Triple product integrals and Rankin-Selberg L-functions

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Abstract. We prove a reciprocity formula that relates a spectral average of products of triple product integrals involving automorphic forms of weights 0 and 1/2 to the classical Rankin-Selberg integrals for automorphic forms of weight 0.

1. Introduction

1.1. Triple product integrals of weight 0 and weight 1/2 Maass forms. Let u and U be two Maass cusp forms of weight 0 for $SL(2,\mathbf{Z})$. This means that u and U are $SL(2,\mathbf{Z})$-invariant functions on the open upper half plane $\mathbb{H}$ decaying exponentially as $\operatorname{Im} z \to \infty$, and u and U are eigenfunctions of the hyperbolic Laplace operator $\Delta_0 := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$. Let D_1 be a fundamental domain of the quotient $SL(2,\mathbf{Z}) \setminus \mathbb{H}$ and $d\mu_z := \frac{dxdy}{y^2}$. We write

$$(f_1, f_2)_1 := \int_{D_1} f_1(z) \overline{f_2(z)} d\mu_z,$$

where $d\mu_z$ is an $SL(2,\mathbf{R})$-invariant measure on $\mathbb{H}$. The triple product integral

$$\left(|U|^2, u \right)_1 \tag{1.1}$$

is an important object of study in the theory of automorphic forms. For example, the famous Quantum Unique Ergodicity (QUE) Conjecture states that if u is fixed, U is a Hecke eigenform satisfying $(U,U)_1 = 1$ and the Laplace eigenvalue of U tends to $-\infty$, then

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(1.1) tends to 0. This conjecture was proved by Lindenstrauss and Soundararajan (see [L] and [S]). However, quantifying the rate of convergence in QUE is still an open problem. Watson (see [Wa]) proved an important identity relating (1.1) to the central value of a degree 8 L-function. This identity shows that the Generalized Riemann Hypothesis for some $GL(2) \times GL(3)$ Rankin-Selberg L-functions would give a quantitative form of QUE. The integrals (1.1) can be expressed in terms of triple product integrals involving weight 1/2 Maass forms, see [B1], Theorem 1.1. This motivates the study of the triple product integrals to be considered in Theorem 1.1 below. To define them properly and to state our main result we need some notations.

1.2. Necessary notations. There is a list of notations at the end of the paper.

We write

$$\Gamma_0(4) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{Z}) : \ c \equiv 0 \pmod 4 \right\}.$$

Let D_4 be a fundamental domain of the quotient $\Gamma_0(4) \setminus \mathbb{H}$ and

$$(f_1, f_2)_4 := \int_{D_4} f_1(z) \overline{f_2(z)} d\mu_z.$$

The hyperbolic Laplace operator of weight l is given by:

$$\Delta_l := y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - ily \frac{\partial}{\partial x}.$$

For a complex number $z \neq 0$, its argument (denoted by $\arg z$) is chosen to be in the range $(-\pi, \pi]$, and we define $\log z := \log |z| + i \arg z$ and $z^s := e^{s \log z}$ for any $s \in \mathbf{C}$.

We write $e(x) := e^{2\pi i x}$. For $z \in \mathbb{H}$, we define

$$B_0(z) := (\operatorname{Im} z)^{\frac{1}{4}} \theta(z) = (\operatorname{Im} z)^{\frac{1}{4}} \sum_{m=-\infty}^{\infty} e(m^2 z). \tag{1.2}$$

We define the symbol $\left(\frac{c}{d}\right)$ where c is an integer and d is an odd integer. For $d > 0$ this is the usual Jacobi symbol, and we extend it by the formulas $\left(\frac{c}{d}\right) := \frac{c}{|c|} \left(\frac{c}{-d}\right)$ for $c \neq 0$, $\left(\frac{0}{d}\right) := 1$ for $d = \pm 1$, $\left(\frac{0}{d}\right) := 0$ for $|d| > 1$. Define $\epsilon_d := 1$ for $d \equiv 1 \pmod 4$, $\epsilon_d := i$ for $d \equiv -1 \pmod 4$. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$ let $\nu(\gamma) := \left(\frac{c}{d}\right) \overline{\epsilon_d}$.

Then for every $z \in \mathbb{H}$ and $\gamma \in \Gamma_0(4)$ we have

$$B_0(\gamma z) = \nu(\gamma) \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^{1/2} B_0(z), \tag{1.3}$$

where $j_\gamma(z) := cz + d$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbf{R})$. It is also known that B_0 has an additional transformation formula

$$B_0 \left(\frac{-1}{4z} \right) = e \left(\frac{-1}{8} \right) \left(\frac{z}{|z|} \right)^{\frac{1}{2}} B_0(z) \tag{1.4}$$

for every $z \in \mathbb{H}$.

In this paper, any automorphic function is of weight $l = \frac{1}{2} + 2n$ or $l = 2n$ with some integer n. A smooth function $f : \mathbb{H} \to \mathbf{C}$ is said to be an automorphic function of weight l for Γ if it has at most polynomial growth at the cusps of Γ and satisfies the transformation formula

$$f(\gamma z) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|} \right)^l f(z) \cdot \begin{cases} 1 & \text{if} \quad l = 2n \\ \nu(\gamma) & \text{if} \quad l = 2n + \frac{1}{2} \end{cases}$$

for any $z \in \mathbb{H}$ and $\gamma \in \Gamma$, where Γ is either $SL(2, \mathbf{Z})$ or $\Gamma_0(4)$. If $l = \frac{1}{2} + 2n$, we can take only $\Gamma = \Gamma_0(4)$. The operator Δ_l acts on automorphic functions of weight l. We say that f is a Maass form of weight l for Γ, if f is an automorphic function and it is an eigenfunction of Δ_l. If a Maass form f has exponential decay at all of the cusps of Γ, it is called a cusp form.

A Maass cusp form f of weight 0 for $SL(2, \mathbf{Z})$ is called *even* if $f(z) = f(-\overline{z})$, and it is called *odd* if $f(z) = -f(-\overline{z})$.

In this work, our weight 0 Maass cusp forms u_1, u_2 for $SL(2, \mathbf{Z})$ are assumed to be

(i) L^2-normalized (i.e. $(u_j, u_j)_1 = 1$ for $j = 1, 2$),

(ii) either orthogonal to each other (i.e. $(u_1, u_2)_1 = 0$) or satisfying $u_1 = u_2$,

(iii) and either both even or both odd.

Assume that $\Delta_0 u_j = s_j(s_j - 1)u_j$, where $s_j = \frac{1}{2} + it_j$ and $t_j > 0$ ($j = 1, 2$). We have the Fourier expansions

$$u_1(z) = \sum_{m \neq 0} \rho_{u_1}(m) W_{0,it_1}(4\pi |m| y) e(mx), \ u_2(z) = \sum_{m \neq 0} \rho_{u_2}(m) W_{0,it_2}(4\pi |m| y) e(mx). \tag{1.5}$$

Here $W_{\alpha,\beta}$ denotes the Whittaker functions, see Section 3.7 for the definition of these functions.

The Rankin-Selberg L-function is defined in terms of an absolutely convegent Dirichlet series

$$L(S) = L(S, u_1 \otimes \overline{u_2}) := \sum_{m>0} \rho_{u_1}(m)\overline{\rho_{u_2}(m)} m^{1-S} \tag{1.6}$$

for $\operatorname{Re} S \gg 1$. It is well-known that $L(S)$ extends meromorphically to the whole complex plane and is regular for $\operatorname{Re} S \geq 1/2$ with at most a simple pole at $S = 1$. Such a simple pole occurs only when $u_1 = u_2$.

The Wilson function $\phi_\lambda(x; a, b, c, d)$ was defined in [G1], we give its definition in Section 3.6. We use the abbreviations $\Gamma(X \pm Y) := \Gamma(X+Y)\Gamma(X-Y)$ and

$$\Gamma(X \pm Y \pm Z) := \Gamma(X+Y+Z)\Gamma(X+Y-Z)\Gamma(X-Y+Z)\Gamma(X-Y-Z).$$

Recalling the notations t_j and s_j $(j = 1, 2)$ from above define

$$N^+(S,t) := \frac{\Gamma(S \pm it_1 + it_2)\Gamma\left(\frac{1}{4} + it_2 \pm it\right)}{\sin\pi(2it_2)} \left(\sin\pi s_1 + \sin\pi(1 - s_2 - S)\right)\phi^+_{i\left(\frac{1}{2}-S\right)}(t),$$

$$N^-(S,t) := \frac{\Gamma(S \pm it_1 - it_2)\Gamma\left(\frac{1}{4} - it_2 \pm it\right)}{\sin\pi(-2it_2)} \left(\sin\pi s_1 + \sin\pi(s_2 - S)\right)\phi^-_{i\left(\frac{1}{2}-S\right)}(t),$$

$$\phi^+_\lambda(x) := \phi_\lambda\left(x; \frac{3}{4} + it_2, \frac{1}{4} + it_1, \frac{1}{4} - it_1, \frac{3}{4} - it_2\right),$$

$$\phi^-_\lambda(x) := \phi_\lambda\left(x; \frac{3}{4} - it_2, \frac{1}{4} + it_1, \frac{1}{4} - it_1, \frac{3}{4} + it_2\right),$$

and let

$$N(S,t) := N^+(S,t) + N^-(S,t).$$

This function was introduced in [B2].

CONVENTION. Since the Maass cusp forms u_1, u_2 and the positive numbers t_1 and t_2 are fixed, we will not denote the dependence on t_1 and t_2 in the sequel.

1.3. The main result.

Denote by $L^2_l(D_4)$ the space of automorphic functions of weight l for $\Gamma_0(4)$ for which $(f,f)_4 < \infty$.

Take $u_{0,1/2} = c_0 B_0$, where c_0 is chosen such that $\left(u_{0,1/2}, u_{0,1/2}\right)_4 = 1$. Let $\left\{u_{j,1/2} :\ j \geq 0\right\}$ be an orthonormal basis of Maass forms for the discrete part of $L^2_{1/2}(D_4)$. Write

$$\Delta_{1/2} u_{j,1/2} = \Lambda_j u_{j,1/2},\ \Lambda_j = S_j(S_j - 1),\ S_j = \frac{1}{2} + iT_j.$$

It is known that $\Lambda_0 = -\frac{3}{16}$ and $\Lambda_j \to -\infty$. It follows from [Sa], Theorem 3.6 that $\Lambda_j < -\frac{3}{16}$ for $j \geq 1$.

For the cusps $\mathfrak{a} = 0, \infty$ denote by $E_{\mathfrak{a}}\left(z, s, \frac{1}{2}\right)$ the Eisenstein series of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$ at the cusp $\mathfrak{a}$. We give its definition for $z \in \mathbb{H}$ and $\operatorname{Re} s > 1$ in Section 2.5. On the one hand, as a function of z it is an eigenfunction of $\Delta_{1/2}$ of eigenvalue $s(s-1)$. On the other hand, for every z the function $E_{\mathfrak{a}}\left(z, s, \frac{1}{2}\right)$ has a meromorphic continuation in s to the whole plane, and this function is regular at every point s with $\operatorname{Re} s = \frac{1}{2}$. If f is an automorphic function of weight $1/2$ and the following integral is absolutely convergent, define

$$\zeta_{\mathfrak{a}}(f, r) := \int_{D_4} f(z) \overline{E_{\mathfrak{a}}\left(z, \frac{1}{2} + ir, \frac{1}{2}\right)} d\mu_z.$$

Let $\beta > 0$. We say that a function χ satisfies condition C_β if χ is an even holomorphic function defined on the strip $|\operatorname{Im} z| < \beta$ and for every fixed $K > 0$ the function

$$|\chi(z)|\, e^{-\pi|z|} \left(1 + |z|\right)^K$$

is bounded on this strip.

Let δ_{u_1,u_2} be Kronecker's symbol. We write $\left(\kappa(u)\right)(z) := u(4z)$. We denote by $\zeta\left(S\right)$ the Riemann zeta function.

THEOREM 1.1. *There is an absolute constant $\beta > 0$ such that if χ is a function satisfying condition C_β, then the sum of*

$$\sum_{j=1}^{\infty} \chi\left(T_j\right) \left(B_0 \kappa\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right)_4 \overline{\left(B_0 \kappa\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)_4} \tag{1.7}$$

and

$$\frac{1}{4\pi} \sum_{\mathfrak{a}=0,\infty} \int_{-\infty}^{\infty} \chi\left(r\right) \zeta_{\mathfrak{a}}\left(B_0 \kappa\left(\overline{u_2}\right), r\right) \overline{\zeta_{\mathfrak{a}}\left(B_0 \kappa\left(\overline{u_1}\right), r\right)} dr \tag{1.8}$$

equals the sum of

$$\frac{3}{2\pi^{3/2}}\frac{\delta_{u_1,u_2}}{\Gamma\left(\frac{1}{2}\pm it_1\right)}\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{4}\pm it\pm it_1\right)}{\Gamma\left(\pm 2it\right)}\chi(t)dt$$

and

$$-\frac{6}{\Gamma\left(\frac{1}{2}\pm it_2\right)}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)}\frac{\zeta\left(2S\right)L\left(S\right)}{\left(2\pi\right)^{2S}}\Gamma\left(S\right)\Gamma\left(1-S\right)H_{\chi}\left(S\right)dS, \tag{1.9}$$

where

$$H_{\chi}\left(S\right):=\int_{-\infty}^{\infty}\frac{\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{4}\pm it\pm it_1\right)}{\Gamma\left(\pm 2it\right)}\chi(t)N\left(S,t\right)dt.$$

The sum in (1.7), and the integrals in (1.8) and (1.9) are absolutely convergent.

1.4. Discussion of the main result.

REMARK 1.1. Many ideas of our proof are present also in papers of Nelson, see [Nel1], [Nel2]. See, in particular, [Nel2, formula (10)], the discussion below that formula and [Nel2, formulas (14), (11)]. Indeed, using our notation, Nelson considered the following quantities:

$$\left|\int_{D_4}B_0\left(z\right)\phi\left(z\right)\overline{h\left(z\right)}d\mu_z\right|^2, \tag{1.10}$$

where $\phi\left(z\right)$ and $h\left(z\right)$ are cusp forms for $\Gamma_0(4)$ of weights 0 and 1/2, respectively. He suggested summing (1.10) over either ϕ or h in an orthonormal basis, and then expressing the resulting sum using Parseval's identity as an inner product involving $|B_0|^2$, i.e.,

$$\int_{D_4}\left|B_0\left(z\right)\right|^2\left|\phi\left(z\right)\right|^2d\mu_z. \tag{1.11}$$

Then he remarks in [Nel2, formula (14)] that $\left|B_0\right|^2$ is orthogonal to cusp forms, which implies that $\left|B_0\right|^2$ can be expressed as a linear combination of Eisenstein series, see [Nel2, formula (11)]. Then one can unfold the integral (1.11), and this leads to Rankin-Selberg L-functions.

In this paper, rather than simply summing (1.10) over h, we insert a weight function that depends on its Laplace eigenvalue. Although Parseval's identity cannot be applied in this case, the resulting sum can still be expressed as a sum of inner products involving $B_0\overline{B_n}$, where the functions B_n are liftings of B_0 via the Maass operators. These products are still linear combinations of Eisensein series, a technical variant of this key fact is proved in Lemma 4.7 below. Then we can apply the unfolding method, getting again an expression involving Rankin-Selberg L-functions.

Many convergence problems occur during this process, but finally we are able to give an explicit class of admissible test functions and an explicit form of the integral transform.

Also, instead of the absolute square in (1.10) we consider the product of two such triple product integrals with two different weight 0 Maass cusp forms u_1, u_2 for $SL(2, \mathbf{Z})$.

Note that the fact that $|B_0|^2$ is orthogonal to cusp forms played a role already in our work [B3] (see Lemma 6.6 there), where a duality relation was proved for the kind of inner products considered also in this paper. The duality relation proved in [B3] involved also holomorphic analogues of the triple product integrals of Theorem 1.1 above. It is possible to prove an analogue of Theorem 1.1 also for such inner products. We will state this holomorphic analogue without proof in Section 1.5.

We will give a bit more detailed sketch of the proof of Theorem 1.1 in Section 1.6.

REMARK 1.2. In this remark we show that it is reasonable to expect that a special case of our formula recovers a particular instance of the spectral reciprocity formulae discovered recently by Humphries-Khan and Kwan in [H-K] and [Kw].

Remark 1.2 can be skipped, the rest of the paper can be understood without reading it. Some notions involved in the present remark will not be used later in the paper, therefore instead of giving every definition here we just refer to the literature. Our main references will be [B1] and [K-S], most of the notions are defined there.

We will consider the cuspidal sum (1.7) of our Theorem 1.1 above in the case $u_1 = u_2$, and assume also that u_1 is a simultaneous Hecke eigenform. We first choose our orthonormal basis $\left\{u_{j,1/2} : \ j \geq 0\right\}$ in a special way. In order to do that we have to define some operators. The Hecke operator T_{p^2} of weight $\frac{1}{2}$ for every prime $p \neq 2$ and the operator L are defined in [K-S], pp 199-200 and p 195, respectively. These operators act on the space $L^2_{1/2}(D_4)$, they are self-adjoint and commute with each other and with $\Delta_{1/2}$. Hence our orthonormal basis $\left\{u_{j,1/2} : \ j \geq 0\right\}$ can be chosen in such a way that every $u_{j,1/2}$ is an eigenfunction of the operators T_{p^2} ($p \neq 2$) and of the operator L (see [K-S], pp 195-196). By Lemma 5.3 and Lemma 5.5 (ii) of [B1] we see that $B_0 \kappa \left(\overline{u_1}\right)$ is an eigenfunction of L of eigenvalue 1. But two L-eigenfunctions with different L-eigenvalues are orthogonal to each other. Therefore we can keep in (1.7) only those $u_{j,1/2}$ having L-eigevalue 1, since the contribution of other terms is 0. For the case $Lu_{j,1/2} = u_{j,1/2}$ we will prove Proposition 1.1 below. We first need some notations.

Let $F \in L^2_{1/2}(D_4)$ be a cusp form of weight 1/2 for $\Gamma_0(4)$ which is an eigenfunction of the Hecke operator T_{p^2} of weight $\frac{1}{2}$ for every prime $p \neq 2$ and satisfies $LF = F$. Assume also $(F, F)_4 = 1$. Assume $\rho_F(1) \neq 0$, where $\rho_F(1)$ is the first Fourier coefficient of F at ∞. Under this assumption the Shimura lift Shim$F$ is defined in [K-S], pp. 196-197. It is an even Maass cusp form of weight 0 for SL(2, Z), it is a simultaneous Hecke eigenform and its first Fourier coefficient is 1. Let U be a cusp form and a simultaneous Hecke eigenform of weight 0 for SL(2, Z) satisfying $(U, U)_1 = 1$.

Assume $\Delta_0(\mathrm{Shim}F) = \left(-\frac{1}{4} - t^2\right)\mathrm{Shim}F$, $\Delta_0 U = \left(-\frac{1}{4} - T^2\right) U$, $\Delta_{1/2}F = \left(-\frac{1}{4} - r^2\right) F$. Note that we have $t = 2r$ e.g. by Theorem 1 of [B4].

PROPOSITION 1.1. *Assume that $\rho_F(1) \neq 0$. Using the notations and assumptions above we have that $\left|\left(B_0\kappa\left(\overline{U}\right), F\right)_4\right|^2$ equals*

$$d\frac{\left|\rho_U(1)\right|^2 L\left(\frac{1}{2}, \mathrm{Shim}F \otimes \mathrm{sym}^2 U\right)}{(\mathrm{Shim}F, \mathrm{Shim}F)_1}\left|\Gamma\left(\frac{\frac{1}{2} + it}{2}\right)\right|^2\left|\Gamma\left(\frac{\frac{1}{2} + 2iT \pm it}{2}\right)\right|^2,$$

where sym^2U *is the symmetric square lift of* U, $L\left(s, \mathrm{Shim}F \otimes \mathrm{sym}^2U\right)$ *is the Rankin-Selberg L-function of the pair* $\left(\mathrm{Shim}F, \mathrm{sym}^2U\right)$*, and $d > 0$ is an absolute constant.*

The Shimura lift Shim$F$ is defined also without the condition $\rho_F(1) \neq 0$ on p 981 of [D-I-T]. It is very likely that using that definition Proposition 1.1 is true without the condition $\rho_F(1) \neq 0$, but we were able to prove it only under this condition.

Assume now that Proposition 1.1 is true without the condition $\rho_F(1) \neq 0$. Let $u_1 = u_2$, and assume also that u_1 is a simultaneous Hecke eigenform. We can then see that choosing the test functions suitably the cuspidal sum (1.7) of Theorem 1.1 above coincides with the cuspidal sum of Theorem 1.1 of [Kw] assuming there that $s = \frac{1}{2}$ and Φ is self-dual.

