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A RELATION BETWEEN TRIPLE PRODUCTS OF WEIGHT 0 AND WEIGHT $\frac{1}{2}$ CUSP FORMS

BY

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ABSTRACT

Let u and U be two Maass–Hecke cusp forms of weight 0 for the full modular group. In this paper we express the inner product of u and $|U|^2$ by a finite linear combination of triple products involving Maass forms of weight 1/2.

1. Introduction

Let u and U be two Maass–Hecke cusp forms of weight 0 for the full modular group $\mathrm{SL}(2,\mathbf{Z})$, i.e., $u(z)$ and $U(z)$ are $\mathrm{SL}(2,\mathbf{Z})$-invariant functions on the open upper half plane H vanishing exponentially as $\mathrm{Im} z \to \infty$, and u and U are eigenfunctions of the hyperbolic Laplace operator of weight 0 and of all the Hecke operators. The triple products

$$\int_{D_1} |U(z)|^2 u(z) d\mu_z, \tag{1.1}$$

where D_1 is a fundamental domain of $\mathrm{SL}(2,\mathbf{Z})$ in H and μ is the invariant measure, are subjects of intensive research in several directions. We just mention a few papers in the next paragraph.

The famous Quantum Unique Ergodicity conjecture of Rudnick and Sarnak, stating that (1.1) tends to 0 if u is fixed and the Laplace eigenvalue of U tends

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to $-\infty$, was very recently proved by Soundararajan (see [So]) proving a theorem on multiplicative functions and using earlier ergodic theoretical work of Lindenstrauss ([L]). In the opposite case when U is fixed, the exact exponential decay of (1.1) depending on the Laplace-eigenvalue of u was determined by Sarnak in [Sa], and even more precise upper bounds were given later by Bernstein and Reznikov (see, e.g., [B-R]) applying tools from representation theory. A very important identity was proved by Watson ([W]), relating the square of the absolute value of (1.1) to the central critical value of an automorphic L-function of degree 8.

In the present paper we prove an identity for (1.1) itself (and not for its absolute square), relating it to weight 1/2 Maass forms in the following way. We show that (1.1) is a finite linear combination of triple products involving U, the classical θ-function and Maass cusp forms of weight 1/2 whose Shimura lift equals u.

In the case when u is an Eisenstein series, a similar expression for (1.1) is well-known; we will explain it in Remark 1. So the meaning of our new identity is that (1.1) is closely related to weight 1/2 Maass forms also in the case when u is a cusp form.

Before stating the theorem precisely, we give the necessary definitions. We write

$$\Gamma_0(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbf{Z}) : \ c \equiv 0 \pmod{m} \right\}.$$

Let D_1 be a fundamental domain of $\mathrm{SL}(2,\mathbf{Z})$, and let D_4 be a fundamental domain of $\Gamma_0(4)$ on H. Define

$$d\mu_z = \frac{dxdy}{y^2};$$

this is the $\mathrm{SL}(2,\mathbf{R})$-invariant measure on H. Introduce the hyperbolic Laplace operator of weight l:

$$\Delta_l := y^2\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) - ily\frac{\partial}{\partial x}.$$

For a complex number $z \neq 0$ we set its argument in $(-\pi, \pi]$, and write $\log z = \log|z| + i \arg z$, where $\log|z|$ is real. We define the power z^s for any $s \in \mathbf{C}$ by $z^s = e^{s \log z}$. We write $e(x) = e^{2\pi i x}$.

For $z \in H$ we define

$$B_0(z) := (\mathrm{Im} z)^{\frac{1}{4}}\theta(z) = (\mathrm{Im} z)^{\frac{1}{4}} \sum_{m=-\infty}^{\infty} e(m^2 z). \tag{1.2}$$

Then

$$B_0(\gamma z) = \nu(\gamma)\Big(\frac{j_\gamma(z)}{|j_\gamma(z)|}\Big)^{1/2} B_0(z) \quad \text{for } \gamma \in \Gamma_0(4)$$

with a well-known multiplier system ν, where, for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbf{R})$ we write $j_\gamma(z) = cz + d$.

Let $l = \frac{1}{2}$ or $l = 0$. We say that a function f on H is a **Maass form** of weight l for $\Gamma = \mathrm{SL}(2,\mathbf{Z})$ or $\Gamma_0(4)$ (but, if $l = \frac{1}{2}$, we can take only $\Gamma = \Gamma_0(4)$), if f is an eigenfunction on H of the operator Δ_l, it satisfies, for every $z \in H$ and $\gamma \in \Gamma$, the transformation formula $f(\gamma z) = f(z)$ in the case $l = 0$, and

$$f(\gamma z) = \nu(\gamma)\Big(\frac{j_\gamma(z)}{|j_\gamma(z)|}\Big)^{\frac{1}{2}} f(z)$$

in the case $l = \frac{1}{2}$, and f has at most polynomial growth in cusps. If f has exponential decay at cusps, it is called a **Maass cusp form.** If f is a Maass form of weight l and $\Delta_l f = s(s-1)f$ with some $\mathrm{Re}s \geq \frac{1}{2}$, $s = \frac{1}{2} + it$, then one has the Fourier expansion

$$f(z) = c_f(y) + \sum_{m \neq 0} \rho_f(m) W_{\frac{l}{2}\mathrm{sgn}(m),it}\left(4\pi\,|m|\,y\right) e\left(mx\right) \tag{1.3}$$

for $z = x + iy \in H$, where $W_{\alpha,\beta}$ is the Whittaker function (see [G-R]), $c_f(y)$ is a linear combination of y^s and y^{1-s}, and $c_f(y) = 0$, if f is a cusp form; $\rho_f(m)$ is called the mth Fourier coefficient of f.

A very important tool in the paper will be the Theorem of [K-S]. Let V and the operator L have the same meaning as on p. 195 of [K-S], and let V^+ be the subspace of V with L-eigenvalue 1. As on p. 224 of [K-S], let F_j $(j = 1, 2, \ldots)$ be an orthonormal basis of V^+ consisting of common eigenfunctions of $\Delta_{\frac{1}{2}}$ and the Hecke operators (of weight $\frac{1}{2}$) T_{p^2}, $p \neq 2$ (p is a prime; see also [K-S] for the definition of these operators). The F_j's are cusp forms of weight $\frac{1}{2}$ for the group $\Gamma_0(4)$. Denote the Fourier coefficients of F_j by $\rho_j(m)$, i.e., $\rho_j(m) = \rho_{F_j}(m)$.

Introduce the weight 0 Hecke operators for every positive integer n:

$$\left(H_n F\right)(z) = \frac{1}{\sqrt{n}} \sum_{ad=n, b \bmod d} F\Big(\frac{az+b}{d}\Big),$$

where a and d run over positive integers.

The Shimura lift $\mathrm{Shim}F_j$ in the case $\rho_j(1) \neq 0$ is defined in [K-S], pp. 196–197. It is a Maass cusp form of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$, which is a simultaneous Hecke eigenform (i.e., an eigenform of every H_n), even and Hecke normalized (i.e., for its Fourier coefficients $a(n)$ we have $a(1)=1$ and $a(n)=a(-n)$).

THEOREM 1.1: *Let u and U be cusp forms and simultaneous Hecke eigenforms of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$ such that*

$$\int_{D_1} |u(z)|^2 \, d\mu_z = 1.$$

Assume that u is even and write $\phi = u/\rho_u(1)$. Then

$$\overline{\rho_u(1)} \int_{D_1} |U(z)|^2 \, u(z) d\mu_z$$

equals

$$\sqrt{2}\pi^{1/4}\overline{\rho_U(1)} \int_{D_4} \overline{B_0(z)} U(4z) \Bigg(\sum_{\mathrm{Shim}F_j=\phi} \overline{\rho_j(1)} F_j(z) \Bigg) d\mu_z. \tag{1.4}$$

Remark 1: If U is as in the Theorem, but $u(z)=E(z,s)$ (an Eisenstein series of weight 0), then (1.1) is a Rankin–Selberg integral, hence it essentially equals (i.e., apart from a well-understood factor) $L(U\otimes U,s)$, where this is the Rankin–Selberg convolution L-function; see [I], (8.10). It is also well-known that if ζ is the Riemann zetafunction, then the quotient $L(U\otimes U,s)/\zeta(s)$ (this quotient is closely related to the symmetric square L-function of U) essentially equals the triple product of U, the classical θ-function, and an Eisenstein series of weight 1/2; see formulas (13.54) and (13.60) of [I2], at least for the completely similar case when U is a holomorphic cusp form. This argument shows that the Eisenstein series analogue of our Theorem was already known.

Remark 2: The sum in (1.4) is finite, and it may well be that it is in fact a one-element sum, see remark (a) on p. 197 of [K-S].

We now give a brief sketch of the proof. We will give two different expressions for an integral

$$\int_{D_4} \overline{B_0(z)} \Bigg(\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} g(\mathrm{Im}\gamma z) e(q\mathrm{Re}\gamma z) \Bigg) f(z) d\mu_z, \tag{1.5}$$

where, using the notations of the Theorem,

$$f(z) = \sum_{\mathrm{Shim} F_j = \phi} \overline{\rho_j(1)} F_j(z), \tag{1.6}$$

g is a smooth test function on $(0, \infty)$ and q is a positive integer. The first expression for (1.5) is a linear combination of triple products involving B_0, f and weight 0 Maass forms, and it is obtained by considering the spectral expansion of the Poincaré series in (1.5). The other expression is obtained by unfolding (1.5), and it has the form

$$\sum_{m \in \mathbf{Z}, m^2 - q \neq 0} \rho_f(m^2 - q) T_g(m^2 - q) \tag{1.7}$$

with some smooth function T_g. Now, we will combine a lemma of ours (see Lemma 3.2 below, first occurring in a slightly different form in [B2]) with the Theorem of [K-S] to express (1.7) as a linear combination of triple products of the form (1.1). The equality of the first expression for (1.5) and this new expression for (1.7) is a sum identity, and it turns out that the same test function appears on both sides. Hence, localizing this sum identity, we get the theorem. Of course there are many details related to the involved function transforms and to oldforms and newforms on $\Gamma_0(4)$.

The structure of the paper is the following. After introducing some notations and quoting some basic facts in Section 2, we prove our most important lemmas in Section 3. The proof of the theorem is completed in Section 4, but a few necessary lemmas are proved only later, in Sections 5 and 6 (on automorphic functions and on function transforms, respectively).

Our paper has two appendices. The reason for writing Appendix 1 is the following. There are two constants in the theorem of [K-S], one for the $d > 0$ case and another for the $d < 0$ case. In our proof the value of the quotient of these two constants is important. We observed that the constant stated in the $d < 0$ case in [K-S] is not correct. We prove the theorem of [K-S] with correct constants in Appendix 1. In fact we just give a modification of the proof in [K-S] at a critical point, otherwise the proof remains the same.

In Appendix 2 we prove the precise exponential decay of a certain triple product involving two half-integral weight forms (one of them is B_0 defined above). Similar results are well-known today, but since we have not found the proof of the needed result in the literature, we present a proof in Appendix 2.

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2. Further notations and preliminaries

Let $\{U_l(z): l \geq 1\}$ be a complete orthonormal system of cusp forms of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$; let $\Delta_0 U_l = S_l(S_l-1)U_l$, where $S_l = \frac{1}{2}+i\tau_l$ and $\tau_l \geq 0$. Let $U_0(z)$ be the constant function normed in such a way that $\{U_l(z): l \geq 0\}$ is still an orthonormal system on D_1. We assume that every U_l is a simultaneous Hecke eigenform with eigenvalues $H_n U_l = \lambda_l(n) U_l$. We denote the Eisenstein series of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$ by $E(z,s)$ (at the cusp ∞; see [I], Chapter 3), and we denote the H_n-eigenvalue of $E\left(z,\frac{1}{2}+it\right)$ by $\eta_t(n)$, as in [I], p. 128.

Let V_m $(m \geq 1)$ be a complete orthonormal system of newforms of weight 0 for $\Gamma_0(2)$ and let W_r $(r \geq 1)$ be a complete orthonormal system of newforms of weight 0 for $\Gamma_0(4)$; see [I], pp. 128–129.

Let $\{u_j(z): j \geq 1\}$ be a complete orthonormal system of cusp forms of weight 0 for $\Gamma_0(4)$, let $\Delta_0 u_j = s_j(s_j-1)u_j$, where $s_j = \frac{1}{2}+it_j$ and $t_j \geq 0$. (We tacitly use the fact, just as above in the case of $\mathrm{SL}(2,\mathbf{Z})$, that there is no exceptional eigenvalue for $\Gamma_0(4)$.) Write $b_j(m) = \rho_{u_j}(m)$. Up to some point in our reasoning the concrete form of this orthonormal system will not be important, but at the end of the proof a special sysem obtained from U_l, V_m, W_r (see Lemma 5.7) will be used.

