. Remarks on relations to other works and on possible future work. We have shown above that there is a strong formal analogy between our summation formula and the Poisson summation formula. I guess that this analogy may be deeper; perhaps there is a common generalization of the two formulas. I think that the explanation of this analogy and the proof of further generalization (perhaps even for groups of higher rank) may come from representation theory. Such an approach could be useful also for understanding the appearance of the Wilson function transform of type II in the formula, which is rather mysterious at the moment. A representation theoretic interpretation of this integral transform was given by Groenevelt himself in [G2], but it does not seem to help in the explanation of our formula. However, it is possible that the general method of [R] for proving spectral identities may be useful in better understanding our formula.

Spectral identities having similarities to our result were proved by several authors. We mention, e.g., the concrete identities proved in the above-mentioned paper [R] (as an application of the general method there), and the paper [B-M], whose method of proof based directly on the spectral structure of the space $L^2\left(SL(2,\mathbf{Z})\setminus SL(2,\mathbf{R})\right)$ may be also important in the context of our formula.

But, as far as I see, the nearest relative of our result is an identity suggested by Kuznetsov in [K] and proved by Motohashi in [M]. The weights are different there than in our case, but the structure of the two formulas are very similar. Indeed, on the one hand, the summation is over Laplace-eigenvalues and integers in both cases. On the other hand, in the case of both identities we have the same type of weights on both sides of the given identity. That formula has been successfully applied already to analytic problems (see [Iv], [J]), so perhaps our formula also may be applied along similar lines for the estimation of the weights w_{u_1,u_2}, hence the estimation of triple products, especially in view of the fact that in the case $u_1 = u_2$ the weights are nonnegative.

We mention finally that the weights $w_{u_1,u_2}(j)$ (or rather their absolute values squared) given at the end of Subsection 1.3 are (at least in some cases, and at least conjecturally) closely related to central values of L-functions. Indeed, let us assume that $u_{j,1/2}$ is an eigenfunction of the Hecke operator T_{p^2} (of weight $1/2$) for every prime $p \neq 2$, and that $u_{j,1/2}$ is an eigenfunction of the operator L of eigenvalue 1 (see [K-S] for the definitions of the operators T_{p^2} and L). Assume also that the first Fourier coefficient at ∞ of $u_{j,1/2}$ is nonzero. Then $\mathrm{Shim}u_{j,1/2}$ (the Shimura lift of $u_{j,1/2}$) is defined in [K-S], pp. 196–197. It is a

Maass cusp form of weight 0 which is a simultaneous Hecke eigenform. If u_1 and u_2 are also simultaneous Hecke eigenforms, then by the Theorem of [B] we see that $w_{u_1,u_2}(j)$ is closely related to

$$\int_{SL(2,\mathbf{Z})\backslash H} |u_1(z)|^2 \left(\mathrm{Shim} u_{j,1/2}\right)(z)\, d\mu_z \int_{SL(2,\mathbf{Z})\backslash H} |u_2(z)|^2 \left(\mathrm{Shim} u_{j,1/2}\right)(z)\, d\mu_z,$$

at least if we accept the unproved but likely statement that the sum in (1.4) of [B] is a one-element sum (see Remark 2 of [B] and Remark (a) on p. 197 of [K-S]). Using the formula of Watson (see [W]) we finally get that $|w_{u_1,u_2}(j)|^2$ is closely related to

$$L\left(\frac{1}{2}, u_1 \times u_1 \times \mathrm{Shim} u_{j,1/2}\right) L\left(\frac{1}{2}, u_2 \times u_2 \times \mathrm{Shim} u_{j,1/2}\right).$$

1.5. Wilson function transform of type II. For the statement and for the proof of our result, we need to quote from [G1] the definition and some important properties of the Wilson function transform of type II.

Let t_1 and t_2 be two real numbers, and write

$$a = \frac{1}{4} + it_1, \quad b = \frac{1}{4} + it_2, \quad c = \frac{1}{4} - it_2, \quad d = \frac{3}{4} + it_1, \quad t = \frac{1}{4}.$$

Then this set of parameters is self-dual, i.e., for the dual parameters $\tilde{a}$, $\tilde{b}$, $\tilde{c}$, $\tilde{d}$, $\tilde{t}$ defined in formula (2.6) and Section 5.1 of [G1] we have

$$\tilde{a} = a, \quad \tilde{b} = b, \quad \tilde{c} = c, \quad \tilde{d} = d, \quad \tilde{t} = t.$$

We use the notation $\Gamma(X \pm Y) = \Gamma(X+Y)\Gamma(X-Y)$ and

$$\Gamma(X \pm Y \pm Z) = \Gamma(X+Y+Z)\Gamma(X+Y-Z)\Gamma(X-Y+Z)\Gamma(X-Y-Z),$$

and define

$$H(x) = \frac{\Gamma\left(\frac{1}{4} \pm it_1 \pm ix\right)\Gamma\left(\frac{1}{4} \pm it_2 \pm ix\right)\Gamma\left(\frac{1}{4} \pm ix\right)\Gamma\left(\frac{3}{4} \pm ix\right)}{\pi^2\Gamma(\pm 2ix)} \tag{1.3}$$

and

$$C = \frac{\pi^2}{\Gamma\left(\frac{1}{2} \pm it_1\right)\Gamma\left(\frac{1}{2} \pm it_2\right)}. \tag{1.4}$$

Let D^+ be as in (1.2), and define the measure dh for functions F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ as

$$\int F(x)dh(x) := \frac{C}{2\pi}\int_0^\infty F(x)H(x)dx + iC\sum_{x \in D^+} F(x)\mathrm{Res}_{z=x}H(z).$$

Explicitly, with the notation

$$R_k = \mathrm{Res}_{z=i\left(\frac{1}{4}-k\right)} H\left(z\right),$$

we have (writing $s_j = \frac{1}{2} + it_j$ for $j = 1, 2$)

$$iR_k = \frac{2k-\frac{1}{2}}{\pi^2} \frac{\left|\Gamma\left(k+it_1\right)\right|^2 \left|\Gamma\left(k+it_2\right)\right|^2}{\left|\left(s_1\right)_k\right|^2 \left|\left(s_2\right)_k\right|^2} \Gamma\left(\frac{1}{2} \pm it_1\right) \Gamma\left(\frac{1}{2} \pm it_2\right). \tag{1.5}$$

(Note that there is a mistake in the concrete expression for this residue in Section 5.1 of [G1]; the formula there should be multiplied by $4t^2$, which is $\frac{1}{4}$ in our case.) This formula means, in particular, that dh is a measure.

The Wilson function

$$\phi_\lambda\left(x\right) = \phi_\lambda\left(x; a, b, c, d\right)$$

is defined in [G1], formula (3.2); we use the parameters a, b, c, d given above.

We define the Hilbert space $\mathcal{H} = \mathcal{H}\left(a, b, c, d; t\right)$ to be the space consisting of functions on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$ that have finite norm with respect to the inner product

$$(f, g)_{\mathcal{H}} = \int f(x)\overline{g(x)} dh(x).$$

Then the Wilson function transform of type II is defined in [G1] as

$$\left(\mathcal{G}F\right)\left(\lambda\right) = \int F(x)\phi_\lambda\left(x\right) dh(x).$$

It is defined first (as in the case of the classical Fourier transform) on the dense subspace of $\mathcal{H}$ where this is absolutely convergent. Then it extends to $\mathcal{H}$, and the following nice theorem is proved in [G1], Theorem 5.10:

The operator $\mathcal{G} : \mathcal{H} \to \mathcal{H}$ is unitary, and $\mathcal{G}$ is its own inverse.

In our proof the second statement will be important, i.e., that $\mathcal{G}$ is its own inverse. We mention two more important facts that will be needed. The first one is $\phi_\lambda\left(x\right) = \phi_x\left(\lambda\right)$; see (3.4) of [G1] and remember that our parameters are self-dual. The second one is that $\phi_\lambda\left(x; a, b, c, d\right)$ is symmetric in $a, b, c, 1-d$ (see Remark 4.5 of [G1]), hence that our Wilson function transform is symmetric in t_1 and t_2.

Since we will work separately with the continuous and discrete part of a function F on $\mathbf{R} \cup D^+$, even on $\mathbf{R}$, we introduce notation for them:

$$f(x) := F(x)\ (x \in \mathbf{R}), \quad a_n := F\left(i\left(\frac{1}{4} - n\right)\right)\ (n \geq 1).$$

So instead of F, we will speak about a pair consisting of an even function f on $\mathbf{R}$ and a sequence $\{a_n\}_{n\geq 1}$. In this language, the Wilson function transform of type II of the pair f, $\{a_n\}_{n\geq 1}$ is the pair of the function g and the sequence $\{b_n\}_{n\geq 1}$ defined by

$$g(\lambda) = \frac{C}{2\pi}\int_0^\infty f(x)\phi_\lambda\left(x\right)H(x)dx + iC\sum_{k=1}^\infty a_k\phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right)R_k \tag{1.6}$$

and

$$b_n = \frac{C}{2\pi}\int_0^\infty f(x)\phi_{i\left(\frac{1}{4}-n\right)}\left(x\right)H(x)dx + iC\sum_{k=1}^\infty a_k\phi_{i\left(\frac{1}{4}-n\right)}\left(i\left(\frac{1}{4}-k\right)\right)R_k \tag{1.7}$$

for $n \geq 1$.

1.6. The Theorem. We now state precisely the summation formula. We still need some notation. If u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u = s(s-1)u$, for $n \geq 0$ define a cusp form $\kappa_n(u)$ of weight $2n$ for the group $\Gamma_0\left(4\right)$ by

$$\left(\kappa_n(u)\right)(z) = \frac{\left(K_{n-1}K_{n-2}\cdots K_1K_0u\right)(4z)}{\left(s\right)_n\left(1-s\right)_n}.$$

Theorem: *Let $u_1(z)$ and $u_2(z)$ be two Maass cusp forms of weight 0 for $SL(2,\mathbf{Z})$ with Laplace-eigenvalues $s_j(s_j-1)$, where $s_j = \frac{1}{2}+it_j$ and $t_j > 0$ ($j = 1,2$). There is a positive constant K depending only on u_1 and u_2 such that property $P\left(f,\{a_n\}\right)$ below is true, if $f(x)$ is an even holomorphic function for $|\mathrm{Im}x| < K$ satisfying that*

$$\left|f(x)e^{-2\pi|x|}\left(1+|x|\right)^K\right|$$

is bounded on the domain $|\mathrm{Im}x| < K$, and $\{a_n\}_{n\geq 1}$ is a sequence satisfying that

$$\left|n^{K+\frac{3}{2}}\left(a_n - \frac{(-1)^n}{n^{3/2}}\sum_{0\leq m<K}\frac{c_m}{n^m}\right)\right|$$

is bounded for $n \geq 1$ with some constants c_m (m runs over integers with $0 \leq m < K$).

Property $P(f,\{a_n\})$: *By g and b_n defined in* (1.6) *and* (1.7) *the sum of the following three lines:*

$$\sum_{j=1}^{\infty} f\left(T_j\right)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0\left(u_1\right),u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(u_2\right),u_{j,\frac{1}{2}}\right)}, \tag{1.8}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} f\left(r\right)\Gamma\left(\frac{3}{4}\pm ir\right)\zeta_a\left(B_0\kappa_0\left(u_1\right),r\right)\overline{\zeta_a\left(B_0\kappa_0\left(u_2\right),r\right)}dr, \tag{1.9}$$

$$\sum_{n=1}^{\infty} a_n\Gamma\left(2n+\frac{1}{2}\right)\sum_{j=1}^{s_n}\left(B_0\kappa_n\left(u_1\right),g_{n,j}\right)\overline{\left(B_0\kappa_n\left(u_2\right),g_{n,j}\right)} \tag{1.10}$$

equals the sum of the following three lines:

$$\sum_{j=1}^{\infty} g\left(T_j\right)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0\left(\overline{u_2}\right),u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(\overline{u_1}\right),u_{j,\frac{1}{2}}\right)}, \tag{1.11}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} g\left(r\right)\Gamma\left(\frac{3}{4}\pm ir\right)\zeta_a\left(B_0\kappa_0\left(\overline{u_2}\right),r\right)\overline{\zeta_a\left(B_0\kappa_0\left(\overline{u_1}\right),r\right)}dr, \tag{1.12}$$

$$\sum_{n=1}^{\infty} b_n\Gamma\left(2n+\frac{1}{2}\right)\sum_{j=1}^{s_n}\left(B_0\kappa_n\left(\overline{u_2}\right),g_{n,j}\right)\overline{\left(B_0\kappa_n\left(\overline{u_1}\right),g_{n,j}\right)}. \tag{1.13}$$

The sums and integrals in (1.6) *and* (1.7) *are absolutely convergent for* $|\mathrm{Im}\lambda|<\frac{3}{4}$ *and* $n\geq 1$, *and every sum and integral in* (1.8)–(1.13) *is absolutely convergent.*

The class of functions appearing in the theorem seems to be sufficiently general, but it may happen that the statement can be extended further for some other functions.

Convention: In what follows, u_1 and u_2 (hence t_1 and t_2) will be fixed. So every variable and every constant (including the constants implied in the $\ll$ and O symbols) may depend on u_1 and u_2, even if we do not denote this dependence.

1.7. Sketch of the proof of the Theorem. In this sketch we ignore problems related to convergence; we just give a formal argument. Assume first that the following special case of the Theorem is already proved:

$$f(x)=0\ (x\in\mathbf{R}),\quad a_n=0\ (n\neq N),\quad a_N=1 \tag{1.14}$$

with a fixed positive integer N. Using Groenevelt's result that the Wilson function transform of type II is its own inverse, we can see that this special case (reading it "in the other direction", and making the changes $u_1 \to \overline{u_2}$, $u_2 \to \overline{u_1}$) proves another case of the Theorem:

$$f(x) = \phi_{i\left(\frac{1}{4}-N\right)}(x) \ \ (x \in \mathbf{R}), \quad a_n = \phi_{i\left(\frac{1}{4}-N\right)}\left(i\left(\frac{1}{4}-n\right)\right) \tag{1.15}$$

with a fixed positive integer N.

There is a special case of the Theorem which is easily seen to be true:

$$f(x) = \frac{1}{\Gamma\left(\frac{3}{4} \pm ix\right)} \quad (x \in \mathbf{R}), \quad a_n = 0 \quad (n \geq 1). \tag{1.16}$$

This special case will follow trivially from the spectral theorem for weight 1/2.

It turns out that the general statement can be proved using these three special cases by purely analytical means. This will follow from Lemma 7.4, which implies that a nice enough even function on $\mathbf{R}$ can be written as a linear combination of the functions

$$\frac{1}{\Gamma\left(\frac{3}{4} \pm ix\right)} \quad \text{and} \quad \phi_{i\left(\frac{1}{4}-N\right)}(x) \ \ (N \geq 1).$$

This will mean that if f is a given nice even function on $\mathbf{R}$, then by (1.15) (using it for every integer $N \geq 1$) and (1.16) we can prove that the Theorem is true for this f and for some sequence $\{a_n\}_{n\geq 1}$. But then, using (1.14) for every integer $N \geq 1$, we can achieve any sequence $\{a_n\}_{n\geq 1}$ without changing f. This will complete the proof of the Theorem.