Indeed, we choose $U = u_1$ in Proposition 1.1. Then U and so T are fixed there, but F may run over those elements of the orthonormal basis $\left\{u_{j,1/2} : j \geq 0\right\}$ having L-eigenvalue 1. Then Shim$F$ runs over an orthogonal basis of even Hecke normalized Maass-Hecke cusp forms of weight 0 for SL(2, Z), see [B-M], Theorem 1.2 and the last lines of p. 982 of [D-I-T]. We see the coincidence with the cuspidal sum of [Kw] in the above-mentioned special case. Note that in the special case $h^{\mathrm{hol}}(k) = 0$ the cuspidal sum of Theorem 3.1 of [H-K] also has this form.

Proof of Proposition 1.1. We apply the Theorem of [B1] for this U and for

$$u := \frac{\mathrm{Shim}F}{\sqrt{(\mathrm{Shim}F, \mathrm{Shim}F)_1}}. \tag{1.12}$$

Theorem 1.2 of [B-M] implies that we have a one-element sum in the Theorem of [B1]. Then we get that

$$|\rho_u(1)|^2 \left| \int_{D_1} |U(z)|^2 u(z) d\mu_z \right|^2 = c_1 |\rho_U(1)|^2 |\rho_F(1)|^2 \left| \left(B_0 \kappa \left(\overline{U} \right), F \right)_4 \right|^2, \tag{1.13}$$

where $c_1 > 0$ is an absolute constant and $\rho_u(1)$, $\rho_U(1)$ are the first Fourier coefficients of u and U, respectively.

Formula (0.19) of [K-S] shows that

$$|\rho_F(1)|^2 = c_2 \left| \Gamma\left(\frac{\frac{1}{2} + it}{2} \right) \right|^2 |\rho_u(1)|^2 L\left(\frac{1}{2}, \mathrm{Shim}F \right), \tag{1.14}$$

where $c_2 > 0$ is an absolute constant and $L(s, \mathrm{Shim}F)$ is the Hecke L-function of $\mathrm{Shim}F$. We applied again Theorem 1.2 of [B-M] to see that we have a one-element sum in [K-S], (0.19). We used also $t = 2r$ and that (1.12) implies $|\rho_u(1)|^2 = \frac{1}{(\mathrm{Shim}F, \mathrm{Shim}F)_1}$.

By (2.4) of [B-K], which is a consequence of Watson's identity (proved in [Wa], Theorem 3) we have that

$$\left| \int_{D_1} |U(z)|^2 u(z) d\mu_z \right|^2 = c_3 \frac{|\rho_U(1)|^4 L\left(\frac{1}{2}, \mathrm{Shim}F\right) L\left(\frac{1}{2}, \mathrm{Shim}F \otimes \mathrm{sym}^2 U\right)}{(\mathrm{Shim}F, \mathrm{Shim}F)_1} G_T(t) \tag{1.15}$$

with

$$G_T(t) := \left| \Gamma\left(\frac{\frac{1}{2} + it}{2} \right) \right|^4 \left| \Gamma\left(\frac{\frac{1}{2} + 2iT \pm it}{2} \right) \right|^2, \tag{1.16}$$

where $c_3 > 0$ is an absolute constant. We used in (2.4) of [B-K] that the expressions

$$|\rho_U(1)|^2 \left| \Gamma\left(\frac{1}{2} + iT \right) \right|^2 L\left(1, \mathrm{sym}^2 U\right), \quad \frac{L\left(1, \mathrm{sym}^2 u\right) \left|\Gamma\left(\frac{1}{2} + it\right)\right|^2}{(\mathrm{Shim}F, \mathrm{Shim}F)_1}$$

are absolute constants, see [I-K], (5.101).

By (1.13), (1.14), (1.15), (1.16) and the fact that $L\left(\frac{1}{2}, \mathrm{Shim}F\right) \neq 0$ by our assumption $\rho_F(1) \neq 0$, and by (1.14) we get the statement.

1.5. Statement of the holomorphic theorem. First we need some further definitions. We introduce the Maass operators

$$K_k := (z-\overline{z})\frac{\partial}{\partial z} + k = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + k, \quad L_k := (\overline{z}-z)\frac{\partial}{\partial \overline{z}} - k = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - k.$$

We will give the basic properties of these operators in Lemma 2.1 below. We just mention here that if f is a Maass form of weight k, then $K_{k/2}f$ and $L_{k/2}f$ are Maass forms of weight $k+2$ and $k-2$, respectively.

If $k \geq 1$ is an integer, let $S_{2k+\frac{1}{2}}$ be the space of holomorphic cusp forms of weight $2k+\frac{1}{2}$ with the multiplier system ν for the group $\Gamma_0(4)$. Let $f_{k,1}, f_{k,2}, ..., f_{k,s_k}$ be an orthonormal basis of $S_{2k+\frac{1}{2}}$, and write $g_{k,j}(z) := (\mathrm{Im}\, z)^{\frac{1}{4}+k} f_{k,j}(z)$. We note that $g_{k,j}$ is a Maass cusp form for $\Gamma_0(4)$ of weight $2k+\frac{1}{2}$, and $\Delta_{2k+\frac{1}{2}} g_{k,j} = \left(k+\frac{1}{4}\right)\left(k-\frac{3}{4}\right) g_{k,j}$ (this follows easily from Lemma 2.1 below, parts (v) and (iii)).

Suppose u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u = s(s-1)u$. For each $n \geq 0$, define

$$\left(\kappa_n(u)\right)(z) := \frac{\left(K_{n-1}K_{n-2}\dots K_1K_0u\right)(4z)}{(s)_n\,(1-s)_n}, \tag{1.17}$$

where $(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)}$. It is easy to check that $\kappa_n(u)$ is a cusp form of weight $2n$ for the group $\Gamma_0(4)$.

THEOREM 1.2. *For every integer $n \geq 1$ we have that*

$$\sum_{j=1}^{s_n} \left(B_0\kappa_n\left(\overline{u_2}\right), g_{n,j}\right)_4 \overline{\left(B_0\kappa_n\left(\overline{u_1}\right), g_{n,j}\right)_4}$$

equals the sum of

$$\frac{6}{\pi^{1/2}}\frac{\delta_{u_1,u_2}\Gamma\left(n \pm it_1\right)}{\Gamma\left(2n-\frac{1}{2}\right)(s_1)_n\,(1-s_1)_n}$$

and

$$-\frac{24\pi\Gamma\left(n\pm it_1\right)\Gamma\left(\frac{1}{2}\pm it_1-n\right)}{\Gamma\left(2n-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\pm it_2\right)}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)}\frac{\zeta(2S)\,L(S)\,\Gamma(S)\,\Gamma(1-S)\,N\left(S, i\left(\frac{1}{4}-n\right)\right)}{(2\pi)^{2S}}dS. \tag{1.18}$$

The integral in (1.18) is absolutely convergent.

REMARK 1.3. This result was informally announced in our paper [B2], see pp 353-354. We decided to prove in this paper only the nonholomorphic case, i.e. Theorem 1.1. Theorem 1.2 can be proved very similarly to the nonholomorphic case.

1.6. Outline of the proof of Theorem 1.1.

We have to give an expression for

$$\sum_{j=1}^{\infty} \chi(T_j) \left(B_0 \kappa(\overline{u_2}), u_{j,\frac{1}{2}}\right)_4 \overline{\left(B_0 \kappa(\overline{u_1}), u_{j,\frac{1}{2}}\right)_4} + \text{Eisenstein part} \tag{1.19}$$

with a weight function χ. We can choose an automorphic kernel $K(z,w)$ such that (1.19) equals

$$\int_{D_4} \left(\int_{D_4} \overline{B_0(z)} u_1(4z) K(z,w) d\mu_z \right) B_0(w) \overline{u_2(4w)} d\mu_w. \tag{1.20}$$

By unfolding the inner integral here can be written as

$$\int_{\mathbb{H}} \overline{B_0(z)} u_1(4z) k(z,w) d\mu_z \tag{1.21}$$

with a kernel function k. We now use geodesic polar coordinates around w, so we have to compute the integral on noneuclidean circles around w. We can determine the Fourier expansion of u_1 on such circles using an important theorem of Fay, which is recorded in the present paper in Lemma 2.2. We get in this way that (1.21) equals

$$\sum_{n=0}^{\infty} a_n \overline{B_n(w)} \left(K_{n-1} K_{n-2} \ldots K_1 K_0 u_1\right)(4w), \tag{1.22}$$

where

$$B_n := \frac{1}{n!} K_{(n-1)+\frac{1}{4}} \ldots K_{\frac{5}{4}} K_{\frac{1}{4}} B_0,$$

and the coefficients a_n are explicitly determined in terms of the weight function χ and the Laplace-eigenvalue of u_1. Inserting (1.22) in place of the inner integral in (1.20) we get a weighted sum of integrals

$$\int_{D_4} \overline{B_n(w)} \left(K_{n-1} K_{n-2} \ldots K_1 K_0 u_1\right)(4w) B_0(w) \overline{u_2(4w)} d\mu_w.$$

This is the inner product involving $B_0\overline{B_n}$ what was mentioned already in Remark 1.1. We show that $B_0\overline{B_n}$ is a linear combination of Eisenstein series. Since the Fourier coefficients of $K_{n-1}K_{n-2}\dots K_1K_0u_1$ can be given explicitly in terms of the Fourier coefficients of u_1, so by unfolding we get an expression which contains the Rankin-Selberg L-function of u_1 and $\overline{u_2}$. Many problems occur concernig convergence and the determination of the involved special functions, but these are the main steps of the proof of the theorem.

To make the convergence problems easier we will first impoose a stronger condition on the weight functions χ than the condition assumed in the theorem. This condition will be the following:

We say that a function χ satisfies condition D if χ is an even entire function satisfying that for every fixed $A, B > 0$ the function $|\chi(z)|\, e^{|z|A}$ is bounded on the strip $|\operatorname{Im} z| \leq B$. If a function χ satisfies Condition D, then it clearly satisfies Condition C_β for every $\beta > 0$. Indeed, Condition D requires that χ decays faster than exponentially on horizontal strips, while Condition C_β allows exponential growth of a certain rate. We will first prove the theorem for χ satisfying Condition D. Then we will show that it is relatively easy to extend the statement for functions satisfying C_β with a suitable $\beta > 0$.

1.7. Structure of the paper. In Section 2 we list the necessary notations and facts on automorphic functions. In Section 3 we define the many types of special functions occurring in the paper, give their properties and prove some necessary lemmas on special functions. We prove some very important lemmas needed for the proof of Theorem 1.1 in Section 4, and we prove Theorem 1.1 in Section 5. However, the proofs of some important lemmas on the kernel function and on the integral transform are postponed to Section 6. We refer to the statements of these lemmas in Section 5.

2. Automorphic preliminaries

2.1. Basic properties of the Maass operators.

LEMMA 2.1. *Let $k, k_1, k_2 \in \mathbf{R}$, $z \in \mathbb{H}$, $\gamma \in SL(2,\mathbf{Z})$, and let $f, g : \mathbb{H} \to \mathbf{C}$ be smooth functions. Then we have the following statements.*

(i) $K_{k_1-k_2}\left(f\overline{g}\right) = \left(K_{k_1}f\right)\overline{g} + fK_{-k_2}\left(\overline{g}\right)$.

(ii) $\overline{\left(K_{-k}\overline{f}\right)} = L_k f$.

(iii) $\Delta_{2k} = L_{k+1}K_k + k(1+k) = K_{k-1}L_k + k(k-1)$, $\Delta_{2k}L_{k+1} = L_{k+1}\Delta_{2k+2}$.

(iv) If $\Delta_{2k}f = s(s-1)f$, *then for every* $n \geq 0$ *we have* $L_{k+1}\dots L_{k+n}K_{k+n-1}\dots K_k f = (-1)^n (s+k)_n (1-s+k)_n f$.

(v) f *is holomorphic if and only if* $K_k\left(y^{-k}\overline{f}\right) = \overline{L_{-k}\left(y^{-k}f\right)} = 0$.

(vi) $K_k\left(f(\gamma z)\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{-2k}\right) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{-2k-2}(K_k f)(\gamma z)$.

Proof. Parts (i), (ii) and (iii) follow by easy computations using the definitions, and part (iv) follows easily from (iii). Statement (iii) and (iv) are mentioned in [F], formulas (6), (7) and (8). Part (v) is proved in Lemma 3.2 of [R], and part (vi) is proved in Lemma 3.1 of [R]. The proof is complete.

2.2. Fourier expansions. We first define the Fourier coefficients of Maass forms. To do that the Whittaker functions $W_{\alpha,\beta}$ are needed. Their definition will be given in Section 3.7.

The three cusps for $\Gamma_0(4)$ are ∞, 0 and $-\frac{1}{2}$. If $\mathfrak{a}$ denotes one of these cusps, we take a scaling matrix $\sigma_\mathfrak{a} \in SL(2,\mathbf{R})$ as it is explained on p. 42 of [I]. We can easily see that one can take

$$\sigma_\infty := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_0 := \begin{pmatrix} 0 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}, \qquad \sigma_{-\frac{1}{2}} := \begin{pmatrix} -1 & \frac{-1}{2} \\ 2 & 0 \end{pmatrix}.$$

The only cusp for $SL(2,\mathbf{Z})$ is ∞, and, of course, we take the identity matrix σ_∞ for scaling matrix also in this case.

If $\mathfrak{a}$ is a cusp for $\Gamma = SL(2,\mathbf{Z})$ or $\Gamma = \Gamma_0(4)$, we define $\chi_\mathfrak{a}$ by

$$\nu\left(\sigma_\mathfrak{a}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\sigma_\mathfrak{a}^{-1}\right) = e(-\chi_\mathfrak{a}), \qquad 0 \leq \chi_\mathfrak{a} < 1.$$

It is easy to check that $\chi_\infty = \chi_0 = 0$, and $\chi_{-\frac{1}{2}} = \frac{3}{4}$. So the cusps 0 and ∞ are said to be singular, and $-1/2$ is said to be nonsingular.

If f is a Maass form of weight l, $\Delta_l f = s(s-1)f$ with some $\operatorname{Re} s \geq \frac{1}{2}$, $s = \frac{1}{2} + it$, and $\mathfrak{a}$ is a cusp of Γ, then $f(\sigma_\mathfrak{a} z)\left(\frac{j_{\sigma_\mathfrak{a}}(z)}{|j_{\sigma_\mathfrak{a}}(z)|}\right)^{-l}$ has the Fourier expansion

$$c_{f,\mathfrak{a}}(y) + \sum_{\substack{m \in \mathbf{Z} \\ m - \chi_\mathfrak{a} \neq 0}} \rho_{f,\mathfrak{a}}(m) W_{\frac{l}{2}\operatorname{sgn}(m-\chi_\mathfrak{a}),it}\left(4\pi\left|m-\chi_\mathfrak{a}\right|y\right)e\left(\left(m-\chi_\mathfrak{a}\right)x\right)$$

for $z = x + iy \in \mathbb{H}$, and $c_{f,\mathfrak{a}}(y) = 0$ if $\chi_{\mathfrak{a}} \neq 0$, while it is a linear combination of y^s and y^{1-s} for $s \neq \frac{1}{2}$ and of $y^{1/2}$ and $y^{1/2}\log y$ for $s = \frac{1}{2}$, if $\chi_{\mathfrak{a}} = 0$.

We will need another type of Fourier expansion, namely Fourier expansion of Laplace-eigenfunctions on noneuclidean circles. We reproduce here a theorem of Fay, which will be important in the present paper. To state this theorem we need geodesic polar coordinates: if $z_0 \in \mathbb{H}$ is fixed, then for every $z \in \mathbb{H}$ we can uniquely write

$$\frac{z - z_0}{z - \overline{z_0}} = \tanh(\frac{r}{2})e^{i\phi} \tag{2.1}$$

with $r > 0$ and $0 \leq \phi < 2\pi$. The invariant measure is expressed in these new coordinates as $d\mu_z = \sinh r dr d\phi$.

LEMMA 2.2. *Let $k \in \mathbf{R}$, $s \in \mathbf{C}$, and let f be a smooth function on H satisfying $\Delta_{2k} f = s(s-1)f$. If $z_0 \in \mathbb{H}$ is given, then for every $z \in \mathbb{H}$ we have the absolutely convergent expansion*

$$f(z)\left(\frac{z - \overline{z_0}}{z_0 - \overline{z}}\right)^k = \sum_{n=-\infty}^{\infty} (f)_n (z_0) P_{s,k}^n(z, z_0) e^{in\phi},$$

where $r = r(z, z_0) > 0$ and $0 \leq \phi = \phi(z, z_0) < 2\pi$ are determined from z by (2.1), and

$$P_{s,k}^n(z, z_0) := \left(\tanh(\frac{r}{2})\right)^{|n|} \left(1 - \tanh^2(\frac{r}{2})\right)^{k_n} F(s - k_n, 1 - s - k_n, 1 + |n|, -y)$$

with $y := \frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$, $k_n := k\frac{n}{|n|}$ for $n \neq 0$, $k_0 := \pm k$,

$$n!\,(f)_n(z_0) := (K_{k+n-1} \dots K_{k+1} K_k f)(z_0) \text{ for } n \geq 0,$$

$$(-n)!\,(f)_n(z_0) := \overline{\left(K_{-k-n-1} \dots K_{1-k} K_{-k} \overline{f}\right)}(z_0) = (L_{k+n+1} \dots L_{k-1} L_k f)(z_0) \text{ for } n \leq 0.$$

This follows from Theorems 1.1 and 1.2 of [F]. Lemma 2.2 was stated also in [B3], see Lemma 3.4 there. It is explained there how to deduce Lemma 2.2 from the theorems of Fay.

2.3. The functions B_n. If $z \in \mathbb{H}$ is arbitrary, let $T_z \in PSL(2, \mathbf{R})$ be such that T_z is an upper triangular matrix and $T_z i = z$. It is clear that T_z is uniquely determined by z, for $z = x + iy$ we have explicitly

$$T_z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{-1}{2}} \\ 0 & y^{\frac{-1}{2}} \end{pmatrix}.$$

If $z \in \mathbb{H}$ is fixed, the function $(\operatorname{Im} z)^{\frac{1}{4}}\, \theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}}$ is holomorphic for $|L| < 1$, so it has a Taylor expansion

$$(\operatorname{Im} z)^{\frac{1}{4}}\, \theta\left(T_z\left(i\frac{1+L}{1-L}\right)\right)(1-L)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} B_n(z) L^n. \tag{2.2}$$

We defined in this way a function $B_n(z)$ ($z \in \mathbb{H}$) for every $n \geq 0$. For $n = 0$ this is in accordance with (1.2). These functions satisfy also

$$\frac{1}{n+1} K_{n+\frac{1}{4}} B_n = B_{n+1} \tag{2.3}$$

for every $n \geq 0$, this is proved in [B3], Lemma 6.1. Indeed, this follows at once from (6.2) of [B3]. Formula (2.3) implies that B_n is a Maass form of weight $2n + \frac{1}{2}$ for $\Gamma_0(4)$ and it has an additional transformation formula

$$B_n\left(\frac{-1}{4z}\right) = e\left(\frac{-1}{8}\right)\left(\frac{z}{|z|}\right)^{\frac{1}{2}+2n} B_n(z) \tag{2.4}$$

for every $z \in \mathbb{H}$, see (6.3), (6.4) and (6.5) of [B3]. These statements follow by induction using (1.3), (1.4), (2.3) and Lemma 2.1 (vi).

2.4. Rankin-Selberg L-functions. It is known that we have the functional equation

$$\frac{\zeta(2S)\, L(S)}{\pi^{2S}} \Gamma\left(\frac{S \pm it_1 \pm it_2}{2}\right) = \frac{\zeta(2(1-S))\, L(1-S)}{\pi^{2(1-S)}} \Gamma\left(\frac{1 - S \pm it_1 \pm it_2}{2}\right)$$

for the Rankin-Selberg L-function defined in (1.6). We see from this functional equation that the function $\zeta(2S)\, L(S)$ is regular for $S \neq 1$, and it has at most polynomial growth in vertical strips.

2.5. Further notations. We now explicitly give closures of fundamental domains of the quotients $SL(2, \mathbf{Z}) \setminus \mathbb{H}$ and $\Gamma_0(4) \setminus \mathbb{H}$.