We introduce also notations for the Fourier coefficients of Eisenstein series for $\Gamma_0(4)$. If a is a cusp of $\Gamma_0(4)$ (i.e., a is ∞, 0 or $-\frac{1}{2}$), then the mth Fourier coefficient $(m \neq 0)$ of the Eisenstein series $E_a^{\Gamma_0(4)}(z,s)$ of weight 0 at the cusp a for $\Gamma_0(4)$ (see again [I], Chapter 3) is denoted by $\beta_{a,s}(m)$. For simplicity, we will write $E_a(z,s)$ in place of $E_a^{\Gamma_0(4)}(z,s)$; it can be distinguished from the Eisenstein series $E(z,s)$ for $\mathrm{SL}(2,\mathbf{Z})$, since in the $\Gamma_0(4)$ case we always denote the dependence on the cusp, but in the $\mathrm{SL}(2,\mathbf{Z})$ case we do not denote it.

We write

$$\Gamma_\infty = \{\gamma \in \mathrm{SL}(2,\mathbf{Z}): \gamma\infty = \infty\}.$$

We use the abbreviation $\Gamma(a \pm b) = \Gamma(a-b)\Gamma(a+b)$.

For $\phi \in [0,2\pi]$, write

$$k_\phi = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix}.$$

These matrices form the stability group of i in $\mathrm{SL}(2,\mathbf{R})$.

If $z \in H$ is arbitrary, let $T_z \in \mathrm{PSL}(2,\mathbf{R})$ be such that T_z is an upper triangular matrix and $T_z i = z$. It is clear that T_z is uniquely determined by z; for $z = x+iy$ we have explicitly

$$T_z = \begin{pmatrix} y^{\frac{1}{2}} & xy^{\frac{-1}{2}} \\ 0 & y^{\frac{-1}{2}} \end{pmatrix}.$$

We note the well-known fact that $B_0(z)$ is not just a Maass form of weight $1/2$, but it satisfies the additional transformation rule

$$B_0\left(\frac{-1}{4z}\right) = e\left(\frac{-1}{8}\right)\left(\frac{z}{|z|}\right)^{\frac{1}{2}} B_0(z). \tag{2.1}$$

If s is a nonzero integer with $s \equiv 0, 1 (\mathrm{mod}\ 4)$, let

$$Q_s = \left\{Q(X,Y) = AX^2 + BXY + CY^2 : A, B, C \in \mathbf{Z}, B^2 - 4AC = s\right\}.$$

As usual, we say that $Q, Q^\star \in Q_s$ are equivalent over $\mathrm{SL}(2,\mathbf{Z})$ if $Q^\star(X,Y) = Q(aX + bY, cX + dY)$ with $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbf{Z})$. Denote by Λ_s a complete set of representatives of these equivalence classes in Q_s.

If $Q(X,Y) = AX^2 + BXY + CY^2$ is an element of Q_s with $s > 0$, and z_1 and z_2 are the roots of $Az^2 + Bz + C$ (if $A = 0$, one root is ∞, otherwise these are real numbers), let l_Q be the noneuclidean line in H connecting z_1 and z_2, let

$$C(Q) = \{\gamma \in \mathrm{PSL}(2,\mathbf{Z}) : \gamma z_1 = z_1, \gamma z_2 = z_2\},$$

and finally let $C_Q = C(Q) \setminus l_Q$, i.e., we factorize by the action of C_Q.

If $Q(X,Y) = AX^2 + BXY + CY^2$ is an element of Q_s with $s < 0$, let z_Q be the unique root in H of $Az^2 + Bz + C$, let

$$C(Q) = \{\gamma \in \mathrm{PSL}(2,\mathbf{Z}) : \gamma z_Q = z_Q\},$$

and $M_Q = |C(Q)|$.

If n, t are integers, $n > 0$, let

$$\Gamma_{n,t} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbf{Z}, ad - bc = n, a + d = t \right\}.$$

The group $\mathrm{SL}(2,\mathbf{Z})$ acts on this set by conjugation. If $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{n,t}$, let $Q_\gamma(X,Y) = cX^2 + (d-a)XY - bY^2$. Then it is easy to see (cf. [B2], p. 119) that this is a one-to-one correspondence between $\Gamma_{n,t}$ and Q_s with $s = t^2 - 4n$, and also between the conjugacy classes of $\Gamma_{n,t}$ over $\mathrm{SL}(2,\mathbf{Z})$ and the $\mathrm{SL}(2,\mathbf{Z})$-equivalence classes of Q_s. We remark that if $s<0$, $\gamma\in\Gamma_{n,t}$, then $z_{Q\gamma}$ is the unique fixed point of γ in H.

As in [B1], for $\lambda<0$ define the two special functions $f_\lambda(\theta)$ and $g_\lambda(r)$ in the following way: $f_\lambda(\theta)$ is the unique even solution of the differential equation

$$f^{(2)}(\theta)=\frac{\lambda}{\cos^2\theta}f(\theta),\quad \theta\in\Big(-\frac{\pi}{2},\frac{\pi}{2}\Big)$$

with $f_\lambda(0)=1$, and $g_\lambda(r)$ ($r\in[0,\infty)$) is the unique solution of

$$g^{(2)}(r)+\frac{\cosh r}{\sinh r}g^{(1)}(r)=\lambda g(r)$$

with $g_\lambda(0)=1$.

3. Basic lemmas

We give two different expressions for a certain inner product on D_4 in the next lemma.

LEMMA 3.1: *Let g be a smooth function on $(0,\infty)$ such that g and every derivative of g vanishes faster than polynomially at ∞ and at 0, i.e., $g^{(j)}(Y)(Y+1/Y)^A$ is bounded on $(0,\infty)$ for every $j\geq 0$ and $A>0$. Let q be a positive integer. For $z=x+iy\in H$ let*

$$V(z)=\overline{B_0(z)}W(z),$$

where

$$W(z)=\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} g(\mathrm{Im}\gamma z)e(q\mathrm{Re}\gamma z).$$

Let f be a cusp form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with

$$\Delta_{\frac{1}{2}}f=s(s-1)f$$

for some $\mathrm{Re}s=\frac{1}{2}$, $s=\frac{1}{2}+it$. Then, on the one hand, $\int_{D_4}V(z)f(z)d\mu_z$ equals

$$\sum_{m\in\mathbf{Z},m^2-q\neq 0}\rho_f(m^2-q)T_g(m^2-q), \tag{3.1}$$

where for $0\neq r\in\mathbf{Z}$ we write

$$T_g(r)=T_{g,q,t}(r)=\int_0^\infty g(y)e^{-2\pi qy}y^{-\frac{7}{4}}W_{\frac{1}{4}\mathrm{sgn}(r),it}\left(4\pi\left|r\right|y\right)e^{-2\pi ry}dy.$$

On the other hand, writing

$$I_\tau = \int_0^\infty g(y) W_{0,i\tau}\left(4\pi q y\right) \frac{dy}{y^2},$$

$\int_{D_4} V(z) f(z) d\mu_z$ *equals the sum of*

$$\sum_{j=1}^{\infty} \overline{b_j(q)} \Big(\int_{D_4} u_j(z) f(z) \overline{B_0(z)} d\mu_z \Big) I_{t_j}$$

and

$$\sum_a \frac{1}{4\pi} \int_{-\infty}^{\infty} \overline{\beta_{a,\frac{1}{2}+ir}(q)} \Big(\int_{D_4} E_a\left(z, \frac{1}{2} + ir\right) f(z) \overline{B_0(z)} d\mu_z \Big) I_r dr.$$

Proof. The function $f(z)\overline{B_0(z)}$ is invariant under $\Gamma_0(4)$, and so

$$\int_{D_4} V(z) f(z) d\mu_z = \int_0^\infty \int_0^1 g(y) e(qx) f(x+iy) \overline{B_0(x+iy)} \frac{dxdy}{y^2}.$$

Taking into account the Fourier expansions of f and B_0 (formulas (1.3) and (1.2)) we obtain (3.1).

To prove the second statement, note that $\int_{D_4} V(z) f(z) d\mu_z$ equals the scalar product (on D_4, with respect to the measure $d\mu_z$)

$$\Big(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} g(\mathrm{Im}\gamma z) e(q \mathrm{Re}\gamma z), B_0(z)\overline{f(z)} \Big).$$

If u is a Maass form of weight 0 for $\Gamma_0(4)$ with $\Delta_0 u = S(S-1)u$ with $S = \frac{1}{2} + iT$, real T, then by unfolding we see that

$$\int_{D_4} \Big(\sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(4)} g(\mathrm{Im}\gamma z) e(q \mathrm{Re}\gamma z) \Big) \overline{u(z)} d\mu_z$$

equals

$$\overline{\rho_u(q)} \int_0^\infty g(y) W_{0,iT}\left(4\pi q y\right) \frac{dy}{y^2}.$$

By the spectral theorem (see [I]), noting that

$$\int_{D_4} f(z) \overline{B_0(z)} d\mu_z = 0,$$

we obtain the second statement, hence the lemma. ∎

The following lemma, together with the theorem of [K-S], will allow us to express the sum (3.1) as an inner product on D_1.

Lemma 3.2: *Let n, t be integers, $n > 0$, and write $s = t^2 - 4n$. Denote by m a continuous function on $[0, \infty)$ such that $m(u)(u+1)^A$ is bounded on $[0, \infty)$ for every $A > 0$. For $z, w \in H$ write*

$$m(z,w) = m\Big(\frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}\Big).$$

For $z \in H$ define

$$M_{n,t}(z) = \sum_{\gamma \in \Gamma_{n,t}} m\left(z, \gamma z\right).$$

Let u be a cusp form of weight 0 for $\mathrm{SL}(2, \mathbf{Z})$ with $\Delta_0 u = \lambda u$, $\lambda < 0$, and let $dS = |dz|/y$ be the hyperbolic arc length. Then, on the one hand, if $s > 0$, we have

$$\int_{D_1} M_{n,t}(z) u(z) d\mu_z = \left(\sum_{Q \in \Lambda_s} \int_{C_Q} u dS \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} m\left(\frac{s}{4n \cos^2 \theta} \right) f_\lambda(\theta) \frac{d\theta}{\cos^2 \theta}.$$

On the other hand, if $s < 0$, then

$$\int_{D_1} M_{n,t}(z) u(z) d\mu_z = \left(\sum_{Q \in \Lambda_s} \frac{2\pi}{M_Q} u\left(z_Q\right) \right) \int_0^\infty m\left(\frac{|s|}{4n} \sinh^2 r \right) g_\lambda(r) \sinh r dr.$$

If $s = 0$, then

$$\int_{D_1} M_{n,t}(z) u(z) d\mu_z = 0.$$

Proof. The case $s > 0$ is essentially proved as [B2], Lemma 2 (take $D = N = 1$ there, and take into account that $m(z, w)$ is defined differently on p. 108 of [B2] than here). The only differences are that in [B2] we assumed that m has compact support and $t > 0$. However, the same proof may be applied under our present conditions, because the convergence is assured also by this condition for m, and $t > 0$ may be achieved by taking $-\gamma$ for every $\gamma \in \Gamma_{n,t}$.

Consider the case $s < 0$. We partition $\Gamma_{n,t}$ into conjugacy classes over $\mathrm{SL}(2, \mathbf{Z})$, for $\gamma \in \Gamma_{n,t}$ let

$$[\gamma] = \left\{ \tau^{-1} \gamma \tau : \tau \in \mathrm{SL}(2, \mathbf{Z}) \right\}.$$

If, for any $\gamma \in \Gamma_{n,t}$, we write $C(\gamma) = \{\tau \in \mathrm{SL}(2, \mathbf{Z}) : \gamma\tau = \tau\gamma\}$ and

$$T_\gamma = \sum_{\delta \in [\gamma]} \int_{D_1} m\left(z, \delta z\right) u(z) d\mu_z,$$

then we have

$$T_\gamma = \int_{C(\gamma) \backslash H} m\left(z, \gamma z\right) u(z) d\mu_z.$$

If $s < 0$, γ has a unique fixed point z_γ in H, and

$$C(\gamma) = \{\tau \in \mathrm{SL}(2, \mathbf{Z}) :\ \tau z_\gamma = z_\gamma\}.$$

This is a finite set; it has an even number of elements (since $\tau \in C(\gamma)$ if and only if $-\tau \in C(\gamma)$). Let $|C_\gamma| = 2M_\gamma$. Choose $h \in \mathrm{SL}(2, \mathbf{R})$ such that $h(i) = z_\gamma$; then there is a $\phi_\gamma \in [0, 2\pi]$ such that

$$h^{-1}\gamma h z = k_{\phi_\gamma} z \tag{3.2}$$

for every $z \in H$. We get

$$T_\gamma = \frac{1}{M_\gamma} \int_H m\left(z, k_{\phi_\gamma} z\right) u(hz) d\mu_z.$$

Then, by the argument on pp. 326–327 of [B1] (see the part from line 13 of p. 326 to line 5 of p. 327) we obtain

$$T_\gamma = \frac{2\pi}{M_\gamma} u(z_\gamma) \int_0^\infty m\left(\left(\sin^2 \phi_\gamma\right) \sinh^2 r\right) g_\lambda(r) \sinh r dr.$$

It follows from (3.2) and $\gamma \in \Gamma_{n,t}$ that $|2\cos\phi_\gamma| = t/\sqrt{n}$, so $\sin^2 \phi_\gamma = |s|/(4n)$. By the remarks in Section 2 on the correspondence between $\Gamma_{n,t}$ and Q_s, we obtain the assertion for $s < 0$.