Hence, it is enough to prove the special case (1.14). We now give a sketch of the proof of this special case.

Observe that we have to give an expression for

$$\sum_{j=1}^{s_N} \left(B_0\kappa_N\left(u_1\right), g_{N,j}\right) \overline{\left(B_0\kappa_N\left(u_2\right), g_{N,j}\right)}, \tag{1.17}$$

which is the inner product of the projection of $B_0\kappa_N(u_1)$ and the projection of $B_0\kappa_N(u_2)$ to the space $(\mathrm{Im} z)^{\frac{1}{4}+N} S_{2N+\frac{1}{2}}$. This is in fact the space of Maass cusp forms of weight $2N+\frac{1}{2}$ and $\Delta_{2N+\frac{1}{2}}$-eigenvalue $\left(N+\frac{1}{4}\right)\left(N-\frac{3}{4}\right)$. We will show that this projection operator can be written as an integral operator: if U is a cusp form of weight $2N$ for $\Gamma_0(4)$, then the projection of B_0U to the above-mentioned space is

$$\int_H B_0(z)U(z)m_N(z,w)d\mu_z$$

with a suitable kernel function m_N. We can apply a theorem of Fay (see our Lemma 3.4) to determine the Fourier expansions of B_0 and U on noneuclidean circles around w. Since the behavior of $m_N(z,w)$ on such circles is well understood, we can compute this integral using geodesic polar coordinates around w, and we get that the projection equals

$$\sum_{l=0}^{\infty} C_{U,l} B_l(w) \left(U\right)_{-l}(w),$$

where the coefficients $C_{U,l}$ are explicitly known, and

$$(U)_{-l} = \frac{1}{l!} L_{N-l+1} \cdots L_{N-1} L_N U, \quad B_l = \frac{1}{l!} K_{(l-1)+\frac{1}{4}} \cdots K_{\frac{5}{4}} K_{\frac{1}{4}} B_0.$$

Hence, applying it with $U = \kappa_N(u_1)$ and also with $U = \kappa_N(u_2)$ we see that for the computation of (1.17) we have to compute integrals of the form

$$\int_{D_4} B_{l_1}(w) \left(\kappa_N(u_1)\right)_{-l_1}(w) \overline{B_{l_2}(w) \left(\kappa_N(u_2)\right)_{-l_2}(w)} d\mu_w.$$

We will consider this integral as the inner product of $B_{l_1}\overline{\left(\kappa_N(u_2)\right)_{-l_2}}$ and $B_{l_2}\overline{\left(\kappa_N(u_1)\right)_{-l_1}}$. These are automorphic forms of weight $\frac{1}{2}+2(l_1+l_2-N)$, and we will compute their inner product using the spectral theorem for this weight (in the form of Corollaries 3.1 or 3.2 below). This leads us to a sum of products of triple products of the form

$$\left(B_{l_1}\overline{\left(\kappa_N(u_2)\right)_{-l_2}}, F\right) \overline{\left(B_{l_2}\overline{\left(\kappa_N(u_1)\right)_{-l_1}}, F\right)},$$

where F is a Maass form of weight $\frac{1}{2}+2(l_1+l_2-N)$. Using partial integration (in the form of Lemmas 3.1 and 3.2) it turns out in Lemma 4.3 that these triple products can be written as linear combinations of such triple products which are present in the Theorem.

This reasoning shows relatively easily that we can get some expression for (1.17) with the products of inner products which are present in the Theorem. However, I cannot give a good explanation of the actual form of the relation, i.e., the occurrence of the Wilson function $\phi_\lambda(x)$, besides the fact that this will be the result of the computation.

1.8. Structure of the paper. In Section 2 we introduce some more notation and gather together some well-known preliminary facts. In Section 3 we prove our most important lemmas, then we prove the special case (1.14) of the Theorem in Section 4, and the general case in Section 5. Some remaining lemmas on automorphic functions are proved in Section 6. We gather together some facts related to the function $\phi_\lambda(x)$ in an Appendix. These facts are used in the proof of our Theorem. However, these lemmas are completely independent of automorphic forms; they belong to the area of special functions. Therefore, we state these lemmas here without proof; their proofs will be published elsewhere.

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2. Further notation and preliminaries

Let D_1 be the closure of the standard fundamental domain of $SL(2,\mathbf{Z})$, hence

$$D_1 = \left\{ z \in H : -\frac{1}{2} \leq \mathrm{Re} z \leq \frac{1}{2}, \ |z| \geq 1 \right\}.$$

Then, it is easy to check that the following set is a closure of a fundamental domain of $\Gamma_0(4)$:

$$D_4 = \bigcup_{j=0}^{5} \gamma_j D_1,$$

where

$$\gamma_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} \quad (0 \leq j \leq 3),$$

and

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

In the sequel D_4 will always denote this fixed fundamental domain of $\Gamma_0(4)$.

The three cusps for $\Gamma_0(4)$ are ∞, 0 and $-\frac{1}{2}$. If a denotes one of these cusps, we take a scaling matrix $\sigma_a \in SL(2,\mathbf{R})$ as explained on p. 42 of [I]. We can easily see that one can take

$$\sigma_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 0 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}, \quad \sigma_{-\frac{1}{2}} = \begin{pmatrix} -1 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}.$$

The only cusp for $SL(2,\mathbf{Z})$ is ∞, and, of course, we take the identity matrix σ_∞ for scaling matrix also in this case.

Let L_n^α and $F(\alpha,\beta,\gamma;z)$ be the usual notation for Laguerre polynomials and Gauss hypergeometric functions, respectively; see [G-R], p. 990 and p. 995.

If a is a cusp for $\Gamma_0(4)$, we define χ_a by

$$\nu\left(\sigma_a\begin{pmatrix}1&1\\0&1\end{pmatrix}\sigma_a^{-1}\right)=e(-\chi_a),\quad 0\le\chi_a<1.$$

It is easy to check that $\chi_\infty=\chi_0=0$, and $\chi_{-\frac12}=\frac34$. So the cusps 0 and ∞ are said to be singular, and $-1/2$ is said to be nonsingular.

If f is a Maass form of weight l, and $\Delta_l f=s(s-1)f$ with some $\mathrm{Re}\, s\ge\frac12$, $s=\frac12+it$, and a is a cusp of Γ, then $f(\sigma_a z)\left(\frac{j_{\sigma_a}(z)}{|j_{\sigma_a}(z)|}\right)^{-l}$ has the Fourier expansion

$$c_{f,a}(y)+\sum_{\substack{m\in\mathbf{Z}\\ m-\chi_a\neq0}}\rho_{f,a}(m)W_{\frac{l}{2}\operatorname{sgn}(m-\chi_a),it}\left(4\pi\left|m-\chi_a\right|y\right)e\left(\left(m-\chi_a\right)x\right)$$

for $z=x+iy\in H$ where $W_{\alpha,\beta}$ is the Whittaker function (see [G-R], p. 1014), and $c_{f,a}(y)=0$ if $\chi_a\neq0$, while it is a linear combination of y^s and y^{1-s} for $s\neq\frac12$ and of $y^{1/2}$ and $y^{1/2}\log y$ for $s=\frac12$, if $\chi_a=0$.

Let $P_l(D_4)$ be the space of such smooth automorphic forms of weight l for $\Gamma_0(4)$ for which we have that for any integers $B,C\ge0$ there is an integer $A=A(B,C)$ such that

$$\left(\max_a \operatorname{Im}\sigma_a^{-1}z\right)^{-A}\left|\left(\frac{\partial^B}{\partial x^B}\frac{\partial^C}{\partial y^C}f\right)(z)\right|$$

is bounded on D_4 (i.e., every partial derivative grows at most polynomially near each cusp on the fixed fundamental domain D_4). We denote by $R_l(D_4)$ the space of such smooth automorphic forms of weight l for $\Gamma_0(4)$ for which we have that for any integers $A,B,C\ge0$ the function

$$\left(\max_a \operatorname{Im}\sigma_a^{-1}z\right)^{A}\left|\left(\frac{\partial^B}{\partial x^B}\frac{\partial^C}{\partial y^C}f\right)(z)\right|$$

is bounded on D_4 (i.e., every partial derivative decays faster than polynomially near each cusp on the fixed fundamental domain D_4).

Let

$$\Gamma_\infty=\left\{\gamma\in SL(2,\mathbf{Z}):\gamma\infty=\infty\right\}.$$

For $z, w \in H$ let

$$H(z,w) = i^{\frac{1}{2}} \left(\frac{|z-\overline{w}|}{(z-\overline{w})} \right)^{\frac{1}{2}} = \left(\frac{z-\overline{w}}{w-\overline{z}} \right)^{-\frac{1}{4}} \tag{2.1}$$

(the last equality holds because the fourth powers are the same, and the arguments of both sides lie in $(-\pi/4, \pi/4)$), as on p. 349 of [H]. It is easy to see that for any $T \in SL(2, \mathbf{R})$ we have

$$\frac{H^2(Tz, Tw)}{H^2(z,w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right) \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-1},$$

so

$$\frac{H(Tz, Tw)}{H(z,w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right)^{\frac{1}{2}} \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-\frac{1}{2}}, \tag{2.2}$$

since both sides lie in the right half-plane. Observe also that

$$H(w,z) = \overline{H(z,w)}. \tag{2.3}$$

If $z \in H$ is arbitrary, let $T_z \in PSL(2, \mathbf{R})$ be such that T_z is an upper triangular matrix and $T_z i = z$. It is clear that T_z is uniquely determined by z; for $z = x + iy$ we have explicitly

$$T_z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{-1}{2}} \\ 0 & y^{\frac{-1}{2}} \end{pmatrix}.$$

If $z \in H$ is fixed, the function $(\operatorname{Im} z)^{\frac{1}{4}} \theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}}$ is holomorphic for $|L| < 1$, so it has a Taylor expansion

$$(\operatorname{Im} z)^{\frac{1}{4}} \theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} B_n(z) L^n. \tag{2.4}$$

We defined in this way a function $B_n(z)$ $(z \in H)$ for every $n \geq 0$. For $n = 0$ this is in accordance with (1.1).

For $\gamma_1, \gamma_2 \in SL(2, \mathbf{R})$, we define

$$w(\gamma_1, \gamma_2) = j_{\gamma_1}(\gamma_2 z)^{1/2} j_{\gamma_2}(z)^{1/2} j_{\gamma_1\gamma_2}(z)^{-1/2};$$

the right-hand side is indeed independent of $z \in H$. Clearly $w = \pm 1$.

For $a=0,\infty$, $\mathrm{Re}s>1$, $z\in H$ and any integer n, define (Γ_a denotes the stability group of a in $\Gamma_0(4)$)

$$E_a\left(z,s,\frac{1}{2}+2n\right)=\sum_{\gamma\in\Gamma_a\backslash\Gamma_0(4)}\overline{\nu(\gamma)w\left(\sigma_a^{-1},\gamma\right)}(\mathrm{Im}\,\sigma_a^{-1}\gamma z)^s\left(\frac{j_{\sigma_a^{-1}\gamma}(z)}{\left|j_{\sigma_a^{-1}\gamma}(z)\right|}\right)^{-\frac{1}{2}-2n}.$$

It follows from [F], formula (5) on p. 145 that for $n\geq 0$ we have

$$E_a\left(z,s,\frac{1}{2}+2n\right)=c_n\left(s\right)K_{n-\frac{3}{4}}\cdots K_{\frac{5}{4}}K_{\frac{1}{4}}E_a\left(z,s,\frac{1}{2}\right),$$

for $n\leq 0$ we have

$$E_a\left(z,s,\frac{1}{2}+2n\right)=c_n\left(s\right)L_{\frac{5}{4}+n}\cdots L_{-\frac{3}{4}}L_{\frac{1}{4}}E_a\left(z,s,\frac{1}{2}\right)$$

(of course s is fixed and we apply the operators in z), where

$$c_n\left(s\right)=\prod_{l=0}^{n-1}\frac{1}{s+\frac{1}{4}+l}$$

for $n\geq 0$, and

$$c_n\left(s\right)=\prod_{l=0}^{-n-1}\frac{1}{s-\frac{1}{4}+l}$$

for $n\leq 0$.

It is known that for every z the function $E_a\left(z,s,\frac{1}{2}\right)$ has a meromorphic continuation in s to the whole plane, and this function is regular at every point s with $\mathrm{Re}s=\frac{1}{2}$.

If $j\geq 0$ and $n\geq 0$ are integers, define

$$u_{j,\frac{1}{2}+2n}(z)=c_{j,n}\left(K_{n-\frac{3}{4}}\cdots K_{\frac{5}{4}}K_{\frac{1}{4}}u_{j,\frac{1}{2}}\right)(z);$$

if $j\geq 1$ and $n<0$, define

$$u_{j,\frac{1}{2}+2n}(z)=c_{j,n}\left(L_{\frac{5}{4}+n}\cdots L_{-\frac{3}{4}}L_{\frac{1}{4}}u_{j,\frac{1}{2}}\right)(z),$$

where the numbers $c_{j,n}$ are chosen in such a way that $(u_{j,\frac{1}{2}+2n},u_{j,\frac{1}{2}+2n})=1$, and, of course, $c_{j,0}=1$ for every $j\geq 0$. We see by [F], pp. 145–146 that this is possible; we have $\Delta_{\frac{1}{2}+2n}u_{j,\frac{1}{2}+2n}=S_j(S_j-1)u_{j,\frac{1}{2}+2n}$, and for a fixed n the functions $u_{j,\frac{1}{2}+2n}$ ($j\geq 0$ for $n\geq 0$, and $j\geq 1$ for $n<0$) form an orthonormal system in $L^2_{\frac{1}{2}+2n}(D_4)$. We also see by (11) of [F] that

$$|c_{j,n}|^2=\frac{1}{\left(S_j+\frac{1}{4}\right)_n\left(\frac{5}{4}-S_j\right)_n} \tag{2.5}$$

for $n \geq 0$, and

$$|c_{j,n}|^2 = \frac{1}{\left(S_j - \frac{1}{4}\right)_{-n}\left(\frac{3}{4} - S_j\right)_{-n}} \tag{2.6}$$

for $n \leq 0$. In this case we used also the general identity

$$\overline{K_k g} = L_{-k}\overline{g}, \tag{2.7}$$

and we will use frequently (and sometimes tacitly) this identity throughout the paper.

For $n \geq k \geq 1$ and $1 \leq j \leq s_k$, let

$$g_{k,j,n} = c_{k,j,n} K_{n-\frac{3}{4}} \cdots K_{k+\frac{5}{4}} K_{k+\frac{1}{4}} g_{k,j},$$

where $c_{k,j,n}$ is chosen such that $(g_{k,j,n}, g_{k,j,n}) = 1$. By [F], pp. 145–146 this is possible, $\Delta_{2n+\frac{1}{2}} g_{k,j,n} = \left(k+\frac{1}{4}\right)\left(k-\frac{3}{4}\right) g_{k,j,n}$, and for a fixed $n > 0$ the functions

$$\left\{u_{j,\frac{1}{2}+2n} :\ j \geq 0\right\} \cup \{g_{k,j,n} :\ 1 \leq k \leq n, 1 \leq j \leq s_k\}$$

form an orthonormal system in $L^2_{\frac{1}{2}+2n}(D_4)$. We also see by (11) of [F] that

$$|c_{k,j,n}|^2 = \frac{1}{\left(2k+\frac{1}{2}\right)_{n-k}(n-k)!} \tag{2.8}$$

for $n \geq k \geq 1$ and $1 \leq j \leq s_k$.