Let D_1 denote the closure of the standard fundamental domain of the quotient $SL(2, \mathbf{Z}) \setminus \mathbb{H}$:

$$D_1 := \left\{ z \in \mathbf{C} :\ \operatorname{Im} z > 0,\ -\frac{1}{2} \leq \operatorname{Re} z \leq \frac{1}{2},\ |z| \geq 1 \right\}.$$

It is easy to check that the following set is a closure of a fundamental domain of $\Gamma_0(4) \setminus \mathbb{H}$:

$$D_4 := \bigcup_{j=0}^{5} \gamma_j D_1,$$

where

$$\gamma_j := \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} \qquad (0 \le j \le 3), \qquad \gamma_4 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \gamma_5 := \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

We always integrate over these fixed sets D_1 and D_4 in the sequel.

We denote by $R_l(D_4)$ the space of such smooth automorphic functions of weight l for $\Gamma_0(4)$ for which we have that for any integers $A, B, C \ge 0$ the function

$$\left(\max_{\mathfrak{a}} \operatorname{Im} \sigma_{\mathfrak{a}}^{-1} z\right)^A \left| \left(\frac{\partial^B}{\partial x^B} \frac{\partial^C}{\partial y^C} f \right)(z) \right|$$

is bounded on D_4 (i.e. every partial derivative decays faster than polynomially near each cusp on the fixed fundamental domain D_4).

For $z, w \in \mathbb{H}$, let

$$H(z,w) := i^{\frac{1}{2}} \left(\frac{|z - \overline{w}|}{(z - \overline{w})} \right)^{\frac{1}{2}}, \tag{2.5}$$

as on p. 349 of [H]. It is easy to see that for any $T \in SL(2, \mathbf{R})$ we have

$$\frac{H^2(Tz, Tw)}{H^2(z,w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right) \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-1},$$

so

$$\frac{H(Tz, Tw)}{H(z,w)} = \left(\frac{j_T(z)}{|j_T(z)|} \right)^{\frac{1}{2}} \left(\frac{j_T(w)}{|j_T(w)|} \right)^{-\frac{1}{2}}, \tag{2.6}$$

since both sides lie in the right half-plane. Observe also that

$$H(w,z) = \overline{H(z,w)}. \tag{2.7}$$

We now give the definition of the Eisenstein series of weight $1/2$. For $\gamma_1, \gamma_2 \in SL(2, \mathbf{R})$, we define

$$w(\gamma_1, \gamma_2) := j_{\gamma_1}(\gamma_2 z)^{1/2} j_{\gamma_2}(z)^{1/2} j_{\gamma_1 \gamma_2}(z)^{-1/2},$$

the right-hand side is independent of $z \in \mathbb{H}$. Clearly $w = \pm 1$. For $\mathfrak{a} = 0, \infty$, $\operatorname{Re} s > 1$, $z \in \mathbb{H}$, define

$$E_{\mathfrak{a}}\left(z, s, \frac{1}{2}\right) := \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma_0(4)} \overline{\nu(\gamma) w\left(\sigma_{\mathfrak{a}}^{-1}, \gamma\right)} (\operatorname{Im} \sigma_{\mathfrak{a}}^{-1} \gamma z)^s \left(\frac{j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)}{\left| j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z) \right|} \right)^{-\frac{1}{2}},$$

where $\Gamma_{\mathfrak{a}}$ denotes the stability group of $\mathfrak{a}$ in $\Gamma_0(4)$.

Finally, we will use the notation $\Gamma_\infty := \{\gamma \in SL(2,\mathbf{Z}) : \ \gamma\infty = \infty\}$. The stability group of ∞ is clearly the same in $\Gamma_0(4)$ and $SL(2,\mathbf{Z})$.

3. Preliminaries on special functions

3.1. Generalized hypergeometric functions. We define these functions in the usual way:

$$ {}_{q+1}F_q\left(\begin{matrix} a_1,\dots,a_{q+1} \\ b_1,\dots,b_q\end{matrix}; z\right) := \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_{q+1})_n}{n!\,(b_1)_n \dots (b_q)_n} z^n. $$

Here the b_i are not nonpsitive integers. We have absolute convergence for $|z| < 1$. The series is also absolutely convergent for $|z| \le 1$ if we assume that Re $\left(\sum b_i - \sum a_i\right) > 0$. We will also use the notation $F(\alpha,\beta,\gamma;z)$ in place of ${}_2F_1\left(\begin{matrix}\alpha,\beta\\ \gamma\end{matrix};z\right)$.

3.2. Properties of ${}_2F_1$ functions. For $\mathrm{Re}\,\alpha$, $\mathrm{Re}\,\beta$, $\mathrm{Re}\,\gamma > 0$, $-1 < z < 0$ and $-\mathrm{Re}\,\alpha, -\mathrm{Re}\,\beta < \sigma < 0$ we see by [S], (1.6.1.6) the Barnes-type integral

$$ F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\gamma+s)} (-z)^s \, ds. \tag{3.1} $$

This shows that $F(\alpha,\beta,\gamma;z)$ extends analytically for $z \notin [1,\infty)$.

For $\mathrm{Re}\, s < 0$, Re $(\alpha+s) > 0$, Re $(\beta+s) > 0$ we have that

$$ \int_0^\infty x^{-s-1} F(\alpha,\beta,\gamma;-x)\, dx = \frac{\Gamma(\gamma)\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(-s)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma+s)}, \tag{3.2} $$

see [G-R], p. 806, 7.511. For $\mathrm{Re}\,\gamma > \mathrm{Re}\,\beta > 0$, $z \notin [1,\infty)$ and any α we have that

$$ F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\, dt, \tag{3.3} $$

see [G-R], p. 995, 9.111. If $\mathrm{Re}\,\gamma > 0$, $z \notin [0,\infty)$, and α and β are any complex numbers satisfying that $\alpha - \beta$ is not an integer, we have (idem (α,β) means the same expression with α and β interchanged) that

$$ F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\gamma-\alpha)} (-z)^{-\alpha} F\left(\alpha, \alpha+1-\gamma, \alpha+1-\beta; \frac{1}{z}\right) + \mathrm{idem}(\alpha,\beta), \tag{3.4} $$

see [S], (1.8.1.11). For $z \notin [1, \infty)$ we have by [G-R], p. 998, 9.131.1 that

$$F(\alpha, \beta, \gamma; z) = (1-z)^{-\alpha} F\left(\alpha, \gamma - \beta, \gamma; \frac{z}{z-1}\right) = (1-z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma; z). \tag{3.5}$$

3.3. Properties of $_3F_2$ functions. Let a, b, c be such that $\operatorname{Re} a$, $\operatorname{Re} b$, $\operatorname{Re} c > 0$ and the set $\{a, b, c\}$ is symmetric with respect to the real axis. We fix three numbers satisfying these conditions throughout this subsection.

If n is a nonnegative integer, the continuous dual Hahn polynomials are defined by

$$S_n\left(x^2\right) = S_n\left(x^2; a, b, c\right) := (a+b)_n (a+c)_n \; {}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}; 1\right), \tag{3.6}$$

see [A-A-R], (6.10.2). Formula (3.6) is symmetric in the parameters a, b, c, this follows from the identity

$${}_3F_2\left(\begin{matrix} A, B, C \\ D, E \end{matrix}; 1\right) = \frac{\Gamma(E)\Gamma(D+E-A-B-C)}{\Gamma(E-A)\Gamma(D+E-B-C)} \; {}_3F_2\left(\begin{matrix} A, D-B, D-C \\ D, D+E-B-C \end{matrix}; 1\right)$$

valid for $\operatorname{Re}(D+E-B-C-A) > 0$, $\operatorname{Re}(E-A) > 0$, see Corollary 3.3.5 of [A-A-R]. These polynomials form a complete orthogonal system in $L^2((0,\infty), w_{a,b,c}(x)\,dx)$ with the weight function

$$w_{a,b,c}(x) := \frac{1}{2\pi} \frac{\Gamma(a \pm ix)\Gamma(b \pm ix)\Gamma(c \pm ix)}{\Gamma(\pm 2ix)}. \tag{3.7}$$

Indeed, we have the relations

$$\int_0^\infty w_{a,b,c}(x) S_m\left(x^2\right) S_n\left(x^2\right) dx = \delta_{mn} \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)\, n!, \tag{3.8}$$

where δ_{mn} is the Kronecker delta symbol, see [A-A-R], (6.10.7). Completeness of the system follows from Theorem 6.5.2 of [A-A-R], taking into account that $w_{a,b,c}(x)$ decays exponentially as $x \to +\infty$.

We can deduce a pointwise upper bound from (3.8). This bound is weak, but it will be enough for our purposes.

LEMMA 3.1. *There is a positive M such that*

$$\left| \frac{S_n\left(x^2; a, b, c\right)}{(a+b)_n (a+c)_n} \right| \le M e^{\frac{\pi}{2}|x|} (1+n)^M$$

for every integer $n \geq 0$ and every real x.

Proof. It is enough to show that for every real y we have

$$\max_{y \leq x \leq y+1} \left| \frac{S_n\left(x^2; a, b, c\right)}{(a+b)_n (a+c)_n} \right| \leq M e^{\frac{\pi}{2}|y|} (1+n)^M \tag{3.9}$$

with a suitable M. The classical Markov inequality states that for any polynomial p of degree n we have

$$\max_{-1 \leq t \leq 1} \left| p^{(1)}(t) \right| \leq n^2 \max_{-1 \leq t \leq 1} |p(t)| .$$

This is proved e.g. in [B-E], Theorem 5.1.8. Then we see that if the left-hand side of (3.9) is m, then there is a subinterval I of $[y, y+1]$ such that the length of I is $\gg n^{-2}$, and $\left| \frac{S_n\left(x^2;a,b,c\right)}{(a+b)_n(a+c)_n} \right| \gg m$ for every $x \in I$. Then we get the lemma by (3.7), the $m = n$ case of (3.8) and the Stirling formula.

LEMMA 3.2. *If* $\operatorname{Re}\gamma$, $\operatorname{Re}A > 0$ *and* $\operatorname{Re}B$ *is large enough in terms of* a, b, c *and* $\operatorname{Re}\gamma$, *then we have for every real x that*

$$\sum_{n=0}^{\infty} \frac{{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix} ; 1 \right)}{n!} \frac{(\gamma)_n (A)_n}{(A+B)_n}$$

equals

$$\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(A+B)}{\Gamma(\gamma)\Gamma(A)\Gamma(B)\Gamma(a \pm ix)\Gamma(A+B-\gamma)}$$

times

$$\frac{1}{2\pi i} \int_{(-C)} \frac{\Gamma(\gamma+s)\Gamma(A+s)\Gamma(a \pm ix + s)\Gamma(-s)\Gamma(B-\gamma-s)}{\Gamma(a+b+s)\Gamma(a+c+s)} ds$$

with $0 < C < \min(\operatorname{Re}a, \operatorname{Re}\gamma, \operatorname{Re}A)$.

Proof. For any complex γ and real x for $|t| < 1/2$ we have

$$\sum_{n=0}^{\infty} \frac{{}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix} ; 1 \right)}{n!} (\gamma)_n t^n = (1-t)^{-\gamma} {}_3F_2\left(\begin{matrix} \gamma, a+ix, a-ix \\ a+b, a+c \end{matrix} ; \frac{t}{t-1} \right). \tag{3.10}$$

Indeed, this follows easily by inserting on the left-hand side the defining series of ${}_3F_2$, changing the summations, and using for every nonnegative integer k that

$$\sum_{n=k}^{\infty} \frac{(-n)_k (\gamma)_n t^n}{n!} = (\gamma)_k (1-t)^{-\gamma} \left(\frac{t}{t-1} \right)^k ,$$

which follows from (3.5). For $0 < t < \frac{1}{2}$ the right-hand side here equals

$$\frac{\Gamma(a+b)\Gamma(a+c)}{\Gamma(\gamma)\Gamma(a\pm ix)}\frac{(1-t)^{-\gamma}}{2\pi i}\int_{(-C)}\frac{\Gamma(\gamma+s)\Gamma(a\pm ix+s)\Gamma(-s)}{\Gamma(a+b+s)\Gamma(a+c+s)}\left(\frac{t}{1-t}\right)^{s}ds, \quad (3.11)$$

which can be seen by shifting the integration to the right. Then using Lemma 3.1 by analytic continuation we see that the left-hand side of (3.10) equals (3.11) for any $0 < t < 1$. Multiplying by $t^{A-1}(1-t)^{B-1}$, integrating from 0 to 1 and using (3.3) with $z = 0$ we obtain the lemma.

We have the difference equation

$$nS_n\left(x^2\right) = B(x)\, S_n\left((x+i)^2\right) - (B(x)+D(x))\, S_n\left(x^2\right) + D(x)\, S_n\left((x-i)^2\right) \quad (3.12)$$

for every $n \geq 0$, where we write

$$B(x) = \frac{(a-ix)(b-ix)(c-ix)}{(-2ix)(1-2ix)},\; D(x) = \frac{(a+ix)(b+ix)(c+ix)}{(2ix)(1+2ix)},$$

see [A-A-R], (6.10.9). This relation has the following consequence.

LEMMA 3.3. *Let χ be a function satisfying Condition D. For any $A > 0$ we have for integers $n \geq 0$ that*

$$C_{n,\chi}(a,b,c) := \int_{-\infty}^{\infty}\chi(x)\frac{S_n\left(x^2;a,b,c\right)}{(a+b)_n(a+c)_n}w_{a,b,c}(x)\,dx \ll_{\chi,a,b,c,A} (1+n)^{-A}. \quad (3.13)$$

Proof. We substitute (3.12) into (3.13), and we shift the integration to $\operatorname{Im} x = 1$ in the case of $S_n\left((x-i)^2\right)$, and to $\operatorname{Im} x = -1$ in the case of $S_n\left((x+i)^2\right)$. We do not cross any pole, and we get for $nC_{n,\chi}(a,b,c)$ an expression of type (3.13), but with a new function in place of χ satisfying Condition D. These facts can be checked using (3.7). We iterate this step many times, and then we apply Cauchy-Schwarz inequality and use (3.8) with $m = n$. By the properties of χ this proves the lemma.

3.4. Some integral formulas. For $0 < \operatorname{Re} a_1$, $\operatorname{Re} a_2$, $\operatorname{Re} b_1$, $\operatorname{Re} b_2$, $\operatorname{Re} b_3$, assuming that $b_4 + a_1$ is not a nonpositive integer, we have that

$$\frac{1}{2\pi i}\int_{(0)}\frac{\Gamma(a_1-s)\Gamma(a_2-s)\Gamma(b_1+s)\Gamma(b_2+s)\Gamma(b_3+s)}{\Gamma(b_4+s)}ds \quad (3.14)$$

equals

$$
{}_3F_2\left(\begin{matrix} a_1+b_1, a_1+b_2, b_4-b_3 \\ a_1+a_2+b_1+b_2, b_4+a_1 \end{matrix}; 1\right) \frac{\Gamma(a_1+b_3)\prod_{k=1}^{2}\left(\Gamma(a_1+b_k)\Gamma(a_2+b_k)\right)}{\Gamma(a_1+a_2+b_1+b_2)\Gamma(b_4+a_1)}, \tag{3.15}
$$

see [S], (4.2.2.1).

In the special case $b_4 = a_1+a_2+b_1+b_2+b_3$ we have the following statement. For $0 < \operatorname{Re} a_1$, $\operatorname{Re} a_2$, $\operatorname{Re} b_1$, $\operatorname{Re} b_2$, $\operatorname{Re} b_3$ we have that

$$
\frac{1}{2\pi i}\int_{(0)} \frac{\Gamma(a_1-s)\Gamma(a_2-s)\Gamma(b_1+s)\Gamma(b_2+s)\Gamma(b_3+s)}{\Gamma(b_1+b_2+b_3+a_1+a_2+s)} ds \tag{3.16}
$$

equals

$$
\frac{\prod_{k=1}^{3}\left(\Gamma(a_1+b_k)\Gamma(a_2+b_k)\right)}{\prod_{1\le k<l\le 3}\Gamma(b_k+b_l+a_1+a_2)}. \tag{3.17}
$$

This is the Second Barnes Lemma, see [S], (4.2.2.2).

LEMMA 3.4. *For* $0 < \operatorname{Re}\alpha \le \operatorname{Re}\beta$, $0 < \operatorname{Re} a \le \operatorname{Re} b$, $0 < \operatorname{Re}\gamma$, $0 < \operatorname{Re} c < \operatorname{Re} a + \operatorname{Re}\alpha$, *if* $\gamma + a - c$ *is not a nonpositive integer, we have that*

$$
\int_0^\infty F(\alpha,\beta,\gamma;-u)\, u^{c-1} F(a,b,c;-u)\, du \tag{3.18}
$$

equals the product of

$$
\frac{\Gamma(\gamma)\Gamma(c)\Gamma(a-c+\alpha)\Gamma(a-c+\beta)\Gamma(b-c+\alpha)\Gamma(b-c+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(b)\Gamma(a+b-2c+\alpha+\beta)\Gamma(\gamma+a-c)}
$$

and

$$
{}_3F_2\left(\begin{matrix} a-c+\alpha, a-c+\beta, \gamma-c \\ a+b-2c+\alpha+\beta, \gamma+a-c \end{matrix}; 1\right).
$$

Proof. Applying (3.1) for the first factor in (3.18) with σ satisfying

$$
-\operatorname{Re}\alpha, -\operatorname{Re} c < \sigma < \operatorname{Re}(a-c), 0
$$

and then using (3.2) we get that (3.18) equals

$$
\frac{\Gamma(\gamma)\Gamma(c)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(a)\Gamma(b)} \frac{1}{2\pi i}\int_{(\sigma)} \frac{\Gamma(a-c-s)\Gamma(b-c-s)\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(c+s)}{\Gamma(\gamma+s)} ds.
$$

By the equality of (3.14) and (3.15) this gives the statement of the lemma.

3.5. A hypergeometric integral transform. Our aim here is to prove Lemma 3.7 below, to prepare its proof we need the identities proved in the next two lemmas.

LEMMA 3.5. *For every integer $n \geq 0$ and for every real t we have that*

$$\int_0^\infty F\left(\tfrac{3}{4}+it, \tfrac{3}{4}-it, 1; -u\right) \frac{1-\left(\frac{u}{1+u}\right)^{n+1}}{n+1} du = \frac{\Gamma\left(\frac{3}{4} \pm it\right) {}_3F_2\left(\begin{matrix}\frac{3}{4}+it, \frac{3}{4}-it, -n \\ \frac{3}{2}, 2\end{matrix}; 1\right)}{\Gamma\left(\frac{3}{2}\right)}.$$

Proof. Let $m \geq 0$ be an integer. Writing $b = c = m+1$, $\gamma = 1$ we get from Lemma 3.4 and from (3.5) that

$$\int_0^\infty F(\alpha, \beta, 1; -u) \frac{u^m}{(1+u)^a} du = \sum_{k=0}^m \frac{(-m)_k \Gamma(a-m-1+\alpha+k) \Gamma(a-m-1+\beta+k)}{k! \Gamma(a-m-1+\alpha+\beta+k) \Gamma(a-m+k)} \tag{3.19}$$

under the conditions $0 < \operatorname{Re} \alpha \leq \operatorname{Re} \beta$, $0 < \operatorname{Re} a \leq m+1$, $m+1-\operatorname{Re} a < \operatorname{Re} \alpha$, assuming that a is not an integer. We estimate the hypergeometric function on the left-hand side by (3.1), and we see by analytic continuation in a that it is enough to assume $0 < \operatorname{Re} \alpha \leq \operatorname{Re} \beta$ and $m+1-\operatorname{Re} a < \operatorname{Re} \alpha$. So assuming $1 < \operatorname{Re} \alpha \leq \operatorname{Re} \beta$ we see that (3.19) is true for $a = m$. Taking the difference of (3.19) for $m = 0$ and $m = n+1$ we get the lemma by analytic continuation in α and β.

LEMMA 3.6. *For any integer $n \geq 0$ and for any $u > 0$ we have that*

$$\int_0^\infty F\left(\frac{3}{4}-it, \frac{3}{4}+it, 1, -u\right) \frac{\Gamma^2\left(\frac{3}{4} \pm it\right) \Gamma\left(\frac{1}{4} \pm it\right)}{\Gamma(\pm 2it)} {}_3F_2\left(\begin{matrix}\frac{3}{4}+it, \frac{3}{4}-it, -n \\ \frac{3}{2}, 2\end{matrix}; 1\right) dt \tag{3.20}$$

equals

$$2\pi \Gamma\left(\frac{3}{2}\right) (1+u)^{-\frac{1}{2}} \frac{1-\left(\frac{u}{1+u}\right)^{n+1}}{n+1}.$$

Proof. First note that for any $\operatorname{Re} s = \frac{1}{2}$ and any integer $k \geq 0$ we have by (3.7) and (3.8) that

$$\int_0^\infty \frac{\Gamma\left(\frac{1}{4} \pm it\right) \Gamma\left(\frac{3}{4} \pm it + k\right) \Gamma\left(\frac{3}{4} \pm it + s\right)}{\Gamma(\pm 2it)} dt = 2\pi \Gamma(1+k) \Gamma(1+s) \Gamma\left(\frac{3}{2}+k+s\right).$$

Then by (3.1) we have that (3.20) equals

$$2\pi \sum_{k=0}^n \frac{(-n)_k}{\left(\frac{3}{2}\right)_k (2)_k} \frac{1}{2\pi i} \int_{(-1/2)} \Gamma(-s) \Gamma\left(\frac{3}{2}+k+s\right) u^s ds.$$

We can compute the integral here by (3.1) and (3.5), and we get that (3.20) equals

$$2\pi\Gamma\left(\frac{3}{2}\right)\sum_{k=0}^{n}\frac{(-n)_k}{(2)_k}(1+u)^{-\frac{3}{2}-k}.$$

The lemma follows by the binomial theorem.