If $s = 0$, then any $\gamma \in \Gamma_{n,t}$ has a unique fixed point $z_\gamma \in \mathbf{Q} \cup \infty$, and

$$C(\gamma) := \{\tau \in \mathrm{SL}(2, \mathbf{Z}) :\ \gamma\tau = \tau\gamma\} = \{\tau \in \mathrm{SL}(2, \mathbf{Z}) :\ \tau z_\gamma = z_\gamma\}$$

is again valid. Choose $h \in \mathrm{SL}(2, \mathbf{Z})$ such that $h(\infty) = z_\gamma$. Then $h^{-1}C(\gamma)h = \Gamma_\infty$, and with some $r \in \mathbf{Z}$ we have

$$h^{-1}\gamma h z = z + r$$

for every $z \in H$. Then (T_γ is defined as in the $s < 0$ case)

$$T_\gamma = \int_{\Gamma_\infty \backslash H} m\left(hz, \gamma hz\right) u(hz) d\mu_z = \int_{\Gamma_\infty \backslash H} m\left(z, z + r\right) u(z) d\mu_z = 0,$$

since u is a cusp form. The lemma is proved. ∎

In the next lemma we give the basic properties of the functions g_A for which we apply Lemma 3.1. Recall the definition of $T_{g,q,t}(k)$ from Lemma 3.1.

Lemma 3.3: *Let $q > 0$ be an integer, and let t and t_0 be real numbers. Assume that $m(Y)$ is a holomorphic function in the half-plane* $\mathrm{Re} Y > 0$*, and*

$m(Y)(1+Y)^K$ is bounded in $Y \in (0,\infty)$ for every fixed $K>0$. Assume also that the function

$$m^*(s) = \int_0^\infty m(Y) Y^{s-1} dY$$

satisfies that $m^*(s)/\Gamma(s)$ is an entire function and, for arbitrary fixed real numbers $\sigma_1 < \sigma_2$, the function $m^*(s)e^{\pi/2|\mathrm{Im}s|}|\mathrm{s}|^K$ is bounded in the strip $\sigma_1 < \mathrm{Re}s < \sigma_2$ for every fixed $K>0$. Then, if $\mathrm{Re}A < 2\pi q$ and ($\sigma > 0$ is arbitrary) for $y>0$ we write

$$g(y) = g_A(y) = \frac{1}{2\pi i} e^{Ay} \int_{(\sigma)} y^s \left(4q\pi\right)^{s-\frac{3}{4}} \frac{m^*(s)}{\Gamma(s)} ds, \tag{3.3}$$

then we have

$$\mathrm{T}_{g,q,t}(k) = \left(\frac{k}{q}\right)^{\frac{3}{4}} \frac{1}{\pi^{\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta) m\Big(\frac{2\pi q - A}{4\pi q} + \frac{k}{q\cos^2\theta}\Big) \frac{d\theta}{\cos^2\theta} \tag{3.4}$$

for every integer $k>0$ with $\lambda = -\left(\frac{1}{4} + (2t)^2\right)$, and

$$\begin{aligned} \int_0^\infty g(Y) W_{0,it_0}(4\pi q Y) \frac{dY}{Y^2} \\ = (4\pi q)^{1/4} \int_0^\infty m\Big(Y + \frac{2\pi q - A}{4\pi q}\Big) F\Big(\frac{1}{2} + it_0, \frac{1}{2} - it_0, 1, -Y\Big) dY. \end{aligned} \tag{3.5}$$

If $A = 2\pi q$ and $g(y)$ is defined by (3.3) (with $\sigma > 0$), then we have

$$T_{g,q,t}(k) = 2\Big(\frac{|k|}{q}\Big)^{\frac{3}{4}} \int_0^\infty m\Big(\frac{|k|}{q}\sinh^2 r\Big) g_\lambda(r) \sinh r dr \tag{3.6}$$

for every integer $k<0$ with $\lambda = -\left(\frac{1}{4} + (2t)^2\right)$.

Proof. By analytic continuation (we use that $f_\lambda(\theta)$ is bounded ([B1], p. 336), we apply trivial upper bounds for g and m using the properties of $m^*(s)$, using the Mellin inversion (the case $l=0$ of (3.8) below) in the case of m, and we estimate the W-functions by [G-R], p. 1015, 9.222.1, the hypergeometric function by [G-R], p. 995, 9.111.1), it is enough to prove (3.4) and (3.5) for real $A < 2\pi q$, and we may assume that $2\pi q - A$ is smaller than any given positive number.

If $\frac{1}{4} < \mathrm{Re}s$, and

$$g(y) = y^s e^{Ay},$$

then for $k > 0$ we have, by [G-R], p. 816, formula 7.621.3, and p. 998, formula 9.131.1, that

$$T_g(k) = \frac{\Gamma(s-\frac14+it)\Gamma(s-\frac14-it)}{\Gamma(s)}(4\pi k)^{\frac34-s} \times F\Big(s-\frac14+it, s-\frac14-it, s, \frac{A-2\pi q}{4\pi k}\Big).$$

Then, if g is given by (3.3), (taking $\frac14 < \sigma$) we see that $T_g(k)$ equals

$$\frac{1}{2\pi i}\int_{(\sigma)}\Big(\frac{k}{q}\Big)^{\frac34-s} m^*(s)\frac{\Gamma(s-\frac14+it)\Gamma(s-\frac14-it)}{\Gamma^2(s)} \times F\Big(s-\frac14+it, s-\frac14-it, s, \frac{A-2\pi q}{4\pi k}\Big)ds.$$

We use the power series representation for the hypergeometric function (this is justified, if $2\pi q - A$ is small enough) and obtain

$$T_g(k) = \sum_{l=0}^{\infty}\frac{1}{l!}\Big(\frac{A-2\pi q}{4\pi k}\Big)^l\frac{1}{2\pi i} \times \int_{(\sigma)}\Big(\frac{k}{q}\Big)^{\frac34-s} m^*(s)\frac{\Gamma(s-\frac14+it+l)\Gamma(s-\frac14-it+l)}{\Gamma(s)\Gamma(s+l)}ds. \tag{3.7}$$

By the definition of $m^*(s)$ for any $l \geq 0$ and $\sigma > 0$ we get

$$(-1)^l m^{(l)}(Y) = \frac{1}{2\pi i}\int_{(\sigma)} Y^{-s-l} m^*(s)(s)_l ds. \tag{3.8}$$

Indeed, for $l = 0$ it is obtained by Mellin inversion, and then it follows for general l by differentiation. By [B1], Lemma 11 we have

$$\frac{\Gamma(s-\frac14+it)\Gamma(s-\frac14-it)}{\Gamma^2(s)} = \frac{1}{\pi^{\frac12}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta)\cos^{2s}\theta\frac{d\theta}{\cos^2\theta} \tag{3.9}$$

for $\frac12 < \mathrm{Re} s$ with $\lambda = -\big(\frac14 + (2t)^2\big)$. So assuming $\sigma > \frac12$, by the substitution $s + l \to s$ in (3.9), and using (3.8) and (3.9), we get for any $l \geq 0$ that

$$\frac{1}{2\pi i}\int_{(\sigma)}\Big(\frac{k}{q}\Big)^{\frac34-s} m^*(s)\frac{\Gamma(s-\frac14+it+l)\Gamma(s-\frac14-it+l)}{\Gamma(s)\Gamma(s+l)}ds$$

equals

$$\Big(\frac{k}{q}\Big)^{\frac34+l}(-1)^l\frac{1}{\pi^{\frac12}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta)m^{(l)}\Big(\frac{k}{q\cos^2\theta}\Big)\frac{d\theta}{\cos^2\theta},$$

and so, by (3.7),

$$T_g(k) = \left(\frac{k}{q}\right)^{\frac{3}{4}} \frac{1}{\pi^{\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta) \times \left(\sum_{l=0}^{\infty} \frac{1}{l!} \left(\frac{A-2\pi q}{4\pi k}\right)^l \left(\frac{k}{q}\right)^l (-1)^l m^{(l)}\left(\frac{k}{q\cos^2\theta}\right)\right) \frac{d\theta}{\cos^2\theta}.$$

Summing the Taylor series (observe that $k/(q\cos^2\theta) \geq 1/q$, but $2\pi q - A$ is small), we obtain (3.4).

If g is given by (3.3) and we take $\frac{1}{2} < \sigma < 1$, then, by [G-R], p. 816, formula 7.621.3 and p. 998, formula 9.131.1,

$$I := \int_0^\infty g(Y) W_{0,it_0}(4\pi q Y) \frac{dY}{Y^2}$$

equals

$$\frac{(4\pi q)^{1/4}}{2\pi i} \int_{(\sigma)} m^*(s) \frac{\Gamma(s-\frac{1}{2}+it_0)\Gamma(s-\frac{1}{2}-it_0)}{\Gamma^2(s)} \times F\left(s-\frac{1}{2}+it_0, s-\frac{1}{2}-it_0, s, \frac{2\pi q + A}{4\pi q}\right) ds.$$

Using [G-R], p. 998, formula 9.131.2, we then obtain

$$I = (4\pi q)^{1/4} \left(\frac{1}{\Gamma(\frac{1}{2}+it_0)\Gamma(\frac{1}{2}-it_0)} I_1 + I_2\right), \tag{3.10}$$

where I_1 equals

$$\frac{1}{2\pi i} \int_{(\sigma)} m^*(s) \frac{\Gamma(s-\frac{1}{2}+it_0)\Gamma(s-\frac{1}{2}-it_0)\Gamma(1-s)}{\Gamma(s)} \times F\left(s-\frac{1}{2}+it_0, s-\frac{1}{2}-it_0, s, \frac{2\pi q - A}{4\pi q}\right) ds$$

and I_2 equals

$$\frac{1}{2\pi i} \int_{(\sigma)} m^*(s) \frac{1}{s-1} \left(\frac{2\pi q - A}{4\pi q}\right)^{1-s} F\left(\frac{1}{2}+it_0, \frac{1}{2}-it_0, 2-s, \frac{2\pi q - A}{4\pi q}\right) ds.$$

In the case of I_1, we use the power series representation for the hypergeometric function (everything can be seen trivially to be absolutely convergent, if $2\pi q - A$ is small enough), and using that

$$\frac{1}{2\pi i} \int_{(\sigma)} \frac{\Gamma(s-\frac{1}{2}+it_0+k)\Gamma(s-\frac{1}{2}-it_0+k)\Gamma(1-s)}{\Gamma(s+k)} Y^{s-1} ds$$

equals

$$\frac{\Gamma(\frac{1}{2}+it_0+k)\Gamma(\frac{1}{2}-it_0+k)}{\Gamma(1+k)}F\Big(\frac{1}{2}+it_0+k,\frac{1}{2}-it_0+k,1+k,-Y\Big)$$

for every integer $k \geq 0$ and for every $Y > 0$ by [G-R], p. 647, formula 6.422.14, we get, by the definition of $m^*(s)$, that $I_1/(\Gamma(\frac{1}{2}+it_0)\Gamma(\frac{1}{2}-it_0))$ equals

(3.11)

$$\int_0^\infty m(Y)\Big(\sum_{k=0}^\infty c(t_0,k)\Big(\frac{2\pi q-A}{4\pi q}\Big)^k F\Big(\frac{1}{2}+it_0+k,\frac{1}{2}-it_0+k,1+k,-Y\Big)\Big)dY$$

with the abbreviation

$$c(t_0,k)=\frac{\left(\frac{1}{2}+it_0\right)_k\left(\frac{1}{2}-it_0\right)_k}{\Gamma^2(1+k)}.$$

Since, in general, differentiating in x,

$$F^{(k)}(\alpha,\beta,\gamma,x)=\frac{(\alpha)_k(\beta)_k}{(\gamma)_k}F(\alpha+k,\beta+k,\gamma+k,x),$$

so, summing the Taylor series, (3.11) equals

$$\int_0^\infty m(Y)F\Big(\frac{1}{2}+it_0,\frac{1}{2}-it_0,1,\frac{2\pi q-A}{4\pi q}-Y\Big)dY. \tag{3.12}$$

Similarly, in the case of I_2 we also use the power series representation for the hypergeometric function, and this is justified if $2\pi q - A$ is small enough. We use the identity

$$\int_0^1 \frac{(1-x)^k}{k!}x^{-s}dx=\frac{1}{(1-s)_{k+1}}$$

(which follows from [G-R], pp. 898–899, formulas 8.380.1 and 8.384.1) and (3.8), and we obtain that I_2 equals

$$-\int_0^1 m\Big(\frac{2\pi q-A}{4\pi q}x\Big)\sum_{k=0}^\infty \frac{\left(\frac{1}{2}+it_0\right)_k\left(\frac{1}{2}-it_0\right)_k}{k!}\frac{(1-x)^k}{k!}\Big(\frac{2\pi q-A}{4\pi q}\Big)^{k+1}dx,$$

which, by writing

$$Y=\frac{2\pi q-A}{4\pi q}x,$$

is the same as

$$-\int_0^{\frac{2\pi q-A}{4\pi q}} m\,(Y)\,F\Big(\frac{1}{2}+it_0,\frac{1}{2}-it_0,1,\frac{2\pi q-A}{4\pi q}-Y\Big)dY.$$

By this relation, by (3.10) and (3.12), we get (3.5).