We will make several times a transition to geodesic polar coordinates: if $z_0 \in H$ is fixed, then for every $z \in H$ we can uniquely write

$$\frac{z - z_0}{z - \overline{z_0}} = \tanh\left(\frac{r}{2}\right)e^{i\phi} \tag{2.9}$$

with $r > 0$ and $0 \leq \phi < 2\pi$. The invariant measure is expressed in these new coordinates as $d\mu_z = \sinh r dr d\phi$.

3. Basic lemmas

3.1. Partial integration. We prove here two simple lemmas, but they will play an important role in the proof of the Theorem, as mentioned in Subsection 1.7.

Lemma 3.1: *Let* $f_1 \in P_{2m_1}(D_4)$ *and* $f_2 \in P_{2m_2}(D_4)$ *with* $m_1 + m_2 = \frac{3}{4}$, *and assume that at least one of* $f_1 \in R_{2m_1}(D_4)$ *and* $f_2 \in R_{2m_2}(D_4)$ *is true. Then we have*

$$\int_{D_4} B_0(z) \left(L_{m_1} f_1\right)(z) f_2(z) d\mu_z = -\int_{D_4} B_0(z) f_1(z) \left(L_{m_2} f_2\right)(z) d\mu_z.$$

Proof. By (9) of [F] (we use a slight extension of that formula, because our functions are not of compact support, but the rapid decay at cusps is sufficient) and (2.7) we have

$$\int_{D_4} B_0(z) \left(L_{\frac{3}{4}}(f_1 f_2)\right)(z) d\mu_z = -\int_{D_4} \left(L_{\frac{1}{4}} B_0\right)(z) \left(f_1 f_2\right)(z) d\mu_z.$$

The right-hand side here is 0, since $L_{\frac{1}{4}} B_0 = 0$ by (4) of [F]. On the other hand,

$$\left(L_{m_1+m_2}(f_1 f_2)\right)(z) = \left(L_{m_1} f_1\right)(z) f_2(z) + f_1(z) \left(L_{m_2} f_2\right)(z)$$

by the definitions, and this proves the lemma. ∎

In the next lemma we deal with the functions B_n defined in (2.4); the basic properties of these functions are given in Lemma 6.1 in Section 6.

Lemma 3.2: *Let* $l \geq 0$ *be an integer, let* $f \in P_{2m}(D_4)$ *and* $g \in P_{2n}(D_4)$ *with* $m+n = -\frac{1}{4} - l$, *and assume that at least one of* $f \in R_{2m}(D_4)$ *and* $g \in R_{2n}(D_4)$ *is true. Then*

$$\int_{D_4} B_l(z) f(z) g(z) d\mu_z$$

equals

$$\frac{(-1)^l}{l!} \sum_{L=0}^{l} \binom{l}{L} \int_{D_4} B_0(z) \left(K_{m+L-1} \cdots K_{m+1} K_m f\right)(z) \times \left(K_{n+l-L-1} \cdots K_{n+1} K_n g\right)(z) d\mu_z.$$

Proof. Using (6.2), and formula (9) of [F] (a slight extension of that formula again), we easily get that

$$\int_{D_4} B_l(z) f(z) g(z) d\mu_z = \frac{(-1)^l}{l!} \int_{D_4} B_0(z) \left(K_{-\frac{5}{4}} \cdots K_{-l+\frac{3}{4}} K_{-l-\frac{1}{4}}(fg)\right)(z) d\mu_z.$$

Using the general identity

$$\left(K_{m_1+m_2}(f_1 f_2)\right)(z) = \left(K_{m_1} f_1\right)(z) f_2(z) + f_1(z) \left(K_{m_2} f_2\right)(z)$$

several times, we get the lemma. ∎

3.2. Inner product of two automorphic forms of weight $\frac{1}{2}+2n$. Here n is any integer. First we give the spectral decomposition of an $f \in R_{\frac{1}{2}+2n}(D_4)$ in Lemma 3.3: in the case $n \geq 0$ we give a complete spectral decomposition (Lemma 3.3 (i)), in the case $n < 0$ a slightly less complete statement will be enough for our purposes (Lemma 3.3 (ii)). We then give two corollaries describing the inner product of two forms. We again give a complete statement in the case $n \geq 0$ (Corollary 3.1); in the case $n < 0$ (Corollary 3.2) the vanishing property (3.3) will suffice instead of a detailed spectral expression for the inner product.

Every statement here is more or less standard, therefore we just give brief indications of the proofs.

Lemma 3.3: *Let n be an integer, and $f \in R_{\frac{1}{2}+2n}(D_4)$. Write*

$$\zeta_a(f,r) := \int_{D_4} f(z)\overline{E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}+2n\right)}d\mu_z$$

for $a = 0,\infty$ and real r. Define

$$g_f = f - \sum_{j=j_0}^{\infty}(f,u_{j,\frac{1}{2}+2n})u_{j,\frac{1}{2}+2n} - \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f,r)E_a\left(*,\frac{1}{2}+ir,\frac{1}{2}+2n\right)dr,$$

where $j_0 = 0$ for $n \geq 0$, and $j_0 = 1$ for $n < 0$.

(i) *If $n \geq 0$, we have*

$$g_f = \sum_{k=1}^{n}\sum_{j=1}^{s_k}(f,g_{k,j,n})g_{k,j,n}. \tag{3.1}$$

(ii) *If $n < 0$, we have*

$$\overline{g_f} = \sum_{k=1}^{-n}K_{-n-\frac{5}{4}}\cdots K_{k+\frac{3}{4}}K_{k-\frac{1}{4}}G_{k,n}, \tag{3.2}$$

where $G_{k,n}(z) = (\operatorname{Im} z)^{-\frac{1}{4}+k}H_{k,n}(z)$ with some $H_{k,n} \in S_{2k-\frac{1}{2}}$.

Remarks on the proof. The case $n = 0$ (where the statement in (3.1) is $g_f = 0$) is well-known, and follows, e.g., from [P], formula (27). For larger $|n|$ we can prove the statements by induction, applying the suitable operator L for the left-hand side of (3.1) and (3.2), and applying [F], formula (4).

Corollary 3.1: *If $f_1, f_2 \in R_{\frac{1}{2}+2n}(D_4)$, then for $n \geq 0$ we have that (f_1, f_2) equals the sum of*

$$\sum_{j=0}^{\infty}(f_1, u_{j,\frac{1}{2}+2n})\overline{(f_2, u_{j,\frac{1}{2}+2n})} + \sum_{k=1}^{n}\sum_{j=1}^{s_k}(f_1, g_{k,j,n})\overline{(f_2, g_{k,j,n})}$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f_1, r)\overline{\zeta_a(f_2, r)}dr.$$

Moreover, we have that the sum of

$$\sum_{j=0}^{\infty}\left|(f_1, u_{j,\frac{1}{2}+2n})\overline{(f_2, u_{j,\frac{1}{2}+2n})}\right| + \sum_{k=1}^{n}\sum_{j=1}^{s_k}\left|(f_1, g_{k,j,n})\overline{(f_2, g_{k,j,n})}\right|$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|\zeta_a(f_1, r)\overline{\zeta_a(f_2, r)}\right|dr$$

is $\leq \left(\int_{D_4}|f_1(z)|^2\,d\mu_z\right)^{\frac{1}{2}}\left(\int_{D_4}|f_2(z)|^2\,d\mu_z\right)^{\frac{1}{2}}$.

Remarks on the proof. The expression for (f_1, f_2) follows at once from Lemma 3.3 (i). The inequality of the lemma follows by Cauchy's inequality.

Corollary 3.2: *If $n < 0$ and $f \in R_{\frac{1}{2}+2n}(D_4)$, then we have ($g_f$ is defined in Lemma 3.3)*

$$K_{-\frac{3}{4}}\cdots K_{n+\frac{5}{4}-r}K_{n+\frac{1}{4}-r}L_{n+\frac{5}{4}-r}\cdots L_{-\frac{3}{4}+n}L_{\frac{1}{4}+n}g_f = 0 \tag{3.3}$$

for every integer $r \geq 0$. If h is another element of $R_{\frac{1}{2}+2n}(D_4)$, then (f, h) equals

$$(g_f, h) + \sum_{j=1}^{\infty}(f, u_{j,\frac{1}{2}+2n})\overline{(h, u_{j,\frac{1}{2}+2n})} + \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\zeta_a(f, r)\overline{\zeta_a(h, r)}dr, \tag{3.4}$$

and

$$|(g_f, h)| + \sum_{j=1}^{\infty}\left|(f, u_{j,\frac{1}{2}+2n})\overline{(h, u_{j,\frac{1}{2}+2n})}\right| + \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|\zeta_a(f, r)\overline{\zeta_a(h, r)}\right|dr$$

is $\leq \left(\int_{D_4}|f(z)|^2\,d\mu_z\right)^{\frac{1}{2}}\left(\int_{D_4}|h(z)|^2\,d\mu_z\right)^{\frac{1}{2}}$.

Remarks on the proof. We see by (2.7) and Lemma 3.3 (ii) that for the proof of (3.3) it is enough to show that

$$L_{\frac{3}{4}}\cdots L_{k-\frac{1}{4}}\left(L_{k+\frac{3}{4}}\cdots L_{-n-\frac{1}{4}+r}K_{-n-\frac{5}{4}+r}\cdots K_{k-\frac{1}{4}}G_{k,n}\right) = 0$$

for every $1 \le k \le -n$. This is true by (8) and (4) of [F], so (3.3) follows. Formula (3.4) follows at once from the definition of g_f. Lemma 3.3 (ii) easily implies

$$(g_f, h) = (g_f, g_h),$$

and then the inequality follows from (3.4) and Cauchy's inequality.

3.3. Fourier expansion of Laplace-eigenforms on noneuclidean circles. We reproduce here an important theorem of Fay, which will be applied several times in the paper.

Lemma 3.4: *Let $k \in \mathbf{R}$, $s \in \mathbf{C}$, and let f be a smooth function on H satisfying $\Delta_{2k} f = s(s-1) f$. If $z_0 \in H$ is given, then for every $z \in H$ we have the absolutely convergent expansion*

$$f(z)\left(\frac{z-\overline{z_0}}{z_0-\overline{z}}\right)^k = \sum_{n=-\infty}^{\infty} (f)_n(z_0) P_{s,k}^n(z,z_0) e^{in\phi}, \tag{3.5}$$

where $r = r(z,z_0) > 0$ and $0 \le \phi = \phi(z,z_0) < 2\pi$ are determined from z by (2.9), and

$$\begin{aligned} P_{s,k}^n(z,z_0) = \\ \left(\tanh\left(\frac{r}{2}\right)\right)^{|n|}\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{k_n} F(s-k_n, 1-s-k_n, 1+|n|, -y)\end{aligned} \tag{3.6}$$

with

$$y = \frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}, \quad k_n = k\frac{n}{|n|} \text{ for } n \ne 0, \quad k_0 = \pm k,$$

$$n!\,(f)_n(z_0) = (K_{k+n-1}\cdots K_{k+1}K_k f)(z_0) \quad \text{for } n \ge 0,$$

$$\begin{aligned}(-n)!\,(f)_n(z_0) &= \overline{(K_{-k-n-1}\cdots K_{-k+1}K_{-k}\overline{f})}(z_0) \\ &= (L_{k+n+1}\cdots L_{k-1}L_k f)(z_0) \quad \text{for } n \le 0.\end{aligned} \tag{3.7}$$

Proof. This follows from Theorems 1.1 and 1.2 of [F]. Formula (3.6) is formally different from (13) of [F], but the right-hand side of (3.6) equals

$$\left(\tanh\left(\frac{r}{2}\right)\right)^{|n|}\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{s} F\left(s-k_n, s+|n|+k_n, 1+|n|, \tanh^2\left(\frac{r}{2}\right)\right)$$

by [G-R], p. 998, 9.131.1. For the second equality in (3.7) we use again (2.7). We remark that for a fixed $r > 0$ the left-hand side of (3.5) is a smooth 2π-periodic function of $\phi \in \mathbf{R}$ (z is determined from ϕ by (2.9)), and the right-hand side is its Fourier expansion, hence it is absolutely convergent. The lemma is proved. ∎

4. Proof of the theorem in a special case

Let $N \geq 1$ be an integer. Our aim in this section is to prove the following special case. See the Theorem for property $P(f, \{a_n\})$.

Lemma 4.1: *Property $P(f, \{a_n\})$ is true if f is identically zero, $a_n = 0$ for $n \neq N$, and $a_N = 1$. We have the estimates*

(4.1)
$$\sum_{j=1}^{\infty} \left| \phi_{T_j} \left(i \left(\frac{1}{4} - N \right) \right) \Gamma \left(\frac{3}{4} \pm i T_j \right) \left(B_0 \kappa_0 \left(\overline{u_2} \right), u_{j,\frac{1}{2}} \right) \overline{\left(B_0 \kappa_0 \left(\overline{u_1} \right), u_{j,\frac{1}{2}} \right)} \right| \leq C N^D,$$

(4.2)
$$\sum_{a=0,\infty} \int_{-\infty}^{\infty} \left| \phi_r \left(i \left(\frac{1}{4} - N \right) \right) \Gamma \left(\frac{3}{4} \pm i r \right) \zeta_a \left(B_0 \kappa_0 \left(\overline{u_2} \right), r \right) \overline{\zeta_a \left(B_0 \kappa_0 \left(\overline{u_1} \right), r \right)} dr \right| \leq C N^D,$$

(4.3)
$$\sum_{k=1}^{\infty} \sum_{j=1}^{s_k} \left| \phi_{i\left(\frac{1}{4}-k\right)} \left(i \left(\frac{1}{4} - N \right) \right) \Gamma \left(2k + \frac{1}{2} \right) \left(B_0 \kappa_k \left(\overline{u_2} \right), g_{k,j} \right) \overline{\left(B_0 \kappa_k \left(\overline{u_1} \right), g_{k,j} \right)} \right| \leq C N^D,$$

with positive constants C and D depending only on u_1, u_2.

In the proof of the general case of the theorem the upper bounds (4.1)–(4.3) will be important.

4.1. Projection to the space $S_{2N+\frac{1}{2}}$. We first construct a kernel function, then we show that the integral operator with this kernel function maps $B_0 U$ (if U is a cusp form of weight $2N$ for $\Gamma_0(4)$) into $S_{2N+\frac{1}{2}}$; finally, we expand this image of $B_0 U$ in our given basis of $S_{2N+\frac{1}{2}}$.