The integral transform (3.21) below is a special case of the so-called *Jacobi transform*, see e.g. [K]. The inversion formula of this transform is also proved there in Theorem 4.2, but since it is not hard to prove it using the results on continuous dual Hahn polynomials mentioned above, so we include a proof.

LEMMA 3.7. *Let χ be a function satisfying Condition D. For $u \geq 0$ define*

$$k_\chi(u) := \frac{1}{\pi}(u+1)^{\frac{1}{4}}\int_0^\infty F\left(\frac{3}{4}-it,\frac{3}{4}+it,1,-u\right)\left|\frac{\Gamma\left(\frac{1}{4}+it\right)\Gamma\left(\frac{3}{4}+it\right)}{\Gamma\left(2it\right)}\right|^2\chi(t)dt. \quad (3.21)$$

Then the following statements hold.

(i) The function $k_\chi^{(j)}(u)(u+1)^A$ is bounded on $[0,\infty)$ for every $A>0$ and $j\geq 0$.

(ii) For every real t we have

$$\chi(t)=\frac{1}{2}\int_0^\infty (u+1)^{\frac{1}{4}}F\left(\frac{3}{4}-it,\frac{3}{4}+it,1,-u\right)k_\chi(u)du.$$

Proof. By (3.4) we know for real t that

$$F\left(\frac{3}{4}-it,\frac{3}{4}+it,1,-u\right)\left|\frac{\Gamma\left(\frac{1}{4}+it\right)\Gamma\left(\frac{3}{4}+it\right)}{\Gamma\left(2it\right)}\right|^2=\phi(u,t)+\phi(u,-t),$$

where

$$\phi(u,t)=\frac{\Gamma\left(\frac{1}{4}-it\right)\Gamma\left(\frac{3}{4}-it\right)}{\Gamma\left(-2it\right)}u^{it-\frac{3}{4}}F\left(\frac{3}{4}-it,\frac{3}{4}-it,1-2it,-\frac{1}{u}\right),$$

hence

$$k_\chi(u)=\frac{1}{\pi}(u+1)^{\frac{1}{4}}\int_{-\infty}^\infty \phi(u,t)\chi(t)dt.$$

Now, if u is large, we push the line of integration upwards to a line $\operatorname{Im} t = B$ with a large positive number B depending on A and j, and using (3.3) we get (i). Indeed, for small u statement (i) is trivial, using the very definition of $k_\chi(u)$ and (3.3).

Let $\{a, b, c\} = \left\{\frac{3}{4}, \frac{3}{4}, \frac{5}{4}\right\}$, and let us write

$$\frac{\chi(t)}{\Gamma\left(\frac{3}{4} \pm it\right)} = \sum_{n=0}^{\infty} a_n \frac{S_n\left(t^2; \frac{3}{4}, \frac{3}{4}, \frac{5}{4}\right)}{\left(\frac{3}{2}\right)_n (2)_n} \tag{3.22}$$

in the space $L^2((0,\infty), w_{a,b,c}(t)\, dt)$. It follows from (3.8) and Lemma 3.3 that $a_n \ll_{\chi,A} (1+n)^{-A}$ for every $A > 0$. Then Lemma 3.1 shows that the right-hand side of (3.22) is a continuous function, so (3.22) is valid pointwise for every $t > 0$. We also see applying Lemma 3.1 and (3.1) that if we express $\chi(t)$ from (3.22) and substitute the obtained expressioon into (3.21), then we can integrate there term by term. Then from (3.6) and Lemma 3.6 we get that

$$k_\chi(u) = 2\Gamma\left(\frac{3}{2}\right)(1+u)^{-\frac{1}{4}} \sum_{n=0}^{\infty} a_n \frac{1 - \left(\frac{u}{1+u}\right)^{n+1}}{n+1}.$$

By Lemma 3.5, (3.6) and (3.22) we obtain (ii). The lemma is proved.

3.6. Properties of ${}_7F_6$ and ${}_4F_3$ functions. For complex A, B, C, D, E and F satisfying $\mathrm{Re}\,(2 + 2A - B - C - D - E - F) > 0$ let

$$W(A; B, C, D, E, F) := {}_7F_6\left(\begin{matrix} A, 1+\frac{A}{2}, B, C, D, E, F \\ \frac{A}{2}, B^*, C^*, D^*, E^*, F^* \end{matrix}; 1\right)$$

where B^*, C^*, D^*, E^*, F^* are given by

$$B + B^* = C + C^* = D + D^* = E + E^* = F + F^* = 1 + A.$$

If a, b, c, d are complex numbers, then writing

$$\tilde{a} := \frac{1}{2}(a + b + c + d - 1), \qquad \tilde{b} := \frac{1}{2}(a + b - c - d + 1), \tag{3.23}$$

$$\tilde{c} := \frac{1}{2}(a - b + c - d + 1), \qquad \tilde{d} := \frac{1}{2}(a - b - c + d + 1), \tag{3.24}$$

we define the Wilson function $\phi_\lambda(x) = \phi_\lambda(x; a, b, c, d)$ by the formula

$$\phi_\lambda(x) := \frac{\Gamma\left(\tilde{a} + \tilde{b} + \tilde{c} + i\lambda\right) W\left(\tilde{a} + \tilde{b} + \tilde{c} + i\lambda - 1; a + ix, a - ix, \tilde{a} + i\lambda, \tilde{b} + i\lambda, \tilde{c} + i\lambda\right)}{\Gamma(a+b)\,\Gamma(a+c)\,\Gamma(1+a-d)\,\Gamma\left(1 - \tilde{d} - i\lambda\right)\Gamma\left(\tilde{b} + c + i\lambda \pm ix\right)},$$

as it was introduced by Groenevelt in [G1] in formula (3.2). This definition is meaningful if the Γ-function is regular at the point $\tilde{a}+\tilde{b}+\tilde{c}+i\lambda$ and $\mathrm{Re}\left(1-\tilde{d}-i\lambda\right)>0$. However, $\phi_\lambda(x)$ is an entire function in $(x,\lambda)\in\mathbf{C}^2$ (see [G1], below formula (3.3)).
The Wilson function $\phi_\lambda(x;a,b,c,d)$ is symmetric in the parameters a, b, c, $1-d$, see [G2], Lemma 5.3 (ii). We have the symmetry relation

$$\phi_\lambda(x;a+i\omega,b+iy,b-iy,1-a+i\omega)=\phi_\omega(y;a+i\lambda,b+ix,b-ix,1-a+i\lambda), \quad (3.25)$$

see [G2], Lemma 5.3 (i).
We have the identity that

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(a\pm ix+R)\,\Gamma(\tilde{a}\pm i\lambda+R)\,\Gamma(-R)\,\Gamma(1-a-d-R)}{\Gamma(a+b+R)\,\Gamma(a+c+R)}dR \quad (3.26)$$

equals

$$\Gamma(a\pm ix)\,\Gamma(\tilde{a}\pm i\lambda)\,\Gamma(1-d\pm ix)\,\Gamma\left(1-\tilde{d}\pm i\lambda\right)\phi_\lambda(x;a,b,c,d), \quad (3.27)$$

assuming that Γ is regular at the points $a\pm ix$, $\tilde{a}\pm i\lambda$, $1-d\pm ix$, $1-\tilde{d}\pm i\lambda$. Here the poles of the functions $\Gamma(a\pm ix+R)$, $\Gamma(\tilde{a}\pm i\lambda+R)$ lie to the left of the path of integration, and the poles of the functions $\Gamma(-R)$, $\Gamma(1-a-d-R)$ lie to the right of it, and $\tilde{a},\tilde{b},\tilde{c},\tilde{d}$ are defined above. This can be seen by shifting the integration to the right in (3.26) above and applying (3.3) of [G1].
We need some important identities, and in order to state them we need further notations. For complex A,B,C,D,E and F let

$$\psi(A;B,C,D,E,F):=\frac{\Gamma(1+A)\,W(A;B,C,D,E,F)}{\Gamma(B^*)\,\Gamma(C^*)\,\Gamma(D^*)\,\Gamma(E^*)\,\Gamma(F^*)\,\Gamma(1-S)}, \quad (3.28)$$

where

$$S:=B+C+D+E+F-2A-1, \quad (3.29)$$

see (2.1) of [W] or p 127 of [S].
We can check that we have

$$\phi_\lambda(x;a,b,c,d)=\psi\left(\tilde{a}+\tilde{b}+\tilde{c}+i\lambda-1;a+ix,a-ix,\tilde{a}+i\lambda,\tilde{b}+i\lambda,\tilde{c}+i\lambda\right) \quad (3.30)$$

with this notation.

Let us write

$$
{}_4F_3^*\left(\begin{matrix}A,B,C,D\\E,F,G\end{matrix};1\right):=\frac{\Gamma(A)\,\Gamma(B)\,\Gamma(C)\,\Gamma(D)}{\Gamma(E)\,\Gamma(F)\,\Gamma(G)}\,{}_4F_3\left(\begin{matrix}A,B,C,D\\E,F,G\end{matrix};1\right). \tag{3.31}
$$

Then assuming $E+F+G-A-B-C-D=1$ and Re $(1+A-G)>0$ we have that

$$
{}_4F_3^*\left(\begin{matrix}A,B,C,D\\E,F,G\end{matrix};1\right)-{}_4F_3^*\left(\begin{matrix}1+A-G,1+B-G,1+C-G,1+D-G\\1+E-G,1+F-G,2-G\end{matrix};1\right)=P_1P_2 \tag{3.32}
$$

with the abbreviations

$$
P_1:=\frac{\Gamma(A)\,\Gamma(B)\,\Gamma(C)\,\Gamma(D)\,\Gamma(1+A-G)\,\Gamma(1+B-G)\,\Gamma(1+C-G)\,\Gamma(1+D-G)}{\Gamma(G)\,\Gamma(1-G)} \tag{3.33}
$$

and

$$
P_2:=\psi\left(B+C+D-G;B,C,D,E-A,F-A\right). \tag{3.34}
$$

This follows by some computations from (2.4.4.3) of [S]. See also (2.3) of [W].

Assuming Re $(2+2A-B-C-D-E-F)>0$ and Re $(B+D-A)>0$ we have that

$$
\frac{\psi(A;B,C,D,E,F)\sin(\pi(D+E+F-A))}{\Gamma(B+D-A)\,\Gamma(B+E-A)\,\Gamma(B+F-A)\,\Gamma(1-C)} \tag{3.35}
$$

equals the sum of

$$
\frac{\psi(E+F-C;E,F,1+A-B-C,1+A-C-D,E+F-A)\sin(\pi(B-A))}{\Gamma(1+A-E-F)\,\Gamma(1+A-D-F)\,\Gamma(1+A-D-E)\,\Gamma(1-S)} \tag{3.36}
$$

and

$$
\frac{\psi(2B-A;B,B+C-A,B+D-A,B+E-A,B+F-A)\sin(\pi(C-S))}{\Gamma(D)\,\Gamma(E)\,\Gamma(F)\,\Gamma(1+A-B-C)}, \tag{3.37}
$$

using the notation (3.29). This is formula (2.7) of [W] (see also (4.3.7.8) of [S]).

3.7. Whittaker functions. For complex numbers α, β satisfying $0<\frac{1}{2}-\operatorname{Re}\alpha-|\operatorname{Re}\beta|$ and for $y>0$ we define the Whittaker function $W_{\alpha,\beta}(y)$ by the formula

$$
W_{\alpha,\beta}(y):=\frac{e^{-\frac{y}{2}}}{2\pi i}\int_{(\sigma)}\frac{\Gamma(v)\,\Gamma\left(\frac{1}{2}-\alpha\pm\beta-v\right)}{\Gamma\left(\frac{1}{2}-\alpha\pm\beta\right)}y^{\alpha+v}dv \tag{3.38}
$$

with $0 < \sigma < \frac{1}{2} - \operatorname{Re}\alpha - |\operatorname{Re}\beta|$, see [G-R], p. 1015, formula 9.223. For given $y > 0$ this function extends to an entire function of $(\alpha, \beta) \in \mathbf{C}^2$. Indeed, this can be seen from the formula

$$W_{\alpha,\beta}(y) = \frac{y^\alpha}{e^{y/2}} \left(\frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(v)\,\Gamma\left(\frac{1}{2} - \alpha \pm \beta - v\right)}{\Gamma\left(\frac{1}{2} - \alpha \pm \beta\right)} y^v dv + \sum_{0 \le j < -\sigma} \frac{(-1)^j \left(\frac{1}{2} - \alpha \pm \beta\right)_j}{j! y^j} \right), \tag{3.39}$$

where $\sigma < \frac{1}{2} - \operatorname{Re}\alpha - |\operatorname{Re}\beta|$ and σ is not a nonnegative integer. This is valid for every $(\alpha, \beta) \in \mathbf{C}^2$ and $y > 0$.

We see from (3.39) that if α and β are fixed, then for $0 < y < 1$ we have $W_{\alpha,\beta}(y) \ll_\delta y^\delta$ for every $\delta < \frac{1}{2} - |\operatorname{Re}\beta|$. We also see that $W_{\alpha,\beta}(y)$ decays exponentially as $y \to \infty$.

The next lemma follows from [G-R], p. 819, 7.625.4 and p. 1022, but since it is very important in our paper we give a proof of it.

LEMMA 3.8. *For any* $\operatorname{Re} S > 0$*, for any positive numbers* t_1*,* t_2*,* M *and for any complex* k *and* λ *such that the function* Γ *is regular at the two points* $\frac{1}{2} - \lambda \pm it_2$*, we have that*

$$\int_0^\infty y^S W_{k,it_1}(My) W_{\lambda,it_2}(My) \frac{dy}{y^2} \tag{3.40}$$

equals

$$\frac{M^{1-S}}{\Gamma\left(\frac{1}{2} - \lambda \pm it_2\right)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right)\Gamma\left(\frac{1}{2} \pm it_1 + s\right)\Gamma(1 - \lambda + s - S)}{\Gamma(1 - k + s)} ds, \tag{3.41}$$

where the path of integration is chosen in such a way that the poles of the functions $\Gamma\left(\frac{1}{2} \pm it_1 + s\right)$ *and* $\Gamma(1 - \lambda + s - S)$ *lie to the left of the path of integration, and the poles of the functions* $\Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right)$ *lie to the right of it.*

Proof. By a substitution we can assume $M = 1$. Using analytic continuation in k, λ and S we may assume $\operatorname{Re} k = 0$, $\operatorname{Re}\lambda = 0$, $\operatorname{Re} S > 1$. By these assumptions, using (3.38) for both Whittaker functions we get that (3.40) equals

$$\frac{1}{2\pi i} \int_{(1/4)} \frac{\Gamma(\mu)\,\Gamma\left(\frac{1}{2} - \lambda \pm it_2 - \mu\right)}{\Gamma\left(\frac{1}{2} - \lambda \pm it_2\right)} I_\mu d\mu \tag{3.42}$$

with

$$I_\mu := \frac{1}{2\pi i} \int_{(1/4)} \frac{\Gamma(v)\,\Gamma\left(\frac{1}{2} - k \pm it_1 - v\right)\Gamma(S + k + v + \lambda + \mu - 1)}{\Gamma\left(\frac{1}{2} - k \pm it_1\right)} dv,$$

because

$$\int_0^\infty e^{-y} y^{S+k+v+\lambda+\mu} \frac{dy}{y^2} = \Gamma\left(S+k+v+\lambda+\mu-1\right)$$

by the definition of the Γ-function. The integral I_μ can be computed by the $b_4 = b_3$ case of the equality of (3.14) and (3.15), and we get that

$$I_\mu = \frac{\Gamma\left(-\frac{1}{2} \pm it_1 + S + \lambda + \mu\right)}{\Gamma\left(S - k + \lambda + \mu\right)}.$$

Substituting it into (3.42) and writing $s = S + \lambda + \mu - 1$ we obtain the lemma.

Part (ii) of the following lemma will be applied directly in this paper.

LEMMA 3.9. *(i) Let $t > 0$ be given and let $n \in \mathbf{Z}$. We have*

$$\int_0^\infty \left| W_{n,it}(y)\Gamma\left(\frac{1}{2} - n\right)\right|^2 \frac{dy}{y} \ll_t 1. \tag{3.43}$$

(ii) Let $t_1, t_2 > 0$, $S > 1/2$ be given and let $n \in \mathbf{Z}$, $M > 0$. Then we have that

$$\int_0^\infty \left| y^S W_{n,it_1}(My) W_{0,it_2}(My)\right| \frac{dy}{y^2} \ll_{t_1,t_2,S} \frac{M^{1-S}}{\Gamma\left(\frac{1}{2} - n\right)}.$$

Proof. To show (i) we apply Lemma 3.8 with $k = \lambda = n$, $t_1 = t_2 = t$, $S = M = 1$. Note that $W_{n,it}(y)$ is real. Then we compute (3.41) using the equality of (3.14) and (3.15), applying it with the parameters

$$a_1 = \frac{1}{2} + it, a_2 = \frac{1}{2} - it, b_1 = \frac{1}{2} - it, b_2 = -n, b_3 = \frac{1}{2} + it, b_4 = 1 - n.$$

We get in this way that the left-hand side of (3.43) equals

$${}_3F_2\left(\begin{matrix} 1, \frac{1}{2} + it - n, \frac{1}{2} - it - n \\ \frac{3}{2} - it - n, \frac{3}{2} + it - n \end{matrix}; 1\right) \frac{\Gamma\left(\frac{1}{2} \pm it - n\right)\Gamma\left(1 \pm 2it\right)}{\Gamma\left(\frac{3}{2} \pm it - n\right)}.$$

We estimate this series trivially and we get (i).

By a substitution we see that $M = 1$ can be assumed in (ii). The statement then follows from Cauchy-Scwarz, applying part (i). The lemma is proved.

We finally note that for $x > 0$ and arbitrary λ and μ we have

$$x\frac{d}{dx} W_{\lambda,\mu}\left(x\right) = \left(\lambda - \frac{1}{2}x\right) W_{\lambda,\mu}\left(x\right) - \left(\mu^2 - \left(\lambda - \frac{1}{2}\right)^2\right) W_{\lambda-1,\mu}\left(x\right), \tag{3.44}$$

see [G-R], p. 1017, 9.234.3.

4. Important lemmas preparing the proof of Theorem 1.1

In this section χ will denote a given function satisfying Condition D.

4.1 Triple product integrals containing an automorphic kernel function. Our goal here is to prove Lemma 4.2, where we give a useful expression for the integral (4.7), which contains $\overline{B_0(z)}$, a cusp form of weight 0 and an automorphic kernel function of weight 1/2.

LEMMA 4.1. *Let U be a cusp form of weight 0 for $\Gamma_0(4)$. For $z = x + iy \in \mathbb{H}$ let*

$$V(z) := \overline{B_0(z)}U(z).$$

Let k be a smooth function on $[0, \infty)$ such that $k^{(j)}(u)(u + 1)^A$ is bounded on $[0, \infty)$ for every $A > 0$ and $j \geq 0$. For $z, w \in \mathbb{H}$ write

$$k(z, w) := k\left(\frac{|z - w|^2}{4\text{Im}\, z\text{Im}\, w}\right) H(z, w) \text{ and } K(z, w) := \sum_{\gamma\in\Gamma_0(4)} k(\gamma z, w)\overline{\nu(\gamma)}\left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{-\frac{1}{2}}.$$

Then for any $w \in \mathbb{H}$ we have

$$\int_{D_4} V(z)K(z, w)d\mu_z = 2\sum_{n=0}^{\infty} \overline{B_n(w)} \int_{\mathbb{H}} \tilde{k}(Z, i)\overline{\left(\frac{Z - i}{Z + i}\right)^n} U(T_w Z)d\mu_Z$$

with

$$\tilde{k}(u) := k(u)\,(u + 1)^{-\frac{1}{4}} \qquad (u \in [0, \infty)), \tag{4.1}$$

and

$$\tilde{k}(z, w) := \tilde{k}\left(\frac{|z - w|^2}{4\text{Im}\, z\text{Im}\, w}\right) \qquad (z, w \in \mathbb{H}). \tag{4.2}$$

The sum

$$\sum_{n=0}^{\infty}\left|\overline{B_n(w)} \int_{\mathbb{H}} \tilde{k}(Z, i)\overline{\left(\frac{Z - i}{Z + i}\right)^n} U(T_w Z)d\mu_Z\right|,$$

as a function of $w \in D_4$, grows at most polynomially at the cusps of $\Gamma_0(4)$.
The integral $\int_{D_4} \overline{V(z)K(z, w)}d\mu_z$, as a function of $w \in D_4$, belongs to $R_{\frac{1}{2}}(D_4)$, and $\int_{D_4} |V(z)K(z, w)|\, d\mu_z$ decays faster than polynomially at the cusps.