Finally, to prove (3.6) we note that if $A = 2\pi q$ in (3.3) and $\frac{1}{2} < \sigma < 1$, then for $k < 0$ we have, by [G-R], p. 817, formula 12 (note that there is a misprint there, the factor $\Gamma(-\kappa-\mu)$ should be read as $\Gamma(-\kappa-\nu)$),

$$T_g(k) = \left(\frac{|k|}{q}\right)^{\frac{3}{4}} \frac{1}{2\pi i} \int_{(\sigma)} \left(\frac{|k|}{q}\right)^{-s} m^*(s) \frac{\Gamma(s-\frac{1}{4}+it)\Gamma(s-\frac{1}{4}-it)}{\Gamma(s)\Gamma(\frac{3}{4}+it)\Gamma(\frac{3}{4}-it)} \Gamma(1-s)ds.$$

Then, using [B1], Lemma 11, we have (using $\Gamma(s)\Gamma(1-s) = \pi(\sin \pi s)^{-1}$)

$$T_g(k) = \left(\frac{|k|}{q}\right)^{\frac{3}{4}} \frac{1}{\pi i} \int_{(\sigma)} \left(\frac{|k|}{q}\right)^{-s} m^*(s) G_\lambda(s) ds$$

for $k < 0$, where $\lambda = -\left(\frac{1}{4} + (2t)^2\right)$. Using [B1], Lemma 11, and (3.8) with $l = 0$, we get (3.6). The lemma is proved. ∎

4. Proof of the theorem

The combination of Lemmas 3.1 and 3.3 gives the following lemma.

LEMMA 4.1: *Let m be a function satisfying the conditions of Lemma 3.3 and Lemma 6.1. Let q be a positive integer, and let f be a fixed cusp form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with $\Delta_{\frac{1}{2}} f = s(s-1)f$ for some $\mathrm{Re} s = \frac{1}{2}$, $s = \frac{1}{2} + it$. Then, writing $\lambda = -\left(\frac{1}{4} + (2t)^2\right)$, the sum of*

$$\sum_{l \in \mathbf{Z}, l^2 < q} \rho_f(l^2 - q) 2 \left(\frac{|l^2 - q|}{q}\right)^{\frac{3}{4}} \int_0^\infty m\left(\frac{|l^2 - q|}{q} \sinh^2 r\right) g_\lambda(r) \sinh r dr \tag{4.1}$$

and

$$\sum_{l \in \mathbf{Z}, l^2 > q} \rho_f(l^2 - q) \left(\frac{l^2 - q}{q}\right)^{\frac{3}{4}} \frac{1}{\pi^{\frac{1}{2}}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta) m\left(\frac{l^2 - q}{q \cos^2 \theta}\right) \frac{d\theta}{\cos^2 \theta} \tag{4.2}$$

equals the sum of (for the notation $H_{m,2\pi q}$, see Lemma 6.1)

$$(4\pi q)^{1/4} \sum_{j=1}^\infty \overline{b_j(q)} \left(\int_{D_4} u_j(z) f(z) \overline{B_0(z)} d\mu_z\right) H_{m,2\pi q}(t_j) \tag{4.3}$$

and

$$(4\pi q)^{1/4} \sum_a \frac{1}{4\pi} \int_{-\infty}^\infty \overline{\beta_{a,\frac{1}{2}+ir}(q)} \times \left(\int_{D_4} E_a\left(z, \frac{1}{2} + ir\right) f(z) \overline{B_0(z)} d\mu_z\right) H_{m,2\pi q}(r)\, dr. \tag{4.4}$$

Proof. Since the function $g = g_A$ defined in (3.3) satisfies the conditions of Lemma 3.1 if $\mathrm{Re}A < 0$, hence applying Lemmas 3.1 and 3.3 we see that if $\mathrm{Re}A < 0$, then the sum of

$$(4.5)\quad \sum_{l\in\mathbf{Z},l^2<q} \rho_f(l^2-q)\int_0^\infty e^{(A-2\pi q-2\pi l^2)y} g_{2\pi q}(y) y^{-\frac{7}{4}} W_{-\frac{1}{4},it}\left(4\pi\left|l^2-q\right|y\right)dy$$

and

$$(4.6)\quad \sum_{l\in\mathbf{Z},l^2>q} \rho_f(l^2-q)\Big(\frac{l^2-q}{q}\Big)^{\frac{3}{4}}\frac{1}{\pi^{\frac{1}{2}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f_\lambda(\theta) m\Big(\frac{2\pi q-A}{4\pi q}+\frac{l^2-q}{q\cos^2\theta}\Big)\frac{d\theta}{\cos^2\theta}$$

equals the sum of

$$(4.7)\qquad (4\pi q)^{1/4}\sum_{j=1}^\infty \overline{b_j(q)}\Big(\int_{D_4} u_j(z) f(z)\overline{B_0(z)}d\mu_z\Big)H_{m,A}(t_j)$$

and

$$(4.8)\quad (4\pi q)^{1/4}\sum_a \frac{1}{4\pi}\int_{-\infty}^\infty \overline{\beta_{a,\frac{1}{2}+ir}(q)}\Big(\int_{D_4} E_a\Big(z,\frac{1}{2}+ir\Big)f(z)\overline{B_0(z)}d\mu_z\Big)H_{m,A}(r)dr,$$

using the notation of Lemma 6.1.

We see that if m, q and f are fixed, then (4.5), (4.6), (4.7) and (4.8) are functions holomorphic for $\mathrm{Re}A < 2\pi q$ and continuous on $(-\infty, 2\pi q]$. In the case of (4.5) this follows from (3.3), the properties of $m^*(s)/\Gamma(s)$, and from the estimate that $W_{-\frac{1}{4},it}(Y)e^{Y/2}$ has at most polynomial growth at $Y = 0$ and at $Y = \infty$; this comes from [G-R], p. 1015, 9.222.1. In the case of (4.6) this follows from (3.8) with $l = 0$, from the boundedness of $f_\lambda(\theta)$ (see [B1], p. 336), and from a polynomial (in l) upper bound for $\rho_f(l^2-q)$ (which can be proved by formula (83) of [P], using there the function ϕ from [D], Section 5). In the case of (4.7) and (4.8) this follows from (6.1), Lemma A2, (8.5) and (8.27) of [I], and the easily checked fact that if a real r is fixed, then $H_{m,A}(r)$ is holomorphic for $\mathrm{Re}A < 2\pi q$ and continuous on $(-\infty, 2\pi q]$.

So using (3.6) and the definition of T_g in Lemma 3.1, by $A \to 2\pi q - 0$ we obtain the lemma. ∎

We now give a new expression for the sum of (4.1) and (4.2) using our Lemma 3.2 and the Theorem of Katok and Sarnak. It is essential that the same test function $H_{m,2\pi q}$ which we had in (4.3) and (4.4) appears also in this new expression.

Lemma 4.2: *Let m and q be as in the previous lemma, but assume now that q is divisible by 4. Let u be a fixed even cusp form of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$ which is a simultaneous Hecke eigenform, let $\int_{D_1}|u(z)|^2\,d\mu_z=1$ and $\Delta_0 u=\lambda u$. Then we have, writing $\phi=u/\rho_u(1)$ and*

$$f(z)=\sum_{\mathrm{Shim}F_j=\phi}\overline{\rho_j(1)}F_j(z), \tag{4.9}$$

that the sum of (4.1) *and* (4.2) *equals the sum of*

$$\frac{q^{-\frac{1}{4}}\overline{\rho_u(1)}}{3}\sum_{l=1}^{\infty}\lambda_l\left(\frac{q}{4}\right)H_{m,2\pi q}(\tau_l)\int_{D_1}|U_l(z)|^2u(z)d\mu_z \tag{4.10}$$

and

$$\frac{q^{-\frac{1}{4}}\overline{\rho_u(1)}}{12\pi}\int_{-\infty}^{\infty}\eta_t\left(\frac{q}{4}\right)H_{m,2\pi q}(t)\Big(\int_{D_1}\Big|E\Big(z,\frac{1}{2}+it\Big)\Big|^2u(z)d\mu_z\Big)dt. \tag{4.11}$$

Proof. By the Theorem of Katok and Sarnak (see our Theorem A1), the sum of (4.1) and (4.2) equals (with f as in (4.9)) the sum of

$$\frac{q^{-\frac{3}{4}}}{12\pi}\sum_{l\in\mathbf{Z},l^2>q}\left(\frac{1}{(\phi,\phi)}\sum_{Q\in\Lambda_{l^2-q}}\int_{C_Q}\phi dS\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}f_\lambda(\theta)m\Big(\frac{l^2-q}{q\cos^2\theta}\Big)\frac{d\theta}{\cos^2\theta}$$

and

$$\frac{q^{-\frac{3}{4}}}{6}\sum_{l\in\mathbf{Z},l^2<q}\left(\frac{1}{(\phi,\phi)}\sum_{Q\in\Lambda_{l^2-q}}\frac{\phi(z_Q)}{M_Q}\right)\int_0^\infty m\Big(\frac{|l^2-q|}{q}\sinh^2 r\Big)g_\lambda(r)\sinh r dr.$$

Since $(\phi,\phi)=1/|\rho_u(1)|^2$, by Lemma 3.2 this sum, and so the sum of (4.1) and (4.2), also equals

$$\frac{q^{-\frac{3}{4}}\overline{\rho_u(1)}}{12\pi}\int_{D_1}\Big(\sum_{l\in\mathbf{Z}}M_{\frac{q}{4},l}(z)\Big)u(z)d\mu_z. \tag{4.12}$$

Since

$$\sum_{l\in\mathbf{Z}}M_{q/4,l}(z)=\sum_{\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right),\,a,b,c,d\in\mathbf{Z},\,\det\gamma=q/4}m(z,\gamma z),$$

so, as on p. 194, lines 6–7 of [I] (we insert a factor 2 in (4.13) below, since for every γ the matrix $-\gamma$ is also present in the above sum, and gives the same transformation; I think that this factor 2 is missing from the cited formula of

[I], but this is not essential in our reasoning), we have, using (1.62′) of [I] and the notation $H_{m,2\pi q}$ from Lemma 6.1, that $(q/4)^{-1/2}\sum_{l\in\mathbf{Z}} M_{q/4,l}(z)$ equals

$$(4.13)\quad 2\sum_{l=0}^{\infty}\lambda_l\left(\frac{q}{4}\right)(4\pi H_{m,2\pi q}(\tau_l))\,|U_l(z)|^2 + \frac{1}{2\pi}\int_{-\infty}^{\infty}\eta_t\left(\frac{q}{4}\right)(4\pi H_{m,2\pi q}(t))\left|E\left(z,\frac{1}{2}+it\right)\right|^2 dt.$$

Together with (4.12) this proves the lemma. ∎

The theorem will follow at once from the next lemma, which will be proved by the combination of the preceding two lemmas.

LEMMA 4.3: *Let u and ϕ be as in Lemma 4.2, and let $l_0 \geq 1$ be a fixed integer. Then*

$$\overline{\rho_u(1)}\int_{D_1}|U_{l_0}(z)|^2\,u(z)d\mu_z$$

equals

$$\sqrt{2}\pi^{1/4}\overline{\rho_{U_{l_0}}(1)}\int_{D_4}\left(\overline{B_0(z)}U_{l_0}(4z)\right)\left(\sum_{\mathrm{Shim}F_j=\phi}\overline{\rho_j(1)}F_j(z)\right)d\mu_z.$$

Proof. Let $T=\tau_{l_0}$. Let Q be a large positive number, and

$$h(t)=h_Q(t)=e^{-Q(t-T)^2}+e^{-Q(t+T)^2}.$$

For a given Q this function satisfies the conditions of Lemma 6.3, and applying Lemma 6.3 we see that for the function m defined there the conditions of Lemmas 3.3 and 6.1 are satisfied. Hence we may apply Lemmas 4.1 and 4.2 for this m and for any positive integer q divisible by 4. We apply Lemma 4.2 with our given u and ϕ, and apply Lemma 4.1 with f defined in (4.9). Note that if $\mathrm{Shim}F_j=\phi$, then the $\Delta_{1/2}$-eigenvalue of F_j is $s(s-1)$ with $s=\frac{1}{2}+it$, where t satisfies $\lambda=-\frac{1}{4}-(2t)^2$ with λ defined in Lemma 4.2 by $\Delta_0 u=\lambda u$ (see, e.g., [B2], Theorem 1). Hence we can indeed apply Lemma 4.1 with this f and λ.