Write
$$k_N(y) = (1+y)^{-N-\frac{1}{4}}, \quad H_N(z,w) = H(z,w)^{4N+1},$$

where $H(z,w)$ is defined in (2.1), and for $z,w\in H$ define

$$k_N(z,w)=k_N\left(\frac{|z-w|^2}{4\operatorname{Im}z\operatorname{Im}w}\right)H_N(z,w)$$

and

$$K_N(z,w)=\sum_{\gamma\in\Gamma_0(4)}k_N(\gamma z,w)\overline{\nu(\gamma)}\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{-\frac{1}{2}-2N};$$

this sum can be seen to be absolutely convergent. It is not hard to check that if $w\in H$ is fixed, then for every $\delta\in\Gamma_0(4)$ and $z\in H$ we have

$$K_N(\delta z,w)=\nu(\delta)\left(\frac{j_\delta(z)}{|j_\delta(z)|}\right)^{\frac{1}{2}+2N}K_N(z,w). \tag{4.4}$$

Let U be a cusp form of weight $2N$ for $\Gamma_0(4)$ with $\Delta_{2N}U=s(s-1)U$. Then we may define

$$F_U(w)=(\operatorname{Im}w)^{-N-\frac{1}{4}}\int_{D_4}B_0(z)U(z)\overline{K_N(z,w)}d\mu_z \tag{4.5}$$

for $w\in H$. We claim that $F_U\in S_{2N+\frac{1}{2}}$. We remark first that it is not hard to check using (2.2) and (2.3) that

$$K_N(w,z)=\overline{K_N(z,w)}. \tag{4.6}$$

So the required transformation property of F_U follows at once from (4.4). It is not hard to check that $(\operatorname{Im}w)^{-N-\frac{1}{4}}k_N(w,z)$ is holomorphic in w for every z, using the identity

$$4\operatorname{Im}z\operatorname{Im}w+|z-w|^2=|z-\overline{w}|^2, \tag{4.7}$$

and then the same is true for $(\operatorname{Im}w)^{-N-\frac{1}{4}}K_N(w,z)$, using

$$\frac{\operatorname{Im}w}{|j_\gamma(w)|^2}=\operatorname{Im}\gamma w. \tag{4.8}$$

Hence $F_U(w)$ is holomorphic. It remains to check the behavior at cusps, i.e., that

$$\left|F_U(\sigma_a w)\left(j_{\sigma_a}(w)\right)^{-2N-\frac{1}{2}}\right|\to 0$$

as $\operatorname{Im}w\to\infty$ for each of the three cusps (in the case of $a=-\frac{1}{2}$ much less would be enough in fact, but it can be proved easily). To see this, we use the trivial estimate

$$|K_N(z,w)|\le\sum_{\gamma\in\Gamma_0(4)}k_N\left(\frac{|\gamma z-w|^2}{4\operatorname{Im}\gamma z\operatorname{Im}w}\right),$$

and the fact that $|B_0(z)U(z)|$ is bounded in z. These bounds together with the definition of $k_N(y)$ imply that the integral in (4.5) is bounded in w, and then the factor $(\operatorname{Im} w)^{-N-\frac{1}{4}}$ assures the required estimate (taking into account (4.8)). Hence indeed, $F_U \in S_{2N+\frac{1}{2}}$.

Consider the inner product

$$\int_{D_4} (\operatorname{Im} w)^{2N+\frac{1}{2}} F_U(w)\overline{f_{N,j}(w)} d\mu_w \tag{4.9}$$

for some $1 \le j \le \mathrm{s}_N$. This is easily seen to be absolutely convergent as a double integral (see (4.5)). Using (4.6) we see by unfolding for any $z \in D_4$ that

(4.10)

$$\int_{D_4} \overline{\overline{K_N(z,w)} f_{N,j}(w) (\operatorname{Im} w)^{N+\frac{1}{4}}} d\mu_w = 2\int_H k_N(w,z) \overline{f_{N,j}(w) (\operatorname{Im} w)^{N+\frac{1}{4}}} d\mu_w.$$

We use geodesic polar coordinates around z:

$$\frac{w-z}{w-\overline{z}} = \tanh\left(\frac{r}{2}\right)e^{i\phi},$$

and since (using (4.7) and the definition of $k_N(y)$) we have

$$\frac{1}{1-\tanh^2(\frac{r}{2})} = \frac{|w-\overline{z}|^2}{4\operatorname{Im} z \operatorname{Im} w} \quad \text{and} \quad k_N\left(\frac{|z-w|^2}{4\operatorname{Im} z \operatorname{Im} w}\right)=\left(\frac{|z-\overline{w}|^2}{4\operatorname{Im} z \operatorname{Im} w}\right)^{-N-\frac{1}{4}},$$

so (taking into account the definition of $k_N(w,z)$ and $H_N(w,z)$) we see that (4.10) equals

$$2i^{\frac{1}{2}+2N}\left(\frac{1}{4\operatorname{Im} z}\right)^{\frac{1}{4}+N} \int_0^\infty \left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{2N+\frac{1}{2}} \overline{\left(\int_0^{2\pi} F_r(\phi)d\phi\right)} \sinh r dr,$$

where we write

$$F_r(\phi) = (w-\overline{z})^{\frac{1}{2}+2N} f_{N,j}(w)$$

using the explicit expression for w in terms of r and ϕ:

$$w = \frac{z-\overline{z}\tanh(\frac{r}{2})Z}{1-\tanh(\frac{r}{2})Z} \quad \text{with } Z := e^{i\phi}.$$

For fixed $0 < r < \infty$ and $z \in D_4$ this last expression is a regular function of Z (with values in H) in a domain containing the unit circle, hence by Cauchy's formula the inner integral is $2\pi(z-\overline{z})^{\frac{1}{2}+2N} f_{N,j}(z)$, so (4.10) equals (recall $g_{N,j}(z) = (\operatorname{Im} z)^{N+\frac{1}{4}} f_{N,j}(z)$)

$$4\pi\overline{g_{N,j}(z)} \int_0^\infty \left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{2N+\frac{1}{2}} \sinh r dr.$$

The integral can be computed; its value is $4/(4N-1)$. So by (4.5) we get that (4.9) equals

$$\frac{16\pi}{4N-1}\int_{D_4} B_0(z)U(z)\overline{g_{N,j}(z)}d\mu_z.$$

Since the functions $f_{N,j}$ form an orthonormal basis of $S_{2N+\frac{1}{2}}$, this implies for any $w \in H$ that

$$F_U(w)\,(\operatorname{Im} w)^{N+\frac{1}{4}} = \frac{16\pi}{4N-1}\sum_{j=1}^{s_N}\left(\int_{D_4} B_0(z)U(z)\overline{g_{N,j}(z)}d\mu_z\right) g_{N,j}(w). \tag{4.11}$$

4.2. Computation in geodesic polar coordinates. We now compute the left-hand side of (4.11) in another way: by unfolding the right-hand side of (4.5). Up to some point, we continue working with a general cusp form U of weight $2N$ for $\Gamma_0(4)$, but then we will specialize to $U = \kappa_N(u)$, where u is a cusp form of weight 0 for $SL(2, \mathbf{Z})$.

By unfolding we see that

$$\int_{D_4} B_0(z)U(z)\overline{K_N(z,w)}d\mu_z = 2\int_H B_0(z)U(z)\overline{k_N(z,w)}d\mu_z \tag{4.12}$$

for any fixed $w \in H$. The integrand here can be written as (see (2.1))

$$\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)\left(U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}\right)k_N\left(\frac{|z-w|^2}{4\operatorname{Im} z\operatorname{Im} w}\right).$$

We now use geodesic polar coordinates around w:

$$\frac{z-w}{z-\overline{w}} = \tanh\left(\frac{r}{2}\right)e^{i\phi},$$

and using the substitution

$$y = \frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$$

we get that (4.12) equals

$$4\int_0^\infty k_N(y)\left(\int_0^{2\pi}\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)\left(U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}\right)d\phi\right)dy, \tag{4.13}$$

where $0 < r = r(y) < \infty$ and $z = z(y,\phi) \in H$ are determined from y and ϕ by the relations above.

For every fixed y we will now compute the inner integral by the Fourier expansions of the two functions there, and then we will integrate in y. To justify this computation, we remark that if

$$B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}=\sum_{l=-\infty}^{\infty}a_l(y)e^{il\phi}\quad\text{and}\quad U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}=\sum_{l=-\infty}^{\infty}b_l(y)e^{il\phi},$$

then for any y, by Cauchy's inequality in l and Parseval's formula in ϕ, we have that (the implied constant in $\ll$ below is absolute)

$$\sum_{l=-\infty}^{\infty}|a_l(y)b_{-l}(y)|\ll\left(\int_0^{2\pi}|B_0(z)|^2\,d\phi\right)^{\frac{1}{2}}\left(\int_0^{2\pi}|U(z)|^2\,d\phi\right)^{\frac{1}{2}},$$

hence by Cauchy's inequality in y we get that

$$\int_0^{\infty}k_N(y)\sum_{l=-\infty}^{\infty}|a_l(y)b_{-l}(y)|\,dy$$

is

$$\ll\left(\int_0^{\infty}k_N(y)\int_0^{2\pi}|B_0(z)|^2\,d\phi dy\right)^{\frac{1}{2}}\left(\int_0^{\infty}k_N(y)\int_0^{2\pi}|U(z)|^2\,d\phi dy\right)^{\frac{1}{2}},$$

which is (making backwards the steps leading from (4.12) to (4.13))

$$\ll M_U(w):=\left(\int_{D_4}K_N^*(z,w)\,|B_0(z)|^2\,d\mu_z\right)^{\frac{1}{2}}\left(\int_{D_4}K_N^*(z,w)\,|U(z)|^2\,d\mu_z\right)^{\frac{1}{2}}$$

with implied absolute constant, where

$$K_N^*(z,w)=\sum_{\gamma\in\Gamma_0(4)}k_N\left(\frac{|\gamma z-w|^2}{4\operatorname{Im}\gamma z\operatorname{Im}w}\right).$$

We get an upper bound for this by extending the summation for $\gamma\in SL(2,\mathbf{Z})$, and then we can see by Lemma 6.3 (using (6.9) and (6.10) for fixed z_1) and the concrete form of k_N that $K_N^*(z,w)$ is bounded in z, so $M_U(w)$ is a finite number for every fixed w, hence we can compute (4.13) as we described above.

We now compute (4.13) explicitly for a given w. By Lemma 3.4 and (6.2), taking into account that $L_{1/4}B_0=0$, we get

$$B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}=\sum_{l=0}^{\infty}\left(\tanh\left(\frac{r}{2}\right)\right)^l\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{\frac{1}{4}}B_l(w)e^{il\phi},$$

and again by Lemma 3.4 we have

$$U(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{N}=\sum_{m=-\infty}^{\infty}(U)_m(w)P^m_{s,N}(z,w)e^{im\phi}$$

with the functions $(U)_m$ defined in Lemma 3.4; we will determine them explicitly later. Using (3.6) we get for any $l\geq 0$ that (recall $y=\frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$)

$$\int_0^\infty k_N(y)\left(\tanh\left(\frac{r}{2}\right)\right)^l\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{\frac{1}{4}}P^{-l}_{s,N}(z,w)dy$$

equals

$$\int_0^\infty y^l(1+y)^{-\frac{1}{2}-l}F(s+N,1-s+N,1+l,-y)\,dy,$$

and by [G-R], p. 807, 7.512.10 the value of this integral is

$$\frac{\Gamma(1+l)\Gamma\left(s-\frac{1}{2}+N\right)\Gamma\left(-s+\frac{1}{2}+N\right)}{\Gamma\left(\frac{1}{2}+l\right)\Gamma\left(\frac{1}{2}+2N\right)}.$$

So $F_U(w)(\operatorname{Im}w)^{N+\frac{1}{4}}$ equals (using (4.5), (4.12) and (4.13))

$$8\pi\frac{\Gamma\left(s-\frac{1}{2}+N\right)\Gamma\left(-s+\frac{1}{2}+N\right)}{\Gamma\left(\frac{1}{2}+2N\right)}\sum_{l=0}^{\infty}\frac{\Gamma(1+l)}{\Gamma\left(\frac{1}{2}+l\right)}B_l(w)(U)_{-l}(w). \tag{4.14}$$

It remains to determine $(U)_{-l}(w)$. By (3.7) for every $l\geq 0$ we have

$$\overline{(U)_{-l}(w)}=\frac{1}{l!}\left(K_{-N+l-1}\cdots K_{-N+1}K_{-N}\left(\overline{U}\right)\right)(w). \tag{4.15}$$

We now assume that $U=\kappa_N(u)$, where u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u=s(s-1)u$, $s=\frac{1}{2}+it$ and $t\geq 0$. Using (4.15), the definition of $\kappa_n(u)$, (2.7) and [F], p. 145, formula (8), we get that

$$(U)_{-l}(w)=\frac{(-1)^l}{l!}(\kappa_{N-l}(u))(w)\quad\text{for } 0\leq l\leq N, \tag{4.16}$$

and then it follows by induction on the basis of (4.15) that

$$\overline{(U)_{-l}(w)}=\frac{(-1)^N}{l!}\left(K_{-N+l-1}\cdots K_1K_0(\overline{u})\right)(4w)\quad\text{for } l\geq N. \tag{4.17}$$

We note a consequence of (4.16) and (4.17), which will be useful later: by the definition of $\kappa_n(u)$ and by [F], formula (11) we can check for every $l\geq 0$ with $v=u$ or $v=\overline{u}$ that

$$\left|(U)_{-l}(w)\right|=\left|\frac{(s-N)_l}{l!\,(s)_N}\left(\frac{1}{(s)_{|l-N|}}K_{|l-N|-1}\cdots K_1K_0(v)\right)(4w)\right|. \tag{4.18}$$

4.3. The inner product of two projections. We now consider two cusp forms of weight $2N$ for $\Gamma_0(4)$, and we substitute the results of the previous two subsections. For proving convergence, we need an upper bound lemma.

Let $U_j(z) = (\kappa_N(u_j))(z)$ for $j = 1, 2$, where u_1, u_2 are as in the Theorem. Then U_1 and U_2 are two cusp forms of weight $2N$ for $\Gamma_0(4)$ with $\Delta_{2N} U_j = s_j(s_j - 1) U_j$ $(j = 1, 2)$, and we have by (4.11), applying it for $U = U_1$ and also for $U = U_2$, that

$$\sum_{j=1}^{s_N} \int_{D_4} B_0(z) U_1(z) \overline{g_{N,j}(z)} d\mu_z \overline{\int_{D_4} B_0(z) U_2(z) \overline{g_{N,j}(z)} d\mu_z} \tag{4.19}$$

equals

$$\left(\frac{4N-1}{16\pi}\right)^2 \int_{D_4} F_{U_1}(w) \left(\operatorname{Im} w\right)^{N+\frac{1}{4}} \overline{F_{U_2}(w) \left(\operatorname{Im} w\right)^{N+\frac{1}{4}}} d\mu_w.$$

Using (4.14) twice in this last expression, we then see that (4.19) equals

$$\frac{\prod_{i=1}^{2} \left(\Gamma\left(s_i - \frac{1}{2} + N\right) \Gamma\left(-s_i + \frac{1}{2} + N\right)\right)}{\Gamma^2\left(-\frac{1}{2} + 2N\right)} \sum_{l_1, l_2 = 0}^{\infty} \frac{\Gamma\left(1 + l_1\right)}{\Gamma\left(\frac{1}{2} + l_1\right)} \frac{\Gamma\left(1 + l_2\right)}{\Gamma\left(\frac{1}{2} + l_2\right)} I_{l_1, l_2}, \tag{4.20}$$

where I_{l_1,l_2} is defined by

$$I_{l_1, l_2} = \int_{D_4} \left(B_{l_1}(w) \overline{\left(U_2\right)_{-l_2}(w)}\right) \overline{\left(B_{l_2}(w) \overline{\left(U_1\right)_{-l_1}(w)}\right)} d\mu_w \tag{4.21}$$

(this depends also on U_1 and U_2, of course, but we do not denote it). This computation is justified by the next lemma, which will be used also later.