Proof. It is clear, using (1.3), that if $w \in \mathbb{H}$ is fixed, then for every $\delta \in \Gamma_0(4)$ and $z \in \mathbb{H}$ we have

$$V(\delta z) = \overline{\nu(\delta)} \left(\frac{j_\delta(z)}{|j_\delta(z)|} \right)^{-\frac{1}{2}} V(z), \qquad K(\delta z, w) = \nu(\delta) \left(\frac{j_\delta(z)}{|j_\delta(z)|} \right)^{\frac{1}{2}} K(z, w). \tag{4.3}$$

Hence $V(z)K(z,w)$ is invariant in z under $\Gamma_0(4)$, and

$$\int_{D_4} V(z)K(z,w) d\mu_z = \sum_{\gamma \in \Gamma_0(4)} \int_{D_4} k(\gamma z, w) V(\gamma z) d\mu_z = 2 \int_{\mathbb{H}} k(z,w) V(z) d\mu_z. \tag{4.4}$$

We have $k(T_w Z, T_w i) = k(Z, i)$ by (2.6), because T_w is upper triangular. Then making the substitution $z = T_w Z$ we get that (4.4) equals

$$2 \int_{\mathbb{H}} k(Z,i) V(T_w Z) d\mu_Z. \tag{4.5}$$

For a given $Z \in \mathbb{H}$ let $Z = i\frac{1+L}{1-L}$, where $|L| < 1$. Then it is easy to see, using also (2.5), that

$$H(Z,i) = \frac{(1-L)^{\frac{1}{2}}}{|1-L|^{\frac{1}{2}}}, \qquad \mathrm{Im} \left(T_w \left(i \frac{1+L}{1-L} \right) \right) = (\mathrm{Im}\, w) \frac{1 - |L|^2}{|1-L|^2}.$$

From the definition of V and from (1.2) we then obtain that $k(Z,i)V(T_w Z)$ equals

$$k \left(\frac{|Z-i|^2}{4 \mathrm{Im}\, Z} \right) \left(1 - |L|^2 \right)^{\frac{1}{4}} \overline{\left((\mathrm{Im}\, w)^{\frac{1}{4}} \theta(T_w Z)(1-L)^{-\frac{1}{2}} \right)} U(T_w Z).$$

It is easy to check that

$$\left(1 - |L|^2 \right)^{\frac{1}{4}} = \left(1 + \frac{|Z-i|^2}{4 \mathrm{Im}\, Z} \right)^{-\frac{1}{4}}, \qquad L = \frac{Z-i}{Z+i}.$$

So, taking the Taylor expansion (2.2) for w in place of z, we get for every $Z \in \mathbb{H}$ that

$$k(Z,i) V(T_w Z) = \tilde{k}(Z,i) U(T_w Z) \sum_{n=0}^{\infty} \overline{B_n(w)} \overline{\left(\frac{Z-i}{Z+i} \right)^n}. \tag{4.6}$$

We need the weak estimate that if $w \in D_4$ and $0 \leq j \leq 5$ is such that $\gamma_j^{-1} w \in D_1$, then $|B_n(w)| \ll (n+1)^{A_0} \left(\mathrm{Im}\, \gamma_j^{-1} w \right)^{A_0}$ with some absolute constant A_0. This follows easily from Lemma 6.2 of [B3]. Using that U is bounded, we then see that inserting (4.6) into (4.5) we can integrate term by term. In this way we get the assertions of the lemma except the last sentence. In the last sentence the transformation property follows easily from (2.6). For the estimates we use

$$|K(z,w)| \leq \sum_{\gamma \in SL(2,\mathbf{Z})} \left| k\left(\frac{|\gamma z - w|^2}{4\mathrm{Im}\,\gamma z \mathrm{Im}\,w}\right)\right|,$$

and we note that if $z, w \in D_1$, then for any $A > 0$ the right-hand side here

$$\ll_A (\mathrm{Im}\,z)^{A_0}\left(1+\frac{\mathrm{Im}\,w}{\mathrm{Im}\,z}\right)^{-A}$$

with some absolute constant A_0. This follows from Lemma 6.3 of [B3]. Using that V decays faster than polynomially at the cusps, the lemma follows.

LEMMA 4.2. *Let $k = k_\chi$ be the function defined in Lemma 3.7, and let $K(z,w)$ be as in Lemma 4.1. Let u be a cusp form of weight* 0 *for $SL(2,\mathbf{Z})$ with $\Delta_0 u = S(S-1)u$, $S = \frac{1}{2} + iT$. Then for any $w \in \mathbb{H}$ we have that*

$$\int_{D_4} \overline{B_0(z)} u(4z) K(z,w) d\mu_z \tag{4.7}$$

equals

$$\frac{4}{\Gamma\left(\frac{1}{2} \pm iT\right)} \sum_{n=0}^{\infty} \frac{C_{n,\chi}(T)}{\Gamma\left(\frac{1}{2}+n\right)} \overline{B_n(w)} \left(K_{n-1}K_{n-2}\dots K_1 K_0 u\right)(4w), \tag{4.8}$$

where

$$C_{n,\chi}(T) := \int_{-\infty}^{\infty} \left| \frac{\Gamma\left(\frac{1}{4}+it\right)\Gamma\left(\frac{1}{4}+it \pm iT\right)}{\Gamma(2it)} \right|^2 \frac{S_n\left(t^2; \frac{1}{4}+iT, \frac{1}{4}, \frac{1}{4}-iT\right)}{\left(\frac{1}{2}+iT\right)_n \left(\frac{1}{2}-iT\right)_n} \chi(t) dt.$$

The sum $\sum_{n=0}^{\infty} \left| \frac{C_{n,\chi}(T)}{\Gamma\left(\frac{1}{2}+n\right)} \overline{B_n(w)} \left(K_{n-1}K_{n-2}\dots K_1 K_0 u\right)(4w) \right|$, as a function of $w \in D_4$, grows at most polynomially at the cusps.

Proof. We will apply Lemma 4.1 with $U(z) = u(4z)$, and we use the notations (4.1), (4.2). Remark that if $n \geq 0$, $w \in \mathbb{H}$, then we get

$$\int_{\mathbb{H}} \tilde{k}(Z,i) \overline{\left(\frac{Z-i}{Z+i}\right)^n} u(4T_w Z) d\mu_Z = \int_{\mathbb{H}} \tilde{k}(z,w) \overline{\left(\frac{z-w}{z-\overline{w}}\right)^n} u(4z) d\mu_z \tag{4.9}$$

by the substitution $z = T_w Z$. We now make a transition to geodesic polar coordinates around w, i.e. we use (2.1) with w in place of z_0. See also the form of the invariant measure given below (2.1). We get in this way that (4.9) equals

$$\int_0^\infty \tilde{k}\left(\frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}\right)\tanh^n(\frac{r}{2})\left(\int_0^{2\pi} u(4z)e^{-in\phi}d\phi\right)\sinh r dr. \tag{4.10}$$

Here we write R in place of $\tanh(\frac{r}{2})$, and use

$$\sinh r dr = \frac{4R}{\left(1-R^2\right)^2}dR. \tag{4.11}$$

We apply also Lemma 2.2 and (4.1), and we get in this way that (4.10) is the same as

$$\frac{8\pi I_k\left(S\right)}{n!}\left(K_{n-1}K_{n-2}\ldots K_1K_0u\right)(4w),$$

where

$$I_k\left(S\right) := \int_0^1 k\left(\tfrac{R^2}{1-R^2}\right)R^{2n+1}\left(1-R^2\right)^{-\frac{7}{4}}F\left(S,1-S,n+1;\tfrac{R^2}{R^2-1}\right)dR.$$

Using here the definition of $k = k_\chi$ from Lemma 3.7 the resulting double integral is easily seen to be absolutely convergent. We get, writing $u = \frac{R^2}{1-R^2}$ and applying (3.5) that

$$I_k\left(S\right) = \frac{1}{2\pi}\int_0^\infty \left|\frac{\Gamma\left(\frac{1}{4}+it\right)\Gamma\left(\frac{3}{4}+it\right)}{\Gamma\left(2it\right)}\right|^2 \chi(t)I_n\left(t,T\right)dt,$$

where

$$I_n\left(t,T\right) := \int_0^\infty F\left(\frac{3}{4}-it,\frac{3}{4}+it,1;-u\right)u^nF(\frac{1}{2}+iT+n,\frac{1}{2}-iT+n,1+n;-u)du.$$

Using Lemma 4.1, Lemma 3.4 and (3.6) we get that (4.7) equals (4.8). The last statement of the lemma follows from the corresponding statement of Lemma 4.1.

4.2. An expression for the spectral sum. We give an expression for the spectral sum of Theorem 1.1 in terms of an automorphic kernel function. Our main result here is Lemma 4.5.

LEMMA 4.3. *The notations and assumptions of Lemma 4.1 are valid. Let f be a Maass form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{1/2}f = s(s-1)f$ for some $\operatorname{Re} s \geq \frac{1}{2}$, $s = \frac{1}{2} + it$. Then*

$$\int_{D_4}\left(\int_{D_4} V(z)K(z,w)d\mu_z\right) f(w)d\mu_w = 16\pi\left(\int_{D_4} V(z)f(z)d\mu_z\right) L_k(s), \tag{4.12}$$

where

$$L_k(s) := \int_0^1 k\left(\frac{R^2}{1-R^2}\right)\left(1-R^2\right)^{-9/4} F\left(s+\frac{1}{4}, \frac{5}{4}-s, 1; \frac{R^2}{R^2-1}\right) RdR.$$

If $k = k_\chi$ is the function defined in Lemma 3.7, then

$$L_k(s) = \chi(t). \tag{4.13}$$

Proof. Taking real and imaginary parts, we may assume that $k(u)$ is real for any $u \in [0,\infty)$. Since k is real, it is not hard to see, using (2.6), (2.7) and (1.3) that $K(z,w) = \overline{K(w,z)}$. Hence by (4.3) and the transformation formulas satisfied by f we see that $K(z,w)f(w)$ is invariant in w under $\Gamma_0(4)$. We also see that

$$\int_{D_4}\left(\int_{D_4} V(z)K(z,w)d\mu_z\right) f(w)d\mu_w = \int_{D_4} V(z)\left(\int_{D_4} f(w)\overline{K(w,z)}d\mu_w\right)d\mu_z,$$

the application of the Fubini theorem is justified by the last statement of Lemma 4.1. By the definition of K we see that

$$\int_{D_4} f(w)\overline{K(w,z)}d\mu_w = 2\int_{\mathbb{H}} f(w)\overline{k(w,z)}d\mu_w. \tag{4.14}$$

We have

$$\overline{H(w,z)} = i^{\frac{-1}{2}}\left(\frac{w-\overline{z}}{|w-\overline{z}|}\right)^{\frac{1}{2}} = \left(\frac{w-\overline{z}}{z-\overline{w}}\right)^{\frac{1}{4}},$$

the last equality holds because the fourth powers are the same, and the arguments of both sides lie in $(-\frac{\pi}{4}, \frac{\pi}{4})$. We use geodesic polar coordinates around z (see (2.1)) and we write

$$F(r,\phi) := f(w)\left(\frac{w-\overline{z}}{z-\overline{w}}\right)^{\frac{1}{4}}.$$

We get in this way that (4.14) equals

$$2\int_0^\infty k\left(\frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}\right)\left(\int_0^{2\pi} F(r,\phi)d\phi\right)\sinh r dr.$$

By Lemma 2.2 we get

$$\int_0^{2\pi} F(r,\phi)d\phi = 2\pi f(z)\left(1-\tanh^2(\frac{r}{2})\right)^{-1/4} F\left(s+\frac{1}{4},\frac{5}{4}-s,1;\frac{\tanh^2(\frac{r}{2})}{\tanh^2(\frac{r}{2})-1}\right).$$

Writing R in place of $\tanh(\frac{r}{2})$, using (4.11) we obtain (4.12). By the substitution $u=\frac{R^2}{1-R^2}$ and by Lemma 3.7 we get (4.13), the lemma is proved.

LEMMA 4.4. *If $f_1, f_2 \in R_{\frac{1}{2}}(D_4)$, then we have that $(f_1,f_2)_4$ equals*

$$\sum_{j=0}^{\infty}\left(f_1,u_{j,\frac{1}{2}}\right)_4\overline{\left(f_2,u_{j,\frac{1}{2}}\right)_4}+\frac{1}{4\pi}\sum_{\mathfrak{a}=0,\infty}\int_{-\infty}^{\infty}\zeta_{\mathfrak{a}}(f_1,r)\overline{\zeta_{\mathfrak{a}}(f_2,r)}dr.$$

Proof. This is well-known, see [P], formula (27).

LEMMA 4.5. *Let $k=k_\chi$ be the function defined in Lemma 3.7, and let $K(z,w)$ be as in Lemma 4.1. Let u_1 and u_2 be two cusp forms of weight 0 for $SL(2,\mathbf{Z})$. Then*

$$\int_{D_4}\left(\int_{D_4}\overline{B_0(z)}u_1(4z)K(z,w)d\mu_z\right)B_0(w)\overline{u_2(4w)}d\mu_w \tag{4.15}$$

equals the sum of

$$16\pi\sum_{j=1}^{\infty}\chi(T_j)\left(B_0\kappa(\overline{u_2}),u_{j,\frac{1}{2}}\right)_4\overline{\left(B_0\kappa(\overline{u_1}),u_{j,\frac{1}{2}}\right)_4} \tag{4.16}$$

and

$$4\sum_{\mathfrak{a}=0,\infty}\int_{-\infty}^{\infty}\chi(r)\zeta_{\mathfrak{a}}(B_0\kappa(\overline{u_2}),r)\overline{\zeta_{\mathfrak{a}}(B_0\kappa(\overline{u_1}),r)}dr. \tag{4.17}$$

Proof. Let

$$f_1(w):=B_0(w)\overline{u_2(4w)},\quad f_2(w):=\int_{D_4}\overline{B_0(z)u_1(4z)K(z,w)}d\mu_z.$$

If f is a Maass form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{1/2}f=s(s-1)f$ for some $s=\frac{1}{2}+it$, then we have by Lemma 4.3 that

$$\overline{(f_2,f)_4}=16\pi\chi(t)\int_{D_4}\overline{B_0(z)}u_1(4z)f(z)d\mu_z.$$

Lemma 4.4 implies that (4.15) equals the sum of (4.16) and (4.17), but at the moment it seems that $j=0$ should be present in the summation in (4.16). However, that term is 0 by Lemma 6.6 of [B3]. The lemma is proved.

4.3. Writing $B_t\overline{B_0}$ as an Eisenstein series. We mentioned in Sections 1.4 that it is very important for our proof that the functions $B_t\overline{B_0}$ are linear combinations of Eisenstein series. We write a certain average of this function as an incomplete Eisenstein series in Lemma 4.7.

LEMMA 4.6. *For $z \in \mathbb{H}$ let*

$$F(z) := \sum_{j=0}^{5} |B_0(\gamma_j z)|^2, \qquad G(z) := \sum_{\gamma \in \Gamma_\infty \backslash SL(2,\mathbf{Z})} \psi\left(\operatorname{Im}(\gamma z)\right),$$

where

$$\psi(y) := \sum_{m=1}^{\infty} e^{-\pi \frac{m^2}{y}}.$$

Then for every $z \in \mathbb{H}$ we have $F(z) = 6G(z) + 3$.

Proof. During the proof of Lemma 6.6 of [B3] (see the last lines of p. 632) it is shown that $F(z) = DG(z) + C$ for $z \in \mathbb{H}$ with some constants C and D. So it is enough to determine these constants.

Recall the definitions of γ_j from Section 2.5. Note first that $B_0\left(Z - \frac{1}{2}\right) = \sqrt{2}B_0(4Z) - B_0(Z)$ for $Z \in \mathbb{H}$ by (1.2). One has $\gamma_5 z = -\frac{1}{w} - \frac{1}{2}$ with $w = 4z - 2$, hence using also (1.4) we get

$$B_0(\gamma_5 z) = \sqrt{2}B_0\left(-\frac{4}{w}\right) - B_0\left(-\frac{1}{w}\right) = e\left(\frac{-1}{8}\right)\left(\frac{w}{|w|}\right)^{\frac{1}{2}}\left(\sqrt{2}B_0\left(\frac{w}{16}\right) - B_0\left(\frac{w}{4}\right)\right)$$

for every $z \in \mathbb{H}$. This shows by (1.2) that $B_0(\gamma_5(iy)) = o(1)$ as $y \to \infty$. For $0 \le j \le 3$ it is clear by (1.4) that we have $|B_0(\gamma_j z)| = \left|B_0\left(\frac{z+j}{4}\right)\right|$. We easily get from these remarks and (1.2) that $F(iy) = 3y^{1/2} + o(1)$ as $y \to \infty$. On the other hand, it is easy to see that $G(iy) = \psi(y) + o(1)$ as $y \to \infty$, and it follows from (1.4) that

$$1 + 2\psi(y) = y^{1/2} \sum_{m=-\infty}^{\infty} e^{-\pi m^2 y} = y^{1/2} + o(1).$$

Letting $y \to \infty$ we get the lemma.

LEMMA 4.7. *If $t \ge 0$ is an integer, for $z \in \mathbb{H}$ let*

$$F_t(z) := \sum_{j=0}^{5} B_t(\gamma_j z)\overline{B_0(\gamma_j z)}\left(\frac{j_{\gamma_j}(z)}{\left|j_{\gamma_j}(z)\right|}\right)^{-2t}$$

and

$$G_t(z) := \sum_{\gamma\in\Gamma_\infty\backslash SL(2,\mathbf{Z})} \psi_t\left(\mathrm{Im}\left(\gamma z\right)\right)\left(\frac{j_\gamma\left(z\right)}{\left|j_\gamma\left(z\right)\right|}\right)^{-2t},$$

where

$$\psi_t\left(y\right) := \frac{1}{t!}\sum_{m=1}^{\infty} e^{-\pi\frac{m^2}{y}}\left(\frac{\pi m^2}{y}\right)^t.$$

Then for every $t > 0$ we have $F_t(z) = 6G_t(z)$.

Proof. We have $F_0 = F$, $G_0 = G$ (see Lemma 4.6). It is easy to see that for every $t \geq 0$ we have

$$\frac{1}{t+1}K_tG_t = G_{t+1},$$

this follows from the identity

$$\frac{1}{t+1}\left(\psi_t^{(1)}\left(y\right)y + t\psi_t\left(y\right)\right) = \psi_{t+1}\left(y\right)$$

and Lemma 2.1 (vi). Using Lemma 4.6 we see that it is enough to prove that

$$\frac{1}{t+1}K_tF_t = F_{t+1} \tag{4.18}$$

for every $t \geq 0$. We use Lemma 2.1 (i) with $k_1 := t + \frac{1}{4}$, $k_2 := \frac{1}{4}$,

$$f(z) := B_t(\gamma_j z)\left(\frac{j_{\gamma_j}\left(z\right)}{\left|j_{\gamma_j}\left(z\right)\right|}\right)^{-2t-\frac{1}{2}}, \qquad g(z) := B_0(\gamma_j z)\left(\frac{j_{\gamma_j}\left(z\right)}{\left|j_{\gamma_j}\left(z\right)\right|}\right)^{-\frac{1}{2}}.$$

Then $K_{-k_2}\left(\overline{g}\right) = 0$ by Lemma 2.1 (vi) and (v). So (4.18) follows using (2.3) and Lemma 2.1 (vi). The lemma is proved.