Using the definition of m and $H_{m,2\pi q}$ in Lemmas 6.3 and 6.1, respectively, by Mehler–Fock inversion (see, e.g., [R], Theorem 2) we have $H_{m,2\pi q}=h$. From the combination of Lemmas 4.1 and 4.2, we see that the sum of (4.3) and (4.4) equals the sum of (4.10) and (4.11). Letting $Q\to\infty$ in (4.3), (4.4), (4.10) and (4.11), we get (using again for the upper estimations Lemma A2, (8.5) and

(8.27) of [I], and we use also (8.33), (8.42) and Proposition 7.2 of [I]) that for any positive integer q divisible by 4,

$$\frac{q^{-\frac{1}{4}}\overline{\rho_u(1)}}{3}\sum_{\tau_l=T}\lambda_l\left(\frac{q}{4}\right)\int_{D_1}|U_l(z)|^2u(z)d\mu_z \tag{4.14}$$

equals

$$(4\pi q)^{1/4}\sum_{t_j=T}\overline{b_j(q)}\left(\int_{D_4}u_j(z)f(z)\overline{B_0(z)}d\mu_z\right). \tag{4.15}$$

Indeed, because of the behaviour of h_Q, (4.4) and (4.11) disappeared completely for $Q\to\infty$, and we are left only with the T-part of (4.3) and (4.10).

Let us consider for a while only the sum (4.15). Up to this point the concrete form of the complete orthonormal system $\{u_j(z):\ j\geq 1\}$ of cusp forms of weight 0 for $\Gamma_0(4)$ was not important, but to handle (4.15) we now take a special system, obtained from newforms on $\mathrm{SL}(2,\mathbf{Z})$, $\Gamma_0(2)$ and $\Gamma_0(4)$.

We first introduce some notation. If $u(z)$ is a cusp form for $\mathrm{SL}(2,\mathbf{Z})$ which is a simultaneous Hecke eigenform with eigenvalues $H_nu=\lambda_nu$, let

$$u^{(1)}(z):=u(4z),\quad u^{(2)}(z):=u(2z)-\frac{\sqrt{2}}{3}\lambda_2u(4z),$$
$$u^{(3)}(z):=u(z)-\frac{\lambda_2}{\sqrt{2}}u\left(2z\right)+\frac{u(4z)}{2}.$$

If $v(z)$ is a newform for $\Gamma_0(2)$ and

$$v\left(\frac{z}{2}\right)+v\left(\frac{z+1}{2}\right)=\mu v(z),$$

which is automatically true with some μ (see [I], (8.38)) let

$$v^{(1)}(z):=v(2z),\quad v^{(2)}(z):=v(z)-\frac{\mu}{2}v(2z).$$

Recall that $\{U_l(z):\ l\geq 1\}$ is a complete orthonormal system of cusp forms of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$ consisting of simultaneous Hecke eigenforms with eigenvalues $H_nU_l=\lambda_l(n)U_l$; V_m $(m\geq 1)$ is a complete orthonormal system of newforms of weight 0 for $\Gamma_0(2)$, and W_r $(r\geq 1)$ is a complete orthonormal system of newforms of weight 0 for $\Gamma_0(4)$. Writing $\|g\|=\left(\int_{D_4}|g(z)|^2d\mu_z\right)^{1/2}$, it is proved in Lemma 5.7 that the following functions together form a complete orthonormal system of cusp forms of weight 0 for $\Gamma_0(4)$:

$$\frac{U_l^{(1)}}{\|U_l^{(1)}\|},\ \frac{U_l^{(2)}}{\|U_l^{(2)}\|},\ \frac{U_l^{(3)}}{\|U_l^{(3)}\|}\quad (l\geq 1),$$

$$\frac{V_m^{(1)}}{\|V_m^{(1)}\|},\ \frac{V_m^{(2)}}{\|V_m^{(2)}\|} \quad (m \geq 1),$$

$$W_r \quad (r \geq 1).$$

Hence we choose $\{u_j(z) :\ j \geq 1\}$ to be this special system.

We now show, using the results of Section 5, that if $j \geq 1$ is such that

$$u_j = \frac{U_l^{(2)}}{\|U_l^{(2)}\|},\ \frac{U_l^{(3)}}{\|U_l^{(3)}\|},\ \frac{V_m^{(1)}}{\|V_m^{(1)}\|},\ \frac{V_m^{(2)}}{\|V_m^{(2)}\|} \text{ or } W_r$$

with some l, m or r, then

$$\overline{b_j(q)}\left(\int_{D_4} u_j(z)f(z)\overline{B_0(z)}d\mu_z\right) = 0, \tag{4.16}$$

where q and f are as above.

Since q is even, by Lemma 5.6 we see that in the cases when u_j is a constant multiple of some $U_l^{(3)}$ or $V_m^{(2)}$, we have $b_j(q) = 0$. But the same is true when u_j equals some W_r. Indeed, the operator $z \to z + \frac{1}{2}$ normalizes $\Gamma_0(4)$, this operator commutes with H_p, $p \neq 2$, and leaves invariant the space of newforms; hence $W_r(z + \frac{1}{2})$ is a constant multiple of $W_r(z)$. But this constant is -1, as can be seen considering the first Fourier coefficients. Hence the even Fourier coefficients of W_r are 0.

We claim that if u_j is a multiple of some $U_l^{(2)}$ or $V_m^{(1)}$, then the integral in (4.16) is 0. Indeed, from Lemma 5.3 and Lemma 5.6 we see, on the one hand, that in these cases $B_0(z)\overline{u_j(z)}$ is an eigenfunction of L of eigenvalue $-\frac{1}{2}$. On the other hand, f is an eigenfunction of L of eigenvalue 1 using (4.9), since every F_j is such an eigenfunction. We use the well-known fact that L is self-adjoint, hence two L-eigenfunctions with different eigenvalues are orthogonal to each other. This proves that the integral in (4.16) is indeed 0 in these cases.

Hence the contribution of a function u_j to the sum (4.15) may be nonzero only in the case when u_j is a multiple of some $U_l^{(1)}$. And if

$$u_j(z) = \frac{U_l^{(1)}(z)}{\|U_l^{(1)}\|} = \frac{U_l(4z)}{\sqrt{6}}$$

(see Lemma 5.1) for some l, then

$$b_j(q) = \frac{1}{\sqrt{6}}\rho_{U_l}\left(\frac{q}{4}\right) = \frac{2}{\sqrt{6q}}\rho_{U_l}(1)\lambda_l\left(\frac{q}{4}\right) \tag{4.17}$$

(see [I], (8.5) and (8.36)). Using these relations (and that the Hecke eigenvalues are real) we finally get that (4.15) equals

$$(4\pi)^{1/4}\frac{q^{-1/4}}{3}\sum_{\tau_l=T}\lambda_l\left(\frac{q}{4}\right)\overline{\rho_{U_l}(1)}\left(\int_{D_4}U_l(4z)f(z)\overline{B_0(z)}d\mu_z\right).$$

We then see by the equality of (4.15) and (4.14), substituting (4.9) for f, that

$$\sum_{\tau_l=T}C_l\lambda_l\left(\frac{q}{4}\right)=0 \tag{4.18}$$

for every positive integer q which is divisible by 4, where C_l is the difference between

$$\overline{\rho_u(1)}\int_{D_1}|U_l(z)|^2u(z)d\mu_z$$

and

$$(4\pi)^{1/4}\overline{\rho_{U_l}(1)}\int_{D_4}\left(\overline{B_0(z)}U_l(4z)\right)\left(\sum_{\mathrm{Shim}F_j=\phi}\overline{\rho_j(1)}F_j(z)\right)d\mu_z.$$

We will prove that (4.18) implies that every C_l is 0, and this will be enough for the proof of the lemma, since $\tau_{l_0}=T$, and $C_{l_0}=0$ is just the statement of the lemma.

Every U_l is either an even or an odd cusp form, since every U_l is a simultaneous Hecke eigenform. We now group together the even and the odd cusp forms in (4.18). We obtain, with the notation

$$u_1:=\sum_{\tau_l=T,U_l\text{ is even}}\frac{1}{\rho_{U_l}(1)}C_lU_l,\quad u_2:=-\sum_{\tau_l=T,U_l\text{ is odd}}\frac{1}{\rho_{U_l}(1)}C_lU_l$$

that u_1 and u_2 are two cusp forms (not necessarily Hecke-eigenforms!) of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$, u_1 is even, u_2 is odd,

$$\Delta_0u_1=-\left(\frac{1}{4}+T^2\right)u_1,\quad \Delta_0u_2=-\left(\frac{1}{4}+T^2\right)u_2,$$

and by (4.18) and the second equality in (4.17), the positive Fourier coefficients of u_1 and u_2 are the same. If $u_1=0$, this implies that every Fourier coefficient of u_2 is 0, so $u_2=0$, hence by linear independence of the functions U_l we get that every C_l is 0. The situation is the same if $u_2=0$, so we may assume that $u_1\neq0$, $u_2\neq0$. We now apply [Bu], p. 107, Proposition 1.9.1. By our condition on the positive Fourier coefficients we have $L(s,u_1)=L(s,u_2)$ for the L-functions defined in (9.7) of [Bu], hence for the quotient of the completed

L-functions (defined in (9.10) of [Bu]; note that we can take this quotient since $\Lambda(s,u_2)$ is not identically 0) we have (using that u_1 is even, u_2 is odd)

$$Q(s) := \frac{\Lambda(s,u_1)}{\Lambda(s,u_2)} = \frac{\Gamma\left(\frac{s+iT}{2}\right)\Gamma\left(\frac{s-iT}{2}\right)}{\Gamma\left(\frac{s-1+iT}{2}\right)\Gamma\left(\frac{s-1-iT}{2}\right)}, \tag{4.19}$$

and $Q(1-s) = -Q(s)$ for every complex s by (9.11) of [Bu], which implies $Q(1/2) = 0$. But $Q(1/2)$ is a nonzero finite number by (4.19), hence this is a contradiction. So $u_1 = 0$ or $u_2 = 0$; the lemma is proved. ∎

5. Lemmas on automorphic functions

Our goal in this section is to prove the basic properties of the special complete orthonormal system of cusp forms for $\Gamma_0(4)$ used in the proof of Lemma 4.3.

Note first that if D_1 is a fundamental domain of $\mathrm{SL}(2,\mathbf{Z})$, then it is easy to see that a fundamental domain D_4 of $\Gamma_0(4)$ can be given in the following way:

$$D_4 = \bigcup_{j=0}^{5} \gamma_j D_1,$$

where

$$\gamma_j = \begin{pmatrix} 0 & -1 \\ 1 & j \end{pmatrix} \quad (0 \le j \le 3),$$

and

$$\gamma_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}.$$

Lemma 5.1: *Let $u(z)$ be a cusp form for $\mathrm{SL}(2,\mathbf{Z})$ which is a simultaneous Hecke eigenform with eigenvalues $H_n u = \lambda_n u$, and let $\int_{D_1} |u(z)|^2\, d\mu_z = 1$. Then*

$$\int_{D_4} |u(z)|^2\, d\mu_z = \int_{D_4} |u(2z)|^2\, d\mu_z = \int_{D_4} |u(4z)|^2\, d\mu_z = 6, \tag{5.1}$$

$$\int_{D_4} u(z)\overline{u(2z)} d\mu_z = \int_{D_4} u(2z)\overline{u(4z)} d\mu_z = 2\sqrt{2}\lambda_2, \tag{5.2}$$

$$\int_{D_4} u(z)\overline{u(4z)} d\mu_z = 2\lambda_4 - 1. \tag{5.3}$$

If $u^*(z)$ *is another cusp form for* $\mathrm{SL}(2,\mathbf{Z})$ *which is a simultaneous Hecke eigenform, and* $\int_{D_1} u(z)\overline{u^*(z(}d\mu_z = 0$*, then*

$$\int_{D_4} u(r_1 z)\overline{u^*(r_2 z)}d\mu_z = 0, \tag{5.4}$$

if $r_1, r_2 \in \{1,2,4\}$.

Proof. We prove just (5.1), (5.2) and (5.3), since the proof of (5.4) is completely similar: one carries out the same steps, writing u^* in place of one copy of u.

