# ON THE GEOMETRIC TRACE OF A GENERALIZED SELBERG TRACE FORMULA 

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#### Abstract

A certain generalization of the Selberg trace formula was proved by the first named author in 1999. In this generalization instead of considering the integral of $K(z, z)$ (where $K(z, w)$ is an automorphic kernel function) over the fundamental domain, one considers the integral of $K(z, z) u(z)$, where $u(z)$ is a fixed automorphic eigenfunction of the Laplace operator. This formula was proved for discrete subgroups of $\operatorname{PSL}(2, \mathbb{R})$, and just as in the case of the classical Selberg trace formula it was obtained by evaluating in two different ways ("geometrically" and "spectrally") the integral of $K(z, z) u(z)$. In the present paper we work out the geometric side of a further generalization of this generalized trace formula: we consider the case of discrete subgroups of $P S L(2, \mathbb{R})^{n}$ where $n>1$. Many new difficulties arise in the case of these groups due to the fact that the classification of conjugacy classes is much more complicated for $n>1$ than in the case $n=1$.


## 1. Introduction

1.1. The Selberg trace formula and its generalizations. The Selberg trace formula (introduced by A. Selberg, see [10) is a particularly important tool in the theory of automorphic functions, it has many applications in different branches of mathematics. Briefly speaking, it is obtained by computing the integral

$$
\operatorname{Tr} K=\int_{F} K(z, z) d \mu(z)
$$

[^0]in two different ways ("geometrically" and "spectrally"), where $d \mu(z)=y^{-2} d x d y$ is the usual measure on the complex upper half-plane $\mathbb{H}, F \subset \mathbb{H}$ is the fundamental domain of a finite volume Fuchsian group $\Gamma \leq P S L(2, \mathbb{R})$ acting on $\mathbb{H}$, and $K(z, w)$ is an appropriate automorphic kernel function (i.e. a function that is invariant under the action of $\Gamma$ ).
A. Biró obtained a generalization of this formula in [1] by evaluating the integral
\[

$$
\begin{equation*}
\operatorname{Tr} K=\int_{F} K(z, z) u(z) d \mu(z) \tag{1}
\end{equation*}
$$

\]

where the weight $u(z)$ is a Maass form (i.e. an automorphic eigenfunction of the Laplace operator). Biró applied his generalization to the the hyperbolic circle problem (see [4]), and some ideas of [1] are applied for cycle integral and triple product identities in [2] and [3].

Selberg's trace formula was developed for a general family of groups, and our aim is to work out the geometric side of Biró's generalized formula for discrete subgroups of $P S L(2, \mathbb{R})^{n}$ where $n>1$. For these, the details of the Selberg trace formula are given in [6].
1.2. Discrete subgroups of $P S L(2, \mathbb{R})^{n}$. The main example of the discrete subgroups of $P S L(2, \mathbb{R})^{n}$ is the Hilbert modular group. For any totally real finite extension $\mathbb{Q} \leq K$ of degree $n>1$ it is defined as

$$
\Gamma_{K}:=\left\{\left(\left[\begin{array}{ll}
a^{(1)} & b^{(1)} \\
c^{(1)} & d^{(1)}
\end{array}\right], \ldots,\left[\begin{array}{ll}
a^{(n)} & b^{(n)} \\
c^{(n)} & d^{(n)}
\end{array}\right]\right):\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in \operatorname{PSL}\left(2, \mathcal{O}_{K}\right)\right\}
$$

where $K^{(1)}, \ldots, K^{(n)}$ are the different embeddings of $K$ into $\mathbb{R}$, and the images of an element $a \in K$ by these embeddings are $a^{(1)}, \ldots, a^{(n)}$. As usually, $\mathcal{O}_{K}$ denotes the ring of integers in $K$.

The generalized trace formula (derived from (11)) is fully worked out in the PhD thesis 13 in the special case when $\Gamma$ is a Hilbert modular group for a quadratic field of class number one. Though a more general situation is handled here, the Hilbert modular groups still play an important role in the following. Before specifying this role, we give a short summary of the main results about discrete subgroups of $\operatorname{PSL}(2, \mathbb{R})^{n}$. Many of these are proved in [7].
1.2.1. Action on the product of upper half-planes. The group $\operatorname{PSL}(2, \mathbb{R})$ acts on $\mathbb{H}$ in the usual way, if $\gamma=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right] \in P S L(2, \mathbb{R})$ and $z \in \mathbb{H}$, then

$$
\begin{equation*}
\gamma z=\frac{a z+b}{c z+d} \tag{2}
\end{equation*}
$$

This induces a coordinate-wise action of $P S L(2, \mathbb{R})^{n}$ on the product space $\mathbb{H}^{n}$. For an element $z \in \mathbb{H}^{n}$ we will use the notation $z=\left(z_{1}, \ldots, z_{n}\right)$ where $z_{k}=x_{k}+i y_{k} \in \mathbb{H}$ with $x_{k} \in \mathbb{R}$ and $y_{k} \in \mathbb{R}^{+}(k=1, \ldots, n)$. That is, if $\gamma=\left(\gamma^{(1)}, \ldots, \gamma^{(n)}\right) \in P S L(2, \mathbb{R})^{n}$ and $z \in \mathbb{H}^{n}$, then $\gamma z=\left(\gamma^{(1)} z_{1}, \ldots, \gamma^{(n)} z_{n}\right)$. It is convenient to represent an element $\gamma \in P S L(2, \mathbb{R})^{n}$ as in the one dimensional case, i.e. as a matrix $\gamma=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$, whose elements are $n$ dimensional row vectors with real coordinates, e.g. $a=\left(a^{(1)}, \ldots, a^{(n)}\right)$. Then the action of $P S L(2, \mathbb{R})^{n}$ on $\mathbb{H}^{n}$ can be written formally as in (2), where the operations are meant to be performed coordinatewise. It is known that $\Gamma \leq P S L(2, \mathbb{R})^{n}$ is discrete if and only if it acts discontinuously on $\mathbb{H}^{n}$ (see Proposition 2.1 in [7]).
1.2.2. Irreducible groups. The groups $\Gamma, \Gamma^{\prime} \leq P S L(2, \mathbb{R})^{n}$ are said to be strictly commensurable if $\Gamma \cap \Gamma^{\prime}$ has finite index in both $\Gamma$ and $\Gamma^{\prime}$. They are said to be commensurable if $\Gamma$ is strictly commensurable with a conjugate of $\Gamma^{\prime}$. A discrete subgroup $\Gamma \leq P S L(2, \mathbb{R})^{n}$ is said to be irreducible if $\Gamma$ is not commensurable with any direct product $\Gamma^{\prime} \times \Gamma^{\prime \prime}$, where $\Gamma^{\prime}$ and $\Gamma^{\prime \prime}$ are discrete subgroups of some non-trivial groups $G^{\prime}$ and $G^{\prime \prime}$, respectively, for
which $\operatorname{PSL}(2, \mathbb{R})^{n}=G^{\prime} \times G^{\prime \prime}$ holds. For a discrete subgroup $\Gamma$ of $\operatorname{PSL}(2, \mathbb{R})^{n}$ we have the following equivalent conditions for irreducibility (see the Corollary after Theorem 2 in [11]):
(i) $\Gamma$ contains no element $\gamma=\left(\gamma^{(1)}, \ldots, \gamma^{(n)}\right)$ such that $\gamma^{(i)}=1$ holds for some $i$ while $\gamma^{(j)} \neq 1$ holds for some $j$,
(ii) there exist no partial product $G^{\prime}$ of $P S L(2, \mathbb{R})^{n}$ such that the projection of $\Gamma$ to $G^{\prime}$ is discrete,
(iii) for every $\gamma \in \Gamma \backslash\{1\}$ the centralizer of $\gamma$ in $\operatorname{PSL}(2, \mathbb{R})^{n}$ is commutative.

In the following $\Gamma$ always denotes an irreducible subgroup. Note that property (i) above implies that the Hilbert modular group is irreducible.
1.2.3. Cusps. The group $\operatorname{PSL}(2, \mathbb{R})^{n}$ and hence its subgroup $\Gamma$ act on the set $(\mathbb{R} \cup\{\infty\})^{n}$. This action is also given by (21) (using the usual extended operations on the set $\mathbb{R} \cup\{\infty\}$ ). Roughly speaking, an element of $(\mathbb{R} \cup\{\infty\})^{n}$ is a cusp if its stabilizer in $\Gamma$ is in some sense as large as possible.

First, let us consider the element $\infty=(\infty, \ldots, \infty)$. The stabilizer $\Gamma_{\infty}$ of $\infty$ contains elements of the form

$$
\gamma=\left[\begin{array}{ll}
a & b \\
0 & d
\end{array}\right]
$$

and among them there are those for which all the coordinates of the vectors $a$ and $d$ are 1 , i.e. the translations. Let us define

$$
\mathbf{t}_{\infty}=\mathbf{t}_{\infty}(\Gamma)=\left\{b \in \mathbb{R}^{n}:\left[\begin{array}{ll}
1 & b \\
0 & 1
\end{array}\right] \in \Gamma\right\}
$$

then, since $\Gamma$ is discrete, $\mathbf{t}_{\infty}$ is a discrete subgroup of $\mathbb{R}^{n}$, and hence it is isomorphic to $\mathbb{Z}^{m}$ for some $1 \leq m \leq n$. The important cases are those, for which $m=n$ holds (i.e. $\mathbf{t}_{\infty}$ is a lattice).

For a general element $\gamma \in \Gamma_{\infty}$ we have $d^{(k)}=\left(a^{(k)}\right)^{-1}(1 \leq k \leq n)$, and therefore $\gamma z=a^{2} z+a b$ holds for any $z \in \mathbb{H}^{n}$ (again, the operations are accomplished coordinatewise). Notice that the the coordinates of the vector $a^{2}$ are all positive. The vectors with this property are called totally positive and they form the group $\left(\mathbb{R}^{+}\right)^{n}$ w.r.t. the coordinate-wise multiplication. An element $\varepsilon \in\left(\mathbb{R}^{+}\right)^{n}$ is called a multiplier for $\Gamma$ if there is a $\gamma \in \Gamma$ for which $\gamma z=\varepsilon z+b$ holds for some $b \in \mathbb{R}^{n}$. The multipliers form a subgroup of $\left(\mathbb{R}^{+}\right)^{n}$, it is denoted by $\Lambda_{\infty}=\Lambda_{\infty}(\Gamma)$.

If $\mathbf{t}_{\infty}$ is a lattice, then $\Lambda_{\infty}$ is a discrete subgroup of $\left(\mathbb{R}^{+}\right)^{n}$ and for each $\varepsilon \in \Lambda_{\infty}$ we have $\varepsilon^{(1)} \ldots \varepsilon^{(n)}=1$ (see Remark 2.3 in [7]). Taking the logarithm coordinate-wise, we obtain that $\log \Lambda_{\infty}$ is a discrete subgroup of $\mathbb{R}^{n}$ contained in the hyperplane

$$
\begin{equation*}
V=\left\{a \in \mathbb{R}^{n}: a^{(1)}+\cdots+a^{(n)}=0\right\} \tag{3}
\end{equation*}
$$

and hence cannot be a lattice. That is, if $\mathbf{t}_{\infty} \cong \mathbb{Z}^{n}$, then $\Lambda_{\infty} \cong \mathbb{Z}^{m}$ for some $0 \leq m \leq n-1$. We say that $\infty$ is a cusp for $\Gamma$ if $\mathbf{t}_{\infty} \cong \mathbb{Z}^{n}$ and $\Lambda_{\infty} \cong \mathbb{Z}^{n-1}$.

Note that $\infty$ is a cusp for the Hilbert modular group. Indeed, in this case the $n$ different embeddings of $K$ induce a map from $\mathcal{O}_{K}$ to $\mathbb{R}^{n}$ whose image is well-known to be a lattice that $\mathbf{t}_{\infty}$ is isomorphic to. Also, $\Lambda_{\infty}$ consists of the squares of the units of $\mathcal{O}_{K}$. By Dirichlet's unit theorem we have that $\mathcal{O}_{K}^{\times} \cong \mathbb{Z}^{n-1} \times\{ \pm 1\}$, hence $\Lambda_{\infty} \cong \mathbb{Z}^{n-1}$ and our claim follow.

We say that an element $\kappa \in(\mathbb{R} \cup\{\infty\})^{n}$ is a cusp for the discrete group $\Gamma \leq \operatorname{PSL}(2, \mathbb{R})^{n}$ if $\infty$ is a cusp for the group $\sigma^{-1} \Gamma \sigma$ for some $\sigma \in \operatorname{PSL}(2, \mathbb{R})^{n}$ with $\sigma \infty=\kappa$. It is not hard to see that in this case $\infty$ is a cusp for $\rho^{-1} \Gamma \rho$ for every element $\rho \in \operatorname{PSL}(2, \mathbb{R})^{n}$ with $\sigma \infty=\rho \infty=\kappa$. Moreover, we have in fact $\Lambda_{\infty}\left(\sigma^{-1} \Gamma \sigma\right)=\Lambda_{\infty}\left(\rho^{-1} \Gamma \rho\right)=: \Lambda_{\kappa}$, hence this group is determined uniquely by the cusp $\kappa$ and it is called the multiplier group for $\kappa$. Note that $\mathbf{t}_{\infty}\left(\sigma^{-1} \Gamma \sigma\right)$ is determined only up to a coordinate-wise scalar multiple. Having this in mind, we will use the notation $\mathbf{t}_{\kappa}$ for such a lattice.
1.2.4. Fundamental domain. Let $X$ be a locally compact second-countable topological space, and assume that $\Gamma$ is a group of topological (i.e. bijective, continuous and open) maps of $X$ onto itself. The set $F \subset X$ is called a fundamental set for $\Gamma$ if

$$
\begin{equation*}
X=\bigcup_{\gamma \in \Gamma} \gamma(F) . \tag{4}
\end{equation*}
$$

For example the space $X$ itself is always a fundamental set. Of course we are interested in fundamental sets that are in some sense as small as possible.

Also, we would like to consider measurable sets, so assume further that a $\Gamma$-invariant Radon measure $\mu$ is given on $X$. Then a measurable set $F \subset X$ is called a fundamental domain for $\Gamma$ if (4) holds, and there is a set $S \subset F$ such that $\mu(S)=0$ and different points of $F \backslash S$ are not on the same $\Gamma$-orbit.

Every measurable fundamental set contains a fundamental domain (see Appendix II in [7), hence there exists a fundamental domain $F \subset \mathbb{H}^{n}$ for every irreducible discrete subgroup $\Gamma \leq \operatorname{PSL}(2, \mathbb{R})^{n}$. In the following we always assume that the volume of $F$ is finite. This holds of course when the quotient space $\mathbb{H}^{n} \backslash \Gamma$ is compact. However, this quotient is not compact once there are cusps for $\Gamma$ since in that case $F$ contains parts that stretch out to the boundary of $\mathbb{H}^{n}$. Now we are in the position to state the following important theorem (see Theorem I.1.5 in [6):

Theorem 1.1. Assume that $n \geq 2$ and $\Gamma \leq P S L(2, \mathbb{R})^{n}$ is an irreducible discrete subgroup whose fundamental domain has finite volume but the quotient space $\mathbb{H}^{n} \backslash \Gamma$ is not compact. Then $\Gamma$ is commensurable with a Hilbert modular group for a number field $K$ of degree $n$.

In addition, the field $K$ in the theorem above is generated by the elements of $\Lambda_{\kappa}$ for any cusp $\kappa$ (see the proof of Theorem 4 in [11). Note that if $\Gamma$ is commensurable with the Hilbert modular group, then its fundamental domain is of finite volume.

From now on, $\Gamma$ always denotes a discrete irreducible subgroup of $\operatorname{PSL}(2, \mathbb{R})^{n}$ with a fundamental domain of finite volume, and we will assume also that $\Gamma$ has at least one cusp. The group $\Gamma$ acts on the set of its cusps. It is well known that the number of the equivalence classes (orbits) of the cusps for a Hilbert modular group is the class number of the corresponding field $K$ (see [12], Proposition 20 on p. 188). It follows from this and from the theorem above, that the number of the equivalence classes of the cusps for $\Gamma$ is finite.

To describe the fundamental domain of $\Gamma$ we first define the cusp regions. We set

$$
U_{C}:=\left\{z \in \mathbb{H}^{n}: N y>C\right\},
$$

where $C>0$ is a positive real number and $N a=\prod_{k=1}^{n} a^{(k)}$ for any real vector $a=$ $\left(a^{(1)}, \ldots, a^{(n)}\right)^{T}$. If $\kappa$ is a cusp for $\Gamma$ and $\sigma \in \operatorname{PSL}(2, \mathbb{R})^{n}$ is a fixed element for which $\sigma \infty=\kappa$ holds, then the sets of the form $\sigma\left(U_{C}\right)$ are called the neighbourhoods of $\kappa$. Note that the set of these neighbourhoods does not depend on the choice of $\sigma$.

The stabilizer $\Gamma_{\kappa}$ of $\kappa$ acts on the sets of the form $U_{\kappa}=\sigma\left(U_{C_{\kappa}}\right)\left(C_{\kappa} \in \mathbb{R}^{+}, \sigma \infty=\kappa\right)$. To construct a fundamental domain for $\Gamma_{\kappa}$ in $U_{\kappa}$, it is sufficient to give a fundamental domain $F_{\kappa}^{\prime}$ for $\sigma^{-1} \Gamma_{\kappa} \sigma=\left(\sigma^{-1} \Gamma \sigma\right)_{\infty}$ in $U_{C_{\kappa}}$, and then the set $F_{\kappa}:=\sigma\left(F_{\kappa}^{\prime}\right)$ is a fundamental domain for $\Gamma_{\kappa}$ in $U_{\kappa}$. Recall that $\mathbf{t}_{\kappa}$ is a lattice in $\mathbb{R}^{n}$ and that $\log \Lambda_{\kappa}$ has rank $n-1$ and it is contained in the hyperplane $V$ defined in (3). Let $P_{\kappa}$ be a fundamental parallelotope for $\mathbf{t}_{\kappa}$ in $\mathbb{R}^{n}$ and let $Q_{\kappa}$ be a fundamental parallelotope for $\log \Lambda_{\kappa}$ in the vector space $V$. It is easy to see that

$$
F_{\kappa}^{\prime}=\left\{z=x+i y \in U_{C_{\kappa}}: x \in P_{\kappa}, \log \frac{y}{(N y)^{1 / n}} \in Q_{\kappa}\right\}
$$

is a fundamental domain for $\sigma^{-1} \Gamma_{\kappa} \sigma$ in $U_{C_{\kappa}}$.

To express $F_{\kappa}$ in a simple way, we introduce the coordinates at the cusp $\kappa$. For this we fix a scaling element $\sigma_{\kappa} \in P S L(2, \mathbb{R})^{n}$ such that $\sigma_{\kappa} \infty=\kappa$, a basis $b_{1}^{\kappa}, \ldots, b_{n}^{\kappa}$ in $\mathbb{R}^{n}$ such that

$$
P_{\kappa}=\left\{b \in \mathbb{R}^{n}: b=\sum_{k=1}^{n} t_{k} b_{k}^{\kappa}, 0 \leq t_{k}<1 \text { for } 1 \leq k \leq n\right\}
$$

is a fundamental parallelotope for $\mathbf{t}_{\kappa}$, and a basis $a_{1}^{\kappa}, \ldots, a_{n-1}^{\kappa}$ in $V$ such that

$$
Q_{\kappa}=\left\{a \in V: a=\sum_{l=1}^{n-1} s_{l} a_{l}^{\kappa}, 0 \leq s_{l}<1 \text { for } 1 \leq l \leq n-1\right\}
$$

is a fundamental parallelotope for $\log \Lambda_{\kappa}$. Note that $a_{l}^{\kappa}=\log \varepsilon_{l}^{\kappa}$, where $\varepsilon_{1}^{\kappa}, \ldots, \varepsilon_{n-1}^{\kappa}$ generate the group $\Lambda_{\kappa}$. Then for any $z \in \mathbb{H}^{n}$ we write $\sigma_{\kappa}^{-1} z=x^{\prime}+i y^{\prime}$ and

$$
\begin{gathered}
x^{\prime}=X_{1}^{\kappa}(z) b_{1}^{\kappa}+\cdots+X_{n}^{\kappa}(z) b_{n}^{\kappa} \\
Y_{0}^{\kappa}(z)=N y^{\prime}, \quad \log \frac{y^{\prime}}{\left(N y^{\prime}\right)^{1 / n}}=Y_{1}^{\kappa}(z) \log \varepsilon_{1}^{\kappa}+\cdots+Y_{n-1}^{\kappa}(z) \log \varepsilon_{n-1}^{\kappa}
\end{gathered}
$$

Once the element $\sigma_{\kappa}$ and the bases above are fixed, the numbers $X_{k}^{\kappa}(z)$ and $Y_{l}^{\kappa}(z)$ are uniquely determined and called the coordinates of $z$ at the cusp $\kappa$. We may simply write $X_{k}^{\kappa}$ and $Y_{l}^{\kappa}$. Note that we also use the notation $Y_{0}(z)=N y$ or simply $Y_{0}$ (which is the same as $Y_{0}^{\infty}(z)$ above once $\infty$ is a cusp and we choose $\left.\sigma_{\infty}=i d\right)$. The fundamental domain of $\Gamma_{\kappa}$ in $U_{\kappa}$ can be expressed in a simple form in terms of these coordinates:

$$
F_{\kappa}=\left\{z \in U_{\kappa}: 0 \leq X_{1}^{\kappa}, \ldots, X_{n}^{\kappa}<1,0 \leq Y_{1}^{\kappa}, \ldots, Y_{n-1}^{\kappa}<1\right\}
$$

If $\kappa$ and $\kappa^{\prime}$ are inequivalent cusps of $\Gamma$, then there exists neighbourhoods $U$ and $U^{\prime}$ of $\kappa$ and $\kappa^{\prime}$, respectively, such that $\gamma(U) \cap U^{\prime}=\emptyset$ holds for any $\gamma \in \Gamma$ (see Lemma $2.9_{2}$ in [7]). Hence, if we fix a maximal set $\mathcal{S}$ of $\Gamma$-inequivalent cusps, then a real number $C>0$ can be chosen such that the sets $U_{\kappa}=\sigma_{\kappa}\left(U_{C}\right)$ are pairwise disjoint for the cusps in $\mathcal{S}$, and the corresponding sets $F_{\kappa}$ contain at most 1 point from every $\Gamma$ orbit. Finally, the fundamental domain for $\Gamma$ is given in the form

$$
F=F_{0} \cup\left(\bigcup_{\kappa \in \mathcal{S}} F_{\kappa}\right)
$$

where $F_{0} \subset \mathbb{H}^{n}$ is compact.
1.2.5. Classification of the elements of $\Gamma$. Recall that an element id $\neq \gamma \in P S L(2, \mathbb{R})$ is called elliptic, parabolic or hyperbolic, if $|\operatorname{tr} \gamma|<2,|\operatorname{tr} \gamma|=2$ or $|\operatorname{tr} \gamma|>2$, respectively. An element of $\Gamma$ is called totally elliptic or totally parabolic, if each of its components are elliptic or parabolic, respectively. If there are elements of different types among the components, then this element is called mixed. Note that if one component of an element is parabolic, then so are the others by Theorem 1.1. Hence a mixed element consists of elliptic and hyperbolic components.

Before we turn to the case when every component is hyperbolic we examine the fixed points of the elements. A totally elliptic element has a single fixed point $z \in \mathbb{H}^{n}$. Since $\Gamma$ acts discontinuously on $\mathbb{H}^{n}, z$ has a neighborhood $U$ such that the set $\{\gamma \in \Gamma: \gamma U \cap U \neq \emptyset\}$ is finite. This means that a totally elliptic element must be of finite order. A totally parabolic element fixes a single point in $(\mathbb{R} \cup\{\infty\})^{n}$. Since $\Gamma$ is irreducible, the parabolic fixed points are exactly the cusps of $\Gamma$ (see Theorem 3 in [11). A mixed element with $1 \leq m<n$ hyperbolic components fixes $2^{m}$ points in $(\mathbb{H} \cup \mathbb{R} \cup\{\infty\})^{n}$. If every component of $\gamma \in \Gamma$ is hyperbolic, then $\gamma$ fixes $2^{n}$ points in $(\mathbb{R} \cup\{\infty\})^{n}$. Such an element is called hyperbolic-parabolic if there is cusp among its fixed points. Otherwise it is called totally hyperbolic.
1.3. Fourier expansion of automorphic forms. A function $f: \mathbb{H}^{n} \rightarrow \mathbb{C}$ is called an automorphic function with respect to the group $\Gamma$ if it is invariant under the action of $\Gamma$, that is, $f(\gamma z)=f(z)$ holds for every $z \in \mathbb{H}^{n}$ and $\gamma \in \Gamma$. An automorphic form $u$ is a smooth automorphic function which is an eigenfunction of the Laplace operators

$$
\Delta_{k}=y_{k}^{2}\left(\frac{\partial^{2}}{\partial x_{k}^{2}}+\frac{\partial^{2}}{\partial y_{k}^{2}}\right), \quad(k=1, \ldots, n)
$$

that is, for which the equations $\left(\Delta_{k}+\lambda_{k}\right) u=0$ hold with some $\lambda_{k} \in \mathbb{C}$. We write these eigenvalues in the form $\lambda_{k}=s_{k}\left(1-s_{k}\right)$ for some $s_{k} \in \mathbb{C}$.