5. Proof of the theorem

5.1. A special case. We first assume that χ is a function satisfying Condition D.

By Lemma 4.5 and Lemma 4.2 we have that the sum of

$$16\pi\sum_{j=1}^{\infty}\chi\left(T_j\right)\left(B_0\kappa\left(\overline{u_2}\right), u_{j,\frac{1}{2}}\right)_4\overline{\left(B_0\kappa\left(\overline{u_1}\right), u_{j,\frac{1}{2}}\right)_4} \tag{5.1}$$

and

$$4\sum_{\mathfrak{a}=0,\infty}\int_{-\infty}^{\infty}\chi\left(r\right)\zeta_{\mathfrak{a}}\left(B_0\kappa\left(\overline{u_2}\right), r\right)\overline{\zeta_{\mathfrak{a}}\left(B_0\kappa\left(\overline{u_1}\right), r\right)}dr \tag{5.2}$$

equals

$$\frac{4}{\Gamma\left(\frac{1}{2}\pm it_1\right)}\sum_{n=0}^{\infty}\frac{C_{n,\chi}\left(t_1\right)}{\Gamma\left(\frac{1}{2}+n\right)}J_n(u_1,u_2), \tag{5.3}$$

where

$$J_n = J_n(u_1,u_2) := \int_{D_4} B_0(w)\overline{B_n(w)u_2\left(4w\right)}\left(K_{n-1}K_{n-2}\ldots K_1K_0u_1\right)(4w)d\mu_w.$$

Let us write

$$f\left(z\right) = f_{n,u_1,u_2}\left(z\right) := \overline{u_2\left(z\right)}\left(K_{n-1}\ldots K_1K_0u_1\right)\left(z\right). \tag{5.4}$$

We then have that

$$f(\gamma z) = \left(\frac{j_\gamma(z)}{|j_\gamma(z)|}\right)^{2n} f(z) \tag{5.5}$$

for every $\gamma\in SL(2,\mathbf{Z})$. Since the substitution $w\to -\frac{1}{4w}$ normalizes $\Gamma_0(4)$, so

$$J_n = \int_{D_4} B_0\left(w\right)\overline{B_n(w)}f\left(4w\right)d\mu_w = \int_{D_4} B_0\left(\frac{-1}{4w}\right)\overline{B_n\left(\frac{-1}{4w}\right)}f\left(\frac{-1}{w}\right)d\mu_w, \tag{5.6}$$

hence by (2.4) and (5.5) we get

$$J_n = \int_{D_4} B_0\left(w\right)\overline{B_n(w)}f\left(w\right)d\mu_w.$$

Using again (5.5), we finally get

$$J_n = \int_{D_1}\overline{F_n(w)}f\left(w\right)d\mu_w, \tag{5.7}$$

with the function F_n defined in Lemma 4.7. Using Lemmas 4.6 and 4.7 we see by unfolding that

$$J_n = 6\int_0^{\infty}\int_0^1\overline{\psi_n(y)}f\left(x+iy\right)\frac{dxdy}{y^2}+3\delta_{0,n}\int_{D_1} f\left(w\right)d\mu_w, \tag{5.8}$$

where $\delta_{0,n}$ is Kronecker's symbol.

It is trivial by our assumptions that if $n=0$, then

$$\int_{D_1} f\left(w\right)d\mu_w = \delta_{u_1,u_2}. \tag{5.9}$$

It is well-known that if u is a cusp form of weight 0 for $SL(2,\mathbf{Z})$ with $\Delta_0 u= s(s-1)u$, where $s=\frac{1}{2}+it$, and

$$u(z)=\sum_{m\neq 0}\rho_u(m)W_{0,it}(4\pi\left|m\right|y)e(mx),$$

then for any $N\geq 0$ we have

$$\left(K_{N-1}K_{N-2}\dots K_1K_0u\right)(z)=\sum_{m\neq 0}\rho_N^u(m)W_{N\mathrm{sgn}(m),it}(4\pi\left|m\right|y)e(mx)$$

with

$$\rho_N^u(m)=(-1)^N\rho_u(m) \tag{5.10}$$

for $m>0$, and

$$\rho_N^u(m)=(s)_N\,(1-s)_N\,\rho_u(m) \tag{5.11}$$

for $m<0$, but we show now these statements for the sake of completeness. Indeed, by (3.44), if $m>0$, then

$$L_k\left(W_{k,it}(4\pi my)e(mx)\right)=-\left((it)^2-\left(k-\frac{1}{2}\right)^2\right)W_{k-1,it}(4\pi my)e(mx), \tag{5.12}$$

and if $m<0$, then

$$K_k\left(W_{-k,it}(4\pi\left|m\right|y)e(mx)\right)=-\left((it)^2-\left(-k-\frac{1}{2}\right)^2\right)W_{-k-1,it}(4\pi\left|m\right|y)e(mx). \tag{5.13}$$

Since, by Lemma 2.1 (iv), we have

$$\left(L_1L_2\dots L_N\left(K_{N-1}K_{N-2}\dots K_1K_0u\right)\right)(z)=\frac{\Gamma(s+N)}{\Gamma(s-N)}u(z),$$

so by repeated application of (5.12) we get (5.10), and by repeated application of (5.13) we get (5.11).

It is easy to see using (5.4) that

$$\int_0^\infty\int_0^1\overline{\psi_n(y)}f\left(x+iy\right)\frac{dxdy}{y^2}=\sum_{m\neq 0}\overline{\rho_{u_2}(m)}\rho_n^{u_1}(m)I_{n,t_1,t_2}\left(m\right) \tag{5.14}$$

with

$$I_{n,t_1,t_2}(m) := \int_0^\infty \psi_n(y) W_{n\mathrm{sgn}(m),it_1}(4\pi |m| y) W_{0,it_2}(4\pi |m| y) \frac{dy}{y^2} \tag{5.15}$$

(remark that $\psi_n(y)$ is real). By the well-known formula

$$\frac{1}{2\pi i} \int_{(\sigma)} \Gamma(S) Y^{-S} dS = e^{-Y}$$

we see for every $l \geq 0$ and $\sigma > \frac{1}{2}$ that

$$\psi_l(y) = \frac{1}{l!} \frac{1}{2\pi i} \int_{(\sigma)} \pi^{-S} \zeta(2S) \Gamma(l+S) y^S dS. \tag{5.16}$$

We will compute (5.3) by (5.8), (5.14), (5.15), (5.16), in this way we get summations over m, n and integration over y and S. We can see that if σ is fixed to be a large enough absolute constant, then these summations and integrations are absolutely convergent together. This can be seen by the definition of $C_{n,\chi}$ in Lemma 4.2, by Lemma 3.3, (5.10), (5.11), estimating the integral involving Whittaker functions by Lemma 3.9 (ii).

Applying Lemma 3.8, we get for any $\mathrm{Re}\, S > 0$ that

$$\int_0^\infty y^S W_{n\mathrm{sgn}(m),it_1}(4\pi |m| y) W_{0,it_2}(4\pi |m| y) \frac{dy}{y^2} \tag{5.17}$$

equals

$$\frac{(4\pi |m|)^{1-S}}{\Gamma\left(\frac{1}{2} \pm it_2\right)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right) \Gamma\left(\frac{1}{2} \pm it_1 + s\right) \Gamma(1+s-S)}{\Gamma(1-n+s)} ds \tag{5.18}$$

in the case $m > 0$, and

$$\frac{(4\pi |m|)^{1-S}}{\Gamma\left(\frac{1}{2} + n \pm it_1\right)} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{\Gamma\left(-\frac{1}{2} \pm it_1 + S - s\right) \Gamma\left(\frac{1}{2} \pm it_2 + s\right) \Gamma(1+n+s-S)}{\Gamma(1+s)} ds \tag{5.19}$$

in the case $m < 0$. Indeed, we obtain it by the choice

$$k = n, \lambda = 0 \text{ in the case } m > 0,$$

$$k = 0, \lambda = -n \text{ in the case } m < 0.$$

In the case $m < 0$ we apply Lemma 3.8 by exchanging t_1 and t_2.

By (5.8), (5.9) and (5.14) we have

$$J_n = 3\delta_{0,n}\delta_{u_1,u_2} + 6\sum_{m\neq 0}\overline{\rho_{u_2}(m)}\rho_n^{u_1}(m) I_{n,t_1,t_2}(m)\,. \tag{5.20}$$

We can determine $\rho_n^{u_1}(m)$ by (5.10) and (5.11). We use that

$$\rho_{u_1}(m)\overline{\rho_{u_2}(m)} = \rho_{u_1}(-m)\overline{\rho_{u_2}(-m)}$$

for every $m \neq 0$, since it is assumed that either u_1 and u_2 are odd, or both of them are even. We then see by (5.15), (5.16), (5.17), (5.18), (5.19) and (1.6) that fixing σ to be a large enough absolute constant,

$$\Gamma\left(\frac{1}{2} \pm it_1\right)\sum_{n=0}^{\infty}\frac{C_{n,\chi}(t_1)}{\Gamma\left(\frac{1}{2}+n\right)}\sum_{m\neq 0}\overline{\rho_{u_2}(m)}\rho_n^{u_1}(m) I_{n,t_1,t_2}(m) \tag{5.21}$$

equals

$$4\pi\frac{1}{2\pi i}\int_{(\sigma)}\left(4\pi^2\right)^{-S}\zeta(2S)\,L(S)\left(E^+\left(S,\frac{1}{2}\right)+\frac{\sin\pi s_2}{\sin\pi s_1}E^-\left(S,\frac{1}{2}\right)\right)dS, \tag{5.22}$$

where $E^+(S,D)$ denotes the sum

$$\sum_{n=0}^{\infty}\frac{C_{n,\chi}(t_1)\,\Gamma(S+n)}{n!\Gamma(D+n)\,2\pi i}\int_{(\tau)}\frac{\Gamma\left(-\frac{1}{2}\pm it_1+S-s\right)\Gamma\left(\frac{1}{2}\pm it_2+s\right)\Gamma(1+n+s-S)}{\Gamma(1+s)}ds,$$

and $E^-(S,D)$ denotes the sum

$$\sum_{n=0}^{\infty}(-1)^n\frac{C_{n,\chi}(t_1)\,\Gamma(S+n)}{n!\Gamma(D+n)\,2\pi i}\int_{(\tau)}\frac{\Gamma\left(-\frac{1}{2}\pm it_2+S-s\right)\Gamma\left(\frac{1}{2}\pm it_1+s\right)\Gamma(1+s-S)}{\Gamma(1-n+s)}ds \tag{5.23}$$

with $\operatorname{Re} S - 1 < \tau < \operatorname{Re} S - \frac{1}{2}$, $\tau > -\frac{1}{2}$. There is such a τ for every $\operatorname{Re} S > 0$. Our computations are justified by the discussion below (5.16). We see by Lemma 3.3 that the summation and integrations in n, s and S are absolutely convergent.

See Section 2.4 for the properties of the function $\zeta(2S)\,L(S)$. This function is regular at $S = 1$ if $u_1 \neq u_2$, and its residue at $S = 1$ in the case $u_1 = u_2$ is

$$\operatorname{res}_{S=1}\zeta(2S)\,L(S) = \zeta(2)\frac{1}{\Gamma\left(\frac{1}{2}\pm it_1\right)|D_1|} = \frac{\pi}{2\Gamma\left(\frac{1}{2}\pm it_1\right)}. \tag{5.24}$$

This follows from [I], (8.12), (8.9) and (8.5), taking into account $\zeta(2) = \frac{\pi^2}{6}$, $|D_1| = \frac{\pi}{3}$. The last relation follows from [I], (6.33), (3.26).

Since $C_{n,\chi}(t_1)$ decreases faster than polynomially in n by Lemma 3.3, so by the properties of $L(S)$ we see shifting the integration to the left that for a small $\epsilon > 0$, e.g. take $\epsilon = \frac{1}{100}$, we have that (5.22) equals the sum of

$$4\pi \frac{1}{2\pi i} \int_{\left(\frac{1}{2}-\epsilon\right)} \left(4\pi^2\right)^{-S} \zeta(2S) L(S) \left(E^+\left(S, \frac{1}{2}\right) + \frac{\sin \pi s_2}{\sin \pi s_1} E^-\left(S, \frac{1}{2}\right)\right) dS \tag{5.25}$$

and

$$\delta_{u_1,u_2} \frac{1}{2\Gamma\left(\frac{1}{2} \pm it_1\right)} \left(E^+\left(1, \frac{1}{2}\right) + E^-\left(1, \frac{1}{2}\right)\right). \tag{5.26}$$

Note that this last term is present only in the case $t_1 = t_2$. We now determine $E^+\left(1, \frac{1}{2}\right) + E^-\left(1, \frac{1}{2}\right)$ in the case $t_1 = t_2$. Since $(-1)^n \frac{\Gamma(s)}{\Gamma(1-n+s)} = -\frac{\Gamma(n-s)}{\Gamma(1-s)}$, so, using the substitution $s \to -s$ in the integral in (5.23), we have that $E^+\left(1, \frac{1}{2}\right) + E^-\left(1, \frac{1}{2}\right)$ equals

$$\sum_{n=0}^{\infty} \frac{C_{n,\chi}(t_1)}{\Gamma\left(\frac{1}{2}+n\right) 2\pi i} \left(\int_{(\tau)} \frac{\Gamma\left(\frac{1}{2} \pm it_1 \pm s\right) \Gamma(n+s)}{\Gamma(1+s)} ds - \int_{(-\tau)} \frac{\Gamma\left(\frac{1}{2} \pm it_1 \pm s\right) \Gamma(n+s)}{\Gamma(1+s)} ds \right)$$

with $0 < \tau < \frac{1}{2}$. For $n > 0$ the difference of these integrals is 0, and for $n = 0$ it is $2\pi i \Gamma^2\left(\frac{1}{2} \pm it_1\right)$. Hence, if $t_1 = t_2$, we have

$$E^+\left(1, \frac{1}{2}\right) + E^-\left(1, \frac{1}{2}\right) = \frac{C_{0,\chi}(t_1)}{\Gamma\left(\frac{1}{2}\right)} \Gamma^2\left(\frac{1}{2} \pm it_1\right). \tag{5.27}$$

It is clear that (5.25) equals

$$\lim_{\delta \to 0+0} \frac{4\pi}{2\pi i} \int_{\left(\frac{1}{2}-\epsilon\right)} e^{\delta S^2} \left(4\pi^2\right)^{-S} \zeta(2S) L(S) \left(E^+\left(S, \frac{1}{2}\right) + \frac{\sin \pi s_2}{\sin \pi s_1} E^-\left(S, \frac{1}{2}\right)\right) dS, \tag{5.28}$$

and for a given $\delta > 0$ we have that

$$\frac{4\pi}{2\pi i} \int_{\left(\frac{1}{2}-\epsilon\right)} e^{\delta S^2} \left(4\pi^2\right)^{-S} \zeta(2S) L(S) \left(E^+(S, D) + \frac{\sin \pi s_2}{\sin \pi s_1} E^-(S, D)\right) dS \tag{5.29}$$

is a regular function of D for $\operatorname{Re} D > 0$. Let us consider this function first for large enough $\operatorname{Re} D$. By the definition of $E^+(S, D)$, $E^-(S, D)$, $C_{n,\chi}(t_1)$, the upper bound for χ and Lemma 3.1 we see that if D has large enough real part, then we can compute (5.29) by

inserting the defining integral for $C_{n,\chi}(t_1)$ in $E^+(S,D)$ and $E^-(S,D)$, since the resulting triple integral in s, S, t and summation in n are absolutely convergent. We will use the following two identities, both of them follow from Lemma 3.2.
For any S and s with $\operatorname{Re} S = \frac{1}{2} - \epsilon$, $\operatorname{Re} s = -\frac{1}{4} - \frac{\epsilon}{2}$ we have that

$$\sum_{n=0}^{\infty} \frac{{}_3F_2\left(\begin{matrix} -n, \frac{1}{4}+it, \frac{1}{4}-it \\ \frac{1}{2}+it_1, \frac{1}{2}-it_1 \end{matrix}; 1\right)}{n!} \frac{\Gamma(S+n)\Gamma(1-S+s+n)}{\Gamma(D+n)}$$

equals

$$\frac{\Gamma\left(\frac{1}{2} \pm it_1\right)}{\Gamma\left(\frac{1}{4} \pm it\right)\Gamma(D-S)\Gamma(D+S-s-1)} F_1(s),$$

defining $F_1(s)$ as

$$\frac{1}{2\pi i}\int_{(-c)} \frac{\Gamma\left(\frac{1}{4} \pm it + T\right)\Gamma(S+T)\Gamma(1-S+s+T)\Gamma(-T)\Gamma(D-1-s-T)}{\Gamma\left(\frac{1}{2} \pm it_1 + T\right)} dT$$

with $\frac{1}{4} - \frac{\epsilon}{2} < c < \frac{1}{4}$; and

$$\sum_{n=0}^{\infty} \frac{{}_3F_2\left(\begin{matrix} -n, \frac{1}{4}+it, \frac{1}{4}-it \\ \frac{1}{2}+it_1, \frac{1}{2}-it_1 \end{matrix}; 1\right)}{n!} \frac{\Gamma(S+n)\Gamma(-s+n)}{\Gamma(D+n)}$$

equals

$$\frac{\Gamma\left(\frac{1}{2} \pm it_1\right)}{\Gamma\left(\frac{1}{4} \pm it\right)\Gamma(D-S)\Gamma(D+s)} F_2(s),$$

defining, again with $\frac{1}{4} - \frac{\epsilon}{2} < c < \frac{1}{4}$,

$$F_2(s) := \frac{1}{2\pi i}\int_{(-c)} \frac{\Gamma\left(\frac{1}{4} \pm it + T\right)\Gamma(S+T)\Gamma(-s+T)\Gamma(-T)\Gamma(D-S+s-T)}{\Gamma\left(\frac{1}{2} \pm it_1 + T\right)} dT.$$

Using these identities and the definition of $E^+(S,D)$, $E^-(S,D)$, $C_{n,\chi}(t_1)$, (3.6) and that (3.6) is symmetric in a, b, c, we get for D with large enough real part that (5.29) equals

$$\frac{4\pi\Gamma\left(\frac{1}{2} \pm it_1\right)}{2\pi i}\int_{\left(\frac{1}{2}-\epsilon\right)}\int_{-\infty}^{\infty} e^{\delta S^2}\left(4\pi^2\right)^{-S} \frac{\zeta(2S)L(S)}{\Gamma(D-S)} \frac{\Gamma\left(\frac{1}{4} \pm it \pm it_1\right)}{\Gamma(\pm 2it)} \chi(t) M(S,t)\, dt dS, \tag{5.30}$$

where

$$M(S,t) := M_1(S,t) + M_2(S,t),$$

and $M_1(S,t)$ denotes

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)}\frac{\Gamma\left(-\frac{1}{2}\pm it_1+S-s\right)\Gamma\left(\frac{1}{2}\pm it_2+s\right)}{\Gamma(1+s)\Gamma(D+S-s-1)}F_1(s)ds,$$

$M_2(S,t)$ denotes

$$\frac{\sin\pi s_2}{\sin\pi s_1}\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)}\frac{\Gamma\left(-\frac{1}{2}\pm it_2+S-s\right)\Gamma\left(\frac{1}{2}\pm it_1+s\right)\Gamma(1+s-S)}{\Gamma(1+s)\Gamma(-s)\Gamma(D+s)}F_2(s)ds.$$

One can check that (5.30) is a regular function of D for $\operatorname{Re} D\geq\frac{1}{2}$, hence by analytic continuation this equals (5.29) also for $D=\frac{1}{2}$. In the case $D=\frac{1}{2}$ we can apply Lemma 6.2 to determine $M(S,t)$. Hence we proved for any $\delta>0$ that in the case $D=\frac{1}{2}$ (5.29) equals (5.30) with $M(S,t)$ given by the sum of (6.21) and (6.22). Recalling the definition of $N(S,t)$ and $H_\chi(S)$ from the Introduction we see that (5.29) for $D=\frac{1}{2}$ equals

$$-\frac{4\pi\Gamma\left(\frac{1}{2}\pm it_1\right)}{2\pi i}\frac{\sin\pi s_2}{\sin\pi s_1}\int_{\left(\frac{1}{2}-\epsilon\right)}e^{\delta S^2}\left(4\pi^2\right)^{-S}\zeta(2S)L(S)\Gamma(S)\Gamma(1-S)H_\chi(S)\,dS. \quad (5.31)$$

Assume that β in Theorem 1.1 is large enough. Applying Lemma 6.3 (ii) and a convexity bound we see that (5.28) equals (5.31) by writing $\delta=0$ there. Using Lemma 6.3 (ii) again we see that we can shift the line of integration to $\operatorname{Re} S=\frac{1}{2}$ in (5.31). Hence we proved finally that (5.25) equals

$$-4\pi\Gamma\left(\frac{1}{2}\pm it_1\right)\frac{\sin\pi s_2}{\sin\pi s_1}\frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)}\left(4\pi^2\right)^{-S}\Gamma(S)\Gamma(1-S)\zeta(2S)L(S)H_\chi(S)\,dS.$$

Using this last relation, (5.1), (5.2), (5.3), (5.20), (5.21), (5.22), (5.25), (5.26) and (5.27), taking into account the definition of $C_{0,\chi}$ in Lemma 4.2 and $\Gamma\left(\frac{1}{2}\right)=\pi^{1/2}$, we get Theorem 1.1 for the case when χ satisfies Condition D.