Let us first remark that if D_{16} is a fundamental domain of $\Gamma_0(16)$, then for $r = 1,2,4$ the group

$$G_r = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \Gamma_0(16) \begin{pmatrix} r^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

is a subgroup of $\Gamma_0(4)$, a fundamental domain for G_r is rD_{16}, and the index of G_r in $\Gamma_0(4)$ is the same for every r (e.g., since the fundamental domains have the same area). Denote this index by m; then

$$\begin{aligned}\int_{D_4} |u(z)|^2 \, d\mu_z &= \frac{1}{m}\int_{rD_{16}} |u(z)|^2 \, d\mu_z = \frac{1}{m}\int_{D_{16}} |u(rz)|^2 \, d\mu_z \\ &= \int_{D_4} |u(rz)|^2 \, d\mu_z,\end{aligned}$$

and similarly $\int_{D_4} u(z)\overline{u(2z)}d\mu_z$ equals

$$\frac{1}{m}\int_{2D_{16}} u(z)\overline{u(2z)}d\mu_z = \frac{1}{m}\int_{D_{16}} u(2z)\overline{u(4z)}d\mu_z = \int_{D_4} u(2z)\overline{u(4z)}d\mu_z.$$

To proceed further, observe that for $r = 1,2,4$ we have

$$\int_{D_4} u(z)\overline{u(rz)}d\mu_z = \int_{D_1} u(z)\overline{\left(\sum_{j=0}^{5} u\left(r(\gamma_j z)\right)\right)}d\mu_z.$$

Then (5.1) is trivial, and we easily see that

$$\sum_{j=0}^{5} u\left(2(\gamma_j z)\right) = 2\left(u\left(2z\right) + u\left(\frac{z}{2}\right) + u\left(\frac{z+1}{2}\right)\right) = 2\sqrt{2}\left(H_2 u\right)(z),$$

$$\sum_{j=0}^{5} u\left(4(\gamma_j z)\right) = u\left(4z\right) + u\left(z + \frac{1}{2}\right) + \sum_{j=0}^{3} u\left(\frac{z+j}{4}\right) = -u(z) + 2\left(H_4 u\right)(z).$$

Since the Hecke eigenvalues are real, the lemma follows. ∎

LEMMA 5.2: *Let $v(z)$ be a newform for $\Gamma_0(2)$, let*

$$v\left(\frac{z}{2}\right)+v\left(\frac{z+1}{2}\right)=\mu v(z) \tag{5.5}$$

(note that it is automatically true with some μ; see [I], (8.38)*), and let*

$$\int_{D_2}|v(z)|^2\,d\mu_z=1$$

where D_2 is a fundamental domain of $\Gamma_0(2)$. Then

$$\int_{D_4}|v(z)|^2\,d\mu_z=\int_{D_4}|v(2z)|^2\,d\mu_z=2, \tag{5.6}$$

$$\int_{D_4}v(z)\overline{v(2z)}d\mu_z=\mu. \tag{5.7}$$

If $v^*(z)$ is another newform for $\Gamma_0(2)$, and $\int_{D_2}v(z)\overline{v^*(z)}d\mu_z=0$, then

$$\int_{D_4}v(r_1z)\overline{v^*(r_2z)}d\mu_z=0, \tag{5.8}$$

if $r_1,r_2\in\{1,2\}$.

Proof. As in the previous proof, we just prove (5.6) and (5.7); the proof of (5.8) is completely similar.

Note first that $\Gamma_0(2)=\Gamma_0(4)\cup\Gamma_0(4)\left(\begin{smallmatrix}1&0\\2&1\end{smallmatrix}\right)$ (disjoint union). The proof of (5.6) is the same as the proof of (5.1). On the other hand, let

$$G=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(4):\ b\text{ is even}\right\}.$$

Then for $j=0,1$ we have that

$$G_j=\begin{pmatrix}1&j\\0&2\end{pmatrix}G\begin{pmatrix}1&j\\0&2\end{pmatrix}^{-1}\le\Gamma_0(4),$$

so, if D is a fundamental domain of G, then $\left(\begin{smallmatrix}1&j\\0&2\end{smallmatrix}\right)D$ is a fundamental domain of G_j. Hence if m is the index of G in $\Gamma_0(4)$ (which is the same as the index of any G_j), then $\int_{D_4}v(z)\overline{v(2z)}d\mu_z$ equals

$$\frac{1}{m}\int_{\left(\begin{smallmatrix}1&j\\0&2\end{smallmatrix}\right)D}v(z)\overline{v(2z)}d\mu_z=\frac{1}{m}\int_D v\left(\frac{z+j}{2}\right)\overline{v(z)}d\mu_z$$

for $j=0$ and for $j=1$, and, averaging over $j=0,1$, using (5.5), this is

$$\frac{\mu}{2m}\int_D v(z)\,\overline{v(z)}d\mu_z=\mu,$$

since the index of G in $\Gamma_0(2)$ is $2m$. The lemma is proved. ∎

Lemma 5.3: *Let f be a Maass form of weight 0 for $\Gamma_0(4)$ with the additional property that $f(z+\frac{1}{2})=f(z)$. Let $g(z)=B_0(z)\overline{f(z)}$. Then we have*

$$(Lg)(z)=\frac{1}{4}B_0(z)\overline{\sum_{j=0}^{3} f\left(\frac{-1/(4z)+j}{4}\right)}.$$

Proof. By (1.2) we have

$$\sum_{j=0}^{3} g\left(\frac{z+j}{4}\right)=\left(\frac{y}{4}\right)^{\frac{1}{4}}\sum_{m=-\infty}^{\infty} e\left(\frac{m^2z}{4}\right)\sum_{j=0}^{3}\overline{f\left(\frac{z+j}{4}\right)}e\left(\frac{m^2j}{4}\right).$$

Here the inner sum is 0 if m is odd, because f is periodic with respect to $\frac{1}{2}$. Hence

$$\sum_{j=0}^{3} g\left(\frac{z+j}{4}\right)=\frac{1}{\sqrt{2}}B_0(z)\sum_{j=0}^{3}\overline{f\left(\frac{z+j}{4}\right)}.$$

From the definition of L in [K-S] and (2.1) we obtain the lemma. ∎

Lemma 5.4: *Let $v(z)$ be a newform for $\Gamma_0(2)$, and $f(z)=v(2z)$. Then*

$$\sum_{j=0}^{3} f\left(\frac{-1/(4z)+j}{4}\right)=-2f(z).$$

Proof. Since the left-hand side is

$$\sum_{j=0}^{3} v\left(-\frac{1}{4(2z)}+\frac{j}{2}\right),$$

it is enough to prove that

$$V(z):=\sum_{j=0}^{1} v\left(-\frac{1}{4z}+\frac{j}{2}\right)=-v(z).$$

Here $V(z)=k\left(-1/(2z)\right)$, where

$$k(z)=\sum_{j=0}^{1} v\left(\frac{z+j}{2}\right).$$

We know by [I], (8.38) that k is a constant multiple of v. It is easy to verify that the operator

$$h\,(z) \to h\left(-\frac{1}{2z}\right)$$

maps $\Gamma_0(2)$-invariant functions to $\Gamma_0(2)$-invariant functions, and it is easy to prove that this operator commutes with the Hecke operators H_p with $p \neq 2$, and leaves invariant the space of newforms. Hence the image of a newform is again a newform with the same H_p-eigenvalues for every $p \neq 2$, so the image of a newform is a constant multiple of this newform. Therefore we see that V is a constant multiple of v. On the other hand,

$$v\left(-\frac{1}{4z}+\frac{1}{2}\right) = v\left(\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\left(z-\frac{1}{2}\right)\right) = v\left(z-\frac{1}{2}\right)$$

and

$$v\Big(-\frac{1}{4\left(z+\frac{1}{2}\right)}\Big) = v\left(\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\left(\frac{-1}{4z}\right)\right) = v\left(\frac{-1}{4z}\right),$$

so

$$\int_0^1 v\left(-\frac{1}{4\,(z+x)}\right) e\,(-x)\,dx = 0,$$

$$\int_0^1 v\left(-\frac{1}{4\,(z+x)}+\frac{1}{2}\right) e\,(-x)\,dx = -\int_0^1 v\,(z+x)\,e\,(-x)\,dx,$$

hence

$$\int_0^1 V\,(z+x)\,e\,(-x)\,dx = -\int_0^1 v\,(z+x)\,e\,(-x)\,dx,$$

and since the first Fourier coefficient of the newform v is nonzero, this implies $V = -v$. The lemma is proved. ∎

Lemma 5.5: *Let $u(z)$ be an $\mathrm{SL}(2,\mathbf{Z})$-invariant function with $H_2 u = \lambda_2 u$.*

(i) *If $f(z) = u(2z)$, then*

$$\sum_{j=0}^{3} f\left(\frac{-1/(4z)+j}{4}\right) = 2\sqrt{2}\lambda_2 u(4z) - 2u(2z). \tag{5.9}$$

(ii) *If $f(z) = u(4z)$, then*

$$\sum_{j=0}^{3} f\left(\frac{-1/(4z)+j}{4}\right) = 4u(4z). \tag{5.10}$$

Proof. By $H_2 u = \lambda_2 u$ we have

$$u\left(\frac{z}{2}\right) + u\left(\frac{z+1}{2}\right) = \sqrt{2}\lambda_2 u(z) - u(2z).$$

So the left-hand side of (5.9) is

$$2\left(u\left(\frac{-1/(4z)}{2}\right) + u\left(\frac{-1/(4z)+1}{2}\right)\right) = 2\sqrt{2}\lambda_2 u\left(-\frac{1}{4z}\right) - 2u\left(-\frac{1}{2z}\right);$$

(5.9) follows. Formula (5.10) is trivial; the lemma is proved. ∎

LEMMA 5.6: (i) *Let $u(z)$ be a cusp form for $\mathrm{SL}(2, \mathbf{Z})$ which is a simultaneous Hecke eigenform with eigenvalues $H_n u = \lambda_n u$. Then*

$$u^{(1)}(z) := u(4z), \quad u^{(2)}(z) := u(2z) - \frac{\sqrt{2}}{3}\lambda_2 u\,(4z)\,,$$
$$u^{(3)}(z) := u(z) - \frac{\lambda_2}{\sqrt{2}} u\,(2z) + \frac{u(4z)}{2}$$

is an orthogonal system on D_4,

(5.11)
$$\sum_{j=0}^{3} u^{(1)}\left(\frac{-1/(4z)+j}{4}\right) = 4u^{(1)}(z), \quad \sum_{j=0}^{3} u^{(2)}\left(\frac{-1/(4z)+j}{4}\right) = -2u^{(2)}(z),$$

and the even Fourier coefficients at ∞ of $u^{(3)}(z)$ are 0.

(ii) *Let $v(z)$ be a newform for $\Gamma_0(2)$ satisfying (5.5). Then*

$$v^{(1)}(z) := v(2z), \quad v^{(2)}(z) := v(z) - \frac{\mu}{2} v(2z)$$

is an orthogonal system on D_4,

$$\sum_{j=0}^{3} v^{(1)}\left(\frac{-1/(4z)+j}{4}\right) = -2v^{(1)}(z), \tag{5.12}$$

and the even Fourier coefficients at ∞ of $v^{(2)}(z)$ are 0.

Proof. The orthogonality statements follow from Lemmas 5.1 and 5.2, using $\lambda_4 = \lambda_2^2 - 1$. Formula (5.11) follows from Lemma 5.5, and formula (5.12) follows from Lemma 5.4. The statement about the even Fourier coefficients follows easily from (5.5) in (ii), and from

$$\rho_u(n) = \rho_u(1)\frac{\lambda_n}{\sqrt{|n|}}, \quad \lambda_2\lambda_n = \sum_{d|(2,n)} \lambda_{2n/d^2}$$

(see [I], (8.5), (8.36) and (8.39)) in (i). The lemma is proved. ∎

LEMMA 5.7: *Write, in general,* $\|g\| = \left(\int_{D_4} |g(z)|^2 \, d\mu_z\right)^{1/2}$. *Then the following functions together form a complete orthonormal system of cusp forms of weight 0 for* $\Gamma_0(4)$*:*

$$\frac{U_l^{(1)}}{\|U_l^{(1)}\|}, \quad \frac{U_l^{(2)}}{\|U_l^{(2)}\|}, \frac{U_l^{(3)}}{\|U_l^{(3)}\|} \quad (l \geq 1),$$

$$\frac{V_m^{(1)}}{\|V_m^{(1)}\|}, \quad \frac{V_m^{(2)}}{\|V_m^{(2)}\|} \quad (m \geq 1),$$

$$W_r \quad (r \geq 1),$$

where we have used the notation of Lemma 5.6.