If $u$ is an automorphic form and $\kappa$ is a cusp, then $u\left(\sigma_{\kappa} z\right)$ is invariant under the action of the translation operator $T_{\alpha} u=u\left(z_{1}+\alpha_{1}, \ldots, z_{n}+\alpha_{n}\right)$ for any $\alpha \in \mathbf{t}_{\kappa}$, hence it has the Fourier expansion

$$
u(z)=\sum_{l \in \mathbf{t}_{\mathrm{k}}^{*}} \phi(y, l) e^{2 \pi i<l, x>},
$$

where $x=\left(x_{1}, \ldots, x_{n}\right), y=\left(y_{1}, \ldots, y_{n}\right)$ and $\mathbf{t}_{\kappa}^{*}=\left\{v \in \mathbb{R}^{n}:\langle v, w\rangle \in \mathbb{Z}\right.$ for any $\left.w \in \mathbf{t}_{\kappa}\right\}$ is the dual lattice of $\mathbf{t}_{\kappa}$. In general, the dual lattice is given in the following way. If $L=A\left(\mathbb{Z}^{n}\right) \subset \mathbb{R}^{n}$ is a lattice, where $A \in G L\left(\mathbb{R}^{n}\right)$, then its dual is given by $L^{*}=\left(A^{-1}\right)^{T}\left(\mathbb{Z}^{n}\right)$. In our case the columns of $A$ are the vectors $b_{1}^{\kappa}, \ldots, b_{n}^{\kappa}$.

For a vector $\alpha \in \mathbb{R}^{n}$ and a lattice $L \subset \mathbb{R}^{n}$ we define

$$
\alpha L=\left\{\left(\alpha_{1} l_{1}, \ldots, \alpha_{n} l_{n}\right):\left(l_{1}, \ldots, l_{n}\right) \in L\right\} .
$$

It is easy to see, that if $a \in \mathbf{t}_{\kappa}$ and $\varepsilon \in \Lambda_{\kappa}$, then $\varepsilon a \in \mathbf{t}_{\kappa}$, and as $\Lambda_{\kappa}$ is a group, we have in fact $\varepsilon \mathbf{t}_{\kappa}=\mathbf{t}_{\kappa}$. If $E$ is the diagonal matrix with the coordinates of $\varepsilon$ in its diagonal, then

$$
\begin{align*}
\varepsilon \mathbf{t}_{\kappa}^{*} & =E\left(A^{-1}\right)^{T}\left(\mathbb{Z}^{n}\right)=E^{T}\left(A^{-1}\right)^{T}\left(\mathbb{Z}^{n}\right)  \tag{5}\\
& =\left(A^{-1} E\right)^{T}\left(\mathbb{Z}^{n}\right)=\left(\left(E^{-1} A\right)^{-1}\right)^{T}\left(\mathbb{Z}^{n}\right)=\left(\varepsilon^{-1} \mathbf{t}_{\kappa}\right)^{*}=\mathbf{t}_{\kappa}^{*} .
\end{align*}
$$

Since the Laplace operator commutes with the action of $\operatorname{PSL}(2, \mathbb{R})^{n}, u\left(\sigma_{\kappa} z\right)$ is still an eigenfunction of the Laplacians, and its Fourier coefficients can be expressed by means of its eigenvalues and the modified Bessel function of the second kind, denoted by $K_{\nu}(z)$ (see Theorem 5.1 in [14]):
Theorem 1.2. Let $u$ be an automorphic form that satisfies the growth condition $u\left(\sigma_{\kappa} z\right)=$ $o\left(e^{2 \pi y_{k}}\right)$ as $y_{k} \rightarrow \infty(k=1, \ldots, n)$ (where $\kappa$ is a cusp for $\Gamma$ ). Then $u\left(\sigma_{\kappa} z\right)$ admits a Fourier expansion of the form

$$
\begin{equation*}
u\left(\sigma_{\kappa} z\right)=\sum_{l \in \mathbf{t}_{\kappa}^{*}} a_{\kappa}(l, y) e^{2 \pi i<l, x>} \tag{6}
\end{equation*}
$$

where

$$
a_{\kappa}(l, y)=c_{\kappa}(l) \sqrt{y_{1} \ldots y_{n}} K_{s_{1}-1 / 2}\left(2 \pi\left|l_{1}\right| y_{1}\right) \ldots K_{s_{n}-1 / 2}\left(2 \pi\left|l_{n}\right| y_{n}\right)
$$

for $l \neq 0$, while $a_{\kappa}(0, y)=: a_{\kappa}(y)$ is the linear combination of two terms of the form $y_{1}^{s_{1}} \ldots y_{n}^{s_{n}}$ and $y_{1}^{1-s_{1}} \ldots y_{n}^{1-s_{n}}$, where the numbers $s_{k} \in \mathbb{C}$ are such that $\left(\Delta_{k}+s_{k}\left(1-s_{k}\right)\right) u=0$.

In the following we always assume that an automorphic form $u$ satisfies the growth condition in Theorem 1.2 and hence admits the Fourier expansion (6). Since $u\left(\sigma_{\kappa} z\right)$ remains unchanged if we substitute $z \mapsto \varepsilon z$ for any $\varepsilon \in \Lambda_{\kappa}$, comparing the Fourier coefficients, using (5) and also that $N \varepsilon=1$ holds, we obtain that $c_{\kappa}(\varepsilon l)=c_{\kappa}(l)$ for every cusp $\kappa$ and for every $\varepsilon \in \Lambda_{\kappa}, l \in \mathbf{t}_{\kappa}^{*} \backslash 0$. Also, well-known bounds for the Bessel function $K_{\nu}(z)$ and the absolute convergence of the sum in (6) easily imply the trivial bound $c_{\kappa}(l) \ll e^{\delta|N(l)|^{1 / n}}$ for any $\delta>0$, where the implied constant depends on $\delta$. From this the exponential decay of $u\left(\sigma_{\kappa} z\right)-a_{\kappa}(y)$ "near the cusp" can be derived. Though we will not detail its (technical but straightforward) proof, the precise statement is given in the following

Proposition 1.3. Let $u$ be an automorphic form with respect to $\Gamma$ with Laplace eigenvalues $s_{k}\left(1-s_{k}\right)$ that satisfies $u\left(\sigma_{\kappa} z\right)=o\left(e^{2 \pi y_{k}}\right)$ for any $1 \leq k \leq n$. Assume that $\log \left(\frac{y}{(N y)^{1 / n}}\right)$ is bounded, then $u\left(\sigma_{\kappa} z\right)-a_{0}(y)=O\left(e^{-C N(y)^{1 / n}}\right)$ for some constant $C>0$ if $N(y)$ is big enough. The implied constant depends on the bounds on $N(y)$ and $\log \left(\frac{y}{(N y)^{1 / n}}\right)$.

Recall that above we fixed the generators $\varepsilon_{1}^{\kappa}, \ldots, \varepsilon_{n-1}^{\kappa}$ of $\Lambda_{\kappa}$. Their coordinates will be denoted by $\left(\varepsilon_{j}^{\kappa}\right)^{(k)}(k=1, \ldots, n)$. If the zeroth Fourier coefficient of $u\left(\sigma_{\kappa} z\right)$ is non-zero, then the comparison of them on both sides of the equation $u\left(\sigma_{\kappa} z\right)=u\left(\sigma_{\kappa}\left(\varepsilon_{j}^{\kappa} z\right)\right)$ gives for each $1 \leq j \leq n-1$ (similarly as in section II. 1 of [6]) that

$$
\begin{equation*}
\prod_{k=1}^{n}\left[\left(\varepsilon_{j}^{\kappa}\right)^{(k)}\right]^{s_{k}}=1 \tag{7}
\end{equation*}
$$

Let us define $s:=\left(s_{1}+\cdots+s_{n}\right) / n$, then by (17) we have

$$
\left(s_{1}, \ldots, s_{n}\right)\left[\begin{array}{cccc}
1 & \log \left(\varepsilon_{1}^{\kappa}\right)^{(1)} & \ldots & \log \left(\varepsilon_{n-1}^{\kappa}\right)^{(1)}  \tag{8}\\
1 & \log \left(\varepsilon_{1}^{\kappa}\right)^{(2)} & \ldots & \log \left(\varepsilon_{n-1}^{\kappa}\right)^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
1 & \log \left(\varepsilon_{1}^{\kappa}\right)^{(n)} & \ldots & \log \left(\varepsilon_{n-1}^{\kappa}\right)^{(n)}
\end{array}\right]=\left(n s, 2 \pi i m_{u, \kappa}^{(1)}, \ldots, 2 \pi i m_{u, \kappa}^{(n-1)}\right)
$$

for some $m_{u, \kappa}=\left(m_{u, \kappa}^{(1)}, \ldots, m_{u, \kappa}^{(n-1)}\right)^{T} \in \mathbb{Z}^{n-1}$. Let us denote the matrix above by $\mathcal{E}_{\kappa}$. Since the vectors $\log \varepsilon_{i}^{\kappa}$ form a basis in the trace 0 subspace of $\mathbb{R}^{n}$ and the first column of $\mathcal{E}_{\kappa}$ is not in that subspace, we get that $\mathcal{E}_{\kappa}$ is invertible. Its inverse is of the form

$$
\mathcal{E}_{\kappa}^{-1}=\left[\begin{array}{cccc}
1 / n & 1 / n & \cdots & 1 / n \\
\left(e_{1}^{\kappa}\right)^{(1)} & \left(e_{1}^{\kappa}\right)^{(2)} & \cdots & \left(e_{1}^{\kappa}\right)^{(n)} \\
\vdots & \vdots & \ddots & \vdots \\
\left(e_{n-1}^{\kappa}\right)^{(1)} & \left(e_{n-1}^{\kappa}\right)^{(2)} & \cdots & \left(e_{n-1}^{\kappa}\right)^{(n)}
\end{array}\right]
$$

and the values $s_{1}, \ldots, s_{n}$ are determined by $s$ and $m_{u, \kappa}$ through

$$
\left(s_{1}, \ldots, s_{n}\right)=\left(n s, 2 \pi i m_{u, \kappa}^{(1)}, \ldots, 2 \pi i m_{u, \kappa}^{(n-1)}\right) \mathcal{E}_{\kappa}^{-1}
$$

That is,

$$
\begin{equation*}
s_{k}=s+\sum_{j=1}^{n-1} 2 \pi i m_{u, \kappa}^{(j)}\left(e_{j}^{\kappa}\right)^{(k)}, \quad y_{k}^{s_{k}}=y_{k}^{s} \exp \left(\sum_{j=1}^{n-1} 2 \pi i m_{u, \kappa}^{(j)}\left(e_{j}^{\kappa}\right)^{(k)} \log y_{k}\right) \tag{9}
\end{equation*}
$$

For a cusp $\kappa$ and for any $m \in \mathbb{Z}^{n-1}$ set

$$
\begin{equation*}
\lambda_{m}^{\kappa}(y)=\exp \left(\sum_{k=1}^{n} \sum_{j=1}^{n-1} 2 \pi i m_{j}\left(e_{j}^{\kappa}\right)^{(k)} \log y_{k}\right)=\prod_{k=1}^{n} \prod_{j=1}^{n-1} y_{k}^{2 \pi i m_{j}\left(e_{j}^{\kappa}\right)^{(k)}} \tag{10}
\end{equation*}
$$

for every $y \in\left(\mathbb{R}^{+}\right)^{n}$. With this notation we may write the zeroth coefficient of $u\left(\sigma_{\kappa} z\right)$ in the following way:

$$
a_{\kappa}(y)=\eta_{\kappa}\left(y_{1} \ldots y_{n}\right)^{s} \lambda_{m_{u, \kappa}}^{\kappa}(y)+\phi_{\kappa}\left(y_{1} \ldots y_{n}\right)^{1-s} \lambda_{-m_{u, \kappa}^{\kappa}}(y)
$$

Later we will see that $m_{u, \kappa}$ can be assumed to be the same vector $m_{u}$ for every cusp, at least if it can be defined. If however $\eta_{\kappa}=\phi_{\kappa}=0$ holds for all $\kappa \in \mathcal{S}$ (i.e. $u$ is a cusp form), then we simply set $m_{u}=0$. Aside from the next paragraph, in the following we always assume that $0<\operatorname{Re} s<1$ holds whenever the number $s$ is associated with the form $u$.

Later we will make use of a specific family automorphic forms, namely the Eisenstein series that are defined as follows. Let $\kappa \in \mathcal{S}$ be a cusp. For an $s \in \mathbb{C}$ and $m \in \mathbb{Z}^{n-1}$ the Eisenstein series belonging to $\kappa$ is given by

$$
\begin{equation*}
E_{\kappa}(z, s, m)=\sum_{\gamma \in \Gamma_{\kappa} \backslash \Gamma} y\left(\gamma^{(1)} z_{1}\right)^{s_{1}} \ldots y\left(\gamma^{(n)} z_{n}\right)^{s_{n}} \tag{11}
\end{equation*}
$$

for any $z \in \mathbb{H}^{n}$, where the exponents $s_{1}, \ldots, s_{n}$ are defined as in (9). This series converges absolutely and uniformly on compact subsets for $\operatorname{Re} s>1$. Also, $E_{\kappa}(z, s, m)$ (as a function in the variable $z$ ) is clearly a $\Gamma$-invariant eigenfunction of the Laplacians and (as a function of $s$ ) it can be continued meromorphically to the whole complex plane. Moreover, for a cusp $\kappa^{\prime} \in \mathcal{S}$ the coefficient $\eta_{\kappa^{\prime}}$ in the Fourier coefficient $a_{\kappa^{\prime}}(y)$ is 1 if $\kappa^{\prime}=\kappa$ and 0 otherwise. For the details see chapter II and also section III. 4 of [6].

## 2. The geometric trace

2.1. The automorphic kernel. In the following we fix a compactly supported smooth function $\psi \in C_{0}^{\infty}\left(\mathbb{R}^{n}\right)$ and define the point pair invariant kernel

$$
k_{\psi}(z, w)=k(z, w)=\psi\left(\frac{\left|z_{1}-w_{1}\right|^{2}}{\operatorname{Im} z_{1} \cdot \operatorname{Im} w_{1}}, \ldots, \frac{\left|z_{n}-w_{n}\right|^{2}}{\operatorname{Im} z_{n} \cdot \operatorname{Im} w_{n}}\right)=\psi\left(\frac{|z-w|^{2}}{\operatorname{Im} z \cdot \operatorname{Im} w}\right)
$$

for every $z, w \in \mathbb{H}^{n}$. Invariance means that $k(z, w)=k(\sigma z, \sigma w)$ holds for every $z, w \in \mathbb{H}^{n}$ and $\sigma \in \operatorname{PSL}(2, \mathbb{R})^{n}$. To avoid long formulae we often use the latter compact notation for $\psi$ and its transforms defined below. In these cases the operations on vectors always indicate coordinate-wise operations. The automorphic kernel $K(z, w)$ is given by the sum

$$
\begin{equation*}
K(z, w)=\sum_{\gamma \in \Gamma} k(z, \gamma w) \tag{12}
\end{equation*}
$$

that clearly defines an automorphic function w.r.t. $\Gamma$.
The following transformations of $\psi$ often occur in computations:

$$
\begin{align*}
Q(w)=Q\left(w_{1}, \ldots, w_{n}\right) & :=\int_{w_{n}}^{\infty} \cdots \int_{w_{1}}^{\infty} \frac{\psi\left(t_{1}, \ldots, t_{n}\right)}{\sqrt{t_{1}-w_{1}} \ldots \sqrt{t_{n}-w_{n}}} d t_{1} \ldots d t_{n}, \\
g(u)=g\left(u_{1}, \ldots, u_{n}\right) & :=Q\left(e^{u_{1}}+e^{-u_{1}}-2, \ldots, e^{u_{n}}+e^{-u_{n}}-2\right),  \tag{13}\\
h(r)=h\left(r_{1}, \ldots, r_{n}\right) & :=\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g\left(u_{1}, \ldots, u_{n}\right) e^{i \sum_{k=1}^{n} r_{k} u_{k}} d u_{1} \ldots d u_{n} .
\end{align*}
$$

Note that this is the multidimensional version of the Harish-Chandra transform. Since $\psi$ is a compactly supported smooth function, $g$ is also a smooth function with compact support and hence $h$ is rapidly decreasing.

The inverses of the transforms above are

$$
\begin{aligned}
g\left(u_{1}, \ldots, u_{n}\right) & =\frac{1}{(2 \pi)^{n}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h\left(r_{1}, \ldots, r_{n}\right) e^{-i \sum_{k=1}^{n} r_{k} u_{k}} d r_{1} \ldots d r_{n} \\
Q\left(w_{1}, \ldots, w_{n}\right) & =g\left(2 \log \left(\sqrt{\frac{w_{1}}{4}+1}+\sqrt{\frac{w_{1}}{4}}\right), \ldots, 2 \log \left(\sqrt{\frac{w_{n}}{4}+1}+\sqrt{\frac{w_{n}}{4}}\right)\right), \\
\psi\left(t_{1}, \ldots, t_{n}\right) & =\frac{(-1)^{n}}{\pi^{n}} \int_{t_{n}}^{\infty} \cdots \int_{t_{1}}^{\infty} \frac{\frac{\partial^{n} Q}{\partial w_{1} \ldots \partial w_{n}}\left(w_{1}, \ldots, w_{n}\right)}{\sqrt{w_{1}-t_{1}} \ldots \sqrt{w_{n}-t_{n}}} d w_{1} \ldots d w_{n}
\end{aligned}
$$

(see Proposition I.2.2 in [6]).
2.2. The geometric trace. Now we turn to the multidimensional version of the generalized Selberg trace formula, more precisely, to the geometric trace that is computed by collecting the terms of the conjugacy classes in the sum (12). As in [1], our starting point is the integral

$$
\operatorname{Tr}_{u} K=\int_{F} K(z, z) u(z) \mu(z)
$$

where $u$ is a fixed automorphic form that satisfies the growth condition $u(z)=o\left(e^{2 \pi y_{k}}\right)$ for $k=1, \ldots, n, F$ is the fundamental domain of $\Gamma$, and $\mu$ is the product measure on $\mathbb{H}^{n}$ obtained from the measure $y^{-2} d x d y$ on $\mathbb{H}$. Note that $\operatorname{Re} s_{k}<1$ is assumed for each $k$ (excluding the case $u(z)=1$, that would yield the trace formula given in [6]). Since this integral is not necessarily convergent, we work with the truncated trace defined by

$$
\begin{equation*}
\operatorname{Tr}_{u}^{A} K:=\int_{F_{A}} K(z, z) u(z) d \mu(z) \tag{15}
\end{equation*}
$$

for every $A>0$, where

$$
F_{A}=F_{0} \cup\left(\bigcup_{\kappa \in \mathcal{S}} F_{\kappa}^{A}\right)
$$

with $F_{\kappa}^{A}=\left\{z \in F_{\kappa}: Y_{0}^{\kappa}(z) \leq A\right\}$.
Substituting the definition of $K(z, w)$ into (15) and summing over the conjugacy classes in $\Gamma$ we get

$$
\operatorname{Tr}_{u}^{A} K=\sum_{\{\gamma\}} \sum_{\sigma \in\{\gamma\}} \int_{F_{A}} k(z, \sigma z) u(z) d \mu(z)
$$

where $\{\gamma\}$ denotes the conjugacy class of an element $\gamma \in \Gamma$. Note that the conjugacy class of the identity element consists only of itself, and the term that belongs to it is a constant multiple of the integral

$$
\int_{F_{A}} u(z) d \mu(z)
$$

This integral converges as $A \rightarrow \infty$ and the limit is zero since the Laplacians are symmetric operators and the eigenvalues of 1 and $u$ are different.

Our aim is to give the contribution of the different types of classes in this trace. The main result can be summarized in the form

$$
\operatorname{Tr}_{u}^{A} K=\Sigma_{\mathrm{ell}}+\Sigma_{\mathrm{mix}}+\Sigma_{\mathrm{par}}+\Sigma_{\mathrm{hyp}-\mathrm{par}}
$$

where the four terms on the right hand side stand for the contribution of totally elliptic, mixed (and totally hyperbolic), totally parabolic and hyperbolic-parabolic classes, respectively. Note that the totally hyperbolic classes can be handled in the same way as the mixed classes, hence they are melted in a single term above. Since the individual terms are given by lengthy and complicated formulae, we do not give the whole sum in one statement, but split the main result into four theorems below instead. We begin with the contribution of elliptic, mixed and totally hyperbolic classes, here the corresponding results are similar to the ones in [1]. Our main focus is therefore on the parabolic and hyperbolic-parabolic classes, that are handled afterwards.
2.3. Contribution of totally elliptic, mixed and totally hyperbolic classes. In these cases the sum

$$
\sum_{\sigma \in\{\gamma\}} \int_{F_{A}} k(z, \sigma z) u(z) d \mu(z)
$$

in $\operatorname{Tr}_{u}^{A} K$ belonging to the class $\{\gamma\}$ actually converges as $A \rightarrow \infty$ (one can see this by analysing the detailed computations in the proofs of the next two theorems). We will also see that there are only finitely many classes for which the sum above is non-zero, hence we can integrate over $F$ instead of $F_{A}$ (by including an $o(A)$ term as well).

Since $\sigma_{1}^{-1} \gamma \sigma_{1}=\sigma_{2}^{-1} \gamma \sigma_{2}$ holds if and only if $\sigma_{2} \sigma_{1}^{-1}$ is in the centralizer $C(\gamma)$ of $\gamma$, and this is equivalent to $\sigma_{2} \in C(\gamma) \sigma_{1}$, we get that

$$
\begin{equation*}
T_{\gamma}:=\sum_{\sigma \in\{\gamma\}} \int_{F} k(z, \sigma z) u(z) d \mu(z)=\sum_{\sigma \in C(\gamma) \backslash \Gamma} \int_{F} k\left(z, \sigma^{-1} \gamma \sigma z\right) u(z) d \mu(z) \tag{16}
\end{equation*}
$$

As $k(\varrho z, \varrho w)=k(z, w)$ holds for every $\varrho \in P S L(2, \mathbb{R})^{n}$ and $u$ is invariant under the action of $\Gamma$, this last sum is

$$
\sum_{\sigma \in C(\gamma) \backslash \Gamma} \int_{F} k(\sigma z, \gamma \sigma z) u(\sigma z) d \mu(z)=\int_{C(\gamma) \backslash \mathbb{H}^{n}} k(z, \gamma z) u(z) d \mu(z),
$$

and for every $\varrho \in P S L(2, \mathbb{R})^{n}$ this can be written as

$$
\begin{equation*}
\int_{\varrho^{-1}\left(C(\gamma) \backslash \mathbb{H}^{n}\right)} k(\varrho z, \gamma \varrho z) u(\varrho z) d \mu(z)=\int_{\left(\varrho^{-1} C(\gamma) \varrho\right) \backslash \mathbb{H}^{n}} k\left(z, \varrho^{-1} \gamma \varrho z\right) u(\varrho z) d \mu(z) \tag{17}
\end{equation*}
$$

since the measure $\mu$ and the function $k$ are $\operatorname{PSL}(2, \mathbb{R})^{n}$ invariant. Note that $\left(\varrho^{-1} C(\gamma) \varrho\right) \backslash \mathbb{H}^{n}$ is nothing else but the fundamental domain of the group $\varrho^{-1} C(\gamma) \varrho$.

Now we turn to the contribution of totally elliptic classes. Let us first note that by Corollary $2.14_{1}$ in [7] there are only finitely many such classes, hence $\Sigma_{\text {ell }}$ is a sum of finitely many terms of the form (17).

Before giving the value $\Sigma_{\text {ell }}$ let us fix the following notations. Every elliptic element $\gamma \in \Gamma$ is conjugate in $\operatorname{PSL}(2, \mathbb{R})^{n}$ to an element of the form $\left(R\left(\theta_{\gamma}^{(1)}\right), \ldots, R\left(\theta_{\gamma}^{(n)}\right)\right)$, where

$$
R(\alpha)=\left[\begin{array}{cc}
\cos \alpha & \sin \alpha  \tag{18}\\
-\sin \alpha & \cos \alpha
\end{array}\right]
$$

and the vector $\left(\theta_{\gamma}^{(1)}, \ldots, \theta_{\gamma}^{(n)}\right) \in[0, \pi)^{n}$ depends only on the conjugacy class of $\gamma$.
Besides, let $g_{\lambda}(r):[0, \infty) \rightarrow \mathbb{C}$ be the unique solution of the differential equation

$$
\begin{equation*}
g^{\prime \prime}(r)+\frac{\cosh r}{\sinh r} g^{\prime}(r)=\lambda g(r) \tag{19}
\end{equation*}
$$

satisfying the initial condition $g(0)=1$.
Theorem 2.1. The contribution of the totally elliptic classes in the truncated trace, i.e., the value of $\Sigma_{\text {ell }}$ is

$$
\sum_{\{\gamma\} \text { t.ell. }} \frac{(2 \pi)^{n}}{m_{\gamma}} u\left(z_{\gamma}\right) \int_{0}^{\infty} \ldots \int_{0}^{\infty} \psi\left(S\left(r_{1}, \theta_{\gamma}^{(1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right)\left(\prod_{k=1}^{n} g_{\lambda_{k}}\left(r_{k}\right) \sinh r_{k} d r_{k}\right)+o(A)
$$

where the sum runs over all totally elliptic classes and for every class $\{\gamma\}$ the point $z_{\gamma} \in \mathbb{H}^{n}$ is the fixed point of $\gamma, m_{\gamma} \in \mathbb{N}^{+}$is the order of the centralizer of $\gamma, S(r, \theta)=(2 \sinh r \sin \theta)^{2}$ for any $r, \vartheta \in \mathbb{R},\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is the Laplacian eigenvalue vector of $u$, and the functions $g_{\lambda_{k}}$ and the vector $\left(\theta_{\gamma}^{(1)}, \ldots, \theta_{\gamma}^{(n)}\right)$ are defined above the theorem. Moreover, there are only finitely many totally elliptic conjugacy classes, hence the sum above is finite.