Lemma 4.2: *We have*

$$J := \sum_{l_1, l_2 = 0}^{\infty} \frac{\Gamma\left(1 + l_1\right)}{\Gamma\left(\frac{1}{2} + l_1\right)} \frac{\Gamma\left(1 + l_2\right)}{\Gamma\left(\frac{1}{2} + l_2\right)} J_{l_1, l_2} \le \frac{1}{\Gamma^2\left(\frac{1}{2} + N\right)} D_1 N^{D_2} 2^{2N}$$

with some positive constants D_1, D_2 depending only on u_1 and u_2, where

$$J_{l_1, l_2} := \left(\int_{D_4} \left|B_{l_1}(w) \left(U_2\right)_{-l_2}(w)\right|^2 d\mu_w\right)^{\frac{1}{2}} \left(\int_{D_4} \left|B_{l_2}(w) \left(U_1\right)_{-l_1}(w)\right|^2 d\mu_w\right)^{\frac{1}{2}}.$$

Proof. Let $1 < K < 3/2$ be fixed. Clearly J_{l_1,l_2} is at most

$$\frac{(1+l_2)^K}{(1+l_1)^K}\int_{D_4}\left|B_{l_1}(w)\,(U_2)_{-l_2}(w)\right|^2 d\mu_w \\ + \frac{(1+l_1)^K}{(1+l_2)^K}\int_{D_4}\left|B_{l_2}(w)\,(U_1)_{-l_1}(w)\right|^2 d\mu_w.$$

Hence, by Lemma 6.4 and (4.18) we have, using $K > 1$, that

$$J \ll_{u_1,u_2} \sum_{i=1}^{2}\left|\frac{1}{(s_i)_N}\right|^2 \sum_{l=0}^{\infty}(1+l)^{\frac{1}{2}+K}\left|\frac{(s_i-N)_l}{l!}\right|^2 \log^2(2+|l-N|).$$

Using $N \geq 1$, $K < 3/2$, by simple estimates (using, e.g., also the summation formula for $F(\alpha,\beta,\gamma;1)$; see [G-R], p. 998, 9.122.1) and Stirling's formula we obtain the lemma. ∎

4.4. Inner products $(B_{l_1}\overline{(U_2)_{-l_2}}, F)$. For the computation of I_{l_1,l_2} (see (4.21)) using Corollaries 3.1 and 3.2, we give expressions for such inner products, mostly with Maass forms F (see (i), (ii) and (iii) of Lemma 4.3 below), but because of Corollary 3.2 we need such inner products also for some automorphic F which are not Laplace eigenfunctions (see (iv) of Lemma 4.3).

Let U_2 be as in Subsection 4.3. The definition of the constants $c_{j,r}$ and $c_{k,j,r}$ can be found above formulas (2.5) and (2.8), respectively. During the proof we will use several times tacitly (2.7) and the general fact that if $\Delta_l g = s(s-1)g$, then $\Delta_{-l}\overline{g} = \overline{s}(\overline{s}-1)\overline{g}$.

Lemma 4.3: *Let $l_1, l_2 \geq 0$, and $m = \frac{1}{4} + (l_1 + l_2 - N)$. Introduce the notation*

$$A_{L_1}(S) = \overline{\Gamma\left(\frac{1}{4}+S\right)}\frac{(-1)^{L_1+l_2}}{l_2!}\frac{\overline{\Gamma(s_2-N+l_2+L_1)}}{\overline{\Gamma(s_2+N-l_2-L_1)}}\frac{\overline{(S-m)_{l_1-L_1}}}{\overline{\Gamma(S+m-l_1+L_1)}}.$$

Let $F \in P_{2m}(D_4)$ satisfy the conditions of (i), (ii), (iii) *or* (iv) *below. Then*

$$\int_{D_4} B_{l_1}(w)\overline{(U_2)_{-l_2}(w)F(w)}d\mu_w = \frac{(-1)^{l_1}}{l_1!}\sum_{L_1=0}^{l_1}\binom{l_1}{L_1}J_{L_1}(F), \tag{4.22}$$

where $J_{L_1}(F)$ is given in the various cases as follows.

(i) *If $F = u_{j,2m}$, where $j \geq 0$ for $m > 0$, and $j \geq 1$ for $m < 0$, then for every $0 \leq L_1 \leq l_1$ we have that $J_{L_1}(F)$ equals*

$$A_{L_1}(S_j)\,\overline{c_{j,l_1+l_2-N}}\left(\overline{S_j}+\frac{1}{4}\operatorname{sgn}\left(m-\frac{1}{4}\right)\right)_{\left|m-\frac{1}{4}\right|}\int_{D_4}B_0(w)\overline{u_2(4w)u_{j,\frac{1}{2}}(w)}d\mu_w.$$

(ii) *If* $F = E_a(*, s, 2m)$ *with* $a = 0$ *or* ∞, $\mathrm{Re}\, s = \frac{1}{2}$, *then for every* $0 \leq L_1 \leq l_1$ *we have that*

$$J_{L_1}(F) = A_{L_1}(s) \int_{D_4} B_0(w) \overline{u_2(4w)} \overline{E_a\left(w, s, \frac{1}{2}\right)} d\mu_w.$$

(iii) *If* $F = g_{k,j,l_1+l_2-N}$ *with some* $1 \leq k \leq l_1 + l_2 - N$, $1 \leq j \leq s_k$, *then for every* $0 \leq L_1 \leq l_1$ *we have that*

$$J_{L_1}(F) = A_{L_1}\left(k+\frac{1}{4}\right) \overline{c_{k,j,l_1+l_2-N}} \left(k+\frac{1}{2}\right)_{m-\frac{1}{4}} \int_{D_4} B_0(w) \left(\kappa_k\left(\overline{u_2}\right)\right)(w) \overline{g_{k,j}(w)} d\mu_w.$$

(iv) *If* $m < 0$, *and* $F \in P_{2m}(D_4)$ *is such that*

$$K_{-\frac{3}{4}} \cdots K_{m-r+1} K_{m-r} L_{m+1-r} \cdots L_{m-1} L_m F = 0$$

for every integer $r \geq 0$, *then for every* $0 \leq L_1 \leq l_1$ *we have that* $J_{L_1}(F) = 0$.

Proof. First we assume only $F \in P_{2m}(D_4)$. By Lemma 3.2 we see that (4.22) holds with

$$(4.23) \quad J_{L_1}(F) = \left(B_0, \left(L_{N-l_2-L_1+1} \cdots L_{N-l_2-1} L_{N-l_2} (U_2)_{-l_2}\right) \left(L_{m-l_1+L_1+1} \cdots L_{m-1} L_m F\right)\right),$$

(the right-hand side denotes an inner product on D_4). It is clear by (4.15) that

$$(4.24) \qquad L_{N-l_2-L_1+1} \cdots L_{N-l_2-1} L_{N-l_2} (U_2)_{-l_2} = \frac{(l_2 + L_1)!}{l_2!} (U_2)_{-l_2-L_1}.$$

For the computation of $J_{L_1}(F)$ we now distinguish between two cases.

Case I. We assume $l_2 + L_1 \leq N$. Then we see by (4.24) and (4.16) that

$$\overline{\left(L_{N-l_2-L_1+1} \cdots L_{N-l_2-1} L_{N-l_2} (U_2)_{-l_2}\right)(w)}$$

equals

$$\frac{(-1)^N}{l_2!} \frac{\overline{\Gamma(s_2 - N + l_2 + L_1)}}{\overline{\Gamma(s_2 + N - l_2 - L_1)}} \left(L_{1-N+l_2+L_1} \cdots L_{-1} L_0 \overline{u_2}\right)(4w).$$

Hence, using Lemma 3.1, we see that if $l_2 + L_1 \leq N$, then $J_{L_1}(F)$ equals

$$(4.25) \qquad \frac{(-1)^{l_2+L_1}}{l_2!} \frac{\overline{\Gamma(s_2 - N + l_2 + L_1)}}{\overline{\Gamma(s_2 + N - l_2 - L_1)}} \int_{D_4} B_0(w) \overline{(u_2)(4w)} F_{l_1,L_1}(w) d\mu_w,$$

where we write

$$(4.26)\quad F_{l_1,L_1} := L_{\frac{3}{4}} \cdots L_{-m+l_1-L_1-1} L_{-m+l_1-L_1} \left(\overline{L_{m-l_1+L_1+1} \cdots L_{m-1} L_m F}\right).$$

By (4.25) and (4.26) we get (iv) of the lemma at once (since if $m < 0$, then we are in Case I for every $L_1 \le l_1$).

Assume that F is a Maass form, and $\Delta_{2m} F = S(S-1)F$. Then, applying (8) of [F], we see that if $l_1 + l_2 \ge N \ge l_2 + L_1$, then

$$(4.27)\qquad F_{l_1,L_1} = \frac{\overline{\Gamma\left(S+\frac{1}{4}\right)}\,\overline{\Gamma\left(S-m+l_1-L_1\right)}}{\overline{\Gamma\left(S-\frac{1}{4}\right)}\,\overline{\Gamma\left(S+m-l_1+L_1\right)}} \overline{L_{\frac{5}{4}} \cdots L_{m-1} L_m F};$$

if $l_1 + l_2 < N$, then

$$(4.28)\qquad F_{l_1,L_1} = \frac{\overline{\Gamma\left(S+m\right)\Gamma\left(S-m+l_1-L_1\right)}}{\overline{\Gamma\left(S-m\right)\Gamma\left(S+m-l_1+L_1\right)}} L_{\frac{3}{4}} \cdots L_{-m-1} L_{-m} \left(\overline{F}\right).$$

And, using (8) and (4) of [F], by (4.25), (4.27) and (4.28) we get, checking every case, that (i), (ii) and (iii) are true for the case $l_2 + L_1 \le N$. (In case (iii) we have that (4.27) is 0, and also $A_{L_1}\left(k+\frac{1}{4}\right) = 0$.)

Case II. Assume now that $l_2 + L_1 > N$. In this case, we need to consider F only of the following form: $F = K_{m-1} K_{m-2} \cdots K_{\frac{5}{4}+t} K_{\frac{1}{4}+t} F_0$ with an integer $0 \le t \le l_1 + l_2 - N$ and a Maass form F_0 of weight $\frac{1}{2} + 2t$ for $\Gamma_0(4)$, such that we have $t = 0$ or $L_{\frac{1}{4}+t} F_0 = 0$. Let $\Delta_{\frac{1}{2}+2t} F_0 = S(S-1)F_0$. It is clear, using (4) and (8) of [F], that if $l_2 + L_1 - N < t$ (hence $m - l_1 + L_1 + 1 \le \frac{1}{4} + t \le m$ and $t > 0$), then

$$(4.29)\qquad L_{m-l_1+L_1+1} \cdots L_{m-1} L_m F = 0.$$

If $l_2 + L_1 - N \ge t$, then $L_{m-l_1+L_1+1} \cdots L_{m-1} L_m F$ equals (by (8) of [F])

$$\frac{\Gamma\left(S-\frac{1}{4}-l_2-L_1+N\right)\Gamma\left(S+m\right)}{\Gamma\left(S+\frac{1}{4}+l_2+L_1-N\right)\Gamma\left(S-m\right)} K_{-\frac{3}{4}+l_2+L_1-N} \cdots K_{\frac{5}{4}+t} K_{\frac{1}{4}+t} F_0,$$

and so, by (4.23), Lemma 3.1 and (4.24), $J_{L_1}(F)$ equals

$$(4.30)\qquad (-1)^{l_2+L_1-N-t} \frac{\overline{\Gamma\left(S-\frac{1}{4}-l_2-L_1+N\right)}\,\overline{\Gamma\left(S+m\right)}}{\overline{\Gamma\left(S+\frac{1}{4}+l_2+L_1-N\right)}\,\overline{\Gamma\left(S-m\right)}} \int_{D_4} B_0(w) V_{l_2,L_1}(w) \overline{F_0(w)} d\mu_w,$$

where we write

$$V_{l_2,L_1} := \frac{(l_2+L_1)!}{l_2!} L_{t+1} \cdots L_{-N+l_2+L_1-1} L_{-N+l_2+L_1} \left(\overline{(U_2)_{-l_2-L_1}}\right).$$

Since $l_2 + L_1 > N$, by (4.17) we get

$$\overline{(U_2)_{-l_2-L_1}}(w) = \frac{(-1)^N}{(l_2+L_1)!}\left(K_{-N+l_2+L_1-1}\cdots K_1K_0\left(\overline{u_2}\right)\right)(4w),$$

hence, again by (8) of [F], for $l_2 + L_1 - N \geq t$ we get
(4.31)

$$V_{l_2,L_1}(w) = \frac{(-1)^N}{l_2!}\overline{\frac{\Gamma(s_2-t)\Gamma(s_2-N+l_2+L_1)}{\Gamma(s_2+t)\Gamma(s_2+N-l_2-L_1)}}\left(K_{t-1}\cdots K_1K_0\left(\overline{u_2}\right)\right)(4w).$$

By (4.23), (4.29), (4.30) and (4.31), checking every case, we get that (i), (ii) and (iii) are true also for $l_2 + L_1 > N$. (In case (iii) and $l_2 + L_1 - N < k$ we have that (4.29) is 0, and also $A_{L_1}\left(k+\frac{1}{4}\right) = 0$.) The lemma is proved. ∎

4.5. Expression for the sum in (4.20). We first compute I_{l_1,l_2} (see (4.21)) on the basis of the previous subsection, using Corollary 3.1 for the case $l_1 + l_2 \geq N$, and Corollary 3.2 for $l_1 + l_2 < N$. Then we substitute the obtained expressions into (4.20).

We first note that

$$\int_{D_4} B_{l_2}(w)\overline{(U_1)_{-l_1}(w)F(w)}d\mu_w \tag{4.32}$$

is the same as the left-hand side of (4.22), if we use the substitutions $l_1 \leftrightarrow l_2$, $U_1 \leftrightarrow U_2$. Hence we can compute also (4.32) using Lemma 4.3.

As in Lemma 4.3, write

$$m = \frac{1}{4} + l_1 + l_2 - N.$$

In fact we should write $m = m_{l_1,l_2}$ to indicate the dependence on l_1 and l_2 (note that N is fixed), but for simplicity we use just the notation m.