Proof. By using (5.4), (5.8) and Lemma 5.6, and also Atkin–Lehmer's theory (see [I], pp. 128–129), we see that the only thing that requires a proof is the following: if $l, m \geq 1$, then

$$\int_{D_4} U_l(r_1 z)\overline{V_m(r_2 z)} d\mu_z = 0,$$

for $r_1 \in \{1, 2, 4\}$, $r_2 \in \{1, 2\}$. If $r_2 = 2$, then for $r_1 = 2, 4$ this follows easily from the newform property of V_m. If $r_2 = 2$, $r_1 = 1$, then

$$\int_{D_4} U_l(z)\overline{V_m(2z)} d\mu_z = \int_{D_4} U_l\Big(z + \frac{1}{2}\Big)\overline{V_m(2z)} d\mu_z$$

(since $z \to z + \frac{1}{2}$ normalizes $\Gamma_0(4)$), and

$$U_l(z) + U_l\Big(z + \frac{1}{2}\Big) + U_l(4z)$$

is a constant multiple of $U_l(2z)$ (using the Hecke operator H_2); hence the statement follows also for $r_2 = 2$, $r_1 = 1$ from the cases already proved. If $r_2 = 1$, then the newform property of V_m gives the statement at once for $r_1 = 1, 2$. Similarly as above,

$$\int_{D_4} U_l(4z)\overline{V_m(z)} d\mu_z = \int_{D_4} U_l(4z)\overline{V_m\Big(z + \frac{1}{2}\Big)} d\mu_z,$$

and since

$$V_m(z) + V_m\Big(z + \frac{1}{2}\Big)$$

is a constant multiple of $V_m(2z)$ by [I], (8.38), the $r_2 = 1$, $r_1 = 4$ case follows from the already proved cases. The lemma is proved. ∎

6. Lemmas on function transforms

LEMMA 6.1: *Let m be a function defined for* Re$Y > 0$ *and for* $Y = 0$ *satisfying that $m(Y)$ is a holomorphic function in the half-plane* Re$Y > 0$, *$m(Y)$ is a smooth function on $[0, \infty)$, and there are constants $c_{l,r} > 0$ for every integer pair $l, r \geq 0$ such that*

$$\left| m^{(l)}(Y)(1 + |Y|)^r \right| \leq c_{l,r}$$

for every Re$Y > 0$ *and also for $Y = 0$. Let q be a fixed positive integer. For* Re$A < 2\pi q$ *and also for $A = 2\pi q$, define*

$$H_{m,A}(t) = \int_0^\infty m\Big(Y + \frac{2\pi q - A}{4\pi q}\Big) F\Big(\frac{1}{2} + it, \frac{1}{2} - it, 1, -Y\Big) dY.$$

Then for every such A the function $H_{m,A}(t)$ is even, entire, and satisfies that for every integer pair $L, R > 0$ we have numbers $d_{L,R} > 0$ such that

$$|H_{m,A}(t)| \leq d_{L,R}(1 + |t|)^{-L} \quad \text{for } |\mathrm{Im}t| \leq R. \tag{6.1}$$

The number $d_{L,R}$ depends only on L, R and the numbers $c_{l,r}$ $(l, r \geq 0)$, hence does not depend on A.

Proof. Everything follows at once from Lemma 6.2 below.

LEMMA 6.2: *Assume that k is a smooth function on $[0, \infty)$ and $c_{l,r} > 0$ is a number for every integer pair $l, r \geq 0$ such that*

$$\left| k^{(l)}(u) \right| \leq c_{l,r}(1 + u)^{-r}$$

for every $u \geq 0$, $l, r \geq 0$. Let

$$h(t) = \int_0^\infty k(u) F\Big(\frac{1}{2} + it, \frac{1}{2} - it, 1, -u\Big) du.$$

Then h is an even entire function, and for every integer pair $L, R > 0$ we have numbers $d_{L,R} > 0$ such that

$$|h(t)| \leq d_{L,R}(1 + |t|)^{-L} \quad \text{for} |\mathrm{Im}t| \leq R.$$

The number $d_{L,R}$ depends only on L, R and the numbers $c_{l,r}$ $(l, r \geq 0)$.

Proof. By (1.62) and (1.62′) of [I] (the function $F_s(u)$ is defined by the formulas on p. 26, line 7, and (B. 23) of [I]), we have

$$h(t) = \frac{1}{4\pi} \int_{-\infty}^\infty e^{i\rho t} g(\rho) d\rho, \tag{6.2}$$

where

$$g(\rho) = 2q\left(\sinh^2 \frac{\rho}{2}\right),$$

$$q(v) = \int_0^\infty k(u+v)u^{-\frac{1}{2}}du.$$

We see, for every $L \geq 0$ and real ρ, that

$$g^{(L)}(\rho) \ll_L \sum_{l=0}^{L} \left| q^{(l)}\left(\sinh^2 \frac{\rho}{2}\right)\right| e^{l|\rho|},$$

and since

$$q^{(l)}(v) = \int_0^\infty k^{(l)}(u+v)u^{-\frac{1}{2}}du,$$

we have

$$\left|q^{(l)}(v)\right| \leq c_{l,r}\int_0^\infty (1+u+v)^{-r}u^{-\frac{1}{2}}du$$

for every $l, r, v \geq 0$. Applying partial integration L times in (6.2), we get the lemma. ∎

Lemma 6.3: *Assume that h is an even entire function satisfying that for every fixed $A, B > 0$ the function $|h(z)|e^{|z|A}$ is bounded on the strip $|\mathrm{Im} z| \leq B$. Then the function*

$$m(Y) := \int_{-\infty}^{\infty} F\left(\frac{1}{2} - it, \frac{1}{2} + it, 1, -Y\right) h(t)\left(\tanh \pi t\right) t dt$$

satisfies the conditions of Lemma 3.3. Moreover, $m(Y)$ is a smooth function on $[0, \infty)$, and there are constants $c_{l,r} > 0$ for every integer pair $l, r \geq 0$ such that

$$\left| m^{(l)}(Y)(1 + |Y|)^r \right| \leq c_{l,r}$$

for every $\mathrm{Re} Y > 0$, and also for $Y = 0$.

Proof. Using [G-R], p. 999, formula 9.132.2 in the case $|Y| \geq \frac{1}{2}$ for our hypergeometric function, and using the integral representation [G-R], p. 995, formula 9.111 for the two new hypergeometric functions, estimating trivially, we easily see that the double integral

$$\int_0^\infty \int_{-\infty}^{\infty} F\left(\frac{1}{2} - it, \frac{1}{2} + it, 1, -Y\right) h(t)\left(\tanh \pi t\right) t Y^{s-1} dt dY$$

is absolutely convergent for $0 < \mathrm{Re}s < \frac{1}{2}$. Hence in this strip we can apply [G-R], p. 806, formula 7.511, and get (with m^* as in Lemma 3.3)

$$m^*(s) = \frac{\Gamma(s)}{\Gamma(1-s)} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2}+it-s\right)\Gamma\left(\frac{1}{2}-it-s\right)}{\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right)} h(t)\left(\tanh\pi t\right) tdt. \tag{6.3}$$

By the identity

$$\frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{2\pi}\Gamma^2(s)\left(f(s,t)+f(s,-t)\right),$$

where

$$f(s,t) = \frac{\Gamma(it)\Gamma(1-it)}{\Gamma\left(\frac{1}{2}+it-s\right)\Gamma\left(\frac{1}{2}-it+s\right)}$$

(this is true, since for fixed t the right-hand side divided by the left-hand side is easily seen to be a bounded entire function of s periodic with respect to 1, and for $s = \frac{1}{2}$ we have equality), using

$$\tanh\pi t = \frac{\Gamma\left(\frac{1}{2}+it\right)\Gamma\left(\frac{1}{2}-it\right)}{i\Gamma(it)\Gamma(1-it)},$$

we also get for $0 < \mathrm{Re}s < \frac{1}{2}$ that

$$m^*(s) = \frac{\Gamma^2(s)}{\pi i} \int_{-\infty}^{\infty} \frac{\Gamma\left(\frac{1}{2}-it-s\right)}{\Gamma\left(\frac{1}{2}-it+s\right)} h(t)tdt. \tag{6.4}$$

When $0 < \mathrm{Re}s < \frac{1}{2}$, we can replace here the line of integration to $\mathrm{Im}t = E$ with any positive number E. Using (6.3) for $\mathrm{Re}s < \frac{1}{2}$, and (6.4) for $\mathrm{Re}s > 0$, we obtain that $\frac{m^*(s)}{\Gamma(s)}$ is entire and, for arbitrary fixed real numbers $\sigma_1 < \sigma_2$, the function $m^*(s)e^{\frac{\pi}{2}|\mathrm{Im}s|}|s|^K$ is bounded in the strip $\sigma_1 < \mathrm{Re}s < \sigma_2$ for every fixed $K > 0$. By Mellin inversion and by differentiating, for any $l \geq 0$ and $\sigma > 0$ we get (3.8) for $Y > 0$. By shifting the contour there appropriately, using the already proved properties of $m^*(s)$ we get the required properties of m. The lemma is proved. ∎

Appendix 1

We show the following form of the theorem of [K-S] (with a modified constant in the $s < 0$ case).

Theorem A1 (Katok–Sarnak): *Let ϕ be an even Hecke normalized Maass cusp form and simultaneous Hecke eigenform of weight 0 for $\mathrm{SL}(2,\mathbf{Z})$ with $\Delta_0\phi = \lambda\phi$, and let s be a nonzero integer, $s \equiv 0, 1 \pmod 4$. If we write*

$(\phi,\phi) = \int_{D_1} |\phi(z)|^2 d\mu_z$, *then, if* $dS = |dz|/y$ *is the hyperbolic arc length, we have*

$$\frac{1}{(\phi,\phi)} \sum_{Q\in\Lambda_s} \int_{C_Q} \phi dS = 12\pi^{\frac{1}{2}} s^{\frac{3}{4}} \sum_{\mathrm{Shim} F_j=\phi} \rho_j(s)\overline{\rho_j(1)}$$

for $s > 0$, *and*

$$\frac{1}{(\phi,\phi)} \sum_{Q\in\Lambda_s} \frac{\phi(z_Q)}{M_Q} = 12\,|s|^{\frac{3}{4}} \sum_{\mathrm{Shim} F_j=\phi} \rho_j(s)\overline{\rho_j(1)}$$

for $s < 0$.

Proof. As in [K-S], formula (2.9), for $z = u+iv \in H$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbf{R})$ let

$$\Theta(z,g) = v^{\frac{3}{4}} \sum_{h=(h_1,h_2,h_3)\in\mathbf{Z}^3} e\left(u\left(h_2^2 - 4h_1h_3\right)\right) f_3\left(\sqrt{v}g^{-1}h\right),$$

where, since

$$g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

by (2.8) of [K-S] we have

$$g^{-1}h = \begin{pmatrix} d^2 & -bd & b^2 \\ -2cd & ad+bc & -2ab \\ c^2 & -ac & a^2 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix},$$

and $f_3(x_1,x_2,x_3) = \exp\left(-2\pi\left(2x_1^2 + x_2^2 + 2x_3^2\right)\right)$.

For $z = u+iv \in H$ and $w = x+iy \in H$ let

$$\Theta(z,w) = \Theta(z,T_w).$$

An easy computation then shows that

$$f_3\Big(\sqrt{v}T_w^{-1}h\Big) = \exp\Big(-2\pi v\Big(2\Big(\frac{h_1 - h_2x + h_3x^2}{y}\Big)^2 + (h_2 - 2xh_3)^2 + 2(yh_3)^2\Big)\Big).$$

It is not hard to check the identity

$$2\left(\frac{h_1 - h_2x + h_3x^2}{y}\right)^2 + (h_2 - 2xh_3)^2 + 2\,(yh_3)^2$$
$$= \frac{2\left|h_3w^2 - h_2w + h_1\right|^2}{\mathrm{Im}^2 w} - \left(h_2^2 - 4h_1h_3\right).$$

Hence, $\Theta(z,w)$ equals

$$v^{\frac{3}{4}} \sum_{h_1,h_2.h_3\in\mathbf{Z}} e\left(u\left(h_2^2-4h_1h_3\right)\right)$$
$$\times \exp\left(2\pi v\left(h_2^2-4h_1h_3-\frac{2\left|h_3w^2-h_2w+h_1\right|^2}{\mathrm{Im}^2 w}\right)\right).$$

For $z\in H$ let

$$F(z)=\int_{D_1}\Theta(z,w)\phi(w)d\mu_w,$$

and for $v>0$ define

$$\mu_s(v)=\int_0^1 F(u+iv)e(-su)du.$$

Then from the above considerations, we have

$$\mu_s(v)=v^{\frac{3}{4}}e^{2\pi sv}\int_{D_1}\Bigg(\sum_{\substack{h_1,h_2,h_3\in\mathbf{Z}\\ h_2^2-4h_1h_3=s}}\exp\left(\frac{-4\pi v\left|h_3w^2-h_2w+h_1\right|^2}{\mathrm{Im}^2 w}\right)\Bigg)\phi(w)d\mu_w.$$

Let t and n be positive integers such that $s=t^2-4n$. Then, from the correspondence between $\Gamma_{n,t}$ and Q_s, using that for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma_{n,t}$ we have

$$\frac{|w-\gamma w|^2}{4\mathrm{Im}w\mathrm{Im}\gamma w}=\frac{\left|cw^2+(d-a)w-b\right|^2}{4n\mathrm{Im}^2 w},$$

we see that

$$\sum_{\substack{h_1,h_2,h_3\in\mathbf{Z}\\ h_2^2-4h_1h_3=s}}\exp\left(-4\pi v\frac{\left|h_3w^2-h_2w+h_1\right|^2}{\mathrm{Im}^2 w}\right)=\sum_{\gamma\in\Gamma_{n,t}} m\left(\frac{|w-\gamma w|^2}{4\mathrm{Im}w\mathrm{Im}\gamma w}\right)$$

with

$$m(t)=\exp\left(-16\pi nvt\right)$$

for $t\in[0,\infty)$. Then, by Lemma 3.2, for $s>0$ we have

$$\mu_s(v)=v^{\frac{3}{4}}e^{2\pi sv}\left(\sum_{Q\in\Lambda_s}\int_{C_Q}\phi dS\right)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\exp\left(-\frac{4\pi sv}{\cos^2\theta}\right)f_\lambda(\theta)\frac{d\theta}{\cos^2\theta},$$

while for $s<0$ we have

$$\mu_s(v)=v^{\frac{3}{4}}e^{2\pi sv}\left(\sum_{Q\in\Lambda_s}\frac{2\pi}{M_Q}\phi\left(z_Q\right)\right)\int_0^\infty \exp\left(-4\pi\left|s\right|v\sinh^2 r\right)g_\lambda(r)\sinh r dr.$$