Next we handle the mixed and totally hyperbolic classes, i.e. the classes whose elements have at least one hyperbolic coordinates. For simplicity, we assume that the first $1 \leq m \leq n$ coordinates of the element are hyperbolic, while the following $n-m$ coordinates are elliptic. The results below can easily be reformulated and proved for other distributions of coordinates of different types. Note also that the case $m=n$ is the totally hyperbolic case, at least if the fixed points of the element are not cusps, which is assumed in this section.

Any mixed or totally hyperbolic element $\gamma$ is conjugate in $P S L(2, \mathbb{R})^{n}$ to an element of the form

$$
\begin{equation*}
\nu=\left(D\left(N_{\gamma}^{(1)}\right), \ldots, D\left(N_{\gamma}^{(m)}\right), R\left(\theta_{\gamma}^{(m+1)}\right), \ldots, R\left(\theta_{\gamma}^{(n)}\right)\right) \tag{20}
\end{equation*}
$$

for some $N_{\gamma}^{(k)}>1$ and $\theta_{\gamma}^{(l)} \in[0, \pi)$, where

$$
D(N)=\left[\begin{array}{cc}
N^{1 / 2} & 0  \tag{21}\\
0 & N^{-1 / 2}
\end{array}\right]
$$

and $R(\theta)$ is defined in (18) above. It is not hard to see that all these numbers are determined uniquely by the class of $\gamma$ (and hence the notations $N_{\gamma}^{(k)}$ and $\theta_{\gamma}^{(l)}$ are justified). The number $N_{\gamma}^{(k)}$ is called the norm of $\gamma^{(k)}$. We also set

$$
\begin{equation*}
N\left(\vartheta, \gamma^{(k)}\right):=\frac{N_{\gamma}^{(k)}+\left(N_{\gamma}^{(k)}\right)^{-1}-2}{\cos ^{2} \vartheta} \tag{22}
\end{equation*}
$$

for any $\vartheta \in\left(-\frac{\pi}{2} ; \frac{\pi}{2}\right)$ and $1 \leq k \leq m$.
Let $\varrho_{\gamma} \in P S L(2, \mathbb{R})^{n}$ be an element such that $\nu=\varrho_{\gamma}^{-1} \gamma \varrho_{\gamma}$ holds. To give the contribution of the class $\{\gamma\}$ we need to describe that centralizer $C(\nu)$ of $\nu$ in $\varrho_{\gamma}^{-1} \Gamma \varrho_{\gamma}$. By the results of section I. 5 in [6] the centralizer $C(\gamma)$ of $\gamma$ is a free abelian group of rank $m$. We fix a set of its generators denoted by $\gamma_{1}, \ldots, \gamma_{m}$, then the centralizer $C(\nu) \leq \varrho_{\gamma}^{-1} \Gamma \varrho_{\gamma}$ is $\varrho_{\gamma}^{-1} C(\gamma) \varrho_{\gamma}$ and it is generated by the elements $\nu_{i}=\varrho_{\gamma}^{-1} \gamma_{i} \varrho_{\gamma}$ for $i=1, \ldots, m$. As the $\gamma_{i}$ 's have the same fixed points as $\gamma$ this is true also for the conjugates and therefore

$$
\nu_{i}=\left(D\left(N_{\gamma_{i}}^{(1)}\right), \ldots, D\left(N_{\gamma_{i}}^{(m)}\right), R\left(\theta_{\gamma_{i}}^{(m+1)}\right), \ldots, R\left(\theta_{\gamma_{i}}^{(n)}\right)\right)
$$

This is a somewhat imprecise notation since $N_{\gamma_{i}}^{(k)}>1$ may not be assured for all $k$. This means that $N_{\gamma_{i}}^{(k)}$ is not necessarily the norm of $\nu_{i}^{(k)}$ in the above sense, but it is still determined by the (fixed) generator $\nu_{i}$ and we keep using this notation. The action of the first $m$ coordinates of the elements $\nu_{i}$ is simple: for every $z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{H}^{n}$ and $k=1, \ldots, m$ we have $\left|\nu_{i}^{(k)} z_{k}\right|=N_{\gamma_{i}}^{(k)}\left|z_{k}\right|$ and $\arg \nu_{i}^{(k)} z_{k}=\arg z_{k}$. The next statement follows now easily:

Proposition 2.2. The fundamental domain of the centralizer $C(\nu)=\varrho_{\gamma}^{-1} C(\gamma) \varrho_{\gamma}$ is

$$
F_{C(\nu)}=\left\{z \in \mathbb{H}^{n}:\left(\log \left|z_{1}\right|, \ldots, \log \left|z_{m}\right|\right) \in P_{\gamma}\right\}
$$

where $P_{\gamma}$ is the fundamental parallelepiped of the lattice in $\mathbb{R}^{m}$ generated by the vectors

$$
\left(\log N_{\gamma_{i}}^{(1)}, \ldots, \log N_{\gamma_{i}}^{(m)}\right) \quad(i=1, \ldots, m)
$$

Before the next theorem we introduce one more notation. Let $f_{\lambda}(\vartheta)$ be the unique solution of the differential equation

$$
\begin{equation*}
F^{\prime \prime}(\vartheta)=\frac{\lambda}{\cos ^{2} \vartheta} F(\vartheta) \quad(\vartheta \in(-\pi / 2, \pi / 2)) \tag{23}
\end{equation*}
$$

with the initial condition $f_{\lambda}(0)=1$ and $f_{\lambda}^{\prime}(0)=0$. We are now ready to state

Theorem 2.3. The contribution of the mixed and totally hyperbolic classes in the truncated trace is

$$
\Sigma_{\text {mix }}=\sum_{\{\gamma\} \text { mixed or totally hyperbolic }} T_{\gamma}+o(A),
$$

where for a class $\{\gamma\}$, for which the first $m$ coordinates of $\gamma$ are hyperbolic and the rest are elliptic, the value of $T_{\gamma}$ is

$$
\begin{aligned}
(2 \pi)^{n-m} F_{\gamma}(0, \ldots, 0) \times & \\
\times \int_{0}^{\infty} \cdots \int_{0}^{\infty} \int_{-\frac{\pi}{2}}^{\infty} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi & \left.\psi\left(\vartheta_{1}, \gamma^{(1)}\right), \ldots, N\left(\vartheta_{m}, \gamma^{(m)}\right), S\left(r_{m+1}, \theta_{\gamma}^{(m+1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right) \\
& \left.\times\left(\prod_{k=1}^{m} f_{\lambda_{k}}\left(\vartheta_{k}\right) \frac{d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}\right)\left(\prod_{k=m+1}^{n} g_{\lambda_{k}}\left(r_{k}\right) \sinh r_{k}\right) d r_{k}\right),
\end{aligned}
$$

where $N\left(\vartheta, \gamma^{(k)}\right)$ is defined in (22), $S(r, \theta)=(2 \sinh r \sin \theta)^{2}$ for any $r, \vartheta \in \mathbb{R},\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is the Laplacian eigenvalue vector of $u$, the functions $f_{\lambda_{k}}$ were defined before the theorem, the functions $g_{\lambda_{k}}$ were defined before Theorem [2.1, the vector $\left(\theta_{\gamma}^{(m+1)}, \ldots, \theta_{\gamma}^{(n)}\right)$ is given by (20) and

$$
F_{\gamma}(0, \ldots, 0)=\int_{\left(\log r_{1}, \ldots, \log r_{m}\right) \in P_{\gamma}} u\left(\varrho_{\gamma}^{(1)}\left(r_{1} i\right), \ldots, \varrho_{\gamma}^{(m)}\left(r_{m} i\right), \varrho_{\gamma}^{(m+1)} i, \ldots, \varrho_{\gamma}^{(n)} i\right) \prod_{k=1}^{m} \frac{d r_{k}}{r_{k}}
$$

Here $\varrho_{\gamma} \in \operatorname{PSL}(2, \mathbb{R})^{n}$ is an element for which $\varrho_{\gamma}^{-1} \gamma \varrho$ is of the form (2Q) and the set $P_{\gamma}$ is given in Proposition 2.2. An analogous formula gives the value of $T_{\gamma}$ when the $m$ hyperbolic and $n-m$ elliptic coordinates are distributed differently. Moreover, the value of $T_{\gamma}$ is zero except for finitely many classes.
2.4. Contribution of hyperbolic-parabolic classes. We continue with the contribution of those classes whose elements have only hyperbolic coordinates but also fix a cusp. Let $\gamma=\left(\gamma^{(1)}, \ldots, \gamma^{(n)}\right)$ be such an element and let $x=\left(x_{1}, \ldots, x_{n}\right)$ be a cusp fixed by $\gamma$. That is, $x_{i}$ is a fixed point of the hyperbolic coordinate $\gamma^{(i)}$ and we denote its other one by $x_{i}^{\prime}$. Then, by the results of $\S 20$ in [11] the fixed point $\left(x_{1}^{\prime}, \ldots, x_{n}^{\prime}\right)$ of $\gamma$ is also a cusp.

Every hyperbolic-parabolic class is represented by an element that fixes a cusp $\kappa \in \mathcal{S}$. An element of this type is conjugated by the scaling element $\sigma_{\kappa} \in P S L(2, \mathbb{R})^{n}$ to an element of the form

$$
\gamma_{m, \alpha}^{\kappa}:=\left[\begin{array}{cc}
\left(u_{m}^{\kappa}\right)^{1 / 2} & \alpha\left(u_{m}^{\kappa}\right)^{-1 / 2} \\
0 & \left(u_{m}^{\kappa}\right)^{-1 / 2}
\end{array}\right]
$$

where $\alpha \in \mathbf{t}_{\kappa}, m=\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1} \backslash\{0\}$ and $u_{m}^{\kappa}=\left(\varepsilon_{1}^{\kappa}\right)^{m_{1}} \ldots\left(\varepsilon_{n-1}^{\kappa}\right)^{m_{n-1}} \in \Lambda_{\kappa}$. The cusp in the notation $u_{m}^{\kappa}$ indicates that this unit depends also on the multiplier group $\Lambda_{\kappa}$ and its generators. However, as a byproduct of the proof of this section's main result we also get the following

Proposition 2.4. The multiplier group $\Lambda_{\kappa}$ is the same for any $\kappa \in \mathcal{S}$.
This fact allows us to drop the index from the notation of the multiplier group and we simply write $\Lambda$ in the following. Also, we can and will fix the same generators $\varepsilon_{1}, \ldots, \varepsilon_{n-1}$ for every cusp and therefore it is legitimate to write $u_{m}$ instead of $u_{m}^{\kappa}$. It follows also that the matrix $\mathcal{E}_{\kappa}$ and consequently the integer vector $m_{u, \kappa}$ (defined in ( 8 ) ) are independent of $\kappa$ and will be denoted simply by $\mathcal{E}$ and $m_{u}$, respectively. Note that the lattice $\mathbf{t}_{\kappa}$ does depend on the cusp $\kappa$.

The element $\gamma_{m, \alpha}^{\kappa}$ fixes the points $\infty$ and $q=\frac{\alpha}{1-u_{m}}$ and according to the first paragraph of this section both points are cusps for $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$. We will denote by $\tilde{\kappa}_{m, \alpha} \in \mathcal{S}$ the cusp for $\Gamma$ that can be taken (by an element of $\Gamma$ ) to $\sigma_{\kappa} q$.

The centralizer $C\left(\gamma_{m, \alpha}^{\kappa}\right)$ of $\gamma_{m, \alpha}^{\kappa}$ in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$ is given in $\S 20$ of [11]:
Proposition 2.5. The centralizer $C\left(\gamma_{m, \alpha}^{\kappa}\right)$ of the element $\gamma_{m, \alpha}^{\kappa}$ is a free abelian group of rank $n-1$ generated by some elements $\gamma\left(l_{1}\right), \ldots, \gamma\left(l_{n-1}\right)$, where $l_{j} \in \mathbb{Z}^{n-1} \backslash\{0\}$ for any $1 \leq j \leq n-1$ and

$$
\gamma\left(l_{j}\right)=\left[\begin{array}{cc}
u_{l_{j}}^{1 / 2} & \frac{u_{l_{j}}-1}{u_{m}-1} \alpha u_{l_{j}}^{-1 / 2} \\
0 & u_{l_{j}}^{-1 / 2}
\end{array}\right]
$$

In the following we fix a generating set of elements described in the proposition above and define the $(n-1) \times(n-1)$ matrix

$$
L_{m, \alpha}^{\kappa}:=\left[\begin{array}{cccc}
l_{1}^{(1)} & l_{2}^{(1)} & \ldots & l_{n-1}^{(1)}  \tag{24}\\
l_{1}^{(2)} & l_{2}^{(2)} & \ldots & l_{n-1}^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
l_{1}^{(n-1)} & l_{2}^{(n-1)} & \ldots & l_{n-1}^{(n-1)}
\end{array}\right]
$$

As before, we need to describe the fundamental domain $F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}$ of $C\left(\gamma_{m, \alpha}^{\kappa}\right)$. One shows by induction that $\gamma\left(l_{j}\right)^{h}=\gamma\left(h l_{j}\right)$ holds for any $h \in \mathbb{Z}$. Let $C$ denote the group generated by the (clearly independent) elements

$$
\rho_{l_{j}}:=\left[\begin{array}{cc}
u_{l_{j}}^{1 / 2} & 0  \tag{25}\\
0 & u_{l_{j}}^{-1 / 2}
\end{array}\right] \quad(1 \leq j \leq n-1)
$$

We set $T=\left[\begin{array}{cc}1 & -\frac{\alpha}{1-u_{m}} \\ 0 & 1\end{array}\right]$, then $C\left(\gamma_{\alpha, m}^{\kappa}\right)=T^{-1} C T$ and hence if $F_{C}$ is a fundamental domain for $C$, then

$$
F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}=T^{-1} F_{C}=F_{C}+\frac{\alpha}{1-u_{m}}=F_{C}+q
$$

is a fundamental domain for $C\left(\gamma_{\alpha, m}^{\kappa}\right)$.
It remains to describe the fundamental domain $F_{C}$. As in the case of mixed and totally hyperbolic elements we use polar coordinates. That is, for a point $z=\left(z_{1}, \ldots, z_{n}\right) \in \mathbb{H}^{n}$ we write $z_{k}=r_{k} e^{i\left(\pi / 2+\vartheta_{k}\right)}$ where $r_{k} \in \mathbb{R}^{+}$and $-\frac{\pi}{2}<\vartheta_{k}<\frac{\pi}{2}(1 \leq k \leq n)$. Let $\tilde{P}_{m, \alpha}^{\kappa}$ be the fundamental domain of the $n-1$ dimensional lattice in $V=\left\{a \in \mathbb{R}^{n}: a_{1}+\cdots+a_{n}=0\right\}$ generated by the vectors $v_{j}=l_{j}^{(1)} \log \varepsilon_{1}+\cdots+l_{j}^{(n-1)} \log \varepsilon_{n-1}(1 \leq j \leq n-1)$. For later purposes we specify the choice of $\tilde{P}_{m, \alpha}^{\kappa}$, namely we take the shifted image of the parallelpiped spanned by $v_{1}, \ldots, v_{n-1}$ (in $V$ ) symmetric to the origin. Let us fix the unit vector $\mathbf{1}=\left(n^{-\frac{1}{2}}, \ldots, n^{-\frac{1}{2}}\right)^{T}$, it spans the subspace $V^{\perp}$. If $P_{m, \alpha}^{\kappa}=\left\{t \mathbf{1}+\tilde{P}_{m, \alpha}^{\kappa}: t \in \mathbb{R}\right\}$, then the fundamental domain $F_{C}$ is given by

$$
\left(\vartheta_{1}, \ldots, \vartheta_{n}\right) \in(-\pi / 2 ; \pi / 2), \quad\left(\log r_{1}, \ldots, \log r_{n}\right) \in P_{m, \alpha}^{\kappa}
$$

The contribution of the class belonging to $\gamma_{m, \alpha}^{\kappa}$ in the truncated trace can be divided into two parts. A main term (that diverges as $A \rightarrow \infty$ ) comes from the zeroth Fourier coefficient of $u\left(\sigma_{\kappa}(z+q)\right)$ and the transformed zeroth coefficient of $u\left(\sigma_{\tilde{\kappa}_{m, \alpha}}(z+q)\right)$ while we obtain the remaining convergent part by subtracting these from $u\left(\sigma_{\kappa}(z+q)\right)$. Note that here the
argument is shifted since we will give the result in terms of the fundamental domain $F_{C}$. For any cusp $\kappa^{\prime} \in \mathcal{S}$ we set

$$
M_{\kappa^{\prime}}(z):=\eta_{\kappa^{\prime}} y_{1}^{s_{1}} \ldots y_{n}^{s_{n}}+\phi_{\kappa^{\prime}} y_{1}^{1-s_{1}} \ldots y_{n}^{1-s_{n}}
$$

and subtract $M_{\kappa}(z+q)=M_{\kappa}(z)$ from $u\left(\sigma_{\kappa}(z+q)\right)$, while in the case of $\tilde{\kappa}_{m, \alpha}$ we first apply a transformation that maps $q$ to $\infty$. This is performed by an element $\sigma_{\tilde{\kappa}_{m, \alpha}}^{-1} \gamma \sigma_{\kappa} \in \sigma_{\tilde{\kappa}_{m, \alpha}}^{-1} \Gamma \sigma_{\kappa}$ having the matrix form $\left[\begin{array}{cc}e & f \\ \frac{u_{m}-1}{\delta} & \frac{\alpha}{\delta}\end{array}\right]$ where $\delta=e \alpha+f\left(1-u_{m}\right)$. Using this notation we define

$$
\tilde{u}_{m, \alpha}^{\kappa}(z):=u\left(\sigma_{\kappa}(z+q)\right)-M_{\kappa}(z)-M_{\tilde{\kappa}_{m, \alpha}}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{z}\right)
$$

where $E_{m}:=u_{m}^{-1 / 2}-u_{m}^{1 / 2}$. Note that it is convenient to work with the quantity $E_{m}$ because of its skew-symmetry in $m$. We mention in advance that though the vector $\delta$ depends on the choice of $\sigma_{\tilde{\kappa}_{m, \alpha}}^{-1} \gamma \sigma_{\kappa}$, but the norm of $\delta^{2}$ depends only on $m$ and $\alpha$. Note also that the translation invariance of $M_{\tilde{\kappa}_{m, \alpha}}(z)$ was also used to simplify the defining formula of $\tilde{u}_{m, \alpha}$.

Before the main statement of the section we define an equivalence relation on the lattice $\mathbf{t}_{\kappa}$ for any $\kappa$ : the elements $\alpha, \beta \in \mathbf{t}_{\kappa}$ are said to be equivalent if $\beta=\left(u_{m}-1\right) a+u_{l} \alpha$ holds for some $l \in \mathbb{Z}^{n-1}$ and $a \in \mathbf{t}_{\kappa}$, that is, if and only if $\beta$ and $u_{l} \alpha$ represent the same element in the finite factor group $\mathbf{t}_{\kappa}^{m}:=\mathbf{t}_{\kappa} /\left(u_{m}-1\right) \mathbf{t}_{\kappa}$. These classes (represented as elements of $\mathbf{t}_{\kappa}^{m} / \Lambda$ ) are used to list the hyperbolic-parabolic conjugacy classes in the next result:

Theorem 2.6. The contribution of the hyperbolic-parabolic classes in the truncated trace is

$$
\Sigma_{\mathrm{hyp}-\mathrm{par}}=\delta_{m_{u}} M(A)+\sum_{\kappa \in \mathcal{S}} \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} \sum_{\alpha \in \mathbf{t}_{k}^{m} / \Lambda} C_{\kappa}(m, \alpha)+o(A),
$$

the main term $M(A)$ is given by

$$
\begin{equation*}
\frac{|\operatorname{det} \mathcal{E}|}{n} \sum_{\kappa \in \mathcal{S}}\left(\frac{\eta_{\kappa} A^{s}}{s}+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} g\left(\log u_{m}\right) \tag{26}
\end{equation*}
$$

where $\mathcal{E}$ and $g$ were defined in (8) and (13), respectively, and the term $C_{\kappa}(m, \alpha)$ is

$$
\frac{1}{2} \int_{\log } \tilde{u}_{r \in P_{m, \alpha}^{\kappa}} \tilde{u}_{m, \alpha}^{\kappa}(r i) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(\frac{E_{m}^{2}}{\cos ^{2} \vartheta}\right)\left(\prod_{k=1}^{n} \frac{f_{\lambda_{k}}\left(\vartheta_{k}\right) d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}\right),
$$

where $\left(\lambda_{1}, \ldots, \lambda_{n}\right)$ is the Laplacian eigenvalue vector of $u$ and $f_{\lambda_{k}}(\vartheta)$ is the unique solution of the differential equation (23)) satisfying the initial condition $f_{\lambda_{k}}(0)=1$ and $f_{\lambda_{k}}^{\prime}(0)=0$. Moreover, the terms $C_{\kappa}(m, \alpha)$ and the terms in (26)) are zero for any cusp $\kappa$ for all but finitely many $m$.
2.5. The $\zeta$-functions at the cusps. In this section we introduce the $\zeta$-function belonging to the lattice $\mathbf{t}_{\kappa}$ and the multiplier group $\Lambda$ where $\kappa$ is any cusp in $\mathcal{S}$, these will be needed for the last part of our result in the next section. In fact we define these objects in the following general situation. Let $L \leq \mathbb{R}^{n}$ be a lattice of full rank for which the following holds: if $l=\left(l^{(1)}, \ldots, l^{(n)}\right)^{T} \in L$ and $l^{(k)}=0$ for any $1 \leq k \leq n$, then $l=0$. We define the norm of $l$ by $N l=\prod_{k=1}^{n} l^{(k)}$. Using this terminology, we assume that any non-zero element of $L$ has non-zero norm. In addition, let $M \leq\left(\mathbb{R}^{+}\right)^{n}$ be a discrete norm-1 multiplicative subgroup of rank $n-1$ that acts on $\mathbb{R}^{n}$ by coordinate-wise multiplication so that $L$ is invariant under this action. That is, let us assume that $M \cong \mathbb{Z}^{n-1}$, for every $\varepsilon \in M N \varepsilon=1$ holds, and finally, for any $\varepsilon \in M$ and $l \in L$ we have $\varepsilon l:=\left(\varepsilon^{(1)} l^{(1)}, \ldots, \varepsilon^{(n)} l^{(n)}\right)^{T} \in L$. We remark that
$N(\varepsilon l)=N l$ holds for every $\varepsilon \in M$ and $l \in \mathbb{R}^{n}$ and hence the norm, restricted to the lattice $L$, is in fact defined on $M$-orbits.

Though it is more convenient to give some required technical statements in the above context, it is important to mention that this generality is illusory. Namely, it only simplifies the notations and helps to focus on the important properties of the underlying objects, but does not give a wider point of view since the setting we talk about is basically the same as in our initial situation above. More concretely, a slight modification of the proof of Theorem 4 in [11] gives the following

Proposition 2.7. Assume that $L$ and $M$ are as above. Then there exists a totally real number field $\mathbb{Q} \leq K$ of degree $n$ with embeddings $K^{(1)} \subset \mathbb{R}, \ldots, K^{(n)} \subset \mathbb{R}$ such that for each $\varepsilon=\left(\varepsilon^{(1)}, \ldots, \varepsilon^{(n)}\right)^{T} \in M$ the coordinate $\varepsilon^{(k)}$ is a totally positive unit in $K^{(k)}$ for all $1 \leq k \leq n$ and the coordinates of $\varepsilon$ are conjugates of each other. Moreover, there is a vector $\nu=\left(\nu_{1}, \ldots, \nu_{n}\right)^{T} \in \mathbb{R}^{n}$ with non-zero coordinates such that for any element $\alpha \in \nu \cdot L$ (where the coordinate-wise product is taken) we have $\alpha^{(k)} \in K^{(k)}$ for every $k$ and the coordinates of $\alpha$ are conjugates of each other.

Let us fix the generators $\varepsilon_{1}, \ldots, \varepsilon_{n-1}$ of the group $M$ and define the matrices $\mathcal{E}_{M}$ and $\mathcal{E}_{M}^{-1}$ analogously as $\mathcal{E}\left(=\mathcal{E}_{\kappa}\right)$ and its inverse were defined in Section 1.3. Then the corresponding Grössencharacter-type exponential sum $\lambda_{M, m}$ is given for every $m \in \mathbb{Z}^{n-1}$ in the same way as $\lambda_{m}^{\kappa}$ was in (10).