5.2. The general case. To extend the theorem for the general case, we first need a lemma.

LEMMA 5.1. *Let $\beta>0$ and let χ be an even holomorphic function on the strip $|\operatorname{Im} z|<\beta$ such that for a fixed $A>0$ the function $|\chi(z)|\,e^{A|z|^2}$ is bounded on the strip $|\operatorname{Im} z|<\beta$. Then for every $0<\gamma<\beta$ there is a sequence χ_n of entire functions, and a nonnegative function M on $[0,\infty)$ with the following properties. The function χ_n satisfies Condition*

D for every n, for every fixed $K > 0$ the function $M(R)e^{KR}$ is bounded on $[0, \infty)$, we have $|\chi_n(z)| \leq M(|z|)$ for every $n \geq 1$ and $|\mathrm{Im}\, z| < \gamma$, and finally, $\chi_n(z) \to \chi(z)$ for every $|\mathrm{Im}\, z| < \gamma$.

Proof. It follows from elementary facts on Fourier transforms that

$$\chi(z)e^{\frac{A}{2}z^2} = \int_{-\infty}^{\infty} h(x)e^{ixz}dx$$

for $|\mathrm{Im}\, z| < \beta$ where h is an even function such that $h(x) \ll_\delta e^{-\delta|x|}$ for every $0 < \delta < \beta$. Define now

$$\chi_n(z) := e^{-\frac{A}{2}z^2} \int_{-n}^{n} h(x)e^{ixz}dx,$$

then for $|\mathrm{Im}\, z| < \gamma$ we have

$$|\chi_n(z)| \leq \left|e^{-\frac{A}{2}z^2}\right| \int_{-\infty}^{\infty} |h(x)|\, e^{\gamma|x|}dx.$$

The lemma follows.

Note that using the convexity bound we see that there is a constant $\beta_0 > 0$ such that

$$\frac{1}{2\pi i}\int_{\left(\frac{1}{2}\right)} |\zeta(2S)\, L(S)|\, |S|^{-\frac{1}{2}-2\beta_0}\, dS < \infty.$$

We choose β such that $\beta > \beta_0$. Let χ be a function satisfying Condition C_β. Then the sum in (1.7) and the integral in (1.8) are absolutely convergent by [B3], formulas (5.2) and (5.3). Then it follows from Lemma 6.3 (ii) and the dominated convergence theorem that it is enough to prove Theorem 1.1 for every function $\chi(z)e^{-z^2/N}$ (N is a positive integer) instead of χ. So we may assume that there is an $A > 0$ such that $\chi(z)e^{A|z|^2}$ is bounded on the strip $|\mathrm{Im}\, z| < \beta$. Finally, for such functions the theorem follows from Lemma 5.1, Lemma 6.3 (ii), the dominated convergence theorem and the already proved special case of Theorem 1.1. The theorem is proved.

6. On the kernel function and the integral transform

In this section t_1 and t_2 are fixed nonzero real numbers, and we write $s_j = \frac{1}{2} + it_j$ for $j = 1, 2$.

6.1. Determination of the kernel function. The first lemma is proved here in a slightly more general form than necessary; in fact, for Theorem 1.1 we use only the $n = 0$

case. The $n \geq 1$ case would be needed for the proof of Theorem 1.2. Our main result in this subsection is Lemma 6.2.

LEMMA 6.1. *Let* $\epsilon = \frac{1}{100}$*, and let* S*,* B *and integer* n *be given such that* $\operatorname{Re} S = \frac{1}{2} - \epsilon$*, and either*

$$n = 0, \qquad \frac{3}{4} - \frac{\epsilon}{2} < \operatorname{Re} B < \frac{3}{4},$$

or

$$n \geq 1, \qquad B = \frac{1}{2}.$$

Let γ_1 *and* γ_2 *be curves (in* s*) connecting* $-i\infty$ *and* $i\infty$ *such that*

$$\text{the poles of } \Gamma\left(\frac{1}{2} \pm it_2 + s\right)\Gamma\left(1 - n + s - S\right) \text{ lie to the left of } \gamma_1,$$

$$\text{the poles of } \Gamma\left(-\frac{1}{2} \pm it_1 + S - s\right)\Gamma\left(n - 1 + B - s\right) \text{ lie to the right of } \gamma_1,$$

$$\text{the poles of } \Gamma\left(\frac{1}{2} \pm it_1 + s\right)\Gamma\left(1 + n + s - S\right)\Gamma\left(B - S + s + n\right) \text{ lie to the left of } \gamma_2,$$

$$\text{the poles of } \Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right) \text{ lie to the right of } \gamma_2.$$

In the case $n = 0$ *both of* γ_1 *and* γ_2 *may be the line with real part* $-\frac{1}{4} - \frac{\epsilon}{2}$*.*

Consider the integrals

$$\frac{1}{2\pi i}\int_{\gamma_1} \frac{\Gamma\left(-\frac{1}{2} \pm it_1 + S - s\right)\Gamma\left(\frac{1}{2} \pm it_2 + s\right)\Gamma\left(1 - n + s - S\right)\Gamma\left(n - 1 + B - s\right)}{\Gamma\left(1 - n + s\right)\Gamma\left(n - 1 + B - s + S\right)} ds \tag{6.1}$$

and

$$\frac{\frac{\sin \pi s_2}{\sin \pi s_1}}{2\pi i}\int_{\gamma_2} \frac{\Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right)\Gamma\left(\frac{1}{2} \pm it_1 + s\right)\Gamma\left(1 + n + s - S\right)\Gamma\left(B - S + s + n\right)}{\Gamma\left(B + n + s\right)\Gamma\left(1 + n + s\right)} ds. \tag{6.2}$$

Then (6.1) equals

$$(-1)^{n-1}\Gamma\left(B - S\right)\Gamma\left(1 - S\right)\Gamma\left(\frac{1}{2} - n \pm it_1\right)\frac{\sin \pi s_2}{\sin \pi s_1}\left(C_1^+Q^+ + C_1^-Q^-\right), \tag{6.3}$$

and (6.2) equals

$$(-1)^{n-1}\Gamma\left(B - S\right)\Gamma\left(1 - S\right)\Gamma\left(\frac{1}{2} - n \pm it_1\right)\frac{\sin \pi s_2}{\sin \pi s_1}\left(C_2^+Q^+ + C_2^-Q^-\right), \tag{6.4}$$

where

$$C_1^+ := \frac{\Gamma\left(B - \frac{1}{2} + n + it_2\right)\Gamma\left(\frac{1}{2} + n + it_2\right)\Gamma\left(S \pm it_1 + it_2\right)}{\sin\pi\left(2it_2\right)} \sin\pi s_1,$$

$$C_1^- := \frac{\Gamma\left(B - \frac{1}{2} + n - it_2\right)\Gamma\left(\frac{1}{2} + n - it_2\right)\Gamma\left(S \pm it_1 - it_2\right)}{\sin\pi\left(-2it_2\right)} \sin\pi s_1,$$

$$C_2^+ := \frac{\Gamma\left(B - \frac{1}{2} + n + it_2\right)\Gamma\left(\frac{1}{2} + n + it_2\right)\Gamma\left(S \pm it_1 + it_2\right)}{\sin\pi\left(2it_2\right)} \sin\pi\left(\frac{1}{2} - it_2 - S\right),$$

$$C_2^- := \frac{\Gamma\left(B - \frac{1}{2} + n - it_2\right)\Gamma\left(\frac{1}{2} + n - it_2\right)\Gamma\left(S \pm it_1 - it_2\right)}{\sin\pi\left(-2it_2\right)} \sin\pi\left(\frac{1}{2} + it_2 - S\right),$$

$$Q^+ := \phi_{i\left(\frac{1}{2}-S\right)}\left(i\left(\frac{1-B}{2} - n\right); 1 - \frac{B}{2} + it_2, \frac{B}{2} + it_1, \frac{B}{2} - it_1, 1 - \frac{B}{2} - it_2\right), \quad (6.5)$$

$$Q^- := \phi_{i\left(\frac{1}{2}-S\right)}\left(i\left(\frac{1-B}{2} - n\right); 1 - \frac{B}{2} - it_2, \frac{B}{2} + it_1, \frac{B}{2} - it_1, 1 - \frac{B}{2} + it_2\right). \quad (6.6)$$

Proof. Formula (6.1) equals, by shifting the integration to the left, the sum of

$$\frac{\Gamma\left(1 + 2it_2\right)\Gamma\left(-2it_2\right)\Gamma\left(\frac{1}{2} - it_2 - n - S\right)\Gamma\left(\frac{1}{2} + n + it_2 + S\right)}{\Gamma\left(\frac{1}{2} - it_2 - n\right)\Gamma\left(\frac{1}{2} + it_2 + n\right)} F^+, \quad (6.7)$$

$$\frac{\Gamma\left(1 - 2it_2\right)\Gamma\left(2it_2\right)\Gamma\left(\frac{1}{2} + it_2 - n - S\right)\Gamma\left(\frac{1}{2} + n - it_2 + S\right)}{\Gamma\left(\frac{1}{2} + it_2 - n\right)\Gamma\left(\frac{1}{2} - it_2 + n\right)} F^- \quad (6.8)$$

and

$$\frac{\Gamma\left(\frac{3}{2} - n \pm it_2 - S\right)\Gamma\left(-\frac{1}{2} \pm it_2 + n + S\right)}{\Gamma\left(S\right)\Gamma\left(1 - S\right)} G, \quad (6.9)$$

where we write

$$F^+ := \sum_{m=0}^{\infty} \frac{\Gamma\left(B - \frac{1}{2} + n + it_2 + m\right)\Gamma\left(\frac{1}{2} + n + it_2 + m\right)\Gamma\left(\pm it_1 + S + it_2 + m\right)}{m!\Gamma\left(1 + 2it_2 + m\right)\Gamma\left(B - \frac{1}{2} + n + it_2 + S + m\right)\Gamma\left(\frac{1}{2} + n + it_2 + S + m\right)},$$

$$F^- := \sum_{m=0}^{\infty} \frac{\Gamma\left(B - \frac{1}{2} + n - it_2 + m\right)\Gamma\left(\frac{1}{2} + n - it_2 + m\right)\Gamma\left(\pm it_1 + S - it_2 + m\right)}{m!\Gamma\left(1 - 2it_2 + m\right)\Gamma\left(B - \frac{1}{2} + n - it_2 + S + m\right)\Gamma\left(\frac{1}{2} + n - it_2 + S + m\right)},$$

$$G := \sum_{m=0}^{\infty} \frac{\Gamma\left(\frac{1}{2} \pm it_1 - n + m\right)\Gamma\left(B - S + m\right)\Gamma\left(1 - S + m\right)}{m!\Gamma\left(\frac{3}{2} - n \pm it_2 - S + m\right)\Gamma\left(B + m\right)}.$$

By (3.31), (3.32), (3.33) and (3.34) we have that

$$F^+ - F^- = \frac{\Gamma\left(B - \frac{1}{2} + n \pm it_2\right)\Gamma\left(\frac{1}{2} + n \pm it_2\right)\Gamma\left(\pm it_1 + S \pm it_2\right)}{\Gamma\left(1 + 2it_2\right)\Gamma\left(-2it_2\right)} P, \quad (6.10)$$

and that $F^{+} - G$ equals

$$\frac{\Gamma\left(B-\frac{1}{2}+n+it_2\right)\Gamma\left(\frac{1}{2}+n+it_2\right)\Gamma\left(S\pm it_1+it_2\right)\Gamma\left(\frac{1}{2}\pm it_1-n\right)\Gamma\left(B-S\right)\Gamma\left(1-S\right)}{\Gamma\left(\frac{1}{2}+n+it_2+S\right)\Gamma\left(\frac{1}{2}-n-it_2-S\right)} \tag{6.11}$$

times Q^{+}, where (the function ψ is defined in (3.28))

$$P := \psi\left(\alpha; B-\frac{1}{2}+n-it_1, \frac{1}{2}+n-it_1, B-\frac{1}{2}+n+it_2, \frac{1}{2}+n+it_2, S+it_2-it_1\right)$$

with the abbreviation

$$\alpha := B-1+2n+S+it_2-it_1,$$

and

$$Q^{+} := \psi\left(S+2it_2; S, \frac{3}{2}+it_2-B-n, S+it_2+it_1, \frac{1}{2}+n+it_2, S+it_2-it_1\right). \tag{6.12}$$

Let

$$Q^{-} := \psi\left(S-2it_2; S, \frac{3}{2}-it_2-B-n, S-it_2+it_1, \frac{1}{2}+n-it_2, S-it_2-it_1\right), \tag{6.13}$$

then by (3.35), (3.36) and (3.37) we have that

$$\frac{\sin\pi\left(\frac{1}{2}+it_2+n+S\right)Q^{+}}{\Gamma\left(\frac{1}{2}+n-it_2\right)\Gamma\left(S-it_2\pm it_1\right)\Gamma\left(B-\frac{1}{2}+n-it_2\right)}+\frac{\sin\pi\left(2it_2\right)P}{\Gamma\left(B-S\right)\Gamma\left(1-S\right)\Gamma\left(\frac{1}{2}-n\pm it_1\right)} \tag{6.14}$$

equals

$$\frac{\sin\pi\left(\frac{1}{2}-it_2+n+S\right)Q^{-}}{\Gamma\left(\frac{1}{2}+n+it_2\right)\Gamma\left(S+it_2\pm it_1\right)\Gamma\left(B-\frac{1}{2}+n+it_2\right)}. \tag{6.15}$$

The identity

$$1=\frac{\sin\pi\left(\frac{1}{2}-it_2-n-S\right)}{\sin\pi\left(\frac{1}{2}+it_2-n-S\right)}+\frac{\sin\pi S\sin\pi\left(1+2it_2\right)}{\sin\pi\left(\frac{1}{2}-it_2-n\right)\sin\pi\left(\frac{3}{2}+it_2-n-S\right)} \tag{6.16}$$

follows from the easily checked fact that the right-hand side is a bounded entire function of S, and its value is 1 at $S=0$. Multiplying (6.7) by the right-hand side of (6.16), we see that the sum of (6.7), (6.8) and (6.9) equals the sum of

$$\frac{\Gamma\left(1-2it_2\right)\Gamma\left(2it_2\right)\Gamma\left(\frac{1}{2}+it_2-n-S\right)\Gamma\left(\frac{1}{2}+n-it_2+S\right)}{\Gamma\left(\frac{1}{2}+it_2-n\right)\Gamma\left(\frac{1}{2}+n-it_2\right)}\left(F^{-}-F^{+}\right)$$

and

$$\frac{\Gamma\left(\frac{3}{2}-n\pm it_2-S\right)\Gamma\left(-\frac{1}{2}\pm it_2+n+S\right)}{\Gamma\left(S\right)\Gamma\left(1-S\right)}\left(G-F^+\right),$$

which sum, by (6.10) and (6.11), equals the sum of

$$\frac{\pi\Gamma\left(B-\frac{1}{2}+n\pm it_2\right)\Gamma\left(\frac{1}{2}+n+it_2\right)\Gamma\left(\pm it_1+S\pm it_2\right)}{\Gamma\left(\frac{1}{2}-n+it_2\right)\sin\pi\left(\frac{1}{2}+it_2-n-S\right)}P \tag{6.17}$$

and

$$\frac{\pi\Gamma\left(B-\frac{1}{2}+n+it_2\right)\Gamma\left(\frac{1}{2}+n+it_2\right)\Gamma\left(S\pm it_1+it_2\right)\Gamma\left(\frac{1}{2}\pm it_1-n\right)\Gamma\left(B-S\right)}{\Gamma\left(S\right)\sin\pi\left(\frac{3}{2}+it_2-n-S\right)}Q^+. \tag{6.18}$$

Hence we proved that (6.1) equals the sum of (6.17) and (6.18).

By shifting the integration to the right, we see that (6.2) equals

$$\frac{\sin\pi s_2}{\sin\pi s_1}\left(\Gamma\left(-2it_2\right)\Gamma\left(1+2it_2\right)F^++\Gamma\left(2it_2\right)\Gamma\left(1-2it_2\right)F^-\right),$$

which, by (6.10), equals

$$\frac{\sin\pi s_2}{\sin\pi s_1}\Gamma\left(B-\frac{1}{2}+n\pm it_2\right)\Gamma\left(\frac{1}{2}+n\pm it_2\right)\Gamma\left(\pm it_1+S\pm it_2\right)P.$$

Hence both (6.1) and (6.2) are linear combinations of P and Q^+. By the equality of (6.14) and (6.15) we can express P by Q^+ and Q^-, and by a tedious, but straightforward calculation we get (6.3) and (6.4), with C_1^+, C_2^+, C_1^-, C_2^- given in the text of the lemma, and Q^+, Q^- given by (6.12) and (6.13). During the calculation we need the identity

$$\frac{\sin\pi S\sin 2\pi it_2}{\sin\pi\left(\frac{1}{2}+it_2-S\right)\sin\pi\left(\frac{1}{2}+it_2+S\right)}+\frac{\sin\pi s_2}{\sin\pi\left(\frac{1}{2}+it_2-S\right)}=\frac{\sin\pi s_2}{\sin\pi\left(\frac{1}{2}-it_2-S\right)};$$

for its proof it is enough to show that the difference of the two sides is a regular function of S, and it is not hard to see.

By (3.30), (3.23) and (3.24) we get the expressions (6.5) and (6.6) for Q^+ and Q^-. The lemma is proved.