In the case $l_1 + l_2 \geq N$, by Corollary 3.1 and (i), (ii) and (iii) of Lemma 4.3, using also (2.5) and (2.8) we get that I_{l_1,l_2} equals the sum of

$$\sum_{j=0}^{\infty} C_{l_1,l_2,j}(v_2, u_{j,\frac{1}{2}})\overline{(v_1, u_{j,\frac{1}{2}})} + \sum_{k=1}^{l_1+l_2-N}\sum_{j=1}^{s_k} C_{l_1,l_2}(k,j)(v_{2,k}, g_{k,j})\overline{(v_{1,k}, g_{k,j})}$$

and

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} C_{l_1,l_2}(r)\zeta_a(v_2,r)\overline{\zeta_a(v_1,r)}dr,$$

where we write

$$v_i = v_{i,0}, \quad v_{i,k} = B_0\kappa_k\left(\overline{u_i}\right) \quad (i = 1, 2 \text{ and } k = 0, 1, 2, \ldots),$$

and the coefficients are defined as follows:

$$C_{l_1,l_2,j} = D_{l_1,l_2,0}(S_j), \tag{4.33}$$

$$C_{l_1,l_2}(k,j) = D_{l_1,l_2,k}\left(k+\frac{1}{4}\right), \tag{4.34}$$

$$C_{l_1,l_2}(r) = D_{l_1,l_2,0}\left(\frac{1}{2}+ir\right), \tag{4.35}$$

with the notation (for general S)

$$D_{l_1,l_2,k}(S) = \frac{\Gamma\left(S+\frac{1}{4}+k\right)\Gamma\left(\frac{5}{4}-S+k\right)}{(l_1!)^2\,(l_2!)^2}\frac{\overline{\Gamma\left(S+m\right)}}{\Gamma\left(1-S+m\right)}\Sigma_{l_1,l_2}(S), \tag{4.36}$$

$$\Sigma_{l_1,l_2}(S) = \sum_{L_1=0}^{l_1}\sum_{L_2=0}^{l_2}(-1)^{L_1+L_2}\binom{l_1}{L_1}\binom{l_2}{L_2}G(S,l_1,l_2,L_1,L_2), \tag{4.37}$$

where $G(S,l_1,l_2,L_1,L_2)$ denotes

$$\overline{\frac{\overline{\Gamma\left(s_2-N+l_2+L_1\right)}}{\Gamma\left(s_2+N-l_2-L_1\right)}}\frac{\Gamma\left(s_1-N+l_1+L_2\right)}{\Gamma\left(s_1+N-l_1-L_2\right)} \tag{4.38}$$
$$\times\overline{\frac{\overline{(S-m)_{l_1-L_1}}}{\Gamma\left(S+m-l_1+L_1\right)}}\frac{(S-m)_{l_2-L_2}}{\Gamma\left(S+m-l_2+L_2\right)}.$$

In the case $l_1+l_2<N$, we apply Corollary 3.2 for the choices $f(w) = B_{l_2}(w)\overline{(U_1)_{-l_1}(w)}$, $h(w) = B_{l_1}(w)\overline{(U_2)_{-l_2}(w)}$. Applying (iv) of Lemma 4.3 and (3.3) we obtain that

$$\int_{D_4} B_{l_1}(w)\overline{(U_2)_{-l_2}(w)g_f(w)}d\mu_w = 0.$$

Then using (i) and (ii) of Lemma 4.3, after some calculations we obtain from Corollary 3.2 (using also (2.6) and the fact that $\mathrm{Re}S_j = \frac{1}{2}$ or S_j is real) that I_{l_1,l_2} equals

$$\sum_{j=1}^{\infty} C_{l_1,l_2,j}(v_2,u_{j,\frac{1}{2}})\overline{(v_1,u_{j,\frac{1}{2}})} + \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} C_{l_1,l_2}(r)\zeta_a(v_2,r)\overline{\zeta_a(v_1,r)}dr$$

for $l_1+l_2<N$, with the above notation.

Then, using that $(v_1, u_{0,\frac{1}{2}}) = 0$ by Lemma 6.6, combining the cases $l_1+l_2 \geq N$ and $l_1 + l_2 < N$, we get that

$$\sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} I_{l_1,l_2} \tag{4.39}$$

equals the sum of

$$\sum_{j=1}^{\infty} C_j (v_2, u_{j,\frac{1}{2}}) \overline{(v_1, u_{j,\frac{1}{2}})} + \sum_{k=1}^{\infty} \sum_{j=1}^{s_k} C(k,j) (v_{2,k}, g_{k,j}) \overline{(v_{1,k}, g_{k,j})} \tag{4.40}$$

and

$$\frac{1}{4\pi} \sum_{a=0,\infty} \int_{-\infty}^{\infty} C(r) \zeta_a(v_2, r) \overline{\zeta_a(v_1, r)} dr, \tag{4.41}$$

where

$$C_j = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2,j}, \tag{4.42}$$

$$C(k,j) = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2}(k,j), \tag{4.43}$$

$$C(r) = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} C_{l_1,l_2}(r) \tag{4.44}$$

(in the case of $C(k,j)$ we used that the factor $1/\Gamma\left(1-S+m\right)$ in (4.34) is 0, if $k > l_1 + l_2 - N$, since $S = k + \frac{1}{4}$). The reordering of the sum is justified by Lemma 4.2 and the inequalities in Corollaries 3.1 and 3.2, and we also see by these statements that if

$$C_j^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2,j}\right|,$$

$$C(k,j)^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2}(k,j)\right|,$$

$$C(r)^* = \sum_{l_1,l_2=0}^{\infty} \frac{\Gamma\left(1+l_1\right)}{\Gamma\left(\frac{1}{2}+l_1\right)} \frac{\Gamma\left(1+l_2\right)}{\Gamma\left(\frac{1}{2}+l_2\right)} \left|C_{l_1,l_2}(r)\right|,$$

then with a constant D_2 depending only on u_1, u_2 we have

$$(4.45)\quad \begin{aligned}\sum_{j=1}^{\infty} C_j^* \left|(v_2, u_{j,\frac{1}{2}})\overline{(v_1, u_{j,\frac{1}{2}})}\right| + \sum_{k=1}^{\infty}\sum_{j=1}^{s_k} C(k,j)^* \left|(v_{2,k}, g_{k,j})\overline{(v_{1,k}, g_{k,j})}\right| \\ \ll_{u_1,u_2} \frac{N^{D_2} 2^{2N}}{\Gamma^2\left(\frac{1}{2}+N\right)}\end{aligned}$$

and

$$(4.46)\quad \frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty} C(r)^* \left|\zeta_a(v_2, r)\overline{\zeta_a(v_1, r)}\right| dr \ll_{u_1,u_2} \frac{N^{D_2} 2^{2N}}{\Gamma^2\left(\frac{1}{2}+N\right)}.$$

We can compute C_j, $C(k,j)$ and $C(r)$ by formulas (4.36)–(4.38) and Lemma 7.1, using (4.42) and (4.33) in the case of C_j, (4.43) and (4.34) in the case of $C(k,j)$, and finally (4.44) and (4.35) in the case of $C(r)$. Then, on the one hand, by (4.19), (4.20), (4.39)–(4.41) and (1.4), (1.5) we get the property $P(f, \{a_n\})$ required in Lemma 4.1; on the other hand, by (4.45) and (4.46) we obtain also the upper bounds (4.1)–(4.3), so Lemma 4.1 is proved. ∎

5. Proof of the general case of the theorem

5.1. Some upper bounds. Lemma 7.2 (i) and (4.3) with $N=1$ implies that

$$(5.1)\quad \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}\Gamma\left(2k+\frac{1}{2}\right)\sum_{j=1}^{s_k}\left|\left(B_0\kappa_k\left(\overline{u_2}\right), g_{k,j}\right)\overline{\left(B_0\kappa_k\left(\overline{u_1}\right), g_{k,j}\right)}\right| < \infty.$$

We now prove that there is a constant $A>0$ depending only on u_1 and u_2 such that

$$(5.2)\quad \sum_{j=1}^{\infty} e^{\pi|T_j|}\left(1+|T_j|\right)^{-A}\left|\left(B_0\kappa_0\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)}\right| < \infty,$$

$$(5.3)\quad \sum_{a=0,\infty}\int_{-\infty}^{\infty}\left|e^{\pi|r|}\left(1+|r|\right)^{-A}\zeta_a\left(B_0\kappa_0\left(\overline{u_2}\right), r\right)\overline{\zeta_a\left(B_0\kappa_0\left(\overline{u_1}\right), r\right)}\right| dr < \infty.$$

To prove this, let k be a large positive integer. It follows from Lemma 7.4 and elementary linear algebra that if $M>0$ is large enough in terms of k, then there is a nonzero vector $(a_m)_{M\le m\le 2M}$ such that for

$$f(x) := \sum_{m=M}^{2M}\frac{a_m}{\Gamma^2\left(m\pm ix\right)}$$

formula (7.4) is true and the coefficients e_j in (7.5) are 0, so we have

$$f(x)=\sum_{N=1}^{\infty} d_N \phi_{i\left(\frac{1}{4}-N\right)}(x)$$

with some coefficients $d_N=O\left(N^{-k}\right)$. If k is large enough in terms of the constant D in (4.1), we get, combining (4.1) for different integers N with coefficients d_N, that

$$\sum_{j=1}^{\infty}\left|f\left(T_j\right)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0\left(\overline{u_2}\right),u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(\overline{u_1}\right),u_{j,\frac{1}{2}}\right)}\right|<\infty,$$

and similarly for Eisenstein series on the basis of (4.2). By the definition of f and Stirling's formula this proves the estimates (5.2) and (5.3).

5.2. A consequence of Lemma 4.1. It is clear, in view of the upper bounds (4.1)–(4.3), that if $\{C_N\}_{N\geq 1}$ is a rapidly decreasing sequence, then we can take the linear combination of the cases of Lemma 4.1 with these coefficients, since everything is absolutely convergent. We will now show that we can take such a linear combination even in some cases when $\{C_N\}_{N\geq 1}$ is not so rapidly decreasing.

Lemma 5.1: *For every A with $\operatorname{Re} A\geq \frac{5}{2}$ we have that*

$$\sum_{n=1}^{\infty}(-1)^n\frac{(1-A)_{n-1}}{\Gamma(n)}\frac{\left|(s_1)_n\right|^2\left|(s_2)_n\right|^2\Gamma\left(2n-\frac{1}{2}\right)}{\left|\Gamma\left(n+it_1\right)\right|^2\left|\Gamma\left(n+it_2\right)\right|^2}\times\sum_{j=1}^{s_n}\left(B_0\kappa_n\left(u_1\right),g_{n,j}\right)\overline{\left(B_0\kappa_n\left(u_2\right),g_{n,j}\right)} \tag{5.4}$$

equals the sum of the following three expressions (see Lemma 7.3 for the definition of $M_\lambda(A)$):

$$\sum_{j=1}^{\infty}M_{T_j}(A)\Gamma\left(\frac{3}{4}\pm iT_j\right)\left(B_0\kappa_0\left(\overline{u_2}\right),u_{j,\frac{1}{2}}\right)\overline{\left(B_0\kappa_0\left(\overline{u_1}\right),u_{j,\frac{1}{2}}\right)}, \tag{5.5}$$

$$\frac{1}{4\pi}\sum_{a=0,\infty}\int_{-\infty}^{\infty}M_r(A)\Gamma\left(\frac{3}{4}\pm ir\right)\zeta_a\left(B_0\kappa_0\left(\overline{u_2}\right),r\right)\overline{\zeta_a\left(B_0\kappa_0\left(\overline{u_1}\right),r\right)}dr, \tag{5.6}$$

$$\sum_{k=1}^{\infty}M_{i\left(\frac{1}{4}-k\right)}(A)\Gamma\left(2k+\frac{1}{2}\right)\sum_{j=1}^{s_k}\left(B_0\kappa_k\left(\overline{u_2}\right),g_{k,j}\right)\overline{\left(B_0\kappa_k\left(\overline{u_1}\right),g_{k,j}\right)}, \tag{5.7}$$

and every sum and integral is absolutely convergent here for every such number A.

Proof. By formulas (1.4)–(1.13) we see that the identity of this lemma is obtained formally by taking a linear combination of the identities of Lemma 4.1 with coefficients $(-1)^N \frac{(1-A)_{N-1}}{iCR_N\Gamma(N)}$. It follows from (1.4), (1.5) and Lemma 4.1 that if $\operatorname{Re} A$ is large enough (depending on u_1 and u_2), then the statement of the present lemma is true (note, in particular, that (5.5) and (5.6) are absolutely convergent if $\operatorname{Re} A$ is large enough). We extend this result to $\operatorname{Re} A \geq 5/2$ by analytic continuation and continuity.

It follows from (5.1) (applying it with $\overline{u_1}$ in place of u_2, and $\overline{u_2}$ in place of u_1, which is possible; these are also fixed cusp forms) that (5.4) extends regularly to $\operatorname{Re} A > \frac{5}{2}$ and extends continuously to $\operatorname{Re} A \geq \frac{5}{2}$. The same assertions are true for (5.7) using Lemma 7.3 (ii) and (5.1).

We claim that the same assertions are true for (5.5) and (5.6) too, but the proof in this case is more complicated. Take any compact subset L of the half-plane $\operatorname{Re} A \geq \frac{5}{2}$, and let K be a large but fixed integer. Take the integer $t > 0$, complex numbers $A_1, A_2, \dots, A_t$ and polynomials $Q_1, Q_2, \dots, Q_t$ as in Lemma 7.3 (iii). Define for $\operatorname{Re} A \geq \frac{5}{2}$ and $|\operatorname{Im} \lambda| < \frac{3}{4}$ (taking into account Lemma 7.3 (i))

$$S_\lambda(A) = M_\lambda(A) - \sum_{i=1}^{t} 2^{A-A_i} Q_i\,(A)\, M_\lambda(A_i). \tag{5.8}$$

We see by (5.2) and Lemma 7.3 (iii) that if K is large enough depending on u_1 and u_2, and we write $S_{T_j}(A)$ in place of $M_{T_j}(A)$ in (5.5), then the sum in T_j will be uniformly absolutely convergent for $A \in L$, and the resulting function of A will be regular on every open subset of L. The same is true for (5.6) if we write $S_r(A)$ in place of $M_r(A)$ there. We have seen in the first paragraph of the proof of the present lemma that (5.5) and (5.6) are absolutely convergent if we write any A_i in place of A (since K is large enough depending on u_1 and u_2 and $\operatorname{Re} A_i > K$). Hence, expressing $M_{T_j}(A)$ and $M_r(A)$ from (5.8), we finally prove that (5.5) and (5.6) are uniformly absolutely convergent for $A \in L$, and the resulting functions are regular on every open subset of L.

By analytic continuation and continuity, these considerations prove the lemma. ∎

5.3. Conclusion. We now finish the proof of the Theorem, combining Lemmas 4.1, 5.1 and 7.4.

We remark first that we have to show that the statement of the Theorem is true if we fix the constant K to be large enough. We will choose K to be larger and larger several times during the proof.

The statement about the absolute convergence in (1.6) and (1.7) follows easily from the absolute convergence of the left-hand side of (7.3), (7.2), (1.5) and Prop. 4.4 of [G1].