We are interested in the sum

$$
\begin{equation*}
Z_{L, M}(s, m):=\sum_{0 \neq l \in L / M} \frac{\lambda_{M,-m}(|l|)}{|N l|^{s}} \tag{27}
\end{equation*}
$$

where $s \in \mathbb{C}, m \in \mathbb{Z}^{n-1}$ and $|l|$ is the vector whose coordinates are the absolute values of the corresponding coordinates of $l$. Since $\lambda_{M,-m}$ is a multiplicative function and it is trivial on $M$, the function $Z_{L, M}(s, m)$ is well-defined and it will be called the zeta function belonging to the lattice $L$ and the group $M$. For $L=\mathbf{t}_{\kappa}$ and $M=\Lambda$ we simply write $Z_{\kappa}(s, m)$ and it will be called the zeta function of $\Gamma$ belonging to the cusp $\kappa$. Note that the index of $\lambda$ in (27) is $-m$ in order to obtain the equivalent form

$$
Z_{L, M}(s, m)=\sum_{0 \neq l \in L / M} \frac{1}{\left|l_{1}\right|^{s_{1}} \ldots\left|l_{n}\right|^{s_{n}}}
$$

where $s_{1}, \ldots, s_{n}$ are defined as in (9).
We will show a few properties of these functions, they are summarized in the following lemma:
Lemma 2.8. The sum in (27) converges absolutely and locally uniformly for $\operatorname{Re} s>1$, hence it defines an analytic function on this half-plane. It can be continued meromorphically to the whole plane $\mathbb{C}$ and has no poles on $\mathbb{C}$ except for the case $m=0$ when $s=1$ and $s=0$ are the only poles of $Z_{L, M}(s, 0)$, they are simple and

$$
\operatorname{Res}_{s=1} Z_{L, M}(s, 0)=\frac{2^{n}\left|\operatorname{det} \mathcal{E}_{M}\right|}{n \cdot \operatorname{vol}\left(\mathbb{R}^{n} / L\right)}
$$

Moreover, the completed function

$$
\Xi_{L, M}(s, m):=\pi^{-\frac{n s}{2}}\left(\prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)\right) Z_{L, M}(s, m)
$$

satisfies the functional equation

$$
\operatorname{vol}\left(\mathbb{R}^{n} / L\right)^{1 / 2} \Xi_{L, M}(s, m)=\operatorname{vol}\left(\mathbb{R}^{n} / L^{*}\right)^{1 / 2} \Xi_{L^{*}, M}(1-s,-m)
$$

where $L^{*}$ is the dual lattice of $L$. The convexity bound

$$
Z_{L, M}(s, m)<_{\varepsilon, m}|I m s|^{n(1-\operatorname{Re} s) / 2+\varepsilon}
$$

holds for any $\varepsilon>0$ if $0 \leq \operatorname{Re} s \leq 1$ and $|\operatorname{Ims}|$ is bounded from below by some positive constant. Also, $Z_{L, M}(s, m)<_{\varepsilon, m}|\operatorname{Ims}|^{\varepsilon}$ holds if $\operatorname{Re} s>1$ and $|\operatorname{Im} s|$ is bounded away from zero.
2.6. Contribution of totally parabolic classes. At last we give the contribution of the totally parabolic classes in the geometric trace. In advance of that we introduce some notations. If $\kappa$ is a cusp of $\Gamma$ and $\sigma_{\kappa}$ is the corresponding scaling element, then the zeroth Fourier coefficient of $u\left(\sigma_{\kappa} z\right)$ is $a_{\kappa}(y)=\eta_{\kappa} y_{1}^{s_{1}} \ldots y_{n}^{s_{n}}+\phi_{\kappa} y_{1}^{1-s_{1}} \ldots y_{n}^{1-s_{n}}$. Recall that if at least one of $\eta_{\kappa}$ and $\phi_{\kappa}$ is non-zero (i.e. when $u$ does not vanish at $\kappa$ ) then the zeroth coefficient $a_{\kappa}(y)$ can be written in the form $\eta_{\kappa}\left(y_{1} \ldots y_{n}\right)^{s} \lambda_{m_{u}}(y)+\phi_{\kappa}\left(y_{1} \ldots y_{n}\right)^{1-s} \lambda_{-m_{u}}(y)$, where $s=\frac{s_{1}+\cdots+s_{n}}{n}$ and $m_{u} \in \mathbb{Z}^{n-1}$. If $\eta_{\kappa}=\phi_{\kappa}=0$ for all $\kappa$, then $m_{u}=0$ holds by definition. Note that in (10) the function $\lambda_{m}=\lambda_{m}^{\kappa}$ was defined in terms of the entries of $\mathcal{E}_{\kappa}=\mathcal{E}$ and hence by Proposition 2.4 and its subsequent paragraph $\lambda_{m}$ is independent of the cusp that (from now on) will not be included in our notation. Now we are in the position to give $\Sigma_{\text {par }}$ explicitly:

Theorem 2.9. The contribution of the totally parabolic classes in the truncated trace, i.e., the value of $\Sigma_{\text {par }}$ is

$$
\begin{aligned}
\sum_{\kappa \in \mathcal{S}} \delta_{m_{u}} \frac{|\operatorname{det} \mathcal{E}|}{n}\left(\frac{\eta_{\kappa} A^{s}}{s}\right. & \left.+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) g(0) \\
& +\operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right)\left(\eta_{\kappa} Z_{\kappa}\left(1-s,-m_{u}\right) F(0)+\phi_{\kappa} Z_{\kappa}\left(s, m_{u}\right) \tilde{F}(0)\right)+o(1)
\end{aligned}
$$

as $A \rightarrow \infty$, where

$$
F(S)=\int_{\left(\mathbb{R}^{+}\right)^{n}} \psi\left(t^{2}\right) \prod_{k=1}^{n} t_{k}^{S-s_{k}} d t_{k} \quad \text { and } \quad \tilde{F}(S)=\int_{\left(\mathbb{R}^{+}\right)^{n}} \psi\left(t^{2}\right) \prod_{k=1}^{n} t_{k}^{S+s_{k}-1} d t_{k} .
$$

The values $F(0)$ and $\tilde{F}(0)$ can be expressed in terms of the function $h$ defined in (13):

$$
F(0)=\left(\frac{i}{2^{2-s} \pi^{2}}\right)^{n}\left(\prod_{k=1}^{n} \Gamma\left(\frac{1-s_{k}}{2}\right)^{2}\right) \int_{\left(\mathbb{R}^{+}\right)^{n}} h(r) \prod_{k=1}^{n} \frac{\Gamma\left(\frac{s_{k}}{2}+i r_{k}\right)}{\Gamma\left(\frac{2-s_{k}}{2}+i r_{k}\right)} r_{k} d r_{k}
$$

and

$$
\tilde{F}(0)=\left(\frac{i}{2^{s+1} \pi^{2}}\right)^{n}\left(\prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)^{2}\right) \int_{\left(\mathbb{R}^{+}\right)^{n}} h(r) \prod_{k=1}^{n} \frac{\Gamma\left(\frac{1-s_{k}}{2}+i r_{k}\right)}{\Gamma\left(\frac{s_{k}+1}{2}+i r_{k}\right)} r_{k} d r_{k}
$$

## 3. Proofs of the theorems

3.1. Proof in the totally elliptic case. We first prove Theorem 2.1. As it was already mentioned, there are only finitely many elliptic conjugacy classes by Corollary $2.14_{1}$ in [7]), hence it remains to show the formula for $T_{\gamma}$ (defined in (161)). We prove by induction. Since our argument is very similar to the one in [1], we only sketch the induction step.

Let $\gamma \in \Gamma$ be a totally elliptic element with the elliptic fixed point $z_{\gamma} \in \mathbb{H}^{n}$. The centralizer $C(\gamma)$ consists of the elements in $\Gamma$ which leave the point $z_{\gamma}$ fixed (see [11], p. 37) and the stabilizer $\Gamma_{z_{\gamma}}$ of $z_{\gamma}$ in $\Gamma$ is a finite cyclic group (see Remark 2.14 in [7]). Let us denote by $m_{\gamma}$ the order of $C(\gamma)$. Every elliptic element in $\operatorname{PSL}(2, \mathbb{R})$ is conjugate to an element of the
form $\left[\begin{array}{cc}\cos \theta & \sin \theta \\ -\sin \theta & \cos \theta\end{array}\right]$, hence the generator $\gamma_{0}$ of $C(\gamma)$ can be chosen so that it is conjugate in $\operatorname{PSL}(2, \mathbb{R})^{n}$ to the element

$$
\gamma_{0}^{\prime}=\left(\left[\begin{array}{cc}
\cos \frac{\pi}{m_{\gamma}} & \sin \frac{\pi}{m_{\gamma}} \\
-\sin \frac{\pi}{m_{\gamma}} & \cos \frac{\pi}{m_{\gamma}}
\end{array}\right],\left[\begin{array}{cc}
\cos \frac{l_{2} \pi}{m_{\gamma}} & \sin \frac{l_{2} \pi}{m_{\gamma}} \\
-\sin \frac{l_{2} \pi}{m_{\gamma}} & \cos \frac{l_{2} \pi}{m_{\gamma}}
\end{array}\right], \ldots,\left[\begin{array}{cc}
\cos \frac{l_{n} \pi}{m_{\gamma}} & \sin \frac{l_{n} \pi}{m_{\gamma}} \\
-\sin \frac{l_{n} \pi}{m_{\gamma}} & \cos \frac{l_{n} \pi}{m_{\gamma}}
\end{array}\right]\right)
$$

where $l_{k} \in \mathbb{Z}$ with $\operatorname{gcd}\left(l_{k}, m_{\gamma}\right)=1$ for every $k=2, \ldots, n$. Let us write $\gamma^{\prime}=\varrho^{-1} \gamma \varrho$, we give the fundamental domain $F_{C\left(\gamma^{\prime}\right)}$ of $C\left(\gamma^{\prime}\right)=\left\langle\gamma_{0}^{\prime}\right\rangle \leq \varrho^{-1} \Gamma \varrho$. The first coordinate of $\gamma_{0}^{\prime}$ is a rotation around the point $i \in \mathbb{H}$ by the angle $2 \pi / m_{\gamma}$, therefore every $C\left(\gamma^{\prime}\right)$-orbit has exactly one point in the set $F_{0} \times \mathbb{H}^{n-1}$, where $F_{0} \subset \mathbb{H}$ is a sector enclosed by two half-lines with endpoint $i$ and angle $2 \pi / m_{\gamma}$. Note that in fact each coordinate is a rotation around $i$ which means that $\varrho$ takes the point $(i, \ldots, i)$ to the fixed point $z_{\gamma}$ of $\gamma$. Now by (17) we have

$$
T_{\gamma}=\int_{F_{C\left(\gamma^{\prime}\right)}} k\left(z, \gamma^{\prime} z\right) u(\varrho z) d \mu(z)=\frac{1}{m_{\gamma}} \int_{\mathbb{H}^{n} n} k\left(z, \gamma^{\prime} z\right) u(\varrho z) d \mu(z),
$$

where we used the $\operatorname{PSL}(2, \mathbb{R})^{n}$-invariance of the function $k$ and the measure $\mu$, the $\Gamma$ invariance of $u$ and that $\gamma^{\prime}$ and $\gamma_{0}^{\prime}$ commute. Writing $z=\left(z_{1}, \ldots, z_{n}\right)$ we have

$$
\begin{equation*}
\int_{\mathbb{H}^{n}} k\left(z, \gamma^{\prime} z\right) u(\varrho z) d \mu(z)=\int_{\mathbb{H}} \cdots \int_{\mathbb{H}} k\left(z, \gamma^{\prime} z\right) u(\varrho z) d \mu\left(z_{1}\right) \ldots d \mu\left(z_{n}\right), \tag{28}
\end{equation*}
$$

where $\mu\left(z_{k}\right)$ denotes the measure $y_{k}^{-2} d x_{k} d y_{k}$. In the inner integral above the coordinates $z_{2}, \ldots, z_{n}$ are fixed, and the function $u(\varrho z)$ can be regarded as a function of $z_{1}$. It is the eigenfunction of the Laplace operator $\Delta_{1}$ (because the operator commutes with the group action), furthermore, the value of $k\left(z, \gamma^{\prime} z\right)$ depends only on the hyperbolic distance of $z_{1}$ and $\gamma^{\prime(1)} z_{1}$. To simplify the notation we write $u(\varrho z)=u_{1}\left(z_{1}\right)$ and $k(z, w)=k_{1}\left(z_{1}, w_{1}\right)$. Furthermore, as $\gamma^{\prime}$ is fixed we can simply write $k\left(z, \gamma^{\prime} z\right)=k_{1}\left(z_{1}, \gamma^{\prime(1)} z_{1}\right)$. With this notation the inner integral becomes

$$
T_{1}:=\int_{\mathbb{H}} k\left(z, \gamma^{\prime} z\right) u(\varrho z) d \mu\left(z_{1}\right)=\int_{\mathbb{H}} k_{1}\left(z_{1}, \gamma^{\prime(1)} z_{1}\right) u_{1}\left(z_{1}\right) d \mu\left(z_{1}\right) .
$$

Recall that in (18) we introduced the notation

$$
R(\alpha)=\left[\begin{array}{cc}
\cos \alpha & \sin \alpha \\
-\sin \alpha & \cos \alpha
\end{array}\right],
$$

for any $\alpha \in \mathbb{R}$. For a vector $\varphi=\left(\varphi^{(1)}, \ldots, \varphi^{(n)}\right) \in \mathbb{R}^{n}$ we set $R(\varphi)=\left(R\left(\varphi^{(1)}\right), \ldots, R\left(\varphi^{(n)}\right)\right)$. The elements of the centralizer $C\left(\gamma^{\prime}\right)$ are of the form $R(\varphi)$, in particular $\gamma^{\prime}=R\left(\theta_{\gamma}\right)$ for some $\theta_{\gamma}=\left(\theta_{\gamma}^{(1)}, \ldots, \theta_{\gamma}^{(n)}\right)$ where $\theta_{\gamma}^{(k)} \in[0, \pi)$. As it was mentioned in Section 2.3] it is not hard to see that the vector $\theta_{\gamma}$ is determined by the class of $\gamma$, i.e. it is independent of the choice of $\varrho$ (at least if every coordinate is chosen from the interval $[0, \pi)$ ). Since $\gamma^{\prime} \in C\left(\gamma^{\prime}\right)$ we have in fact $\theta_{\gamma}^{(k)}=l_{k} \pi / m_{\gamma}$ for some integer $0<l_{k}<m_{\gamma}(k=1, \ldots, n)$.

Next we use geodesic polar coordinates (see [8], section 1.3), i.e. we make the substitution $z_{1}=R\left(\varphi_{1}\right) e^{-r_{1}} i$ where $r_{1} \in(0, \infty)$ is the hyperbolic distance of $i$ and $z_{1}$ and $\varphi_{1} \in[0, \pi)$. Then we have $d \mu\left(z_{1}\right)=\left(2 \sinh r_{1}\right) d r_{1} d \varphi_{1}$ and

$$
\left.T_{1}=\int_{0}^{\infty} \int_{0}^{\pi} k_{1}\left(R\left(\varphi_{1}\right) e^{-r_{1}} i, R\left(\theta_{\gamma}^{(1)}\right) R\left(\varphi_{1}\right) e^{-r_{1}} i\right)\right) u_{1}\left(R\left(\varphi_{1}\right) e^{-r_{1}} i\right) 2 \sinh r_{1} d \varphi_{1} d r_{1}
$$

As the elements $R\left(\theta_{\gamma}^{(1)}\right)$ and $R\left(\varphi_{1}\right)$ commute and $k_{1}$ depends only on the hyperbolic distance of the variables, we get that

$$
\left.T_{1}=\int_{0}^{\infty} k_{1}\left(e^{-r_{1}} i, R\left(\theta_{\gamma}^{(1)}\right) e^{-r_{1}} i\right)\right)\left(\int_{0}^{\pi} u_{1}\left(R\left(\varphi_{1}\right) e^{-r_{1}} i\right) d \varphi_{1}\right)\left(2 \sinh r_{1}\right) d r_{1} .
$$

Recall that $k_{1}\left(z_{1}, w_{1}\right)=\psi\left(\rho\left(z_{1}, w_{1}\right), \ldots, \rho\left(z_{n}, w_{n}\right)\right)=: \psi_{1}\left(\rho\left(z_{1}, w_{1}\right)\right)$, where

$$
\rho\left(z_{k}, w_{k}\right)=\frac{\left|z_{k}-w_{k}\right|^{2}}{\operatorname{Im} z_{k} \operatorname{Im} w_{k}}
$$

for $k=1, \ldots, n$. One gets by a computation that

$$
\rho\left(e^{-r_{k}} i, R\left(\theta_{\gamma}^{(k)}\right) e^{-r_{k}} i\right)=\frac{\left|-e^{-2 r_{k}}+1\right|^{2} \sin ^{2} \theta_{\gamma}^{(k)}}{e^{-2 r_{k}}}=\left(2 \sinh r_{k} \sin \theta_{\gamma}^{(k)}\right)^{2}
$$

hence

$$
\begin{equation*}
T_{1}=\int_{0}^{\infty} \psi_{1}\left(\left(2 \sinh r_{1} \sin \theta_{\gamma}^{(1)}\right)^{2}\right)\left(\int_{0}^{\pi} u_{1}\left(R\left(\varphi_{1}\right) e^{-r_{1}} i\right) d \varphi_{1}\right)\left(2 \sinh r_{1}\right) d r_{1} \tag{29}
\end{equation*}
$$

Let us define the function

$$
G_{1}(w)=\frac{1}{\pi} \int_{0}^{\pi} u_{1}\left(R\left(\varphi_{1}\right) w\right) d \varphi_{1}
$$

where $w \in \mathbb{H}$. By Lemma 1.10 in 8 the value of $G_{1}$ depends only on the hyperbolic distance $r$ of $w$ and $i$. Moreover, $G_{1}$ is the eigenfunction of the (one dimensional) Laplace operator $\Delta$ with eigenvalue $\lambda_{1}$, where $\lambda_{1}$ is the first coordinate of the eigenvalue vector of $u$. Now by Lemma 1.12 of [8] this function is unique up to a constant factor. Furthermore

$$
\Delta=\frac{\partial^{2}}{\partial r^{2}}+\frac{\cosh r}{\sinh r} \frac{\partial}{\partial r}+\frac{1}{4 \sinh ^{2} r} \frac{\partial^{2}}{\partial \varphi^{2}}
$$

hence the function $G_{1}$ (as a function of $r$ ) satisfies the differential equation (19) with the constant $\lambda=\lambda_{1}$, and consequently

$$
G_{1}(w)=g_{\lambda_{1}}(r) u_{1}(i)=g_{\lambda_{1}}(r) u\left(\varrho^{(1)} i, \varrho^{(2)} z_{2}, \ldots, \varrho^{(n)} z_{n}\right),
$$

where $g_{\lambda_{1}}(r):[0, \infty) \rightarrow \mathbb{C}$ is the solution of (19) with $\lambda=\lambda_{1}$ satisfying the initial condition $g_{\lambda_{1}}(0)=1$. By substituting this in (29), then interchanging the integrals in (28) and proceeding by induction one gets the statement of the theorem.
3.2. Proof in the mixed and totally hyperbolic cases. We continue with the proof of Theorem 2.3. Let $\gamma \in \Gamma$ a mixed or a totally hyperbolic element. We assume that the first $1 \leq m \leq n$ coordinate of $\gamma$ are hyperbolic while the following $n-m$ coordinates are elliptic, the proof of the statement is similar in the other cases. We have seen in Section 2.3 that such an element is conjugated by an element $\varrho \in P S L(2, \mathbb{R})^{n}$ to an element of the form $\nu=\left(D\left(N_{\gamma}^{(1)}\right), \ldots, D\left(N_{\gamma}^{(m)}\right), R\left(\theta_{\gamma}^{(m+1)}\right), \ldots, R\left(\theta_{\gamma}^{(n)}\right)\right)$ for some $N_{\gamma}^{(k)}>1$ and $\theta_{\gamma}^{(l)} \in[0 ; 2 \pi)$, where $D(N)$ and $R(\theta)$ were defined in (21) and (18), respectively. An easy computation shows that these numbers are uniquely defined by the class $\{\gamma\}$.

As in the totally elliptic case, by (17) we need to consider the integral

$$
T_{\gamma}=\int_{F_{C(\nu)}} k(z, \nu z) u(\varrho z) d \mu(z)
$$

where $F_{C(\nu)}$ is the fundamental domain for the centralizer $C(\nu) \leq \varrho^{-1} \Gamma \varrho$ of $\nu=\varrho^{-1} \gamma \varrho$. This domain was described in Proposition 2.2 and the notations introduced there will be used in the following.

For the first $m$ coordinates of $z$ we change to polar coordinates, i.e. make the substitution $z_{k}=r_{k} e^{i\left(\pi / 2+\vartheta_{k}\right)}$ where $r_{k} \in(0, \infty)$ and $\vartheta_{k} \in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)(k=1, \ldots, m)$, while for the last $n-m$ coordinates we change to geodesic polar coordinates as in the previous proof. A simple computation shows that $\rho\left(z_{k}, \nu^{(k)} z_{k}\right)=N\left(\vartheta_{k}, \gamma^{(k)}\right)$ for $k=1, \ldots, m$ (where $N\left(\vartheta_{k}, \gamma^{(k)}\right)$ was defined in (22) ), and by $y_{k}^{-2} d x_{k} d y_{k}=\left(r_{k} \cos ^{2} \vartheta_{k}\right)^{-1} d r_{k} d \vartheta_{k}$ and the results of the previous proof we obtain that $T_{\gamma}$ is

$$
\begin{aligned}
(2 \pi)^{n-m} \int_{0}^{\infty} \cdots & \int_{0}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \ldots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(N\left(\vartheta_{1}, \gamma^{(1)}\right), \ldots, N\left(\vartheta_{m}, \gamma^{(m)}\right), S\left(r_{m+1}, \theta_{\gamma}^{(m+1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right) \\
& \times F\left(e^{i\left(\frac{\pi}{2}+\vartheta_{1}\right)}, \ldots, e^{i\left(\frac{\pi}{2}+\vartheta_{m}\right)}\right) \frac{d \vartheta_{1}}{\cos ^{2} \vartheta_{1}} \ldots \frac{d \vartheta_{m}}{\cos ^{2} \vartheta_{m}}\left(\prod_{k=m+1}^{n} g_{\lambda_{k}}\left(r_{k}\right) \sinh r_{k} d r_{k}\right)
\end{aligned}
$$

where $S(r, \theta)=(2 \sinh r \sin \theta)^{2}$ and

$$
F(z)=\int_{\left(\log r_{1}, \ldots, \log r_{m}\right) \in P_{\gamma}} u\left(\varrho^{(1)}\left(r_{1} z_{1}\right), \ldots, \varrho^{(m)}\left(r_{m} z_{m}\right), \varrho^{(m+1)} i, \ldots, \varrho^{(n)} i\right) \prod_{k=1}^{m} \frac{d r_{k}}{r_{k}}
$$

for any $z \in \mathbb{H}^{m}$. Since $u(\varrho z)$ is invariant under the action of the centralizer $C(\nu)$, one sees easily that the function $F$ is invariant under is coordinate-wise scalar multiplication, i.e. $F\left(R_{1} z_{1}, \ldots, R_{m} z_{m}\right)=F\left(z_{1}, \ldots, z_{m}\right)$ holds for any $R_{1}, \ldots, R_{m} \in(0, \infty)$ and $z_{1}, \ldots, z_{m} \in \mathbb{H}$. This means that $F$ depends only on the vector $\left(\vartheta_{1}, \ldots, \vartheta_{m}\right)\left(\right.$ where $\left.z_{k}=r_{k} e^{i\left(\frac{\pi}{2}+\vartheta_{k}\right)}\right)$.

Moreover, since $u$ is the eigenfunction of every $\Delta_{k}$ with eigenvalue $\lambda_{k}$ and these operators commute with the group action, we infer that $F(z)$ is also an eigenfunction of the Laplacians $\Delta_{1}, \ldots, \Delta_{m}$ with the same corresponding eigenvalues. As

$$
\Delta_{k}=\left(r_{k} \cos \vartheta_{k}\right)^{2}\left(\frac{\partial^{2}}{\partial r_{k}^{2}}+r_{k}^{-1} \frac{\partial}{\partial r_{k}}+r_{k}^{-2} \frac{\partial^{2}}{\partial \vartheta_{k}^{2}}\right)
$$

we obtain the differential equations

$$
\begin{equation*}
\frac{\partial^{2} F}{\partial \vartheta_{k}^{2}}\left(\vartheta_{1}, \ldots, \vartheta_{m}\right)=\frac{\lambda_{k}}{\cos ^{2} \vartheta_{k}} F\left(\vartheta_{1}, \ldots, \vartheta_{m}\right) \quad\left(\vartheta_{k} \in(-\pi / 2, \pi / 2), k=1, \ldots, m\right) \tag{30}
\end{equation*}
$$

Let $f_{\lambda_{k}}(\vartheta)$ be the unique solution of the differential equation (23) with $\lambda=\lambda_{k}$ and the initial conditions $f_{\lambda_{k}}(0)=1$ and $f_{\lambda_{k}}^{\prime}(0)=0$, and $\tilde{f}_{\lambda_{k}}(\vartheta)$ the one with $\tilde{f}_{\lambda_{k}}(0)=0$ and $\tilde{f}_{\lambda_{k}}^{\prime}(0)=1$. Note that $f_{\lambda_{k}}(-\vartheta)$ satisfies (23) and the initial conditions of $f_{\lambda_{k}}(\vartheta)$ and hence they agree, i.e. $f_{\lambda_{k}}$ is an even function. Similarly, $\tilde{f}_{\lambda_{k}}$ is an odd function.