LEMMA 6.2. *Let* $\epsilon=\frac{1}{100}$, *and let* t *and* S *be given such that* t *is real and* $\operatorname{Re}S=\frac{1}{2}-\epsilon$. *Consider the integrals*

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)}\frac{\Gamma\left(-\frac{1}{2}\pm it_1+S-s\right)\Gamma\left(\frac{1}{2}\pm it_2+s\right)}{\Gamma\left(1+s\right)\Gamma\left(S-s-\frac{1}{2}\right)}F_1(s)ds \tag{6.19}$$

and

$$\frac{\sin\pi s_2}{\sin\pi s_1}\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)}\frac{\Gamma\left(-\frac{1}{2}\pm it_2+S-s\right)\Gamma\left(\frac{1}{2}\pm it_1+s\right)\Gamma\left(1+s-S\right)}{\Gamma\left(1+s\right)\Gamma\left(-s\right)\Gamma\left(\frac{1}{2}+s\right)}F_2(s)ds, \quad (6.20)$$

where $F_1(s)$ *denotes*

$$\frac{1}{2\pi i}\int_{(-c)}\frac{\Gamma\left(\frac{1}{4}\pm it+T\right)\Gamma\left(S+T\right)\Gamma\left(-T\right)\Gamma\left(1-S+s+T\right)\Gamma\left(-\frac{1}{2}-s-T\right)}{\Gamma\left(\frac{1}{2}\pm it_1+T\right)}dT,$$

and $F_2(s)$ *denotes*

$$\frac{1}{2\pi i}\int_{(-c)}\frac{\Gamma\left(\frac{1}{4}\pm it+T\right)\Gamma\left(S+T\right)\Gamma\left(-T\right)\Gamma\left(-s+T\right)\Gamma\left(\frac{1}{2}-S+s-T\right)}{\Gamma\left(\frac{1}{2}\pm it_1+T\right)}dT$$

with $\frac{1}{4}-\frac{\epsilon}{2}<c<\frac{1}{4}$.
Then (6.19) equals

$$-\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{2}-S\right)\Gamma\left(S\right)\Gamma\left(1-S\right)\left(A^{+}\left(S,t\right)+A^{-}\left(S,t\right)\right)\sin\pi s_2 \quad (6.21)$$

with

$$A^{+}\left(S,t\right):=\frac{\Gamma\left(S\pm it_1+it_2\right)\Gamma\left(\frac{1}{4}+it_2\pm it\right)}{\sin\pi\left(2it_2\right)}\phi^{+}_{i\left(\frac{1}{2}-S\right)}\left(t\right),$$

$$A^{-}\left(S,t\right):=\frac{\Gamma\left(S\pm it_1-it_2\right)\Gamma\left(\frac{1}{4}-it_2\pm it\right)}{\sin\pi\left(-2it_2\right)}\phi^{-}_{i\left(\frac{1}{2}-S\right)}\left(t\right),$$

and (6.20) equals

$$-\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{2}-S\right)\Gamma\left(S\right)\Gamma\left(1-S\right)\frac{\sin\pi s_2}{\sin\pi s_1}\left(B^{+}\left(S,t\right)+B^{-}\left(S,t\right)\right) \quad (6.22)$$

with

$$B^{+}\left(S,t\right):=\frac{\Gamma\left(S\pm it_1+it_2\right)\Gamma\left(\frac{1}{4}+it_2\pm it\right)}{\sin\pi\left(2it_2\right)}\left(\sin\pi\left(\frac{1}{2}-it_2-S\right)\right)\phi^{+}_{i\left(\frac{1}{2}-S\right)}\left(t\right),$$

$$B^{-}\left(S,t\right):=\frac{\Gamma\left(S\pm it_1-it_2\right)\Gamma\left(\frac{1}{4}-it_2\pm it\right)}{\sin\pi\left(-2it_2\right)}\left(\sin\pi\left(\frac{1}{2}+it_2-S\right)\right)\phi^{-}_{i\left(\frac{1}{2}-S\right)}\left(t\right).$$

Proof. We see by (3.26) and (3.27) that $F_1(s)$ equals

$$\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(-\frac{1}{4}-s\pm it\right)\Gamma\left(S\right)\Gamma\left(1+s-S\right)\Gamma\left(\frac{1}{2}-S\right)\Gamma\left(S-s-\frac{1}{2}\right)$$

times

$$\phi_{i\left(\frac{1+s}{2}-S\right)}\left(t;\frac{1}{4},\frac{1}{4}+it_1,\frac{1}{4}-it_1,\frac{5}{4}+s\right). \tag{6.23}$$

Similarly, we see that

$$\frac{1}{2\pi i}\int_{\left(-\frac{\epsilon}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\pm it_1+A\right)\Gamma\left(\frac{1}{4}+A\right)\Gamma\left(-\frac{1}{4}+A-s\right)\Gamma\left(-A\pm it\right)}{\Gamma\left(\frac{3}{4}+A-S\right)\Gamma\left(-\frac{1}{4}+A+S-s\right)}dA \tag{6.24}$$

equals

$$\Gamma\left(\frac{1}{4}\pm it\pm it_1\right)\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(-\frac{1}{4}\pm it-s\right)$$

times

$$\phi_{i\left(\frac{1}{4}+\frac{s}{2}\right)}\left(t_1;\frac{1}{4}+it,\frac{1}{2}-S,-\frac{1}{2}+S-s,\frac{3}{4}+it\right). \tag{6.25}$$

We see by (3.25) and by the symmetry of the Wilson function in its parameters (see the sentence above (3.25)) that (6.23) equals (6.25). Hence $F_1(s)$ equals

$$\frac{\Gamma\left(S\right)\Gamma\left(1+s-S\right)\Gamma\left(\frac{1}{2}-S\right)\Gamma\left(S-s-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\pm it\pm it_1\right)}$$

times (6.24). This means that (6.19) equals

$$\frac{1}{2\pi i}\frac{\Gamma\left(S\right)\Gamma\left(\frac{1}{2}-S\right)}{\Gamma\left(\frac{1}{4}\pm it\pm it_1\right)}\int_{\left(-\frac{\epsilon}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\pm it_1+A\right)\Gamma\left(\frac{1}{4}+A\right)\Gamma\left(-A\pm it\right)}{\Gamma\left(\frac{3}{4}+A-S\right)}L_1(A)dA, \tag{6.26}$$

where $L_1(A)$ denotes

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)}\frac{\Gamma\left(-\frac{1}{2}\pm it_1+S-s\right)\Gamma\left(\frac{1}{2}\pm it_2+s\right)\Gamma\left(1+s-S\right)\Gamma\left(-\frac{1}{4}+A-s\right)}{\Gamma\left(1+s\right)\Gamma\left(-\frac{1}{4}+A-s+S\right)}ds.$$

We see by (3.26) and (3.27) that $F_2(s)$ equals

$$\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{3}{4}-S+s\pm it\right)\Gamma\left(S\right)\Gamma\left(-s\right)\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-S\right)$$

times

$$\phi_{i\left(\frac{s+S}{2}\right)}\left(t;\frac{1}{4},\frac{1}{4}+it_1,\frac{1}{4}-it_1,\frac{1}{4}+S-s\right). \tag{6.27}$$

Similarly, we see that

$$\frac{1}{2\pi i}\int_{\left(-\frac{\epsilon}{4}\right)}\frac{\Gamma\left(\frac{1}{4}\pm it_1+A\right)\Gamma\left(\frac{1}{4}+A\right)\Gamma\left(\frac{3}{4}+A-S+s\right)\Gamma\left(-A\pm it\right)}{\Gamma\left(\frac{3}{4}+A-S\right)\Gamma\left(\frac{3}{4}+A+s\right)}dA \tag{6.28}$$

equals

$$\Gamma\left(\frac{1}{4} \pm it \pm it_1\right)\Gamma\left(\frac{1}{4} \pm it\right)\Gamma\left(\frac{3}{4} \pm it + s - S\right)$$

times

$$\phi_{i\left(\frac{1}{4}+\frac{s-S}{2}\right)}\left(t_1; \frac{1}{4} + it, \frac{1}{2} - S, \frac{1}{2} + s, \frac{3}{4} + it\right). \tag{6.29}$$

We see again by (3.25) and by the symmetry of the Wilson function in its parameters that (6.27) equals (6.29). Hence $F_2(s)$ equals

$$\frac{\Gamma(S)\,\Gamma(-s)\,\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(\frac{1}{2}-S\right)}{\Gamma\left(\frac{1}{4} \pm it \pm it_1\right)}$$

times (6.28). This means that (6.20) equals

$$\frac{1}{2\pi i}\frac{\sin \pi s_2}{\sin \pi s_1}\frac{\Gamma(S)\,\Gamma\left(\frac{1}{2}-S\right)}{\Gamma\left(\frac{1}{4} \pm it \pm it_1\right)}\int_{\left(-\frac{\epsilon}{4}\right)} \frac{\Gamma\left(\frac{1}{4} \pm it_1 + A\right)\Gamma\left(\frac{1}{4}+A\right)\Gamma(-A \pm it)}{\Gamma\left(\frac{3}{4}+A-S\right)} L_2(A)\,dA, \tag{6.30}$$

where $L_2(A)$ denotes

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{4}-\frac{\epsilon}{2}\right)} \frac{\Gamma\left(-\frac{1}{2} \pm it_2 + S - s\right)\Gamma\left(\frac{1}{2} \pm it_1 + s\right)\Gamma(1+s-S)\,\Gamma\left(\frac{3}{4}+A-S+s\right)}{\Gamma\left(\frac{3}{4}+A+s\right)\Gamma(1+s)}ds.$$

Applying Lemma 6.1 with $n = 0$, $B = A + \frac{3}{4}$ we see for $\operatorname{Re} A = -\frac{\epsilon}{4}$ that $L_1(A)$ equals (6.3), and $\frac{\sin \pi s_2}{\sin \pi s_1} L_2(A)$ equals (6.4) with $n = 0$, $B = A + \frac{3}{4}$ there.

We see by (6.5), (3.26) and (3.27) (using again that the Wilson function is symmetric in the parameters a, b, c and $1-d$) that if $n = 0$, $B = A + \frac{3}{4}$, $\operatorname{Re} A = -\frac{\epsilon}{4}$, then Q^+ equals

$$\frac{1}{\Gamma\left(\frac{1}{2} \pm ix\right)\Gamma\left(\frac{1}{2} \pm ix - it_1 + it_2\right)\Gamma\left(\frac{1}{2}+it_1\right)\Gamma\left(\frac{1}{2}+it_2\right)\Gamma\left(\frac{1}{4}+A+it_1\right)\Gamma\left(\frac{1}{4}+A+it_2\right)}$$

times

$$\frac{1}{2\pi i}\int_{(d)} \frac{\Gamma\left(\frac{1}{2} \pm ix + R\right)\Gamma\left(\frac{1}{2}+it_1+R\right)\Gamma\left(\frac{1}{4}+A+it_1+R\right)\Gamma(-R)\,\Gamma(it_2-it_1-R)}{\Gamma\left(\frac{3}{4}+A+R\right)\Gamma(1+it_1+it_2+R)}dR$$

with $d = -1/8$, where we write $S = \frac{1}{2} + ix$.

Observe that

$$\frac{1}{2\pi i}\int_{\left(-\frac{\epsilon}{4}\right)} \frac{\Gamma\left(\frac{1}{4}-it_1+A\right)\Gamma\left(\frac{1}{4}+A\right)\Gamma(-A \pm it)\,\Gamma\left(\frac{1}{4}+A+it_1+R\right)}{\Gamma\left(\frac{3}{4}+A+R\right)}dA \tag{6.31}$$

equals

$$\frac{\Gamma\left(\frac{1}{4}-it_1\pm it\right)\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{4}+it_1+R\pm it\right)}{\Gamma\left(\frac{1}{2}+R\right)\Gamma\left(\frac{1}{2}+it_1+R\right)\Gamma\left(\frac{1}{2}-it_1\right)} \tag{6.32}$$

by (3.16) and (3.17).

We claim that (6.19) equals

$$-\frac{\Gamma\left(S+it_1+it_2\right)\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{2}-S\right)}{\Gamma\left(1-S-it_1+it_2\right)\sin\pi\left(2it_2\right)\Gamma\left(\frac{1}{4}\pm it+it_1\right)}\sin\pi s_2 \tag{6.33}$$

times

$$\frac{1}{2\pi i}\int_{\left(-\frac{1}{8}\right)}\frac{\Gamma\left(S+R\right)\Gamma\left(1-S+R\right)\Gamma\left(\frac{1}{4}+it_1+R\pm it\right)\Gamma\left(-R\right)\Gamma\left(-it_1+it_2-R\right)}{\Gamma\left(1+it_1+it_2+R\right)\Gamma\left(\frac{1}{2}+R\right)}dR \tag{6.34}$$

plus the similar product obtained by writing $-t_2$ in place of t_2 in (6.33) and (6.34). Indeed, we can see it by (6.26), (6.31), (6.32), by the above-mentioned fact that $L_1(A)$ equals (6.3) writing $n=0$, $B=A+\frac{3}{4}$ there, by the above expression for Q^+, and by the fact that Q^- and C_1^- are obtained from Q^+ and C_1^+ by writing $-t_2$ in place of t_2.

Similarly, but using (6.30) in place of (6.26), we see that (6.20) equals

$$-\frac{\Gamma\left(S+it_1+it_2\right)\Gamma\left(\frac{1}{4}\pm it\right)\Gamma\left(\frac{1}{2}-S\right)}{\Gamma\left(1-S-it_1+it_2\right)\sin\pi\left(2it_2\right)\Gamma\left(\frac{1}{4}\pm it+it_1\right)}\frac{\sin\pi s_2}{\sin\pi s_1}\sin\pi\left(\frac{1}{2}-it_2-S\right) \tag{6.35}$$

times (6.34) plus the similar product obtained by writing $-t_2$ in place of t_2 in (6.35) and (6.34).

By (3.26) and (3.27) we see that (6.34) equals

$$\Gamma\left(S\right)\Gamma\left(1-S\right)\Gamma\left(S-it_1+it_2\right)\Gamma\left(1-S-it_1+it_2\right)\Gamma\left(\frac{1}{4}+it_1\pm it\right)\Gamma\left(\frac{1}{4}+it_2\pm it\right)$$

times

$$\phi_{i\left(\frac{1}{2}-S\right)}\left(t;\frac{3}{4}+it_2,\frac{1}{4}+it_1,\frac{1}{4}-it_1,\frac{3}{4}-it_2\right).$$

This proves the lemma.

6.2. An estimation for $H_\chi\left(S\right)$. During the proof of Theorem 1.1 we need an upper bound for $H_\chi\left(S\right)$ defined in Theorem 1.1 not only for an individual χ but also for a function series χ_n, assuming a universal upper bound for every $|\chi_n|$. The most important aspect of the lemma below is that the estimate (6.36) depends only on the upper bound M for $|\chi|$.

LEMMA 6.3. *(i) Recall the notations $\phi_\lambda^+(x)$, $\phi_\lambda^-(x)$ from Section 1.2. There is an absolute constant $C>0$ such that we have*

$$\left|\phi^+_{i\left(\frac{1}{2}-S\right)}(t)\right| + \left|\phi^-_{i\left(\frac{1}{2}-S\right)}(t)\right| \ll e^{\pi(|S|+|t|)}\left(1+|S|\right)^C\left(1+|t|\right)^C$$

with an implied absolute constant for every S with $-1 \le \operatorname{Re} S \le 2$ and for every real t.
(ii) Let $\beta>0$ be a given number and let M be a given nonnegative function on $[0,\infty)$ satisfying that for every fixed $K>0$ the function $M(R)e^{-\pi R}(1+R)^K$ is bounded on $[0,\infty)$. Then, if χ is any even holomorphic function on the strip $|\operatorname{Im} z|<\beta$ with $|\chi(z)|\le M(|z|)$ on this strip, then for every $0<B<\frac{1}{2}+2\beta$ we have that $H_\chi(S)$ is regular in the strip $\frac{1}{2}-\frac{1}{100}\le \operatorname{Re} S\le \frac{1}{2}$, and for every S in this strip we have

$$\Gamma^2(1-S)\,H_\chi(S) \ll_{B,M} (1+|S|)^{-B}. \tag{6.36}$$

Proof. To show (i) note that for every fixed real t the functions $\phi^+_{i\left(\frac{1}{2}-S\right)}(t)$ and $\phi^-_{i\left(\frac{1}{2}-S\right)}(t)$ are entire in S. Combining this fact with (3.26) and (3.27) we get (i) by trivial estimates. The regularity statement in (ii) follows then at once from (i) and from the definition.
By the definition of $N^+(S,t)$ in Section 1, and by (3.26), (3.27) we see for any real t and for any S with $\frac{1}{2}-\frac{1}{100}\le \operatorname{Re} S\le\frac{1}{2}$ that

$$\frac{\Gamma(S)\,\Gamma(1-S)\,N^+(S,t)}{\Gamma(S+it_1\pm it_2)}$$

equals

$$\frac{\sin\pi(S+it_1-it_2)}{\pi\sin\pi(2it_2)\,\Gamma\left(\frac{1}{4}+it_1\pm it\right)}\left(\sin\pi s_1+\sin\pi\left(\frac{1}{2}-it_2-S\right)\right) \tag{6.37}$$

times

$$\frac{1}{2\pi i}\int_{(-1/8)}\frac{\Gamma(S+R)\,\Gamma(1-S+R)\,\Gamma\left(\frac{1}{4}+it_1+R\pm it\right)\Gamma(-R)\,\Gamma(-it_1+it_2-R)}{\Gamma(1+it_1+it_2+R)\,\Gamma\left(\frac{1}{2}+R\right)}dR. \tag{6.38}$$

We get

$$\frac{\Gamma(S)\,\Gamma(1-S)\,N^-(S,t)}{\Gamma(S+it_1\pm it_2)}$$

by writing $-t_2$ in place of t_2 in (6.37) and (6.38).

We now show (6.36). It is clear that taking the term $\sin \pi s_1$ from the bracket in (6.37) we get expressions acceptable in (6.36). On the other hand, we have that

$$\sin \pi \left(S + it_1 - it_2\right) \sin \pi \left(\frac{1}{2} - it_2 - S\right) = \frac{\cos \pi \left(2S + it_1 - \frac{1}{2}\right)}{2} - \frac{\cos \pi \left(-2it_2 + it_1 + \frac{1}{2}\right)}{2}$$

by [G-R], p. 29, 1.313.5. Taking the second term from here in (6.37) gives again an acceptable contribution in (6.36); the first term is independent of t_2. So defining

$$G(R) := \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{4} \pm it\right) \Gamma\left(\frac{1}{4} - it_1 \pm it\right) \Gamma\left(\frac{1}{4} + it_1 + R \pm it\right)}{\Gamma\left(\pm 2it\right)} \chi(t) dt, \tag{6.39}$$

it is enough to prove for $\frac{1}{2} - \frac{1}{100} \leq \operatorname{Re} S \leq \frac{1}{2}$ that the difference of

$$\frac{1}{2\pi i} \int_{(-1/8)} \frac{\Gamma\left(S + R\right) \Gamma\left(1 - S + R\right) G\left(R\right) \Gamma\left(-R\right) \Gamma\left(-it_1 + it_2 - R\right)}{\Gamma\left(1 + it_1 + it_2 + R\right) \Gamma\left(\frac{1}{2} + R\right)} dR$$

and the same integral with $-t_2$ in place of t_2 is $\ll_{B,M} e^{-\pi|S|} \left(1 + |S|\right)^{-B}$. We claim that

$$\frac{\Gamma\left(-it_1 + it_2 - R\right) \Gamma\left(1 + it_1 - it_2 + R\right)}{\Gamma\left(1 + it_1 + it_2 + R\right) \Gamma\left(-it_1 - it_2 - R\right)} - 1$$

equals

$$\frac{\Gamma\left(-it_1 + it_2 - R\right) \Gamma\left(1 + it_1 - it_2 + R\right) \Gamma\left(\frac{1}{2} + it_2\right) \Gamma\left(\frac{1}{2} - it_2\right)}{\Gamma\left(\frac{1}{2} + it_1 + R\right) \Gamma\left(\frac{1}{2} - it_1 - R\right) \Gamma\left(1 + 2it_2\right) \Gamma\left(-2it_2\right)}.$$

This is true because the difference of these two functions is a bounded entire function of R which vanishes at $R = \frac{1}{2} - it_1$. Using this identity we see that it is enough to prove for $\frac{1}{2} - \frac{1}{100} \leq \operatorname{Re} S \leq \frac{1}{2}$ that

$$\frac{1}{2\pi i} \int_{(-1/8)} \frac{\Gamma\left(S + R\right) \Gamma\left(1 - S + R\right) G\left(R\right) \Gamma\left(-R\right) \Gamma\left(-it_1 \pm it_2 - R\right)}{\Gamma\left(\frac{1}{2} + it_1 + R\right) \Gamma\left(\frac{1}{2} - it_1 - R\right) \Gamma\left(\frac{1}{2} + R\right)} dR$$

is $\ll_{B,M} e^{-\pi|S|} \left(1 + |S|\right)^{-B}$. By shifting the R-integration to the left, we see then that it is enough to prove that

$$H\left(R\right) := \frac{G\left(R\right)}{\Gamma\left(\frac{1}{2} + it_1 + R\right) \Gamma\left(\frac{1}{2} + R\right)}$$

is holomorphic for $\operatorname{Re} R > -\frac{1}{4} - \beta$ and satisfies

$$\frac{G\left(R\right)}{\Gamma\left(\frac{1}{2} + it_1 + R\right) \Gamma\left(\frac{1}{2} + R\right)} \ll_{K,\rho,M} e^{\pi|R|} \left(1 + |R|\right)^{-K}$$

for every $K > 0$ and $0 \le \rho < \beta$ on the strip $-\frac{1}{4} - \rho \le \operatorname{Re} R \le 1$. We now prove this statement. It is clear that we may assume that $\frac{1}{4} + \rho$ and $\frac{1}{4} - \rho$ are not integers.
Let b be a large positive integer. There are constants $c_{a,b}$ such that

$$\frac{\Gamma(z)}{\Gamma(z+b)} = \sum_{a=0}^{b-1} c_{a,b} (z+a)^{-1}.$$

Applying it for $z = \frac{1}{4} + it_1 + R \pm it$, we see that $\Gamma\left(\frac{1}{4} + it_1 + R \pm it\right)$ equals

$$\Gamma\left(\frac{1}{4} + it_1 + R \pm it + b\right) \sum_{0 \le a_1, a_2 \le b-1} \frac{c_{a_1,b} c_{a_2,b}}{\left(\frac{1}{4} + it_1 + R + it + a_1\right)\left(\frac{1}{4} + it_1 + R - it + a_2\right)}.$$

We use that

$$\frac{-2it + a_2 - a_1}{\left(\frac{1}{4} + it_1 + R + it + a_1\right)\left(\frac{1}{4} + it_1 + R - it + a_2\right)}$$

equals

$$\frac{1}{\frac{1}{4} + it_1 + R + it + a_1} - \frac{1}{\frac{1}{4} + it_1 + R - it + a_2}, \tag{6.40}$$

and because of the presence of the factor $\frac{1}{\Gamma(\pm 2it)}$, shifting the line of integration in (6.39) to $\operatorname{Im} t = \pm\rho$ (the minus sign is used in the case of the first term in (6.40), and the plus sign in the case of the second term), we get such an expression for $G(R)$ which proves the above statement for the function $H(R)$. We cross some poles when we shift the t-integration, but the residues also give holomorphic expressions for $H(R)$ in the required strip, because of the factor $\Gamma\left(\frac{1}{2} + it_1 + R\right)\Gamma\left(\frac{1}{2} + R\right)$ in the denominator of $H(R)$. The lemma is proved.

Important notations

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