When f is identically 0, the statement follows at once from Lemma 4.1 and from the cases $A = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \dots$ of Lemma 5.1 (a finite number of them suffice). Indeed, by subtracting a suitable finite linear combination of these cases of Lemma 5.1, we can achieve that $a_n = O\left(n^{-R}\right)$ for any given $R > 0$ (we use for this Stirling's formula in the form [G-R], p. 889, 8.344), and then we can apply Lemma 4.1.

In the case when $f(x) = 1/\Gamma\left(\frac{3}{4} \pm ix\right)$ and $a_n = 0$ for every n, we have $g(x) \equiv f(x)$ and $b_n \equiv 0$ by the formula in the proof of Theorem 6.5 of [G1] with $n = 0$ and $g = 1/4$ there. Then by Corollary 3.1 and Lemma 6.6 we see that both sides equal

$$\int_{D_4} \left|B_0\left(z\right)\right|^2 u_1\left(4z\right) \overline{u_2\left(4z\right)} d\mu_z .$$

Hence the statement is true for this case, and so we may assume that f satisfies (7.4) by subtracting a suitable constant multiple of $1/\Gamma\left(\frac{3}{4} \pm ix\right)$.

Let f be a function satisfying (7.4) and the conditions of the theorem; then we can apply Lemma 7.4. Define now sequences b_n and a_n ($n \geq 1$) in the following way: $iCb_nR_n = d_n$, i.e.,

$$f(x) = iC \sum_{k=1}^{\infty} b_k \phi_x \left(i\left(\frac{1}{4} - k\right)\right) R_k$$

for $|\operatorname{Im} x| < \frac{3}{4}$ on the basis of (7.6), and

$$a_n := iC \sum_{k=1}^{\infty} b_k \phi_{i\left(\frac{1}{4} - n\right)} \left(i\left(\frac{1}{4} - k\right)\right) R_k .$$

Observe that the pair f, $\{a_n\}$ is the Wilson function transform of type II of the pair g, $\{b_n\}$, where $g \equiv 0$. The sequences a_n and b_n satisfy the condition given for a_n in the theorem (the constant K there may be different than the original K, but it is still large), for b_n it follows from (7.5) and (1.5), and for

a_n it follows from (7.3). We claim that with this b_n, a_n and f formula (1.13) equals the sum of (1.8), (1.9) and (1.10). Indeed, this follows from an already proved special case of our Theorem, the $P(g, \{b_n\})$ case (this is really proved already, since $g \equiv 0$), writing in this special case $\overline{u_1}$ in place of u_2, $\overline{u_2}$ in place of u_1, and taking into account that $\phi_\lambda(x; a, b, c, d)$ is symmetric in $a, b, c, 1-d$, hence that our Wilson function transform is symmetric in t_1 and t_2.

Since our Wilson function transform is its own inverse by Theorem 5.10 of [G1] (note that our functions are square integrable with respect to the measure dh of [G1]), we get that (1.6) and (1.7) are true with $g \equiv 0$ and with b_n, a_n and f above. Hence the fact (proved above) that (1.13) equals the sum of (1.8), (1.9) and (1.10) implies that our theorem is true with the given f and with this sequence a_n.

Since we proved the $f \equiv 0$ case already, the theorem is proved.

6. Lemmas on automorphic functions

6.1. The functions B_n. We prove in Lemmas 6.1 and 6.2 basic identities and estimates for the functions B_n defined in (2.4). Lemma 6.3 is needed for Lemma 6.2 but it is used also at another point in the paper. Recall that L_n^α denotes Laguerre polynomials.

Lemma 6.1: *We have*

$$B_n(z) = y^{\frac{1}{4}} \sum_{m=-\infty}^{\infty} L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right) e\left(m^2 z\right) \tag{6.1}$$

for every $n \geq 0$ and $z = x + iy \in H$, and

$$\frac{1}{n!} K_{(n-1)+\frac{1}{4}} \cdots K_{\frac{5}{4}} K_{\frac{1}{4}} B_0 = B_n \tag{6.2}$$

for every $n \geq 1$. We also have the following relations for every $n \geq 0$:

$$\Delta_{2n+\frac{1}{2}} B_n = \frac{1}{4}\left(\frac{1}{4} - 1\right) B_n, \tag{6.3}$$

$$B_n(\gamma z) = \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{2n+\frac{1}{2}} B_n(z) \tag{6.4}$$

for every $\gamma \in \Gamma_0(4)$,

$$B_n\left(\frac{-1}{4z}\right) = e\left(\frac{-1}{8}\right)\left(\frac{z}{|z|}\right)^{2n+\frac{1}{2}} B_n(z), \tag{6.5}$$

and finally, for every $z=x+iy\in H$ and $n\geq 0$ we have that

$$B_n(\sigma_{-\frac{1}{2}}z)\left(\frac{j_{\sigma_{-1/2}}(z)}{\left|j_{\sigma_{-1/2}}(z)\right|}\right)^{-\frac{1}{2}-2n}$$

equals

$$e\Big(-\frac{1}{8}\Big)y^{\frac{1}{4}}\sum_{m=-\infty}^{\infty}L_n^{-\frac{1}{2}}\left(4\pi\left(m+\frac{1}{2}\right)^2 y\right)e\left(\left(m+\frac{1}{2}\right)^2 z\right). \tag{6.6}$$

Proof. Using [G-R], p. 992, formula 8.975.1, we have

$$\sum_{n=0}^{\infty}L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right)L^n=(1-L)^{-\frac{1}{2}}e^{\frac{4\pi m^2 yL}{L-1}}$$

for $y>0$ and $|L|<1$, from which it follows for $z=x+iy\in H$ and $|L|<1$ that

$$\sum_{n=0}^{\infty}\left(\sum_{m=-\infty}^{\infty}L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right)e\left(m^2 z\right)\right)L^n=\theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}},$$

which, together with (2.4), proves (6.1). To prove (6.2), it is enough to show that

$$\frac{1}{n+1}K_{n+\frac{1}{4}}B_n=B_{n+1} \tag{6.7}$$

for every $n\geq 0$. By the definition of the operators K and by (6.1) we have that $(K_{n+\frac{1}{4}}B_n)(z)$ equals (here $(L_n^{-\frac{1}{2}})^{(1)}$ denotes the derivative of $L_n^{-\frac{1}{2}}$)

$$y^{\frac{1}{4}}\sum_{m=-\infty}^{\infty}\left(\left(-4\pi m^2 y+n+\frac{1}{2}\right)L_n^{-\frac{1}{2}}\left(4\pi m^2 y\right)+4\pi m^2 y\left(L_n^{-\frac{1}{2}}\right)^{(1)}\!\left(4\pi m^2 y\right)\right)e\left(m^2 z\right),$$

and applying [G-R], p. 991, 8.971.3 we get (6.2). Formula (6.3) can be checked directly for $n=0$, and then it follows for larger n from (6.2) and [F], p. 145, formula (6). Similarly, (6.4) and (6.5) are well-known for $n=0$, and they follow for larger n from (6.2) and [F], p. 145, formula (5). The case $n=0$ of (6.6) is known (and not hard to prove), and the general case follows by induction, using again (6.2), [G-R], p. 991, 8.971.3 and [F], formula (5). The lemma is proved. ∎

Lemma 6.2: *Let $z\in D_4$, and let $0\leq j\leq 5$ be such that $\gamma_j^{-1}z\in D_1$.*

(i) *There is an absolute constant $A > 0$ such that if $n \geq 0$ is an integer and* $\operatorname{Im}\left(\gamma_j^{-1} z\right) \geq An$, *then*

$$|B_n(z)| \leq A \left(\operatorname{Im}\left(\gamma_j^{-1} z\right)\right)^{\frac{1}{4}} (n+1)^{-\frac{1}{2}}.$$

(ii) *If $N \geq 0$ is an integer and* $\operatorname{Im}\left(\gamma_j^{-1} z\right) \ll N+1$ *with implied absolute constant, then for any $\epsilon > 0$ we have*

$$\sum_{n=N}^{2N} |B_n(z)|^2 \ll_\epsilon (N+1)^{\frac{1}{2}+\epsilon}.$$

Proof. Part (i) follows easily from (6.1), (6.5), (6.6) and [G-R], p. 990, formula 8.970.1, since $L_n^{-\frac{1}{2}}(0) \ll (n+1)^{-\frac{1}{2}}$.

For the proof of (ii) let $n \geq 0$, and write

$$h_z(L) = (\operatorname{Im} z)^{\frac{1}{4}} \, \theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}}.$$

Then

$$B_n(z) = \frac{1}{2\pi i} \int_{|L|=r} \frac{h_z(L)}{L^{n+1}} dL \tag{6.8}$$

for any $0 < r < 1$. Now

$$\operatorname{Im}\left(T_z\left(i\frac{1+L}{1-L}\right)\right) = (\operatorname{Im} z)\frac{1-|L|^2}{|1-L|^2},$$

so

$$h_z(L) = B_0\left(T_z\left(i\frac{1+L}{1-L}\right)\right)\left(1-|L|^2\right)^{-\frac{1}{4}} \frac{|1-L|^{\frac{1}{2}}}{(1-L)^{\frac{1}{2}}}.$$

Hence, using Parseval's identity and (6.8) for a fixed r, and then averaging over

$$1 - \frac{2}{N+2} \leq r \leq 1 - \frac{1}{N+2},$$

we get

$$\sum_{n=N}^{2N} |B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}} \int_{1-\frac{2}{N+2}}^{1-\frac{1}{N+2}} \int_0^{2\pi} \left|B_0\left(T_z\left(i\frac{1+re^{i\phi}}{1-re^{i\phi}}\right)\right)\right|^2 \frac{r\,d\phi\,dr}{(1-r^2)^2},$$

hence, using a substitution,

$$\sum_{n=N}^{2N} |B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}} \int_{w\in H, \left|\frac{w-i}{w+i}\right| \leq 1-\frac{1}{N+2}} |B_0(T_z w)|^2 \, d\mu_w$$

with implied absolute constant. For simplicity, instead of $|B_0|^2$, we take an $SL(2,\mathbf{Z})$-invariant majorant and write

$$F(Z)=\sum_{j=0}^{5}|B_0(\gamma_j Z)|^2 .$$

Since

$$\frac{\left|\frac{w-i}{w+i}\right|^2}{1-\left|\frac{w-i}{w+i}\right|^2}=\frac{|w-i|^2}{4\operatorname{Im} w},$$

hence

$$\sum_{n=N}^{2N}|B_n(z)|^2 \ll (N+1)^{-\frac{1}{2}}\int_{D_1} K(z,w;N+2)\,F(w)d\mu_w,$$

where we write

$$K(z,w;x)=\sum_{\gamma\in SL(2,\mathbf{Z}),\,\frac{|\gamma z-w|^2}{4\operatorname{Im}\gamma z\operatorname{Im} w}\le x} 1.$$

Since we have $F(w)\ll(\operatorname{Im} w)^{\frac{1}{2}}$ for $w\in D_1$ (which follows from the $n=0$ case of (i)), Lemma 6.3 below proves the present lemma. ∎

LEMMA 6.3: *Let $z_1,z_2\in D_1$, write $y_1=\operatorname{Im} z_1$, $y_2=\operatorname{Im} z_2$, and let $x\ge 2$. Then for every $\epsilon>0$ we have*

$$K(z_1,z_2;x)\ll_\epsilon x^{1+\epsilon}+(xy_1y_2)^{\frac{1}{2}}, \tag{6.9}$$

and if E is a large enough absolute constant and $y_2\ge Exy_1$, then

$$K(z_1,z_2;x)=0. \tag{6.10}$$

Proof. It is easy to see by (1.2), (1.3) of [I], and by the triangle inequality (for the hyperbolic distance function on H), that if

$$\gamma\in\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL(2,\mathbf{Z})$$

and

$$\frac{|\gamma z_1-z_2|^2}{4\operatorname{Im}(\gamma z_1)\operatorname{Im} z_2}\le x,$$

then

$$\frac{|\gamma(iy_1)-iy_2|^2}{4\operatorname{Im}(\gamma(iy_1))\,y_2}\le Cx$$

with some absolute constant $C > 0$. The left-hand side here is

$$\frac{(ay_1 - dy_2)^2 + (b + cy_1y_2)^2}{4y_1y_2},$$

hence we need

$$a^2y_1^2 + d^2y_2^2 + b^2 + c^2y_1^2y_2^2 \leq (4Cx + 2)\, y_1y_2. \tag{6.11}$$

This implies

$$|b| \leq ((4Cx + 2)\, y_1y_2)^{\frac{1}{2}}, \quad |c| \leq \left(\frac{4Cx + 2}{y_1y_2}\right)^{\frac{1}{2}}. \tag{6.12}$$

If $y_1y_2 \leq 4Cx + 2$, then the number of possible (b, c) pairs is $\ll x$. If b and c are given and $bc \neq -1$, then $ad = 1 + bc$ is also given, and $0 \neq |ad| \ll x$, hence the number of possible (a, d) pairs is $\ll_\epsilon x^\epsilon$. If $bc = -1$, then the number of possible (b, c) pairs is $\ll 1$, and $a = 0$ or $d = 0$, and we also see by (6.11) and the relations $y_1y_2 \leq 4Cx + 2$ and $y_1, y_2 \gg 1$ that $a^2 + d^2 \ll x^2$. This proves (6.9) for the case $y_1y_2 \leq 4Cx + 2$.