The equation

$$
F\left(\vartheta_{1}, \ldots, \vartheta_{m}\right)=F\left(0, \vartheta_{2}, \ldots, \vartheta_{m}\right) f_{\lambda_{1}}\left(\vartheta_{1}\right)+\frac{\partial F}{\partial \vartheta_{1}}\left(0, \vartheta_{2}, \ldots, \vartheta_{m}\right) \tilde{f}_{\lambda_{1}}\left(\vartheta_{1}\right)
$$

holds by (30) for every fixed $\vartheta_{2}, \ldots, \vartheta_{m}$, hence the inner integral in $T_{\gamma}$ is

$$
\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(N\left(\vartheta_{1}, \gamma^{(1)}\right), \ldots,\right. & \left.N\left(\vartheta_{m}, \gamma^{(m)}\right), S\left(r_{m+1}, \theta_{\gamma}^{(m+1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right) \\
& \times\left(F\left(0, \vartheta_{2}, \ldots, \vartheta_{m}\right) f_{\lambda_{1}}\left(\vartheta_{1}\right)+\frac{\partial F}{\partial \vartheta_{1}}\left(0, \vartheta_{2}, \ldots, \vartheta_{m}\right) \tilde{f}_{\lambda_{1}}\left(\vartheta_{1}\right)\right) \frac{d \vartheta_{1}}{\cos ^{2} \vartheta_{1}}
\end{aligned}
$$

$$
\begin{aligned}
&=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(N\left(\vartheta_{1}, \gamma^{(1)}\right), \ldots, N\left(\vartheta_{m}, \gamma^{(m)}\right), S\left(r_{m+1}, \theta_{\gamma}^{(m+1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right) \\
& \times F\left(0, \vartheta_{2}, \ldots, \vartheta_{m}\right) f_{\lambda_{1}}\left(\vartheta_{1}\right) \frac{d \vartheta_{1}}{\cos ^{2} \vartheta_{1}}
\end{aligned}
$$

because $N\left(\vartheta_{1}, \gamma^{(1)}\right)$ and $\cos ^{-2}\left(\vartheta_{1}\right)$ are even and $\tilde{f}_{\lambda_{1}}\left(\vartheta_{1}\right)$ is an odd function. Then, by induction (using also that $f_{\lambda_{k}}$ is even) we infer

$$
\begin{aligned}
& T_{\gamma}=(2 \pi)^{n-m} F(0, \ldots, 0) \times \\
& \times \int_{0}^{\infty} \cdots \int_{0}^{\infty} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi( \left.N\left(\vartheta_{1}, \gamma^{(1)}\right), \ldots, N\left(\vartheta_{m}, \gamma^{(m)}\right), S\left(r_{m+1}, \theta_{\gamma}^{(m+1)}\right), \ldots, S\left(r_{n}, \theta_{\gamma}^{(n)}\right)\right) \\
&\left.\times\left(\prod_{k=1}^{m} f_{\lambda_{k}}\left(\vartheta_{k}\right) \frac{d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}\right)\left(\prod_{k=m+1}^{n} g_{\lambda_{k}}\left(r_{k}\right) \sinh r_{k}\right) d r_{k}\right)
\end{aligned}
$$

where

$$
F(0, \ldots, 0)=\int_{\left(\log r_{1}, \ldots, \log r_{m}\right) \in P_{\gamma}} u\left(\varrho^{(1)}\left(r_{1} i\right), \ldots, \varrho^{(2)}\left(r_{m} i\right), \varrho^{(m+1)} i, \ldots, \varrho^{(n)} i\right) \prod_{k=1}^{m} \frac{d r_{k}}{r_{k}}
$$

It remains to show that there are only finitely many mixed or totally hyperbolic equivalence classes for which $T_{\gamma}$ is non-zero. Note that since $\psi$ has compact support and

$$
N\left(\vartheta_{k}, \gamma^{(k)}\right)=\frac{N_{\gamma}^{(k)}+\left(N_{\gamma}^{(k)}\right)^{-1}-2}{\cos ^{2} \vartheta_{k}} \geq N_{\gamma}^{(k)}+\left(N_{\gamma}^{(k)}\right)^{-1}-2=\left|\operatorname{tr}\left[\gamma^{(k)}\right]\right|-2
$$

we get $T_{\gamma}=0$ once $\left|\operatorname{tr}\left[\gamma^{(k)}\right]\right|$ is big enough for some $k$. Hence it is enough to show that there are only finitely many classes whose representatives have hyperbolic coordinates of bounded norm (and trace).

By Theorem $1.1 \Gamma$ is commensurable with a Hilbert modular group $\Gamma_{K}$, so there is an $M_{\gamma} \in \mathbb{N}^{+}$such that $\gamma^{M_{\gamma}}$ is conjugate to an element in $\Gamma_{K}$, moreover, the exponent $M_{\gamma}$ is bounded by a constant depending on $\Gamma$. It follows that it is enough to show that there are only finitely many mixed or totally hyperbolic classes in $\Gamma_{K}$ with coordinates of bounded trace.

Note that the trace of $\left(\gamma^{M_{\gamma}}\right)^{(k)}$ is in $\mathcal{O}_{K^{(k)}}$ for every $1 \leq k \leq n$, and these values are conjugates of each other. Hence if each of them is bounded, then the norm of them is bounded as well, so there are finitely many possibilities for the values of these traces. Finally, by Proposition I.7.1 and the paragraph after Definition I.7.2 in [6], there are only finitely many totally hyperbolic conjugacy classes with fixed traces, and this completes the proof of Theorem 2.3.
3.3. Proof in the hyperbolic-parabolic case. In this section we prove Theorem 2.6. In the course of the following proof the statement of Proposition 2.4 will also be verified, but until that point of our argument we always indicate any possible or evident dependence on a cusp. Accordingly, we temporarily use the notations $\Lambda_{\kappa}$ for the multiplier group and $\varepsilon_{j}^{\kappa}$ for its generators.

Recall that every hyperbolic-parabolic class is represented by an element that fixes a cusp $\kappa \in \mathcal{S}$ and such an element is conjugated by the scaling element $\sigma_{\kappa} \in P S L(2, \mathbb{R})^{n}$ to an
element of the form

$$
\gamma_{m, \alpha}^{\kappa}=\left[\begin{array}{cc}
\left(u_{m}^{\kappa}\right)^{1 / 2} & \alpha\left(u_{m}^{\kappa}\right)^{-1 / 2}  \tag{31}\\
0 & \left(u_{m}^{\kappa}\right)^{-1 / 2}
\end{array}\right]
$$

where $\alpha \in \mathbf{t}_{\kappa}, m=\left(m_{1}, \ldots, m_{n-1}\right) \in \mathbb{Z}^{n-1} \backslash\{0\}$ and $u_{m}^{\kappa}=\left(\varepsilon_{1}^{\kappa}\right)^{m_{1}} \ldots\left(\varepsilon_{n-1}^{\kappa}\right)^{m_{n-1}} \in \Lambda_{\kappa}$. To simplify the notation, we often write $u_{m}$ instead of $u_{m}^{\kappa}$ in the following, at least when $\kappa$ is fixed. Note that the action of $\gamma_{m, \alpha}^{\kappa}$ on a point $z \in \mathbb{H}^{n}$ can be written as $\gamma_{m, \alpha}^{\kappa} z=u_{m}^{\kappa} z+\alpha$ and hence $q=\frac{\alpha}{1-u_{m}^{\kappa}}$ is the real fixed vector of $\gamma_{m, \alpha}^{\kappa}$ that is also a cusp for $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$ (as it was already mentioned in Section 2.4).

A simple computation shows that if elements $\gamma_{m, \alpha}$ and $\gamma_{m^{\prime}, \beta}$ of the form (31) are conjugate in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$, then $m^{\prime}=m$ or $m^{\prime}=-m$ holds. Assume first that the elements $\gamma_{m, \alpha}$ and $\gamma_{m, \beta}$ are conjugate to each other. Again, it follows by a straightforward calculation that

$$
\beta=\left(u_{m}^{\kappa}-1\right) a+\left(u_{l}^{\kappa}\right)^{-1} \alpha
$$

holds for some $l \in \mathbb{Z}^{n-1}$ and $a \in \mathbf{t}_{\kappa}$ in this case. This means exactly that $\beta$ represents the same element in the finite factor group $\mathbf{t}_{\kappa}^{m}:=\mathbf{t}_{\kappa} /\left(u_{m}^{\kappa}-1\right) \mathbf{t}_{\kappa}$ as $\left(u_{l}^{\kappa}\right)^{-1} \alpha$, and hence for a fixed $m$ (and $\kappa$ ) the hyperbolic-parabolic classes are represented by the equivalence classes of $\mathbf{t}_{\kappa}^{m} / \Lambda_{\kappa}$.

Now assume that the elements $\gamma_{m, \alpha}$ and $\gamma_{-m, \beta}$ are conjugate for some $m \in \mathbb{Z}^{n-1} \backslash\{0\}$ and $\alpha, \beta \in \mathbf{t}_{\kappa}$, i.e. $\tau^{-1} \gamma_{m, \alpha} \tau=\gamma_{-m, \beta}$ for some $\tau \in \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$. Since $\tau^{-1} \gamma_{m, \alpha} \tau$ fixes $\tau^{-1} \infty$ and also $\tau^{-1} q$, one of these points must be $\infty$. If $\tau^{-1} \infty=\infty$ was true, then the conjugate would be of the form $\gamma_{m, \beta}$, which is impossible (since $m \neq 0$ ). It follows that $\tau^{-1}$ takes $q$ to $\infty$, hence these cusps are equivalent in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$. Similarly, if these cusps are equivalent then $\gamma_{m, \alpha}$ is conjugate to an element $\gamma_{-m, \beta}$.

Based on this the contribution of the hyperbolic-parabolic classes in the trace can be written as

$$
\begin{align*}
& \frac{1}{2} \sum_{\kappa \in \mathcal{S}} \sum_{m \in \mathbb{Z}^{n-1} \backslash 0} \sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda_{\kappa}} \sum_{\substack{\sigma \in C(\gamma) \backslash \Gamma \\
\gamma \sim \gamma_{m, \alpha}^{\kappa}}} \int_{F_{A}} k\left(z, \sigma^{-1} \gamma \sigma z\right) u(z) d \mu(z)= \\
& \quad=\frac{1}{2} \sum_{\kappa \in \mathcal{S}} \sum_{m \in \mathbb{Z}^{n-1} \backslash 0} \sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda_{\kappa}} \sum_{\sigma \in C\left(\gamma_{m, \alpha}^{\kappa}\right) \backslash \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}} \int_{\sigma\left(\sigma_{\kappa}^{-1} F_{A}\right)} k\left(z, \gamma_{m, \alpha}^{\kappa} z\right) u\left(\sigma_{\kappa} z\right) d \mu(z), \tag{32}
\end{align*}
$$

where $C\left(\gamma_{m, \alpha}^{\kappa}\right)$ is the centralizer of $\gamma_{m, \alpha}^{\kappa}$ in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$. We multiply the whole sum by $1 / 2$ since every class is taken into account for both fixed cusps of their elements except for those whose fixed points are equivalent in $\Gamma$. But in the latter case we count these classes twice for an $m$ and $-m$ as well.

Let us focus on the inner sum

$$
\begin{equation*}
\sum_{\sigma \in C\left(\gamma_{m, \alpha}^{\kappa}\right) \backslash \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}} \int_{\sigma\left(\sigma_{\kappa}^{-1} F_{A}\right)} k\left(z, \gamma_{m, \alpha}^{\kappa} z\right) u\left(\sigma_{\kappa} z\right) d \mu(z) \tag{33}
\end{equation*}
$$

First, note that $k\left(z, \gamma_{m, \alpha}^{\kappa} z\right)$ can be written as

$$
\psi\left(\frac{\left(E_{m}^{(1)} x_{1}-\alpha_{1}\left(u_{m}^{(1)}\right)^{-1 / 2}\right)^{2}}{y_{1}^{2}}+\left(E_{m}^{(1)}\right)^{2}, \ldots, \frac{\left(E_{m}^{(n)} x_{n}-\alpha_{n}\left(u_{m}^{(n)}\right)^{-1 / 2}\right)^{2}}{y_{n}^{2}}+\left(E_{m}^{(n)}\right)^{2}\right)
$$

where $E_{m}=E_{m}^{\kappa}=u_{m}^{-1 / 2}-u_{m}^{1 / 2}$. Since $\psi$ is compactly supported, it follows immediately that (33) is zero for all but finitely many $m$, and hence the sum in (32) is finite.

The union of the sets $\sigma\left(\sigma_{\kappa}^{-1} F_{A}\right)$ in the integrals above, where $\sigma$ runs through the right cosets of the centralizer $C\left(\gamma_{m, \alpha}^{\kappa}\right)$, makes up the fundamental domain $F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}$ of $C\left(\gamma_{m, \alpha}^{\kappa}\right)$
except for the images of the part $F \backslash F_{A}=: F_{A}^{*}$. We will now show that for some cosets it is unnecessary to omit the images of $F_{A}^{*}$, since there we integrate only the zero function. For this we write the kernel function in the form

$$
\begin{equation*}
k\left(z, \gamma_{m, \alpha}^{\kappa} z\right)=\psi\left(\frac{\left|z_{1}-q_{1}\right|^{2}}{\left(E_{m}^{(1)}\right)^{-2} y_{1}^{2}}, \ldots, \frac{\left|z_{n}-q_{n}\right|^{2}}{\left(E_{m}^{(n)}\right)^{-2} y_{n}^{2}}\right) \tag{34}
\end{equation*}
$$

The part $\sigma\left(\sigma_{\kappa}^{-1} F_{A}^{*}\right)$ is the same as

$$
\bigcup_{\kappa^{\prime}}\left\{z \in F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}: \sigma_{\kappa} \sigma^{-1} z \in F, Y_{0}^{\kappa^{\prime}}\left(\sigma_{\kappa} \sigma^{-1} z\right) \geq A\right\}
$$

The condition $Y_{0}^{\kappa^{\prime}}\left(\sigma_{\kappa} \sigma^{-1} z\right) \geq A$ means that $z \in \sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} U_{A}$, that is, there is a $w \in U_{A}$ such that $z=\sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} w$. Let us define $\nu=\sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}}=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$. If $w=u+i v$, then the product of the expressions in the arguments on the right hand side of (34) is

$$
\begin{aligned}
\prod_{k=1}^{n} \frac{\left|\nu_{k} w_{k}-q_{k}\right|^{2}}{\left(E_{m}^{(k)}\right)^{-2} y_{k}(\nu w)^{2}} & =\frac{N\left(E_{m}^{2}\right)}{Y_{0}(\nu w)^{2}} \prod_{k=1}^{n}\left|\nu_{k} w_{k}-\frac{\alpha_{k}}{1-u_{m}^{(k)}}\right|^{2} \\
& =\frac{N\left(\left(1-u_{m}\right)^{2}\right)}{Y_{0}(\nu w)^{2}} \prod_{k=1}^{n}\left|\frac{a_{k} w_{k}+b_{k}}{c_{k} w_{k}+d_{k}}-\frac{\alpha_{k}}{1-u_{m}^{(k)}}\right|^{2} \\
& \geq \frac{N\left(\left(1-u_{m}\right)^{2}\right)}{Y_{0}(\nu w)^{2}} \prod_{k=1}^{n} \frac{\left(\left(1-u_{m}^{(k)}\right) a_{k}-\alpha_{k} c_{k}\right)^{2} v_{k}^{2}}{\left(1-u_{m}^{(k)}\right)^{2}\left|c_{k} w_{k}+d_{k}\right|^{2}} \\
& =N\left(\left(1-u_{m}\right) a-\alpha c\right)^{2} \prod_{k=1}^{n}\left|c_{k} w_{k}+d_{k}\right|^{2} \\
& \geq N\left(\left(1-u_{m}\right) a-\alpha c\right)^{2} N(c)^{2} Y_{0}(w)^{2}
\end{aligned}
$$

The first two factors of the last product are bounded away from zero by Lemma $2.9_{1}$ in [7], at least if they are non-zero. This follows for the second factor easily but requires some explanation in the case of the first factor. The point $q$ is a cusp for $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$, hence $q=\sigma_{\kappa}^{-1} \beta$ for some cusp $\beta$ of $\Gamma$. Let $\lambda \in \mathcal{S}$ denote the base element of the $\Gamma$-equivalence class of $\beta$, then $q=\sigma_{\kappa}^{-1} \gamma_{0} \lambda$ for some $\gamma_{0} \in \Gamma$, i.e. $\sigma_{\lambda}^{-1} \gamma_{0}^{-1} \sigma_{\kappa} \in \sigma_{\lambda}^{-1} \Gamma \sigma_{\kappa}$ takes $q$ to $\infty$, hence it is of the form $\left[\begin{array}{cc}e & f \\ \frac{u_{m}-1}{\delta} & \frac{\alpha}{\delta}\end{array}\right]$, where $\delta=e \alpha+f\left(1-u_{m}\right)$.

We first show that $N\left(\delta^{2}\right)$ depends only on $q$, i.e. on $m$ and $\alpha$. Assume that $\sigma_{\lambda}^{-1} \gamma^{\prime} \sigma_{\kappa}$ also takes $q$ to $\infty$ for some $\gamma^{\prime} \in \Gamma$, and therefore it is of the form $\left[\begin{array}{cc}e^{\prime} & f^{\prime} \\ \frac{u_{m}-1}{\delta^{\prime}} & \frac{\alpha}{\delta^{\prime}}\end{array}\right]$. Now

$$
\left(\sigma_{\lambda}^{-1} \gamma^{\prime} \sigma_{\kappa}\right)\left(\sigma_{\lambda}^{-1} \gamma_{0}^{-1} \sigma_{\kappa}\right)^{-1}=\sigma_{\lambda}^{-1} \gamma^{\prime} \gamma_{0} \sigma_{\lambda} \in \sigma_{\lambda}^{-1} \Gamma \sigma_{\lambda}
$$

fixes $\infty$ (which is a cusp for $\sigma_{\lambda}^{-1} \Gamma \sigma_{\lambda}$ ), and then

$$
\left[\begin{array}{cc}
e^{\prime} & f^{\prime} \\
\frac{u_{m}-1}{\delta^{\prime}} & \frac{\alpha}{\delta^{\prime}}
\end{array}\right]\left[\begin{array}{cc}
e & f \\
\frac{u_{m}-1}{\delta} & \frac{\alpha}{\delta}
\end{array}\right]^{-1}=\left[\begin{array}{cc}
e^{\prime} & f^{\prime} \\
\frac{u_{m}-1}{\delta^{\prime}} & \frac{\alpha}{\delta^{\prime}}
\end{array}\right]\left[\begin{array}{cc}
\frac{\alpha}{\delta} & -f \\
\frac{1-u_{m}}{\delta} & e
\end{array}\right]=\left[\begin{array}{cc} 
\pm u^{\frac{1}{2}} & t \\
0 & \pm u^{-\frac{1}{2}}
\end{array}\right]
$$

holds for some $u \in \Lambda_{\lambda}$ and $t \in \mathbf{t}_{\lambda}$. It follows that $\delta^{2}=u \delta^{2}$, hence $N\left(\delta^{2}\right)=N\left(\delta^{2}\right)$.

Now

$$
\sigma_{\lambda}^{-1} \gamma_{0}^{-1} \sigma_{\kappa} \nu=\left[\begin{array}{cc}
* & * \\
-\frac{\left(1-u_{m}\right) a-\alpha c}{\delta} & *
\end{array}\right] \in \sigma_{\lambda}^{-1} \Gamma \sigma_{\kappa^{\prime}}
$$

and choosing $\Delta:=\sigma_{\lambda}^{-1} \Gamma \sigma_{\kappa^{\prime}}$ in Lemma $2.9_{1}$ of [7] we get that $N\left(\left(1-u_{m}\right) a-\alpha c\right)^{2} \geq N\left(\delta^{2}\right) C_{0}$ for some positive constant $C_{0}>0$ (assuming that $\left.\left(1-u_{m}\right) a-\alpha c \neq 0\right)$.

Next we handle the cases where one of the factors $N(c)^{2}$ and $N\left(\left(1-u_{m}\right) a-\alpha c\right)^{2}$ is 0 . Since $\sigma=\sigma_{\kappa}^{-1} \gamma \sigma_{\kappa}$ for some $\gamma \in \Gamma$, hence $c=0$ holds if and only if

$$
\infty=\nu \infty=\sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} \infty=\sigma_{\kappa}^{-1} \gamma \kappa^{\prime}
$$

that is, $\kappa=\sigma_{\kappa} \infty=\gamma \kappa^{\prime}$. But $\Gamma$ permutes the elements of the class of $\kappa^{\prime}$, so $c=0$ can hold only if $\kappa=\kappa^{\prime}$, and then $\nu=\sigma$, and therefore $\infty=\sigma \infty$ holds. Consequently, the condition $Y_{0}^{\kappa^{\prime}}\left(\sigma_{\kappa} \sigma^{-1} z\right) \geq A$ reduces to $Y_{0}\left(\sigma^{-1} z\right)=Y_{0}(z) \geq A$ in this case.

Assume now that $\left(1-u_{m}\right) a-\alpha c=0$, that is, $\frac{a}{c}=q$ holds. This means that

$$
\sigma_{\kappa}^{-1} \gamma_{0} \lambda=q=\nu \infty=\sigma_{\kappa}^{-1} \gamma \kappa^{\prime}
$$

hence $\kappa^{\prime}=\lambda$ must hold. This means that the exceptional set $\sigma\left(\sigma_{\kappa}^{-1} F_{A}^{*}\right)$ can be reduced to

$$
\left\{z \in F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}: Y_{0}\left(\sigma_{\lambda}^{-1} \sigma_{\kappa} \sigma^{-1} z\right) \geq A\right\}
$$

The element $\sigma_{\lambda}^{-1} \sigma_{\kappa} \sigma^{-1}$ takes $q$ to $\infty$, hence - as we have already seen above - it is of the form $\left[\begin{array}{cc}e & f \\ \frac{u_{m}-1}{\delta} & \frac{\alpha}{\delta}\end{array}\right]$, where $\delta=e \alpha+f\left(1-u_{m}\right)$, and then

$$
Y_{0}\left(\sigma_{\lambda}^{-1} \sigma_{\kappa} \sigma^{-1} z\right)=\frac{N\left(\delta^{2}\right) Y_{0}(z)}{\prod_{k=1}^{n}\left|\left(u_{m}^{(k)}-1\right) z_{k}+\alpha_{k}\right|^{2}}=\frac{N\left(\delta E_{m}^{-1}\right)^{2} Y_{0}(z)}{\prod_{k=1}^{n}\left|z_{k}-q_{k}\right|^{2}}
$$

All this shows that the expression in (33) can be written as

$$
\int_{S_{A}} k\left(z, \gamma_{m, \alpha}^{\kappa} z\right) u\left(\sigma_{\kappa} z\right) d \mu(z)
$$

where

$$
S_{A}=\left\{z \in F_{C\left(\gamma_{m, \alpha}^{\kappa}\right)}: Y_{0}(z) \leq A, \frac{N\left(\delta E_{m}^{-1}\right)^{2} Y_{0}(z)}{\prod_{k=1}^{n}\left|z_{k}-q_{k}\right|^{2}} \leq A\right\}
$$

at least when $A$ is big enough. Recall that the centralizer $C\left(\gamma_{m, \alpha}^{\kappa}\right)$ and its fundamental domain was described in Section 2.4. In the following we also use some relating notations defined there. A direct calculation shows now that the last integral above is

$$
\begin{aligned}
\int_{S_{A}} \psi\left(\frac{\left|z_{1}-q_{1}\right|^{2}}{\left(E_{m}^{(1)}\right)^{-2} y_{1}^{2}}\right. & \left., \ldots, \frac{\left|z_{n}-q_{n}\right|^{2}}{\left(E_{m}^{(n)}\right)^{-2} y_{n}^{2}}\right) u\left(\sigma_{\kappa} z\right) d \mu(z)= \\
& =\int_{S_{A}-q} \psi\left(\frac{\left|E_{m}^{(1)} z_{1}\right|^{2}}{y_{1}^{2}}, \ldots, \frac{\left|E_{m}^{(n)} z_{n}\right|^{2}}{y_{n}^{2}}\right) u\left(\sigma_{\kappa}(z+q)\right) d \mu(z)
\end{aligned}
$$

where

$$
S_{A}-q=\left\{z \in F_{C}: Y_{0}(z) \leq A, \frac{N\left(\delta E_{m}^{-1}\right)^{2} Y_{0}(z)}{\prod_{k=1}^{n}\left|z_{k}\right|^{2}} \leq A\right\}
$$

The two inequalities above can be written in terms of the polar coordinates as follows:

$$
A_{\vartheta}:=\frac{N\left(\delta E_{m}^{-1}\right)^{2} \prod_{k=1}^{n} \cos \vartheta_{k}}{A} \leq \prod_{k=1}^{n} r_{k} \leq \frac{A}{\prod_{k=1}^{n} \cos \vartheta_{k}}=: A^{\vartheta}
$$

Since $\frac{d x d y}{y^{2}}=\frac{d r d \vartheta}{r \cos ^{2} \vartheta}$, the integral above becomes after the change of variables

$$
\begin{equation*}
\int_{-\pi / 2}^{\pi / 2} \ldots \int_{-\pi / 2}^{\pi / 2} \psi\left(\frac{\left(E_{m}^{(1)}\right)^{2}}{\cos ^{2} \vartheta_{1}}, \ldots, \frac{\left(E_{m}^{(n)}\right)^{2}}{\cos ^{2} \vartheta_{n}}\right) I_{u}(A, \vartheta, m, \alpha) \prod_{k=1}^{n} \frac{d \vartheta_{k}}{\cos ^{2} \vartheta_{k}} \tag{35}
\end{equation*}
$$

where

$$
I_{u}(A, \vartheta, m, \alpha)=\int_{\substack{\log r \in P_{m, \alpha}^{\kappa} \\ A_{\vartheta} \leq N r \leq A^{\vartheta}}} u\left(\sigma_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}
$$

We handle the zeroth term and the remaining terms of the Fourier expansion of the function $u\left(\sigma_{\kappa}(z+q)\right)$ separately. To this end, for any cusp $\kappa^{\prime}$ we write

$$
u\left(\sigma_{\kappa^{\prime}} z\right)=\eta_{\kappa^{\prime}} y_{1}^{s_{1}} \ldots y_{n}^{s_{n}}+\phi_{\kappa^{\prime}} y_{1}^{1-s_{1}} \ldots y_{n}^{1-s_{n}}+R_{\kappa^{\prime}}(z)=M_{\kappa^{\prime}}(z)+R_{\kappa^{\prime}}(z)
$$

Subtracting the contribution of the zeroth term $M_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)$ from the integral $I_{u}(A, \vartheta, m, \alpha)$ one obtains

$$
\int_{\substack{\log g \in P_{m, \alpha}^{\kappa} \\ A_{\vartheta} \leq N \leq A^{\vartheta}}} R_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}
$$

However, this integral does not converge as $A \rightarrow \infty$, but Proposition 1.3 gives that it does converge if one integrates only over $\left\{\log r \in P_{m, \alpha}^{\kappa}: 1 \leq N r\right\}$. Note that by the compact support of $\psi$ the coordinates of the vector $\cos \vartheta$ can be assumed to be bounded away from zero and hence $R_{\kappa}$ in the above integral can be bounded uniformly exponentially using Proposition 1.3. To ensure convergence on the other half of the set $\left\{\log r \in P_{m, \alpha}^{\kappa}\right\}$ we will subtract the main term of $u$ at $q$. By the $\Gamma$-invariance of $u$ we have

$$
u\left(\sigma_{\kappa}(z+q)\right)=u\left(\sigma_{\lambda}\left(\sigma_{\lambda}^{-1} \gamma_{0}^{-1} \sigma_{\kappa}(z+q)\right)\right)=u\left(\sigma_{\lambda}\left(-e(e q+f)-\frac{(e q+f)^{2}}{z}\right)\right)
$$

hence (after the substitution $\left.r \mapsto \frac{1}{r}\right)$ the integral of $u\left(\sigma_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)\right)$ over the set $\{\log r \in$ $\left.P_{m, \alpha}^{\kappa}: A_{\vartheta} \leq N r \leq 1\right\}$ becomes

$$
\begin{aligned}
& \int_{\substack{\log r \in P_{m, \alpha}^{\kappa} \\
1 \leq N r \leq A_{\vartheta}^{-1}}} u\left(\sigma_{\lambda}\left(-e(e q+f)+(e q+f)^{2} r e^{i\left(\frac{\pi}{2}-\vartheta\right)}\right)\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}= \\
& \quad=\int_{\substack{\log \\
\\
\\
1 \leq N \in P_{m, \alpha}^{k}}} M_{\lambda}\left((e q+f)^{2} r e^{i\left(\frac{\pi}{2}-\vartheta\right)}\right)+R_{\lambda}\left((e q+f)^{2} r e^{i\left(\frac{\pi}{2}-\vartheta\right)}-e(e q+f)\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}} .
\end{aligned}
$$

Here we also used the translation invariance of $M_{\lambda}$. It is now clear that the integral of the second term above converges, i.e.

$$
u\left(\sigma_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)\right)-M_{\lambda}\left(-\frac{(e q+f)^{2}}{r e^{i\left(\frac{\pi}{2}+\vartheta\right)}}\right)=u\left(\sigma_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)\right)-M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{r e^{i\left(\frac{\pi}{2}+\vartheta\right)}}\right)
$$

is integrable over $\left\{\log r \in P_{m, \alpha}^{\kappa}: N r \leq 1\right\}$. A straightforward computation (detailed below) shows the same for $M_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right)$ and then consequently for the function

$$
\tilde{u}_{m, \alpha}^{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right)=u\left(\sigma_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}+q\right)\right)-M_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right)-M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{r e^{i\left(\frac{\pi}{2}+\vartheta\right)}}\right) .
$$

A similar argument gives that $\tilde{u}_{m, \alpha}^{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right)$ is integrable over $\left\{\log r \in P_{m, \alpha}^{\kappa}: 1 \leq N r\right\}$ and hence over the whole set $\left\{\log r \in P_{m, \alpha}^{\kappa}\right\}$.

Now $I_{u}(A, \vartheta, m, \alpha)$ can be written as

$$
\begin{equation*}
\int_{\substack{\log r \in P_{m, \alpha}^{k} \\ A_{\vartheta} \leq N r \leq A^{\vartheta}}}\left[M_{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right)+M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{r e^{i\left(\frac{\pi}{2}+\vartheta\right)}}\right)\right] \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}+\int_{\log r \in P_{m, \alpha}^{\kappa}} \tilde{u}_{m, \alpha}^{\kappa}\left(r e^{i\left(\frac{\pi}{2}+\vartheta\right)}\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}+o(A) . \tag{36}
\end{equation*}
$$

First we turn to the second integral above. Observe that the function $U(z):=u\left(\sigma_{\kappa}(z+q)\right)$ is invariant under the action of $\rho_{l_{j}}$ (defined in (251)):

$$
\begin{aligned}
U\left(\rho_{l_{j}} z\right) & =u\left(\sigma_{\kappa}\left(u_{l_{j}} z+q\right)\right)=u\left(\sigma_{\kappa}\left(u_{l_{j}}(z+q)+\left(1-u_{l_{j}}\right) q\right)\right) \\
& =u\left(\sigma_{\kappa} \gamma\left(l_{j}\right)(z+q)\right)=u\left(\sigma_{\kappa}(z+q)\right)=U(z),
\end{aligned}
$$

because the element $\gamma\left(l_{j}\right)$ is in the centralizer $C\left(\gamma_{\alpha, m}^{\kappa}\right) \leq \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$, and $u$ is invariant under the action of $\Gamma$. The same invariance holds for $M_{\kappa}(z)$ and $M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{z}\right)$ as well, hence the invariance of $\tilde{u}_{m, \alpha}^{k}(z)$ under the action of $\rho_{l j}$ follows.

Let us define the function

$$
F_{m, \alpha}^{\kappa}(z)=\int_{\log r \in P_{m, \alpha}^{\kappa}} \tilde{u}_{m, \alpha}^{\kappa}(r z) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}} .
$$

By the observation of the last paragraph we can see as in the case of mixed elements that $F_{m, \alpha}^{\kappa}$ is invariant under coordinate-wise scalar multiplication, i.e. $F_{m, \alpha}^{\kappa}(z)=F_{m, \alpha}^{\kappa}(\vartheta)$ depends only on $\vartheta$ where $z=r e^{i\left(\frac{\pi}{2}+\vartheta\right)}$.

Since the Laplacian $\Delta_{k}$ is an invariant operator (i.e. it commutes with the action of $\left.\operatorname{PSL}(2, \mathbb{R})^{n}\right)$, we get that $U(z)$ is an eigenfunction of it with the eigenvalue $\lambda_{k}$. But the same is true $M_{\kappa}(z)$ and $M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{z}\right)$, and therefore $\tilde{u}_{m, \alpha}^{\kappa}(z)$ and also $F_{m, \alpha}^{\kappa}(z)$ are eigenfunctions of $\Delta_{k}$ with the eigenvalue $\lambda_{k}$. In the same way as in the case of mixed elements we conclude that the contribution of the second integral of (36) in (35) is

$$
F_{m, \alpha}^{\kappa}(0, \ldots, 0) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(\frac{\left(E_{m}^{(1)}\right)^{2}}{\cos ^{2} \vartheta_{1}}, \ldots, \frac{\left(E_{m}^{(n)}\right)^{2}}{\cos ^{2} \vartheta_{n}}\right)\left(\prod_{k=1}^{n} \frac{f_{\lambda_{k}}\left(\vartheta_{k}\right) d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}\right)
$$

where $f_{\lambda_{k}}(\vartheta)$ is the unique solution of the differential equation (23) with the initial condition $f_{\lambda_{k}}(0)=1$ and $f_{\lambda_{k}}^{\prime}(0)=0$ and

$$
F_{m, \alpha}^{\kappa}(0, \ldots, 0)=\int_{\log r \in P_{m, \alpha}^{\kappa}} \tilde{u}\left(r_{1} i, \ldots, r_{n} i\right) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}}
$$

Finally we calculate the first integral in (36). The map $r \mapsto\left(\log r_{1}, \ldots, \log r_{n}\right)$ maps the set we integrate on to

$$
\mathcal{F}_{A}:=\bigcup_{n^{-\frac{1}{2}} \log A_{\vartheta} \leq t \leq n^{-\frac{1}{2}} \log A^{\vartheta}}\{t \mathbf{1}\} \times \tilde{P}_{m, \alpha}^{\kappa}
$$

and the determinant of its Jacobian is $\prod_{k=1}^{n} r_{k}^{-1}$, so the integral is

$$
\begin{equation*}
\int_{\mathcal{F}_{A}} M_{\kappa}\left(e^{x+i\left(\frac{\pi}{2}+\vartheta\right)}\right)+M_{\lambda}\left(-\frac{\delta^{2} E_{m}^{-2} u_{m}^{-1}}{e^{x+i\left(\frac{\pi}{2}+\vartheta\right)}}\right) \prod_{k=1}^{n} d x_{k} . \tag{37}
\end{equation*}
$$

The first term of the zeroth Fourier coefficient $M_{\kappa}$ gives the following contribution to the integral above:

$$
\begin{equation*}
\eta_{\kappa} \prod_{k=1}^{n}\left(\cos \vartheta_{k}\right)^{s_{k}} \int_{\mathcal{F}_{A}} \exp \left(\sum_{k=1}^{n} s_{k} x_{k}\right) \prod_{k=1}^{n} d x_{k} \tag{38}
\end{equation*}
$$

Let $\mathcal{L}$ be the linear map that maps the standard basis of $\mathbb{R}^{n}$ to $1, v_{1}^{\kappa}, \ldots, v_{n-1}^{\kappa}$ (see Section 2.4 for the definitions) so that $\mathcal{L}\left(\left[n^{-\frac{1}{2}} \log A_{\vartheta} ; n^{-\frac{1}{2}} \log A^{\vartheta}\right] \times\left[-\frac{1}{2} ; \frac{1}{2}\right]^{n-1}\right)=\mathcal{F}_{A}$. The matrix of this map w.r.t. the standard basis is $[\mathcal{L}]=\mathcal{E}_{\kappa}\left[\begin{array}{cc}n^{-\frac{1}{2}} & \\ & L_{m, \alpha}^{\kappa}\end{array}\right]$, where $\mathcal{E}_{\kappa}$ is defined in (8) and $L_{m, \alpha}^{\kappa}$ is defined in (24). Hence, after a change of variables, the integral in (38) becomes

$$
\begin{aligned}
& |\operatorname{det}[\mathcal{L}]| \int_{n^{-\frac{1}{2}} \log A_{\vartheta}}^{n^{-\frac{1}{2}} \log A^{\vartheta}} e^{n^{-\frac{1}{2}} t_{0} \sum_{k=1}^{n} s_{k}} d t_{0} \prod_{q=1}^{n-1} \int_{-1 / 2}^{1 / 2} \exp \left(t_{q} \sum_{j=1}^{n-1} l_{q}^{(j)} \sum_{k=1}^{n} s_{k} \log \varepsilon_{j}^{\kappa(k)}\right) d t_{q}= \\
& \quad=|\operatorname{det}[\mathcal{L}]| \int_{n^{-\frac{1}{2}} \log A_{\vartheta}}^{n^{-\frac{1}{2}} \log A^{\vartheta}} e^{t_{0} n^{\frac{1}{2}} s} d t_{0} \prod_{q=1}^{n-1} \int_{-1 / 2}^{1 / 2} \exp \left(2 \pi i t_{q} \sum_{j=1}^{n-1} l_{q}^{(j)} m_{u, \kappa}^{(j)}\right) d t_{q}
\end{aligned}
$$

by (8). This expression is zero unless $\left(L_{m, \alpha}^{\kappa}\right)^{T} m_{u, \kappa}=0$ holds. Since

$$
0 \neq \operatorname{det}[\mathcal{L}]=n^{-\frac{1}{2}} \operatorname{det} \mathcal{E}_{\kappa} \operatorname{det} L_{m, \alpha}^{\kappa}
$$

i.e. det $L_{m, \alpha}^{\kappa} \neq 0$, this can hold only if $m_{u, \kappa}=0$. In the latter case (38) becomes
$\frac{\eta_{\kappa}\left|\operatorname{det} \mathcal{E}_{\kappa}\right|\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|}{n s}\left[A^{s}-\frac{N\left(\delta E_{m}^{-1}\right)^{2 s} \prod_{k=1}^{n}\left(\cos \vartheta_{k}\right)^{2 s}}{A^{s}}\right]=\frac{\eta_{\kappa}\left|\operatorname{det} \mathcal{E}_{\kappa}\right|\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right| A^{s}}{n s}+o(A)$.
Here we used that since $\psi$ has a compact support, the values $\cos \vartheta_{k}$ are bounded away from zero by a constant depending on $\psi$ and $\Gamma$. The same argument gives that the contribution of the second term of the zeroth Fourier coefficient $M_{\kappa}$ in $I_{u}(A, \vartheta, m, \alpha)$ is

$$
\begin{equation*}
\frac{\phi_{\kappa}\left|\operatorname{det} \mathcal{E}_{\kappa}\right|\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right| A^{1-s}}{n(1-s)}+o(A) \tag{40}
\end{equation*}
$$

Now we turn to the integral of the second term in (37). The first term of this Fourier coefficient contributes

$$
\eta_{\lambda} \prod_{k=1}^{n}\left(\left(\delta^{(k)}\right)^{2}\left(E_{m}^{(k)}\right)^{-2}\left(u_{m}^{(k)}\right)^{-1} \cos \vartheta_{k}\right)^{s_{k}} \int_{\mathcal{F}_{A}} \exp \left(-\sum_{k=1}^{n} s_{k} x_{k}\right) \prod_{k=1}^{n} d x_{k}
$$

The same computation as above shows that this is zero unless $m_{u, \kappa}=0$, in which case one gets

$$
\eta_{\lambda}|\operatorname{det}[\mathcal{L}]| N\left(\delta^{2} E_{m}^{-2}\right)^{s} \prod_{k=1}^{n}\left(\cos \vartheta_{k}\right)^{s} \int_{n^{-\frac{1}{2}} \log \left(A^{\vartheta}\right)^{-1}}^{n^{-\frac{1}{2}} \log A_{\vartheta}^{-1}} e^{t_{0} n^{\frac{1}{2} s}} d t_{0}=\frac{\eta_{\lambda}\left|\operatorname{det} \mathcal{E}_{\kappa}\right|\left|\operatorname{det} L_{m, \alpha}\right|}{n s} A^{s}+o(A) .
$$

Finally, we get one more term which is of the form as the one in (40), replacing $\phi_{\kappa}$ by $\phi_{\lambda}$.
We conclude that (37) contributes in (35) (aside from an $o(A)$ term) as

$$
\left(\frac{\left(\eta_{\kappa}+\eta_{\lambda}\right) A^{s}}{s}+\frac{\left(\phi_{\kappa}+\phi_{\lambda}\right) A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|\left|\operatorname{det} \mathcal{E}_{\kappa}\right|}{n} \int_{-\pi / 2}^{\pi / 2} \ldots \int_{-\pi / 2}^{\pi / 2} \psi\left(\frac{E_{m}^{2}}{\cos ^{2} \vartheta}\right) \prod_{k=1}^{n} \frac{d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}
$$

where the last integral is

$$
\begin{aligned}
& 2^{n} \int_{0}^{\frac{\pi}{2}} \cdots \int_{0}^{\frac{\pi}{2}} \psi\left(\frac{\left(E_{m}^{(1)}\right)^{2}}{\cos ^{2} \vartheta_{1}}, \ldots, \frac{\left(E_{m}^{(n)}\right)^{2}}{\cos ^{2} \vartheta_{n}}\right) \prod_{k=1}^{n} \frac{d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}= \\
& =\frac{1}{N\left(\left|E_{m}\right|\right)} \int_{\left(E_{m}^{(1)}\right)^{2}}^{\infty} \ldots \int_{\left(E_{m}^{(n)}\right)^{2}}^{\infty} \frac{\psi\left(t_{1}, \ldots, t_{n}\right)}{\prod_{k=1}^{n} \sqrt{t_{k}-\left(E_{m}^{(k)}\right)^{2}}} \prod_{k=1}^{n} d t_{k}=\frac{1}{N\left(\left|E_{m}\right|\right)} g\left(\log u_{m}^{(1)}, \ldots, \log u_{m}^{(n)}\right)
\end{aligned}
$$

We summarize the results obtained so far. The contribution of the hyperbolic-parabolic classes in the truncated trace is

$$
\begin{equation*}
\Sigma_{\mathrm{hyp}-\mathrm{par}}=\frac{1}{2} \sum_{\kappa \in \mathcal{S}} \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} \sum_{\alpha \in \mathbf{t}_{k}^{m} / \Lambda_{\kappa}}\left[\delta_{m_{u, \kappa}} M_{\kappa}(m, \alpha, A)+C_{\kappa}(m, \alpha)\right]+o(A), \tag{41}
\end{equation*}
$$

where the main term $M_{\kappa}(m, \alpha, A)$ is given by

$$
\left(\frac{\left(\eta_{\kappa}+\eta_{\tilde{\kappa}_{m, \alpha}}\right) A^{s}}{s}+\frac{\left(\phi_{\kappa}+\phi_{\tilde{\kappa}_{m, \alpha}}\right) A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|\left|\operatorname{det} \mathcal{E}_{\kappa}\right|}{n N\left(\left|E_{m}^{\kappa}\right|\right)} g\left(\log u_{m}^{\kappa}\right),
$$

where $\tilde{\kappa}_{m, \alpha} \in \mathcal{S}$ is the cusp for $\Gamma$ that can be taken (by an element of $\Gamma$ ) to $\sigma_{\kappa} \frac{\alpha}{1-u_{m}^{\kappa}}$, and the term $C_{\kappa}(m, \alpha)$ is

$$
\begin{equation*}
\int_{\log }^{r \in P_{m, \alpha}^{\kappa}}, \tilde{u}_{m, \alpha}^{\kappa}(r i) \prod_{k=1}^{n} \frac{d r_{k}}{r_{k}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cdots \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \psi\left(\frac{\left(E_{m}^{\kappa}\right)^{2}}{\cos ^{2} \vartheta}\right)\left(\prod_{k=1}^{n} \frac{f_{\lambda_{k}}\left(\vartheta_{k}\right) d \vartheta_{k}}{\cos ^{2} \vartheta_{k}}\right) \tag{42}
\end{equation*}
$$

Moreover, the terms above are zero for any cusp $\kappa$ for all but finitely many $m$.
Let us fix a $\kappa \in \mathcal{S}$, an $m \in \mathbb{Z}^{n-1} \backslash\{0\}$ and an $\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda_{\kappa}$ in the sum (41) above. The corresponding term is counted twice, it occurs also in the case of the cusp $\kappa^{\prime}=\tilde{\kappa}_{m, \alpha}$ for an appropriate $m^{\prime} \in \mathbb{Z}^{n-1} \backslash\{0\}$ and a class $\beta$. It follows that the main terms $\delta_{m_{u, \kappa}} M_{\kappa}(m, \alpha, A)$ and $\delta_{m_{u, k^{\prime}}} M_{\kappa^{\prime}}\left(m^{\prime}, \beta, A\right)$ are equal.

At this point we specify the function $u(z)$, namely, we work with the Eisenstein series $E_{\kappa}(z, s, 0)$ for some fixed $\frac{1}{2}<s<1$ (defined in (117). It is not a cusp form and $s_{1}=\ldots=s_{n}$ hold for its eigenvalues, and therefore $m_{u, \kappa}=m_{u, \kappa^{\prime}}=0$ must hold by (8) which implies $M_{\kappa}(m, \alpha, A)=M_{\kappa^{\prime}}\left(m^{\prime}, \beta, A\right)$. Also, since $\eta_{\kappa}=1$ and $\eta_{\kappa^{\prime}}=0$, the first factor of each of these main terms is non-zero.

Assume that $u_{m}^{\kappa} \neq u_{m}^{\kappa_{m}^{\prime}, \alpha}$ holds, then we can choose the function $g$ so that exactly one of $g\left(\log u_{m}^{\kappa}\right)$ and $g\left(\log u_{m}^{\kappa_{m, \alpha}}\right)$ is zero. This yields that exactly one of $M_{\kappa}(m, \alpha, A)$ and $M_{\tilde{\kappa}_{m, \alpha}}\left(m^{\prime}, \beta, A\right)$ is zero, which is impossible, and hence $u_{m}^{\kappa}=u_{m^{\prime}}^{\tilde{\kappa}_{m, \alpha}}$ must hold.

Let us choose the vector $m=\mathbf{e}_{j}=(0, \ldots, 0,1,0, \ldots, 0)^{T}$ whose the $j$ th coordinate is 1 and the others are zero $(1 \leq j \leq n-1)$. Then $u_{\mathbf{e}_{j}}^{\kappa}=\varepsilon_{j}^{\kappa}$, and it follows that $\varepsilon_{j}^{\kappa} \in \Lambda_{\tilde{\kappa}_{m, \alpha}}$, i.e. $\Lambda_{\kappa} \subset \Lambda_{\tilde{\kappa}_{m, \alpha}}$. Changing the role of $\kappa$ and $\tilde{\kappa}_{m, \alpha}$ in the previous argument and specifying $m^{\prime}$ instead of $m$ we infer that $\Lambda_{\tilde{\kappa}_{m, \alpha}} \subset \Lambda_{\kappa}$ hence these groups are identical. Hence to conclude the proof of Proposition [2.4, that is, to show that the multiplier group $\Lambda_{\kappa}$ is independent of $\kappa$ it is enough to prove the following:

Lemma 3.1. For any two different cusps $\kappa$, $\kappa^{\prime}$ there is a hyperbolic-parabolic element in $\Gamma$ with fixed points $\kappa$ and $\kappa^{\prime}$.

Proof. In the first step we show the analogous statement for the Hilbert modular group $\Gamma_{K}$ and any two different cusps $\kappa$ and $\kappa^{\prime}$ for $\Gamma_{K}$. These cusps can and will be represented by an element of the field $K$ and the corresponding vector is obtained via the different embeddings of $K$ into $\mathbb{R}$. It is well-known that the number of the equivalence classes of cusps for $\Gamma_{K}$ is the class number $h=h(K)$ of $K$ (see Proposition 20 on page 188 in [12]). These classes are represented by a fixed set of integer ideals $\mathfrak{a}_{1}, \ldots, \mathfrak{a}_{h} \subset \mathcal{O}_{K}$ such that the corresponding cusps are written in the form $\lambda_{j}=\rho_{j} / \sigma_{j}$ where $\rho_{j}, \sigma_{j} \in \mathcal{O}_{K}$ and $\mathfrak{a}_{j}=\left(\rho_{j}, \sigma_{j}\right)$.

Assume first that $\kappa=\lambda_{j}$ for some $1 \leq j \leq h$. Let us fix the elements $\eta_{j}, \xi_{j} \in \mathfrak{a}_{j}^{-1}$ such that $\rho_{j} \eta_{j}-\xi_{j} \sigma_{j}=1$ holds, then the matrix

$$
A_{j}=\left[\begin{array}{ll}
\rho_{j} & \xi_{j} \\
\sigma_{j} & \eta_{j}
\end{array}\right] \in S L(2, K)
$$

takes $\infty$ to $\lambda_{j}$, hence $\infty$ is a cusp of $A_{j}^{-1} \Gamma_{K} A_{j}$. The stabilizer of $\infty$ in this group consists of elements of the form

$$
\left[\begin{array}{cc}
u & \zeta u^{-1} \\
0 & u^{-1}
\end{array}\right]
$$

with $u \in \mathcal{O}_{K}^{\times}$and $\zeta \in \mathfrak{a}_{j}^{-2}$ (see [12]). For a $\kappa^{\prime} \in(K \cup\{\infty\}) \backslash\{\kappa\}$, let $c=A_{j}^{-1} \kappa^{\prime}$ be a cusp of $A_{j}^{-1} \Gamma_{K} A_{j}$ different from $\infty$. We show that the latter matrix above can be chosen so that its other fixed cusp is $c$. For this, it is enough to choose the unit $u$ so that $c\left(1-u^{2}\right) \in \mathfrak{a}_{j}^{-2}$ holds. But this can be reached since for an arbitrary integral ideal $\mathfrak{a}$ one can choose $u$ so that $\mathfrak{a} \mid\left(1-u^{2}\right) \Longleftrightarrow 1-u^{2} \in \mathfrak{a}$ holds.

It follows that there is a hyperbolic-parabolic element in $\Gamma$ with fixed points $\lambda_{j}$ and $\kappa^{\prime}$. But any cusp $\kappa$ can be written as $\gamma_{\kappa}^{-1} \lambda_{j}$ for some $j$ and $\gamma_{\kappa}^{-1} \in \Gamma$, so if $\gamma \in \Gamma$ is a hyperbolicparabolic element that fixes $\lambda_{j}$ and $\gamma_{\kappa} \kappa^{\prime}$, then $\gamma_{\kappa}^{-1} \gamma \gamma_{\kappa}$ fixes $\kappa$ and $\kappa^{\prime}$. Since the cusps are the same for any finite index subgroup of $\Gamma_{K}$ (though their equivalence classes are not), the claim of the lemma follows now from Theorem [1.1,

From now on, we drop the index in the notation of the multiplier group and assume that the generating set $\varepsilon_{1}^{\kappa}, \ldots, \varepsilon_{n-1}^{\kappa}$ is the same for any $\kappa \in \mathcal{S}$. Hence the matrices $\mathcal{E}_{\kappa}$ are identical for any $\kappa$, so we omit the indices here as well. Note that the integer vectors $m_{u, \kappa}$ are also defined in terms of $\mathcal{E}$ and therefore their common value will be denoted by $m_{u}$.