If $y_1y_2 > 4Cx + 2$, then (6.12) implies $c = 0$, hence the number of possible (a, d) pairs is $\ll 1$, and the number of possible numbers b is $\ll (xy_1y_2)^{\frac{1}{2}}$. The inequality (6.9) is proved. Since $d^2 + c^2y_1^2 \gg 1$, (6.11) implies (6.10). ∎

6.2. An upper bound for an integral of Maass forms.

Lemma 6.4: *Let $C > 1/2$, and let u be a cusp form of weight 0 for $SL(2, \mathbf{Z})$ with $\Delta_0 u = s(s-1)u$, where $s = \frac{1}{2} + it$ and $t > 0$. Then for integers $n \geq 0$ we have, by the notation*

$$u_{(n)}(z) = \left(\prod_{l=0}^{n-1} \frac{1}{s + l}\right) (K_{n-1}K_{n-2} \cdots K_1K_0u)\,(z),$$

the inequality

$$\int_{D_4} \left(\sum_{l=0}^{\infty} (1 + l)^{-C} |B_l(z)|^2\right) \left|u_{(n)}\,(4z)\right|^2 d\mu_z \ll_{u,C} \log^2 (n + 2)\,.$$

Proof. We use the substitution $z \to -1/4z$, which normalizes $\Gamma_0(4)$. By (6.5) we see that $|B_l\,(-1/4z)|^2 = |B_l(z)|^2$, and $u_{(n)}\,(4\,(-1/4z)) = u_{(n)}\,(z)$ by the $SL(2, \mathbf{Z})$-invariance of u. For $z \in D_1$ we have

$$\sum_{j=0}^{5} \sum_{l=0}^{\infty} (1 + l)^{-C} |B_l(\gamma_j z)|^2 \ll_C (\operatorname{Im} z)^{1/2}$$

by Lemma 6.2, so it is enough to prove that

$$\int_{D_1} (\operatorname{Im} z)^{1/2} \left|u_{(n)}(z)\right|^2 d\mu_z \ll_u \log^2(n+2).$$

We will give an upper bound by extending the integration to $\operatorname{Im} z \geq \sqrt{3}/2$, $|\operatorname{Re} z| \leq 1/2$, and using Parseval's formula. Consider the Fourier expansion

$$u_{(n)}(z) = \sum_{m \neq 0} b_{u,n}(m) W_{n\,\mathrm{sgn}(m), it}(4\pi |m| y) e(mx).$$

It is well-known (see [D], formulas (2.4) and (2.6), and take into account our formula (2.7)) that for $m > 0$ we have

$$b_{u,n}(m) = (-1)^n \left(\prod_{l=0}^{n-1} \frac{1}{s+l}\right) b_{u,0}(m),$$

and for $m < 0$ we have

$$b_{u,n}(m) = (-1)^n \left(\prod_{l=1}^{n} (s-l)\right) b_{u,0}(m).$$

By [G-R], p. 814, 7.611.4 and p. 893, 8.362.1 we see for real $t \neq 0$ and any integer m (note that $W_{m,it}(y)$ is real) that

$$\int_0^\infty \left| W_{m,it}(y) \Gamma\left(\frac{1}{2} - m + it\right)\right|^2 \frac{dy}{y}$$
$$= \frac{\pi}{\sin 2\pi t} \sum_{k=0}^{\infty} \left(\frac{1}{\frac{1}{2} - it - m + k} - \frac{1}{\frac{1}{2} + it - m + k}\right),$$

which is $\ll_t \log(|m| + 2)$. By these relations, Lemma 6.5 below and formulas (8.17) and (8.5) of [I] we easily get the lemma. ∎

Lemma 6.5: *There are positive absolute constants C_1, C_2, C_3 such that if $n \in \mathbf{Z}$, $t \geq 0$, then*

$$\left| W_{n,it}(y) \Gamma\left(\frac{1}{2} - n + it\right)\right| \leq C_1 e^{-C_2 y} \quad \text{for } y \geq C_3 \max(1+t, n). \tag{6.13}$$

Proof. By [G-R], p. 1015, formula 9.223 we have for $y > 0$ and $t \geq 0$ that

$$W_{n,it}(y) \Gamma\left(\frac{1}{2} - n + it\right) - \frac{e^{-\frac{y}{2}}}{2\pi i} \int_{(1/4)} \frac{\Gamma(u-n)\Gamma\left(\frac{1}{2} - u - it\right)\Gamma\left(\frac{1}{2} - u + it\right)}{\Gamma\left(\frac{1}{2} - n - it\right)} y^u du$$

is $\ll$ than

$$e^{-\frac{y}{2}}\sum_{j=1}^{n}\left|\operatorname{Res}_{u=j}\frac{\Gamma(u-n)\,\Gamma\left(\frac{1}{2}-u-it\right)\Gamma\left(\frac{1}{2}-u+it\right)}{\Gamma\left(\frac{1}{2}-n-it\right)}y^{u}\right|$$

(this is of course 0 for $n\le 0$). We use the well-known statement that if σ is a real number which is not a nonpositive integer, then $\max_{\tau\in\mathbf{R}}|\Gamma(\sigma+i\tau)|=|\Gamma(\sigma)|$. We apply this statement to estimate $\Gamma(u-n)$ if $\operatorname{Re}u=1/4$, and $\Gamma\left(\frac{1}{2}-u-it\right)$ if $u=j$. Then Stirling's formula easily implies (6.13); the lemma is proved. ∎

6.3. An orthogonality relation.

Lemma 6.6: *If u is a cusp form of weight* 0 *for $SL(2,\mathbf{Z})$, then*

$$\int_{D_4}|B_0(z)|^2\,u(4z)d\mu_z=0. \tag{6.14}$$

Proof. By the substitution $z\to -1/4z$ and by (6.5) with $n=0$ we get (as in the proof of Lemma 6.4) that the left-hand side of (6.14) equals

$$\int_{D_4}|B_0(z)|^2\,u(z)d\mu_z=\int_{D_1}\left(\sum_{j=0}^{5}|B_0(\gamma_j z)|^2\right)u(z)d\mu_z. \tag{6.15}$$

We now determine the Fourier expansion of $F(z):=\sum_{j=0}^{5}|B_0(\gamma_j z)|^2$. We use that

$$\begin{pmatrix}0&-1\\1&j\end{pmatrix}=\begin{pmatrix}0&-1/2\\2&0\end{pmatrix}\begin{pmatrix}1/2&j/2\\0&2\end{pmatrix},\quad \begin{pmatrix}1&0\\-2&1\end{pmatrix}=\begin{pmatrix}-1&-1/2\\2&0\end{pmatrix}\begin{pmatrix}-1&1/2\\0&-1\end{pmatrix}.$$

It is also not hard to see for any integer n and $y>0$ that

$$\int_0^1\left(\sum_{j=0}^{3}\left|B_0\left(\frac{x+iy+j}{4}\right)\right|^2\right)e(-nx)dx=4\int_0^1\left|B_0\left(x+\frac{iy}{4}\right)\right|^2e(-4nx)dx,$$

hence, using (6.5) and (6.6), we get that $\int_0^1 F(x+iy)\,e(-nx)dx$ equals

$$y^{\frac{1}{2}}\sum_{\substack{m_1,m_2\in\frac{1}{2}\mathrm{Z}\\ m_1^2-m_2^2=n}}e^{-2\pi\left(m_1^2+m_2^2\right)y}+2y^{\frac{1}{2}}\sum_{\substack{m_1,m_2\in\mathrm{Z}\\ m_1^2-m_2^2=4n}}e^{-2\pi\left(m_1^2+m_2^2\right)\frac{y}{4}}.$$

One easily checks that the two sums above have the same value, and, writing $a=m_1-m_2$, $d=m_1+m_2$ in the first sum, we finally get for any $y>0$ and integer n that

$$\int_0^1 F\left(x+iy\right)e(-nx)dx = 3y^{\frac{1}{2}} \sum_{\substack{a,d\in \mathrm{Z}\\ ad=n}} e^{-\pi\left(a^2+d^2\right)y}. \tag{6.16}$$

We now show that a certain incomplete Eisenstein series has the same Fourier coefficients. Indeed, for $z \in H$ let

$$G(z) := E(z,\psi) = \sum_{\gamma\in\Gamma_\infty\backslash SL_2(\mathrm{Z})} \psi\left(\mathrm{Im}(\gamma z)\right),$$

where

$$\psi\left(y\right) = \sum_{m=1}^{\infty} e^{-\pi\frac{m^2}{y}}.$$

Then by [I], (3.17) we have for $y>0$ and $n\neq 0$ that

$$\int_0^1 G\left(x+iy\right)e(-nx)dx = \sum_{c=1}^{\infty} S(0,n,c)\int_{-\infty}^{\infty} \psi\left(\frac{yc^{-2}}{t^2+y^2}\right)e(-nt)dt,$$

where $S(0,n,c)$ is given by [I], (2.26). We can compute easily that

$$\int_{-\infty}^{\infty} \psi\left(\frac{yc^{-2}}{t^2+y^2}\right)e(-nt)dt = d\sqrt{y}\sum_{m=1}^{\infty}\frac{1}{cm}e^{-\pi y\left(m^2c^2+\frac{n^2}{m^2c^2}\right)}$$

with a nonzero absolute constant d. Since for any positive integer a we have

$$\sum_{c|a} S(0,n,c) = \begin{cases} a & \text{if } a|n,\\ 0 & \text{otherwise,}\end{cases}$$

we get finally for any $y>0$ and nonzero integer n that

$$\int_0^1 G\left(x+iy\right)e(-nx)dx = d\sqrt{y}\sum_{a|n} e^{-\pi y\left(a^2+\frac{n^2}{a^2}\right)}.$$

This and (6.16) imply that there is a nonzero absolute constant D such that $F(z)-DG(z)$ depends only on $\mathrm{Im}\, z$. Since $F(z)-DG(z)$ is $SL(2,\mathrm{Z})$-invariant, it is a constant. This implies that (6.15) is 0 (since cusp forms are orthogonal to incomplete Eisenstein series and constants); the lemma is proved. ∎

7. Appendix: lemmas on Wilson functions

Let $t_1, t_2 \in \mathbf{R}$ be given nonzero numbers. See Subsection 1.5 for the notation s_1, s_2, $\phi_\lambda(x)$ and $H(x)$.

Lemma 7.1: *Let n be a fixed positive integer, and write*

$$m_{l_1,l_2} = \frac{1}{4} + l_1 + l_2 - n.$$

For nonnegative integers l_1, l_2 and complex S define

$$\Sigma_{l_1,l_2}(S) = \sum_{L_1=0}^{l_1} \sum_{L_2=0}^{l_2} (-1)^{L_1+L_2} \binom{l_1}{L_1} \binom{l_2}{L_2} G(S, l_1, l_2, L_1, L_2),$$

where $G(S, l_1, l_2, L_1, L_2)$ denotes

$$\frac{\overline{\Gamma(s_2 - n + l_2 + L_1)}}{\overline{\Gamma(s_2 + n - l_2 - L_1)}} \frac{\Gamma(s_1 - n + l_1 + L_2)}{\Gamma(s_1 + n - l_1 - L_2)} \times \frac{\overline{(S - m_{l_1,l_2})_{l_1 - L_1}}}{\overline{\Gamma(S + m_{l_1,l_2} - l_1 + L_1)}} \frac{(S - m_{l_1,l_2})_{l_2 - L_2}}{\Gamma(S + m_{l_1,l_2} - l_2 + L_2)}.$$

Note that since n is given, we have not denoted the dependence on n in m_{l_1,l_2}, $\Sigma_{l_1,l_2}(S)$ and $G(S, l_1, l_2, L_1, L_2)$.

Then, if $S = \frac{1}{2} + i\tau$, where τ is either real or purely imaginary, we have that the sum

$$\sum_{l_1,l_2=0}^{\infty} \frac{\overline{\Gamma(S + m_{l_1,l_2})}}{\Gamma(1 - S + m_{l_1,l_2})} \frac{\Sigma_{l_1,l_2}(S)}{\Gamma\left(\frac{1}{2} + l_1\right)\Gamma(1 + l_1)\Gamma\left(\frac{1}{2} + l_2\right)\Gamma(1 + l_2)}$$

is absolutely convergent, and equals

$$\frac{|\Gamma(s_1)|^2 |\Gamma(s_2)|^2 \Gamma\left(-\frac{1}{2} + 2n\right)}{|\Gamma(s_1 + n)|^2 |\Gamma(s_2 + n)|^2} \phi_\tau\left(i\left(\frac{1}{4} - n\right)\right).$$

Lemma 7.2: (i) *We have*

$$\phi_{-\frac{3}{4}i}\left(i\left(\frac{1}{4} - k\right)\right) = c\frac{(-1)^k}{k^{3/2}}(1 + o(1)) \tag{7.1}$$

with a nonzero constant c as $k \to \infty$.

(ii) *For any given compact set L on the complex plane and any $\epsilon > 0$ we have*

$$\left|\phi_\lambda\left(i\left(\frac{1}{4} - k\right)\right)\right| \ll_{\epsilon,L} k^{\epsilon + 2|\mathrm{Im}\,\lambda|} \tag{7.2}$$

for positive integers k and $\lambda \in L$.

(iii) *If a_n $(n \geq 1)$ is any given sequence satisfying $a_n = O\left(n^d\right)$ with a number $d < \frac{1}{2}$, then for any positive integer M there are constant coefficients b_m such that*

$$\sum_{n=1}^{\infty} a_n \phi_{i\left(\frac{1}{4}-n\right)}\left(i\left(\frac{1}{4}-k\right)\right) = \frac{(-1)^k}{k^{3/2}}\left(\sum_{m=0}^{M-1} \frac{b_m}{k^m} + O\left(k^{-M}\right)\right) \tag{7.3}$$

as $k \to \infty$ over positive integers, and the left-hand side here is absolutely convergent for every integer $k \geq 1$.

Lemma 7.3: *Write*

$$M_\lambda(A) = \frac{1}{\Gamma(1-A)} \sum_{k=1}^{\infty} (-1)^k \frac{\Gamma(k-A)}{\Gamma(k)} \phi_\lambda\left(i\left(\frac{1}{4}-k\right)\right).$$

(i) *$M_\lambda(A)$ is absolutely convergent if $\mathrm{Re} A > 1 + 2\,|\mathrm{Im}\,\lambda|$, or if $\lambda = i\left(\frac{1}{4}-k\right)$ with a positive integer k and $\mathrm{Re}\,A \geq 2$.*

(ii) *If a compact set L on the complex plane is given such that $2 \leq \mathrm{Re}\,A$ for every$A \in L$, then*

$$M_{i\left(\frac{1}{4}-k\right)}(A) = O_L(k^{-3/2})$$

for any $A \in L$ and positive integer k. The left-hand side here is regular in A on every open subset of L for every fixed positive integer k.

(iii) *If a compact set L on the complex plane and an integer $K \geq 2$ are given such that $2 \leq \mathrm{Re}\,A$ for every $A \in L$, then we can find an integer $t > 0$, complex numbers $A_1, A_2, \ldots, A_t$ with $\mathrm{Re}\,A_i > K$ $(i = 1, 2, \ldots, t)$ and polynomials $Q_1, Q_2, \ldots, Q_t$ such that*

$$M_\lambda(A) - \sum_{i=1}^{t} 2^{A-A_i} Q_i(A)\, M_\lambda(A_i) = O_{L,K}\left(e^{2\pi|\lambda|}(1+|\lambda|)^{-K}\right)$$

for any $A \in L$ and real λ. The left-hand side here is regular in A on every open subset of L for every fixed real λ.

Lemma 7.4: *Assume that K is a positive number, and $f(x)$ is an even holomorphic function for $|\mathrm{Im}\,x| < K$ satisfying*

$$\int_{-\infty}^{\infty} f(\tau)\, H(\tau) \frac{1}{\Gamma\left(\frac{3}{4} \pm i\tau\right)} d\tau = 0, \tag{7.4}$$

and that

$$\left| f(x) e^{-2\pi|x|} \left(1+|x|\right)^K \right|$$

is bounded on the domain $|\mathrm{Im}\, x| < K$*. If* k *is a positive integer and* K *is large enough in terms of* k*, then we have a sequence* d_n *satisfying*

$$d_n = \frac{(-1)^n}{n^{5/2}} \left(\sum_{j=0}^{k} \frac{e_j}{n^j} + O\left(\frac{1}{n^{k+1}}\right) \right) \tag{7.5}$$

with some constants e_j *and*

$$f(x) = \sum_{n=1}^{\infty} d_n \phi_{i\left(\frac{1}{4}-n\right)}(x) \tag{7.6}$$

for every $|\mathrm{Im}\, x| < \frac{3}{4}$*, and the sum on the right-hand side of (7.6) is absolutely convergent for every such* x*.*

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