Returning to the main terms $M_{\kappa}(m, \alpha, A)$ and $M_{\tilde{\kappa}_{m, \alpha}}\left(m^{\prime}, \beta, A\right)$ in our argument, from $u_{m}^{\kappa}=u_{m^{\prime}}^{\tilde{\kappa}_{m, \alpha}}$ we infer that $m=m^{\prime}$ and we simply write $u_{m}$ and $E_{m}$ in the following. Finally, the equality of the main terms implies that $\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|=\left|\operatorname{det} L_{m, \beta}^{\kappa_{m, \alpha}}\right|$.

Now we return to a general form $u$, and we only assume that it is not a cusp form and $m_{u}=0$ holds (otherwise there are no main terms in (41)). We split each main term
$M_{\kappa}(m, \alpha, A)$ into two parts:

$$
\begin{aligned}
\left(\frac{\eta_{\kappa} A^{s}}{s}+\right. & \left.\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right||\operatorname{det} \mathcal{E}|}{n N\left(\left|E_{m}\right|\right)} g\left(\log u_{m}\right)+ \\
& +\left(\frac{\eta_{\tilde{\kappa} m, \alpha} A^{s}}{s}+\frac{\phi_{\tilde{\kappa}_{m, \alpha}} A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right||\operatorname{det} \mathcal{E}|}{n N\left(\left|E_{m}\right|\right)} g\left(\log u_{m}\right) .
\end{aligned}
$$

This main term has the (equal) pair $M_{\tilde{\kappa}_{m, \alpha}}(m, \beta, A)$, and if $\lambda=\tilde{\kappa}_{m, \alpha}$ then clearly $\tilde{\lambda}_{m, \beta}=\kappa$ and the pair of $M_{\lambda}(m, \beta, A)$ is $M_{\kappa}(m, \alpha, A)$. That is, this pairing gives a bijection for every $m$ on the set of pairs $(\kappa, \alpha)$, where $\kappa \in \mathcal{S}$ and $\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda$. Moreover, the term $M_{\tilde{\kappa}_{m, \alpha}}(m, \beta, A)$ has the split form

$$
\begin{aligned}
& \left(\frac{\eta_{\tilde{\kappa}, \alpha} A^{s}}{s}+\frac{\phi_{\tilde{\kappa}_{m, \alpha}} A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \beta}^{\tilde{\kappa}_{m, \alpha}}\right||\operatorname{det} \mathcal{E}|}{n N\left(\left|E_{m}\right|\right)} g\left(\log u_{m}\right)+ \\
& \quad+\left(\frac{\eta_{\kappa} A^{s}}{s}+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \frac{\left|\operatorname{det} L_{m, \beta}^{\tilde{\kappa}_{m, \alpha}}\right||\operatorname{det} \mathcal{E}|}{n N\left(\left|E_{m}\right|\right)} g\left(\log u_{m}\right) .
\end{aligned}
$$

By the last remark of the previous paragraph we have that the first term of $M_{\kappa}(m, \alpha, A)$ is the second term of $M_{\tilde{\kappa}_{m, \alpha}}(m, \beta, A)$ and vice versa. These simple observations imply immediately that if we sum the main terms in (41) obtaining

$$
\begin{aligned}
& \frac{\delta_{m_{u}}|\operatorname{det} \mathcal{E}|}{2 n}\left[\sum_{\kappa \in \mathcal{S}}\left(\frac{\eta_{\kappa} A^{s}}{s}+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} \frac{g\left(\log u_{m}\right)}{N\left(\left|E_{m}\right|\right)} \sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda}\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|\right. \\
&\left.+\sum_{\kappa \in \mathcal{S}} \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} \frac{g\left(\log u_{m}\right)}{N\left(\left|E_{m}\right|\right)} \sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda}\left(\frac{\eta_{\tilde{\kappa}_{m, \alpha}} A^{s}}{s}+\frac{\phi_{\tilde{\kappa}_{m, \alpha}} A^{1-s}}{1-s}\right)\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|\right],
\end{aligned}
$$

then the two triple sums above are equal, hence this expression simply becomes

$$
\begin{equation*}
\frac{\delta_{m_{u}}|\operatorname{det} \mathcal{E}|}{n}\left[\sum_{\kappa \in \mathcal{S}}\left(\frac{\eta_{\kappa} A^{s}}{s}+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} \frac{g\left(\log u_{m}\right)}{N\left(\left|E_{m}\right|\right)} \sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda}\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|\right] . \tag{43}
\end{equation*}
$$

Next we give group theoretic interpretations of the quantities $N\left(\left|E_{m}\right|\right)$ and $\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|$. The sublattice $\left(u_{m}-1\right) \mathbf{t}_{\kappa}$ of $\mathbf{t}_{\kappa}$ is obtained by coordinate-wise multiplication, i.e. via multiplication by a diagonal matrix with entries $\left(u_{m}-1\right)^{(k)}$ in its diagonal. It is well-known that the index of this sublattice in $\mathbf{t}_{\kappa}$, i.e. the order of the factor group $\mathbf{t}_{\kappa}^{m}=\mathbf{t}_{\kappa} /\left(u_{m}-1\right) \mathbf{t}_{\kappa}$ is the absolute value of the determinant of this matrix, that is simply $\left|N\left(u_{m}-1\right)\right|=N\left(\left|E_{m}\right|\right)$.

Now we consider the $\Lambda$-equivalent elements of $\mathbf{t}_{\kappa}^{m}$. Assume that for an $\alpha \in \mathbf{t}_{\kappa}^{m}$ we have $u_{l} \alpha=\alpha$ (in $\mathbf{t}_{\kappa}^{m}$ ) for some $l \in \mathbb{Z}^{n-1}$. This means exactly that $\frac{u_{l}-1}{u_{m}-1} \alpha \in \mathbf{t}_{\kappa}$. A simple computation shows that in this case the element

$$
\gamma(l, m, \alpha):=\left[\begin{array}{cc}
u_{l}^{1 / 2} & \frac{u_{l}-1}{u_{m}-1} \alpha u_{l}^{-1 / 2} \\
0 & u_{l}^{-1 / 2}
\end{array}\right] \in C\left(\gamma_{m, \alpha}^{\kappa}\right) .
$$

Every element of the centralizer $C\left(\gamma_{m, \alpha}^{\kappa}\right)$ has this form by Proposition 2.5, and hence $u_{l} \alpha=\alpha$ is equivalent to $\gamma(l, m, \alpha) \in C\left(\gamma_{m, \alpha}^{\kappa}\right)$. Again, by Proposition 2.5 this holds if and only if $\log u_{l}$ is in the lattice spanned by $v_{1}^{\kappa}, \ldots, v_{n-1}^{\kappa}$ in the subspace $V=\left\{a \in \mathbb{R}^{n}: a_{1}+\cdots+a_{n}=0\right\}$, where $v_{j}^{\kappa}=l_{j}^{(1)} \log \varepsilon_{1}+\cdots+l_{j}^{(n-1)} \log \varepsilon_{n-1}$ and the integer vectors $l_{j} \in \mathbb{Z}^{n-1}$ are defined in Proposition 2.5. Hence, for a fixed $\alpha$, the number of inequivalent points $u_{l} \alpha \in \mathbf{t}_{\kappa}^{m}$ is exactly
the index of the before-mentioned sublattice in the lattice generated by $\log \varepsilon_{1}, \ldots, \log \varepsilon_{n-1}$ in $V$, and this is exactly $\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|$. It follows that

$$
\sum_{\alpha \in \mathbf{t}_{\kappa}^{m} / \Lambda}\left|\operatorname{det} L_{m, \alpha}^{\kappa}\right|=\left|\mathbf{t}_{\kappa}^{m}\right|=N\left(\left|E_{m}\right|\right)
$$

and hence (43) becomes

$$
\frac{\delta_{m_{u}}|\operatorname{det} \mathcal{E}|}{n} \sum_{\kappa \in \mathcal{S}}\left(\frac{\eta_{\kappa} A^{s}}{s}+\frac{\phi_{\kappa} A^{1-s}}{1-s}\right) \sum_{m \in \mathbb{Z}^{n-1} \backslash\{0\}} g\left(\log u_{m}\right)
$$

and this (together with (41) and (42)) completes the proof of Theorem 2.6.
3.4. Extension of $\zeta$-functions corresponding to lattices. In this section we prove Lemma 2.8. We use the notations of Section 2.5 and note that the following argument is a standard one in analytic number theory (we basically copy the proof of Theorem 1.7.2 in (5) and hence some details will be omitted.

It is easy to see that the sum in (27) converges absolutely and locally uniformly for $\operatorname{Re} s>1$, and this latter condition will be assumed in the first part of the proof. We write the terms of the sum in (27) as follows: let $0 \neq l \in L, l=\left(l^{(1)}, \ldots, l^{(n)}\right)$, then

$$
\int_{0}^{\infty} e^{-\pi x\left(l^{(k)}\right)^{2}} x^{\frac{s_{k}}{2}} \frac{d x}{x}=\frac{1}{\pi^{\frac{s_{k}}{2}}\left(l^{(k)}\right)^{s_{k}}} \int_{0}^{\infty} e^{-u} u^{\frac{s_{k}}{2}} \frac{d u}{u}=\frac{\Gamma\left(\frac{s_{k}}{2}\right)}{\pi^{\frac{s_{k}}{2}}\left|l^{(k)}\right|^{s_{k}}}
$$

and hence

$$
\int_{\left(\mathbb{R}^{+}\right)^{n}} e^{-\pi \operatorname{tr}\left(\mathbf{x} l^{2}\right)} \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}}=\pi^{-\frac{n s}{2}} \frac{\lambda_{M,-m}(|l|)}{|N l|^{s}} \prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)
$$

where $\mathbf{x}=\left(x_{1}, \ldots, x_{n}\right)^{T}, \mathbf{x} l^{2}$ is the coordinate-wise product of $\mathbf{x}$ and $l^{2}$, and the trace $\operatorname{tr}(\cdot)$ of a vector is the sum of its coordinates. Then

$$
\begin{aligned}
\pi^{-\frac{n s}{2}}\left(\prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)\right) Z_{L, M}(s, m) & =\sum_{0 \neq l \in L / M} \int_{\left(\mathbb{R}^{+}\right)^{n}} e^{-\pi \operatorname{tr}\left(\mathbf{x} l^{2}\right)} \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}} \\
& =\sum_{\varepsilon \in M^{2}} \int_{\left(\mathbb{R}^{+}\right)^{n} / M^{2}}\left(\sum_{0 \neq l \in L / M} e^{-\pi \operatorname{tr}\left(\varepsilon x l^{2}\right)}\right) \prod_{k=1}^{n}\left(\varepsilon^{(k)} x_{k}\right)^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}} \\
& =\int_{\left(\mathbb{R}^{+}\right)^{n} / M^{2}}\left(\sum_{0 \neq l \in L} e^{-\pi \operatorname{tr}\left(\mathbf{x} l^{2}\right)}\right) \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}} .
\end{aligned}
$$

Let us define the following theta function for the lattice $L$ :

$$
\Theta_{L}(\mathbf{x}):=\sum_{l \in L} e^{-\pi \operatorname{tr}\left(\mathbf{x} l^{2}\right)}
$$

Using this notation we have

$$
\begin{equation*}
\Xi_{L, M}(s, m):=\pi^{-\frac{n s}{2}}\left(\prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)\right) Z_{L, M}(s, m)=\int_{\left(\mathbb{R}^{+}\right)^{n} / M^{2}}\left(\Theta_{L}(\mathbf{x})-1\right) \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}} \tag{44}
\end{equation*}
$$

For a fixed $\mathbf{x} \in\left(\mathbb{R}^{+}\right)^{n}$ we set $f_{\mathbf{x}}(\mathbf{y})=e^{-\pi \operatorname{tr}\left(\mathbf{x y}^{2}\right)}$, its Fourier transform is

$$
\hat{f}_{\mathbf{x}}(\xi)=\int_{\mathbb{R}^{n}} f_{\mathbf{x}}(\mathbf{y}) e^{-2 \pi i\langle\mathbf{y}, \xi\rangle} \mathrm{d} \mathbf{y}=\prod_{k=1}^{n} \frac{1}{\sqrt{x_{k}}} e^{-\pi \xi_{k}^{2} / x_{k}}
$$

By the Poisson summation formula we have

$$
\begin{equation*}
\Theta_{L}(\mathbf{x})=\sum_{l \in L} f_{\mathbf{x}}(l)=\frac{1}{\operatorname{vol}\left(\mathbb{R}^{n} / L\right)} \sum_{\beta \in L^{*}} \hat{f}_{\mathbf{x}}(\beta)=\frac{1}{\operatorname{vol}\left(\mathbb{R}^{n} / L\right)(N \mathbf{x})^{1 / 2}} \Theta_{L^{*}}(1 / \mathbf{x}) . \tag{45}
\end{equation*}
$$

Now we split the integral in (44) into two parts depending on $N \mathbf{x}$. If $N \mathbf{x}<1$, we use (45) and then substitute $1 / \mathbf{x}$ to obtain that $\Xi_{L, M}(s, m)$ is
$\int_{\substack{\left(\mathbb{R}^{+}\right)^{n} / M^{2} \\ N \mathbf{x}>1}}\left(\Theta_{L}(\mathbf{x})-1\right) \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}}+\frac{1}{\operatorname{vol}\left(\mathbb{R}^{n} / L\right)} \int_{\substack{\left(\mathbb{R}^{+}\right)^{n} / M^{2} \\ N \mathbf{x}>1}}\left(\Theta_{L^{*}}(\mathbf{x})-1\right) \prod_{k=1}^{n} x_{k}^{\frac{1-s_{k}}{2}} \frac{d x_{k}}{x_{k}}$
$N \mathbf{x}>1$ $N \mathbf{x}>1$

$N \mathbf{x}<1$
$N \mathbf{x}>1$

A straightforward computation (similar to the one that led to (39) in the previous proof) shows that

$$
\int_{\substack{\left(\mathbb{R}^{+}\right)^{n} / M^{2} \\ N \mathbf{x}<1}} \prod_{k=1}^{n} x_{k}^{\frac{s_{k}}{2}} \frac{d x_{k}}{x_{k}}=2^{n-1}\left|\operatorname{det} \mathcal{E}_{M}\right| \int_{-\infty}^{0} e^{n s y_{0} / 2} d y_{0} \prod_{j=1}^{n-1} \int_{0}^{1} e^{2 \pi m_{j} i y_{j}} d y_{j}
$$

This expression is 0 unless all coordinates of $m$ are zero, in which case it is $2^{n}\left|\operatorname{det} \mathcal{E}_{M}\right| /(n s)$. Similarly,

$$
\int_{\left(\mathbb{R}^{+}\right)^{n} / M^{2}} \prod_{k=1}^{n} x_{k}^{\frac{1-s_{k}}{2}} \frac{d x_{k}}{x_{k}}=2^{n-1}\left|\operatorname{det} \mathcal{E}_{M}\right| \int_{0}^{\infty} e^{n(1-s) y_{0} / 2} d y_{0} \prod_{j=1}^{n-1} \int_{0}^{1} e^{-2 \pi m_{j} i y_{j}} d y_{j}
$$

$N \mathbf{x}>1$
and as above, this is 0 if $m \neq 0$ and otherwise we get $\frac{2^{n}\left|\operatorname{det} \mathcal{E}_{M}\right|}{n(s-1)}$ (note that $\operatorname{Re} s>1$ is still assumed here).

Therefore if $m \neq 0$, then $\Xi_{L, M}(s, m)$ is entire and

$$
\begin{equation*}
\operatorname{vol}\left(\mathbb{R}^{n} / L\right)^{1 / 2} \Xi_{L, M}(s, m)=\operatorname{vol}\left(\mathbb{R}^{n} / L^{*}\right)^{1 / 2} \Xi_{L^{*}, M}(1-s,-m) \tag{46}
\end{equation*}
$$

holds. If $m=0$, then $\Xi_{L, M}(s, m)$ is holomorphic except for $s=1$ and $s=0$, where it has simple poles with residues $\frac{2^{n}\left|\operatorname{det} \mathcal{E}_{M}\right|}{n \cdot \operatorname{vol}\left(\mathbb{R}^{n} / L\right)}$ and $-\frac{2^{n}\left|\operatorname{det} \mathcal{E}_{M}\right|}{n}$, respectively. The functional equation (46) holds also in this case for any $s \neq 0,1$.

One can reorder the equation (46) asymmetrically:

$$
Z_{L, M}(s, m)=\operatorname{vol}\left(\mathbb{R}^{n} / L^{*}\right) \pi^{n s-\frac{n}{2}}\left(\prod_{k=1}^{n} \frac{\Gamma\left(\frac{1-s_{k}}{2}\right)}{\Gamma\left(\frac{s_{k}}{2}\right)}\right) Z_{L^{*}, M}(1-s,-m) .
$$

Here

$$
\frac{\Gamma\left(\frac{1-s_{k}}{2}\right)}{\Gamma\left(\frac{s_{k}}{2}\right)}=\Gamma\left(\frac{1-s_{k}}{2}\right) \Gamma\left(1-\frac{s_{k}}{2}\right) \frac{\sin \frac{\pi s_{k}}{2}}{\pi}=2^{s_{k}} \Gamma\left(1-s_{k}\right) \frac{\sin \frac{\pi s_{k}}{2}}{\pi^{\frac{1}{2}}},
$$

i.e.

$$
Z_{L, M}(s, m)=\operatorname{vol}\left(\mathbb{R}^{n} / L^{*}\right) 2^{n s} \pi^{n(s-1)}\left(\prod_{k=1}^{n} \Gamma\left(1-s_{k}\right) \sin \frac{\pi s_{k}}{2}\right) Z_{L^{*}, M}(1-s,-m) .
$$

If $\operatorname{Re} s$ is bounded and $|\operatorname{Im} s| \geq t_{0}$ for some big enough $t_{0}>0$ then by Stirling's formula

$$
\left|\Gamma\left(1-s_{k}\right) \sin \frac{\pi s_{k}}{2}\right| \asymp|\operatorname{Im} s|^{1 / 2-\operatorname{Re} s}
$$

and hence $\left|Z_{L, M}(s, m)\right| \asymp(\operatorname{Im} s)^{n\left(\frac{1}{2}-\operatorname{Re} s\right)}\left|Z_{L^{*}, M}(1-s,-m)\right|$. By the Phragmén-Lindelöf principle, it follows from this and from the trivial bound $\left|Z_{L, M}(s, m)\right| \leq\left|Z_{L, M}(\operatorname{Re} s, 0)\right|$ for Re $s>1$ that

$$
Z_{L, M}(s, m)<_{\varepsilon, m}|\operatorname{Im} s|^{n(1-\operatorname{Re} s) / 2+\varepsilon}
$$

holds for $0 \leq \operatorname{Re} s \leq 1$ and $|\operatorname{Im} s| \geq t_{0}>0$. Similarly, one can bound $Z_{L, M}(s, m)$ by $|\operatorname{Im} s|^{\varepsilon}$ once $\operatorname{Re} s>1$ and $|\operatorname{Im} s| \geq t_{0}$ (or by a constant if $\operatorname{Re} s>1+\delta$ for some $\delta>0$ ). This completes the proof of Lemma [2.8,
3.5. Proof in the totally parabolic case. We proceed by calculating the part of the trace where we sum over parabolic classes. Every such class is represented by an element that fixes a cusp $\kappa \in \mathcal{S}$. An element of this type is conjugated by $\sigma_{\kappa} \in \operatorname{PSL}(2, \mathbb{R})^{n}$ to an element of the form

$$
\gamma_{\alpha}^{\kappa}:=\left[\begin{array}{cc}
1 & \alpha \\
0 & 1
\end{array}\right]
$$

where $0 \neq \alpha \in \mathbf{t}_{\kappa}$. A simple computation shows that two such elements $\gamma_{\alpha}^{\kappa}$ and $\gamma_{\beta}^{\kappa}$ are conjugate in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$ if and only if $\alpha=\varepsilon \beta$ for some $\varepsilon \in \Lambda$ (see also [6], section III.2). Hence summation over parabolic classes means a double summation over the elements of $\mathcal{S}$ and the non-zero elements of $\mathbf{t}_{\kappa} / \Lambda$. Therefore the contribution of the parabolic classes in the trace can be written as

$$
\begin{align*}
& \sum_{\kappa \in \mathcal{S}} \quad \sum_{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda} \sum_{\substack{\sigma \in C(\gamma) \backslash \Gamma}} \int_{F_{A}} k\left(z, \sigma^{-1} \gamma \sigma z\right) u(z) d \mu(z)= \\
& \quad=\sum_{\kappa \in \mathcal{S}} \sum_{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda}^{\gamma \sim \gamma_{\alpha}^{\kappa}} \sum_{\sigma \in C\left(\gamma_{\alpha}^{\kappa}\right) \backslash \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}} \int_{\sigma\left(\sigma_{\kappa}^{-1} F_{A}\right)} k\left(z, \gamma_{\alpha}^{\kappa} z\right) u\left(\sigma_{\kappa} z\right) d \mu(z) \tag{47}
\end{align*}
$$

where $C\left(\gamma_{\alpha}^{\kappa}\right)$ is the centralizer of $\gamma_{\alpha}^{\kappa}$ in $\sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$ given by

$$
C\left(\gamma_{\alpha}^{\kappa}\right):=\left\{\left[\begin{array}{cc}
1 & \beta \\
0 & 1
\end{array}\right] \in \operatorname{PSL}(2, \mathbb{R})^{n}: \beta \in \mathbf{t}_{\kappa}\right\}
$$

and its fundamental domain is $F_{C\left(\gamma_{\alpha}^{\kappa}\right)}=\left\{z \in \mathbb{H}^{n}: 0 \leq X_{1}^{\kappa}(z), \ldots, X_{n}^{\kappa}(z)<1\right\}$.
The union of the sets $\sigma\left(\sigma_{\kappa}^{-1} F_{A}\right)$ in (47) makes up the set $F_{C\left(\gamma_{\alpha}^{\kappa}\right)}$ except for the images of the part $F \backslash F_{A}=F_{A}^{*}$. As in the hyperbolic-parabolic case, for some cosets the images of $F_{A}^{*}$ can be added to the domain we integrate over because the kernel function vanishes on those sets. If $\sigma \in \sigma_{\kappa}^{-1} \Gamma \sigma_{\kappa}$ leaves $\infty$ fixed, then so does every element in its coset, and the part $\sigma\left(\sigma_{\kappa}^{-1} F_{A}^{*}\right)$ is the same as

$$
\left\{z \in F_{C\left(\gamma_{\kappa}^{\kappa}\right)}: \sigma^{-1} z \in \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} U_{A}, \text { for some cusp } \kappa^{\prime} \in \mathcal{S}\right\},
$$

at least if $A$ is big enough. If $\kappa \neq \kappa^{\prime}$, then $\sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}}=\left[\begin{array}{ll}a & b \\ c & d\end{array}\right]$ does not fix the point $\infty$ and hence $c \neq 0$. Since $\sigma^{-1} \infty=\infty$, the values $Y_{0}\left(\sigma^{-1} z\right)$ and $Y_{0}(z)$ are the same. Therefore, if $\sigma^{-1} z \in \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} U_{A}$, then there is a $w \in U_{A}$ such that $\sigma^{-1} z=\sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} w$, and hence

$$
Y_{0}(z)=Y_{0}\left(\sigma^{-1} z\right)=Y_{0}\left(\sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} w\right) \leq \frac{1}{(N c)^{2} A} .
$$

The inequality above follows easily from the identity $\left|c_{k} z+d_{k}\right|^{2} \cdot \operatorname{Im}\left(\sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} z\right)_{k}=\operatorname{Im} z_{k}$ that holds for every $k=1, \ldots, n$. The function $\psi$ is compactly supported, hence for a large enough $A$ the kernel

$$
k\left(z, \gamma_{\alpha}^{\kappa} z\right)=\psi\left(\frac{\left|z_{1}-\left(z_{1}-\alpha_{1}\right)\right|^{2}}{y_{1}^{2}}, \ldots, \frac{\left|z_{n}-\left(z_{n}-\alpha_{n}\right)\right|^{2}}{y_{n}^{2}}\right)=\psi\left(\frac{\alpha_{1}^{2}}{y_{1}^{2}}, \ldots, \frac{\alpha_{n}^{2}}{y_{n}^{2}}\right)
$$

vanishes for every $z \in F_{C\left(\gamma_{\alpha}\right)}$ for which $\sigma^{-1} z \in \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} U_{A}$ holds for some $\kappa \neq \kappa^{\prime}$, hence these parts can be simply added to the domain we integrate over.

Now assume that $\sigma$ does not fix the cusp $\infty$. Then $\sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}}$ cannot fix $\infty$, because this would be equivalent to

$$
\kappa=\left(\sigma_{\kappa} \sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}}\right) \infty=\left(\sigma_{\kappa} \sigma \sigma_{\kappa}^{-1}\right) \kappa^{\prime} .
$$

But $\sigma_{\kappa} \sigma \sigma_{\kappa}^{-1} \in \Gamma$ and hence $\kappa \neq \kappa^{\prime}$ cannot hold, because these two cusps are not equivalent. Also, $\sigma_{\kappa} \sigma \sigma_{\kappa}^{-1}$ does not fix $\kappa$, so (similarly as above) the parts $\sigma \sigma_{\kappa}^{-1} \sigma_{\kappa^{\prime}} U_{A}$ can be added.

Hence (47) becomes

$$
\begin{aligned}
& \sum_{\kappa \in \mathcal{S}} \sum_{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda} \int_{z \in F_{C\left(\gamma_{\alpha}^{\kappa}\right), Y_{0}(z) \leq A}} \psi\left(\frac{\alpha_{1}^{2}}{y_{1}^{2}}, \ldots, \frac{\alpha_{n}^{2}}{y_{n}^{2}}\right) u\left(\sigma_{\kappa} z\right) d \mu(z)= \\
& =\sum_{\kappa \in \mathcal{S}} \sum_{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda} \int_{0 \leq X_{1}^{\kappa}, \ldots, X_{n}^{\kappa}<1} \int_{Y_{0} \leq A} \psi\left(\frac{\alpha_{1}^{2}}{y_{1}^{2}}, \ldots, \frac{\alpha_{n}^{2}}{y_{n}^{2}}\right) u\left(\sigma_{\kappa} z\right) \frac{d y_{1} \ldots d y_{n}}{y_{1}^{2} \ldots y_{n}^{2}} d x_{1} \ldots d x_{n} .
\end{aligned}
$$

Using the Fourier expansion of $u\left(\sigma_{\kappa} z\right)$ and that for an $l \in \mathbf{t}_{\kappa}^{*}$ we have

$$
\int_{0 \leq X_{1}^{\kappa}, \ldots, X_{n}^{\kappa}<1} e^{2 \pi i<l, x>} d x_{1} \ldots d x_{n}= \begin{cases}\operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right), & \text { if } l=0, \\ 0 & \text { otherwise },\end{cases}
$$

the sum above can be written in the following way:

$$
\sum_{\kappa \in \mathcal{S}} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) \sum_{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda} \int_{Y_{0} \leq A} \psi\left(\frac{\alpha_{1}^{2}}{y_{1}^{2}}, \ldots, \frac{\alpha_{n}^{2}}{y_{n}^{2}}\right)\left(\eta_{\kappa} y_{1}^{s_{1}} \ldots y_{n}^{s_{n}}+\phi_{\kappa} y_{1}^{1-s_{1}} \ldots y_{n}^{1-s_{n}}\right) \frac{d y_{1} \ldots d y_{n}}{y_{1}^{2} \ldots y_{n}^{2}}
$$

The substitution $u_{k}=\left|\alpha_{k}\right| / y_{k}$ gives then

$$
\begin{aligned}
& \sum_{\kappa \in \mathcal{S}} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) \sum_{\substack{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda}} \frac{1}{|N \alpha|} \int_{\substack{0<u_{1}, \ldots, u_{n}<\infty \\
|N \alpha| \leq A u_{1} \ldots u_{n}}} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) \times \\
& \times\left[\eta_{\kappa} \lambda_{m_{u}}(|\alpha|) \frac{|N \alpha|^{s}}{u_{1}^{s_{1}} \ldots u_{n}^{s_{n}}}+\phi_{\kappa} \lambda_{-m_{u}}(|\alpha|) \frac{|N \alpha|^{1-s}}{u_{1}^{1-s_{1}} \ldots u_{n}^{1-s_{n}}}\right] d u_{1} \ldots d u_{n}
\end{aligned}
$$

where $|\alpha|$ denotes the coordinate-wise absolute value of the vector $\alpha$. Hence we have to examine two terms:

$$
\begin{equation*}
\eta_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) \int_{0<u_{1}, \ldots, u_{n}<\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{-s_{1}} \ldots u_{n}^{-s_{n}} \sum_{\substack{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda \\|N \alpha| \leq A u_{1} \ldots u_{n}}} \frac{\lambda_{m_{u}}(|\alpha|)}{|N(\alpha)|^{1-s}} d u_{1} \ldots d u_{n} \tag{48}
\end{equation*}
$$

and
(49)

$$
\phi_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) \int_{0<u_{1}, \ldots, u_{n}<\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{s_{1}-1} \ldots u_{n}^{s_{n}-1} \sum_{\substack{0 \neq \alpha \in \mathbf{t}_{\kappa} / \Lambda \\|N \alpha| \leq A u_{1} \ldots u_{n}}} \frac{\lambda_{-m_{u}}(|\alpha|)}{|N(\alpha)|^{s}} d u_{1} \ldots d u_{n}
$$

We express the sums in (48) and (49) in terms of the zeta functions $Z_{\kappa}\left(1-s,-m_{u}\right)$ and $Z_{\kappa}\left(s, m_{u}\right)$, respectively. By Lemma $2.8 Z_{\kappa}\left(s, m_{u}\right)$ can be continued meromorphically to $\mathbb{C}$ with simple poles at 0 and 1 if and only if $m_{u}=0$, and in the latter case its residue at 1 is $\frac{2^{n}|\operatorname{det} \mathcal{E}|}{n \cdot v o l\left(\mathbb{R}^{n} / \mathbf{t}_{k}\right)}$. By Proposition 2.7 there is a vector $\nu \in \mathbb{R}^{n}$ with non-zero coordinates such that the coordinates of $\nu \cdot \mathbf{t}_{\kappa}$ are conjugate integers. Hence we may write

$$
Z_{\kappa}\left(s, m_{u}\right)=\sum_{0 \neq \alpha \in \nu \cdot \mathbf{t}_{\kappa} / \Lambda} \frac{\lambda_{-m_{u}}(|\alpha| / \nu)}{|N(\alpha / \nu)|^{s}}=|N \nu|^{s} \sum_{0 \neq \alpha \in \nu \cdot \mathbf{t}_{\kappa} / \Lambda} \frac{\lambda_{-m_{u}}(|\alpha / \nu|)}{|N(\alpha)|^{s}}=|N \nu|^{s} \sum_{k=1}^{\infty} \frac{a_{m_{u}}(k)}{k^{s}},
$$

where

$$
a_{m_{u}}(k)=\sum_{0 \neq \alpha \in \nu \cdot \mathbf{t}_{\kappa} / \Lambda,|N \alpha|=k} \lambda_{-m_{u}}(|\alpha| / \nu) .
$$

Since $\Lambda$ is isomorphic to a finite index subgroup of the multiplicative group of the units in $\mathcal{O}_{K}$, the latter sum can be estimated from above by the number of integer ideals of norm $k$ in $K$ and hence by $\tau(k)^{[K: \mathbb{Q}]}<_{\delta} k^{\delta}$ for any $\delta>0$, where $\tau(k)$ is the number of divisors of the rational integer $k$.

Now we can apply Theorem 5.2 and Corollary 5.3 in [9] for the function

$$
\alpha_{s, m_{u}}(S)=\sum_{k=1}^{\infty} \frac{a_{m_{u}}(k)}{k^{s+S}}=\frac{Z_{\kappa}\left(s+S, m_{u}\right)}{|N \nu|^{s+S}} .
$$

If $0<\operatorname{Re} s<1$ and $\sigma_{0}>1-\operatorname{Re} s$, then

$$
\sum_{k \leq A} \frac{a_{m_{u}}(k)}{k^{s}}=\frac{1}{2 \pi i} \int_{\sigma_{0}-i T}^{\sigma_{0}+i T} \alpha_{s, m_{u}}(S) \frac{A^{S}}{S} d S+R_{s, m_{u}}
$$

where $\sum^{\prime}$ indicates that if $A$ is an integer, then the last term is to be counted with half weight, further

$$
R_{s, m_{u}} \ll \sum_{A / 2<k<2 A, k \neq A}\left|a_{m_{u}}(k)\right| k^{-\mathrm{Re} s} \min \left(1, \frac{A}{T|A-k|}\right)+\frac{4^{\sigma_{0}}+A^{\sigma_{0}}}{T} \sum_{k=1}^{\infty} \frac{\left|a_{m_{u}}(k)\right|}{k^{\sigma_{0}+\mathrm{Re} s}} .
$$

Hence for any $\sigma_{0}>\operatorname{Re} s$ the integral in (48) can be rewritten as

$$
\begin{aligned}
& |N \nu|^{1-s} \int_{0<u_{1}, \ldots, u_{n}<\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{-s_{1}} \ldots u_{n}^{-s_{n}} \times \\
& \quad \times\left[\frac{1}{2 \pi i} \int_{\sigma_{0}-i T}^{\sigma_{0}+i T} \alpha_{1-s,-m_{u}}(S) \cdot \frac{\left(|N \nu| A u_{1} \ldots u_{n}\right)^{S}}{S} d S+R_{1-s,-m_{u}}\right] d u_{1} \ldots d u_{n},
\end{aligned}
$$

where with the notation $B=|N \nu| A u_{1} \ldots u_{n}$ we have that $R_{1-s,-m_{u}}$ is bounded by

$$
\begin{equation*}
\sum_{B / 2<k<2 B, k \neq B}\left|a_{-m_{u}}(k)\right| k^{\mathrm{Re} s-1} \min \left(1, \frac{B}{T|B-k|}\right)+\frac{4^{\sigma_{0}}+B^{\sigma_{0}}}{T} \sum_{k=1}^{\infty} \frac{\left|a_{-m_{u}}(k)\right|}{k^{\sigma_{0}+1-\operatorname{Res} s}} . \tag{50}
\end{equation*}
$$

Let us fix an $0<\delta_{0}<1-\operatorname{Re} s$ and use the estimate $a_{-m_{u}}(k) \ll k^{\delta_{0}}$. Also, we set the values $\sigma_{0}=\operatorname{Re} s+\delta_{0}+\frac{1}{\log A}$ and $T=A^{\operatorname{Re} s+\delta_{1}}$ for some $\delta_{0}<\delta_{1}<1-\operatorname{Re} s$. Since $u_{k}$ is bounded from above for any $1 \leq k \leq n$, the second term on the right hand side of (50) is bounded by $A^{\delta_{0}-\delta_{1}} \log A=o(1)$ (as $A \rightarrow \infty$ ), and the implied constant depends on $\delta_{0}$.

Turning to the first term we divide the sum in it into three parts. The first one is where $|B-k|<B T$. The terms of this part are of the form $\left|a_{-m_{u}}(k)\right| k^{\operatorname{Res} s-1} \ll k^{\delta_{0}+\operatorname{Re} s-1}$ hence they give at most a constant times $(B T) \cdot B^{\delta_{0}+\operatorname{Re} s-1}=\left(|N \nu| u_{1} \ldots u_{n}\right)^{\delta_{0}+\operatorname{Res}} A^{\delta_{0}-\delta_{1}}$, and since $\psi$ is compactly supported this gives an $o(1)$ term in the last integral above.

The second part is where $B T \leq|B-k|<B T+1$, i.e. it consists of at most two terms bounded by a constant times

$$
k^{\delta_{0}+\operatorname{Re} s-1} \frac{B}{T|B-k|} \ll \frac{B^{\delta_{0}+\operatorname{Re} s}}{T^{2} B} \leq B^{\delta_{0}+\operatorname{Re} s-1}=\left(|N \nu| A u_{1} \ldots u_{n}\right)^{\delta_{0}+\operatorname{Re} s-1} .
$$

Hence we obtain a term in the integral above that can be bounded by

$$
A^{\delta_{0}+\operatorname{Re} s-1} \int_{0<u_{1}, \ldots, u_{n}<C}\left(u_{1} \ldots u_{n}\right)^{\delta_{0}-1} d u_{1} \ldots d u_{n}
$$

for some $C>0$, and the latter integral converges at 0 giving an $o(1)$ term as $A \rightarrow \infty$.
The third part is where $|B T+1| \leq|B-k|$. Note that in this case $1 \leq|B-k| \leq B$, hence this error term is bounded by

$$
\begin{aligned}
B^{\delta_{0}+\operatorname{Re} s-1} \cdot \frac{B}{T} \sum_{1 \leq|B-k| \leq B} \frac{1}{|B-k|} & \ll\left(|N \nu| u_{1} \ldots u_{n}\right)^{\delta_{0}+\operatorname{Re} s} A^{\delta_{0}-\delta_{1}} \sum_{1 \leq k \leq B} \frac{1}{k} \\
& \ll A^{\delta_{0}-\delta_{1}} \max (0, \log B) \ll A^{\delta_{0}-\delta_{1}} \log A=o(1) .
\end{aligned}
$$

Finally, to cover also those cases when $k=|N \nu| A u_{1} \ldots u_{n}$ is an integer we may add

$$
\frac{1}{2} \cdot \frac{a_{-m_{u}}(k)}{k^{1-s}} \ll\left(|N \nu| A u_{1} \ldots u_{n}\right)^{\delta_{0}+\operatorname{Re} s-1}
$$

to the error term $R_{1-s,-m_{u}}$, which also gives an $o(1)$ term.
It follows that aside from an $o(1)$ term the expression in (48) is

$$
\begin{aligned}
\eta_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right)|N \nu|^{1-s} & \int_{0<u_{1}, \ldots, u_{n}<\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{-s_{1}} \ldots u_{n}^{-s_{n}} \times \\
& \times\left(\frac{1}{2 \pi i} \int_{\sigma_{0}-i T}^{\sigma_{0}+i T} \alpha_{1-s,-m_{u}}(S) \cdot \frac{\left(|N \nu| A u_{1} \ldots u_{n}\right)^{S}}{S} d S\right) d u_{1} \ldots d u_{n}
\end{aligned}
$$

where $\sigma_{0}=\operatorname{Re} s+\delta_{0}+\frac{1}{\log A}$ and $T=A^{\operatorname{Re} s+\delta_{1}}$ for some $\delta_{0}<\delta_{1}<1-\operatorname{Re} s$. Substituting the definition of $\alpha_{1-s,-m_{u}}(S)$ and interchanging the order of integration this becomes

$$
\frac{\eta_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right)}{2 \pi i} \int_{\sigma_{0}-i T}^{\sigma_{0}+i T} F(S) Z_{\kappa}\left(1-s+S,-m_{u}\right) \frac{A^{S}}{S} d S
$$

where

$$
F(S)=\int_{0}^{\infty} \cdots \int_{0}^{\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{S-s_{1}} \ldots u_{n}^{S-s_{n}} d u_{1} \ldots d u_{n}
$$

Let us choose a number $\operatorname{Re} s-1<\sigma_{1}<0$ and set $G(S)=F(S) Z_{\kappa}\left(1-s+S,-m_{u}\right) \frac{A^{S}}{S}$. We shift the line of integration to the line $\sigma_{1}+i t$, then by the residue theorem

$$
\begin{align*}
\frac{1}{2 \pi i} \int_{\sigma_{0}-i T}^{\sigma_{0}+i T} G(S) d S= & Z_{\kappa}\left(1-s,-m_{u}\right) F(0)+\delta_{m_{u}} \cdot \frac{2^{n}|\operatorname{det} \mathcal{E}|}{n \cdot \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right)} \cdot \frac{A^{s}}{s} F(s) \\
& -\frac{1}{2 \pi i}\left(\int_{\sigma_{0}+i T}^{\sigma_{1}+i T} G(S) d S+\int_{\sigma_{1}+i T}^{\sigma_{1}-i T} G(S) d S+\int_{\sigma_{1}-i T}^{\sigma_{0}-i T} G(S) d S\right) \tag{51}
\end{align*}
$$

We show that the last three integral above is $o(1)$ as $A \rightarrow \infty$. Firstly, repeated integration by parts in $F(S)$ with respect to $u_{1}$ (for example) gives

$$
G(S)=Z_{\kappa}\left(1-s+S,-m_{u}\right) \frac{A^{S}}{S^{N+1}} \int_{0}^{\infty} \ldots \int_{0}^{\infty} H_{s_{1}}^{(N)}\left(u_{1}, \ldots, u_{n}\right) u_{1}^{S-s_{1}} \ldots u_{n}^{S-s_{n}} d u_{1} \ldots d u_{n}
$$

where $N$ is any positive integer and $H_{s_{1}}^{(N)}$ is a compactly supported smooth function.
To estimate the integrals on the right hand side of (51) we apply Lemma 2.8, if $0 \leq \operatorname{Re} s$, then we have $Z_{\kappa}\left(s,-m_{u}\right) \ll|\operatorname{Im} s|^{n / 2+\varepsilon}$ for any $\varepsilon>0$ as $|t| \rightarrow \infty$, hence on the horizontal segments we have

$$
Z_{\kappa}\left(1-s+S,-m_{u}\right) \frac{\left(u_{1} \ldots u_{n} A\right)^{S}}{S^{N+1}} \ll \frac{A^{\operatorname{Re} S}}{T^{N+1}} \cdot T^{n / 2+\varepsilon} \ll A^{\operatorname{Re} S+\left(\operatorname{Re} s+\delta_{1}\right)(n / 2+\varepsilon-N-1)}
$$

Here (by the compact support of $\left.H_{s_{1}}^{(N)}\right) u_{1}, \ldots, u_{n}$ can be bounded from above by a constant. Choosing an appropriate $N$ it follows that these integrals give $o(1)$ terms. On the vertical line we have

$$
Z_{\kappa}\left(1-s+S,-m_{u}\right) \frac{\left(u_{1} \ldots u_{n} A\right)^{S}}{S^{N+1}} \ll \frac{A^{\sigma_{1}}}{(1+|\operatorname{Im} S|)^{N+1}} \cdot|\operatorname{Im} S|^{n / 2+\varepsilon} \ll A^{\sigma_{1}}|\operatorname{Im} S|^{n / 2+\varepsilon-N-1}
$$

if $|\operatorname{Im} S|$ is big enough and hence (choosing an $N>n / 2+\varepsilon-1$ )

$$
\int_{\sigma_{1}+i T}^{\sigma_{1}-i T} G(S) d S \ll A^{\sigma_{1}}=o(1)
$$

The expression (48) becomes

$$
\eta_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) Z_{\kappa}\left(1-s,-m_{u}\right) F(0)+\delta_{m_{u}} \cdot \frac{\eta_{\kappa} 2^{n}|\operatorname{det} \mathcal{E}|}{n} \cdot \frac{A^{s}}{s} F(s)+o(1)
$$

One can show similarly that (49) is

$$
\phi_{\kappa} \operatorname{vol}\left(\mathbb{R}^{n} / \mathbf{t}_{\kappa}\right) Z_{\kappa}\left(s, m_{u}\right) \tilde{F}(0)+\delta_{m_{u}} \cdot \frac{\phi_{\kappa} 2^{n}|\operatorname{det} \mathcal{E}|}{n} \cdot \frac{A^{1-s}}{1-s} \tilde{F}(1-s)+o(1)
$$

where

$$
\tilde{F}(S)=\int_{0}^{\infty} \ldots \int_{0}^{\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{S+s_{1}-1} \ldots u_{n}^{S+s_{n}-1} d u_{1} \ldots d u_{n}
$$

Next we calculate the values $2^{n} F(s)$ and $2^{n} \tilde{F}(s)$ in the case $m_{u}=0$, when

$$
\begin{aligned}
2^{n} F(s)=2^{n} \tilde{F}(1-s)= & 2^{n} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) \prod_{k=1}^{n} d u_{k} \\
& =\int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{\psi\left(t_{1}, \ldots, t_{n}\right)}{\prod_{k=1}^{n} \sqrt{t_{k}}} \prod_{k=1}^{n} d t_{k}=g(0, \ldots, 0) .
\end{aligned}
$$

Finally, we evaluate $F$ and $\tilde{F}$ at 0 :

$$
\begin{aligned}
F(0) & =\int_{0}^{\infty} \ldots \int_{0}^{\infty} \psi\left(u_{1}^{2}, \ldots, u_{n}^{2}\right) u_{1}^{-s_{1}} \ldots u_{n}^{-s_{n}} d u_{1} \ldots d u_{n} \\
& =\frac{1}{2^{n}} \int_{0}^{\infty} \ldots \int_{0}^{\infty} \frac{\psi\left(u_{1}, \ldots, u_{n}\right)}{\sqrt{u_{1} \ldots u_{n}}} u_{1}^{-\frac{s_{1}}{2}} \ldots u_{n}^{-\frac{s_{n}}{2}} d u_{1} \ldots d u_{n} .
\end{aligned}
$$

Using (14) we get that this is

$$
\begin{aligned}
& \frac{(-1)^{n}}{(2 \pi)^{n}} \int_{0}^{\infty} \cdots \int_{0}^{\infty}\left(\int_{u_{n}}^{\infty} \cdots \int_{u_{1}}^{\infty} \frac{\frac{\partial^{n} Q}{\partial w_{1} . . w_{n}}}{\sqrt{w_{1}-u_{1}} \ldots \sqrt{w_{n}-u_{n}}} d w_{1}, \ldots w_{n} \ldots d w_{n}\right) \frac{u_{1}^{-\frac{s_{1}}{2}} \ldots u_{n}^{-\frac{s_{n}}{2}}}{\sqrt{u_{1} \ldots u_{n}}} d u_{1} \ldots d u_{n} \\
& =\frac{(-1)^{n}}{(2 \pi)^{n}} \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{\partial^{n} Q}{\partial w_{1} \ldots \partial w_{n}}\left(w_{1}, \ldots, w_{n}\right)\left(\prod_{k=1}^{n} \int_{0}^{w_{k}} \frac{u_{k}^{-\frac{s_{k}}{2}}}{\sqrt{w_{k}-u_{k}} \sqrt{u_{k}}} d u_{k}\right) d w_{1} \ldots d w_{n} .
\end{aligned}
$$

We have

$$
\int_{0}^{w} \frac{u^{-\frac{\alpha}{2}}}{\sqrt{w-u} \sqrt{u}} d u=w^{-\frac{\alpha}{2}} B\left(\frac{1-\alpha}{2}, \frac{1}{2}\right)=w^{-\frac{\alpha}{2}} \frac{\Gamma\left(\frac{1-\alpha}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}+\frac{1}{2}\right)}=w^{-\frac{\alpha}{2}} 2^{-\alpha} \frac{\Gamma\left(\frac{1-\alpha}{2}\right)^{2}}{\Gamma(1-\alpha)}
$$

for any $w>0$ and $\alpha \in \mathbb{C}$ with $0<\operatorname{Re} \alpha<1$, where $B$ is the beta function and we used the following relations:

$$
\Gamma(1 / 2)=\sqrt{\pi}, \quad \Gamma(z) \Gamma\left(z+\frac{1}{2}\right)=2^{1-2 z} \sqrt{\pi} \Gamma(2 z) .
$$

Then $F(0)$ is

$$
\frac{(-1)^{n}}{\left(2^{1-s} \pi\right)^{n}}\left(\prod_{k=1}^{n} \frac{\Gamma\left(\frac{1-s_{k}}{2}\right)^{2}}{\Gamma\left(1-s_{k}\right)}\right) \int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{\partial^{n} Q}{\partial w_{1} \ldots \partial w_{n}}\left(w_{1}, \ldots, w_{n}\right) w_{1}^{-\frac{s_{1}}{2}} \ldots w_{n}^{-\frac{s_{n}}{2}} d w_{1} \ldots d w_{n}
$$

We substitute $w_{k}=e^{x_{k}}+e^{-x_{k}}-2$ to express this in terms of the function $g$, and then the integral above becomes

$$
\int_{0}^{\infty} \cdots \int_{0}^{\infty} \frac{\partial^{n} g}{\partial x_{1} \ldots \partial x_{n}}\left(x_{1}, \ldots, x_{n}\right) \prod_{k=1}^{n}\left(e^{x_{k}}+e^{-x_{k}}-2\right)^{-\frac{s_{k}}{2}} d x_{k}
$$

Now

$$
\frac{\partial^{n} g}{\partial x_{1} \ldots \partial x_{n}}\left(x_{1}, \ldots, x_{n}\right)=\frac{(-i)^{n}}{(2 \pi)^{n}} \int_{0}^{\infty} \cdots \int_{0}^{\infty} h\left(r_{1}, \ldots, r_{n}\right) r_{1} \ldots r_{n} e^{-i\left(r_{1} x_{1}+\cdots+r_{n} x_{n}\right)} d r_{1} \ldots d r_{n}
$$

and

$$
\begin{aligned}
\int_{0}^{\infty}\left(e^{x}+e^{-x}-2\right)^{-\alpha} e^{-i r x} d x & =\int_{0}^{1}\left(y^{-\frac{1}{2}}-y^{\frac{1}{2}}\right)^{-2 \alpha} y^{i r-1} d y=\int_{0}^{1}(1-y)^{-2 \alpha} y^{\alpha+i r-1} d y \\
& =B(\alpha+i r, 1-2 \alpha)=\frac{\Gamma(\alpha+i r) \Gamma(1-2 \alpha)}{\Gamma(1-\alpha+i r)}
\end{aligned}
$$

We conclude that $F(0)$ is

$$
\left(\frac{i}{2^{2-s} \pi^{2}}\right)^{n}\left(\prod_{k=1}^{n} \Gamma\left(\frac{1-s_{k}}{2}\right)^{2}\right) \int_{0}^{\infty} \ldots \int_{0}^{\infty} h\left(r_{1}, \ldots, r_{n}\right) \prod_{k=1}^{n} \frac{r_{k} \Gamma\left(\frac{s_{k}}{2}+i r_{k}\right)}{\Gamma\left(\frac{2-s_{k}}{2}+i r_{k}\right)} d r_{k}
$$

Similarly, $\tilde{F}(0)$ is

$$
\left(\frac{i}{2^{s+1} \pi^{2}}\right)^{n}\left(\prod_{k=1}^{n} \Gamma\left(\frac{s_{k}}{2}\right)^{2}\right) \int_{0}^{\infty} \cdots \int_{0}^{\infty} h\left(r_{1}, \ldots, r_{n}\right) \prod_{k=1}^{n} \frac{r_{k} \Gamma\left(\frac{1-s_{k}}{2}+i r_{k}\right)}{\Gamma\left(\frac{s_{k}+1}{2}+i r_{k}\right)} d r_{k}
$$

and this completes the proof of Theorem 2.9.

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