Now, by the substitution $x = \sqrt{4\pi sv}\frac{\sin\theta}{\cos\theta}$ we have

$$\int_0^{\frac{\pi}{2}} \exp\left(-\frac{4\pi sv}{\cos^2\theta}\right) f_\lambda(\theta)\frac{d\theta}{\cos^2\theta} = \frac{e^{-4\pi sv}}{\sqrt{4\pi sv}}\int_0^\infty e^{-x^2} f_\lambda\left(\arctan\frac{x}{\sqrt{4\pi sv}}\right)dx,$$

and by the substitution $x = \sqrt{4\pi\,|s|\,v}\sinh r$ we have

$$\int_0^\infty \exp\left(-4\pi\,|s|\,v\sinh^2 r\right) g_\lambda(r)\sinh r dr$$
$$= \frac{1}{4\pi\,|s|\,v}\int_0^\infty e^{-x^2} g_\lambda^\star\left(\operatorname{arsh}\frac{x}{\sqrt{4\pi\,|s|\,v}}\right)xdx,$$

where $g_\lambda^\star(r) = \frac{g_\lambda(r)}{\cosh r}$. Since

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}, \qquad \int_0^\infty e^{-x^2}xdx = \frac{1}{2},$$

using that the functions $f_\lambda(\theta)$ and $g_\lambda(r)$ are bounded (see [B1], p. 336), we finally get, as $v\to\infty$:

$$\mu_s(v) = (1+o(1))\frac{1}{2\sqrt{s}}v^{\frac{1}{4}}e^{-2\pi sv}\left(\sum_{Q\in\Lambda_s}\int_{C_Q}\phi dS\right)$$

for $s>0$, and

$$\mu_s(v) = (1+o(1))\frac{1}{4\,|s|}v^{-\frac{1}{4}}e^{-2\pi|s|v}\left(\sum_{Q\in\Lambda_s}\frac{\phi(z_Q)}{M_Q}\right)$$

for $s<0$. The relation for $s>0$ is in accordance with (3.28) of [K-S]. However, we see that the relation for $s<0$ differs from (3.13) of [K-S]. Following the steps on pp. 224–225 of [K-S], we get the theorem. ∎

Appendix 2

In this appendix we use the Maass operators

$$K_k = (z-\overline{z})\frac{\partial}{\partial z} + k = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + k,$$

$$L_k = (\overline{z}-z)\frac{\partial}{\partial \overline{z}} - k = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - k,$$

and the identities involving these operators in [F], pp. 145–146.

Lemma A2: *Let f be a fixed cusp form of weight $\frac{1}{2}$ for $\Gamma_0(4)$ with*

$$\Delta_{\frac{1}{2}} f = s(s-1)f$$

for some $\mathrm{Re}s = \frac{1}{2}$, $s = \frac{1}{2} + it$. *Then there is a constant C such that*

$$\int_{D_4} u_j(z) f(z)\overline{B_0(z)} d\mu_z \ll e^{-\frac{\pi}{2}t_j} (1+t_j)^C$$

as $j \to \infty$, and for any cusp a

$$\int_{D_4} E_a\left(z, \frac{1}{2} + ir\right) f(z)\overline{B_0(z)} d\mu_z \ll e^{-\frac{\pi}{2}r} (1+r)^C$$

as $r \to +\infty$.

Proof. Let

$$K(z,w) = \sum_{\gamma \in \Gamma_0(4)} k(\gamma z, w),$$

where

$$k(\mathrm{z}, w) = k\left(\frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}\right)$$

with

$$k(y) = \frac{1}{4\pi}\int_{-\infty}^{\infty} F\left(\frac{1}{2} - i\tau, \frac{1}{2} + i\tau, 1, -y\right) h(\tau)\tau \tanh \pi\tau d\tau \tag{A2.1}$$

as in [I], (1.64′), and let here

$$h(\tau) = e^{-(\tau - T)^2} + e^{-(\tau + T)^2}$$

with a fixed (large) real T. It is not hard to see that it is enough to prove that if

$$M(w) := \int_{D_4} f(z)\overline{B_0(z)} K(z,w) d\mu_z, \tag{A2.2}$$

then for $T \le t_j, r \le T+1$ we have

$$\int_{D_4} M(w) u_j(w) d\mu_w \ll e^{-\frac{\pi}{2}T} (1+T)^C, \tag{A2.3}$$

and we have

$$\int_{D_4} M(w) E_a\left(w, \frac{1}{2} + ir\right) d\mu_w \ll e^{-\frac{\pi}{2}T} (1+T)^C. \tag{A2.4}$$

By unfolding we see that (A2.2) equals 2 times

$$\int_H f(z)\overline{B_0(z)} k(\mathrm{z}, w) d\mu_z. \tag{A2.5}$$

The integrand here can be written as

$$\overline{\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)}\left(f(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)k\left(\frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w}\right).$$

We now use geodesic polar coordinates around w:

$$\frac{z-w}{z-\overline{w}}=\tanh\left(\frac{r}{2}\right)e^{i\phi}.$$

Using the substitution

$$y=\frac{\tanh^2(\frac{r}{2})}{1-\tanh^2(\frac{r}{2})}$$

we get that (A2.5) equals

$$\text{(A2.6)}\qquad 2\int_0^\infty k\left(y\right)\left(\int_0^{2\pi}\overline{\left(B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)}\left(f(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}\right)d\phi\right)dy,$$

where $0<r<\infty$ and $z\in H$ are determined from y and ϕ by the above relations. By [F], Theorems 1.1 and 1.2 we have

$$B_0(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}=\sum_{l=0}^{\infty}\left(\tanh\left(\frac{r}{2}\right)\right)^l\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^{\frac{1}{4}}B_l(w)e^{il\phi}$$

and

$$f(z)\left(\frac{z-\overline{w}}{w-\overline{z}}\right)^{\frac{1}{4}}=\sum_{m=-\infty}^{\infty}\left(f\right)_m(w)P^m_{s,\frac{1}{4}}(z,w)e^{im\phi},$$

where

$$B_n=\frac{1}{n!}K_{(n-1)+\frac{1}{4}}\cdots K_{\frac{5}{4}}K_{\frac{1}{4}}B_0;$$

and for any $l\geq 0$ we have

$$\begin{aligned}P^l_{s,\frac{1}{4}}(z,w)=\left(\tanh\left(\frac{r}{2}\right)\right)^l&\left(1-\tanh^2\left(\frac{r}{2}\right)\right)^s\\&\times F\left(s-\frac{1}{4},s+\frac{1}{4}+l,1+l,\tanh^2\left(\frac{r}{2}\right)\right),\end{aligned}$$

and so by [G-R], p. 998, 9.131.1 we finally get that (A2.6), and so (A2.5), equals

$$\text{(A2.7)}\qquad 4\pi\sum_{l=0}^{\infty}\overline{B_l(w)}\left(f\right)_l(w)\int_0^\infty y^lk(y)F\left(\frac{3}{4}-it+l,\frac{3}{4}+it+l,1+l,-y\right)dy.$$

If τ is a real number, then in the integral

(A2.8)
$$\int_0^\infty y^l F\left(\frac{1}{2}-i\tau,\frac{1}{2}+i\tau,1,-y\right) F\left(\frac{3}{4}-it+l,\frac{3}{4}+it+l,1+l,-y\right)dy$$

we express the first hypergeometric function by [G-R], p. 995, 9.113 as

$$\frac{1}{\Gamma\left(\frac{1}{2}\pm i\tau\right)}\frac{1}{2\pi i}\int_{(\sigma)}\frac{\Gamma\left(\frac{1}{2}\pm i\tau+S\right)\Gamma\left(-S\right)}{\Gamma\left(1+S\right)}y^S dS$$

with a $-\frac{1}{2}<\sigma<-\frac{1}{4}$. Then we compute the resulting integral in y by [G-R], p. 806, 7.511, and attain that (A2.8) equals

$$\frac{\Gamma\left(1+l\right)}{2\pi i\Gamma\left(\frac{1}{2}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm it+l\right)} \times\int_{(\sigma)}\frac{\Gamma\left(\frac{1}{2}\pm i\tau+S\right)\Gamma\left(-\frac{1}{4}\pm it-S\right)}{\Gamma\left(1+S\right)}\Gamma\left(1+l+S\right)dS.$$

We use the identity

$$\frac{\Gamma\left(1+l+S\right)}{\Gamma\left(1+S\right)}=\sum_{j=0}^{l}(-1)^j\binom{l}{j}\frac{\Gamma\left(\frac{3}{4}+it+l\right)\Gamma\left(-\frac{1}{4}+it+j-S\right)}{\Gamma\left(\frac{3}{4}+it+j\right)\Gamma\left(-\frac{1}{4}+it-S\right)},$$

which follows from [S], (1.7.7), and computing the integral for every given j by [G-R], p. 644, 6.412, we get that (A2.8) equals

$$\frac{\Gamma\left(1+l\right)\Gamma\left(\frac{1}{4}\pm i\tau\pm it\right)\left(\frac{3}{4}+it\right)_l}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\pm i\tau\right)\Gamma\left(\frac{3}{4}\pm it+l\right)}F_{3,2}\left(\begin{matrix}-l,\frac{1}{4}-i\tau+it,\frac{1}{4}+i\tau+it\\ \frac{1}{2},\frac{3}{4}+it\end{matrix};1\right).$$

It then follows by (A2.7), (A2.1), (A2.2), (A2.5), [F], Theorems 1.1 and 1.2 and by Corollary 3.3.5 of [A-A-R] that

$$M(w)=C\sum_{l=0}^{\infty}\overline{B_l(w)}\frac{\left(K_{l-1+\frac{1}{4}}\dots K_{\frac{5}{4}}K_{\frac{1}{4}}f\right)(w)}{\Gamma\left(\frac{1}{2}+l\right)\Gamma\left(\frac{3}{4}\pm it\right)}M_l \tag{A2.9}$$

with a nonzero absolute constant C, and

$$M_l=\int_{-\infty}^{\infty}F_{3,2}\left(\begin{matrix}-l,\frac{1}{2}-i\tau,\frac{1}{2}+i\tau\\ \frac{3}{4}-it,\frac{3}{4}+it\end{matrix};1\right)\frac{\Gamma\left(\frac{1}{4}\pm i\tau\pm it\right)}{\Gamma\left(\frac{1}{2}\pm i\tau\right)}h(\tau)\tau\tanh\pi\tau d\tau.$$

Since

$$\tau\tanh\pi\tau=\frac{\Gamma\left(\frac{1}{2}\pm i\tau\right)}{\Gamma\left(\pm i\tau\right)},$$

we have that M_l equals a nonzero absolute constant times
(A2.10)
$$\frac{1}{\left(\frac{3}{4}\pm it\right)_l}\int_{-\infty}^{\infty} h(\tau)\frac{\Gamma\left(\frac{1}{2}\pm i\tau\right)\Gamma\left(\frac{1}{4}\pm i\tau\pm it\right)}{\Gamma\left(\pm 2i\tau\right)} S_l\left(\tau^2;\frac{1}{2},\frac{1}{4}-it,\frac{1}{4}+it\right)d\tau$$
with a continuous dual Hahn polynomial (see, e.g., [N], the beginning of Section 4, or [Ko-S]). We use the well-known difference equation for continuous dual Hahn polynomials (see [N], (4.1)):
$$lS_l\left(\tau^2\right)=B(\tau)S_l\left((\tau+i)^2\right)-\left(B(\tau)+D(\tau)\right)S_l\left(\tau^2\right)+D(\tau)S_l\left((\tau-i)^2\right),$$
where we write
$$S_l\left(\tau^2\right)=S_l\left(\tau^2;\frac{1}{2},\frac{1}{4}-it,\frac{1}{4}+it\right),$$
$$B(\tau)=\frac{(a-i\tau)(b-i\tau)(c-i\tau)}{(-2i\tau)(1-2i\tau)},\quad D(\tau)=\frac{(a+i\tau)(b+i\tau)(c+i\tau)}{(2i\tau)(1+2i\tau)},$$
$$a=\frac{1}{2},\quad b=\frac{1}{4}-it,\quad c=\frac{1}{4}+it.$$
Substitution into (A2.10) and shifting the integration to $\mathrm{Im}\tau=-1$ in the case of $S_l\left((\tau+i)^2\right)$, and to $\mathrm{Im}\tau=1$ in the case of $S_l\left((\tau-i)^2\right)$ (we do not cross any pole), we can express lM_l by an expression of type (A2.10), but with a new (even entire) function in place of h. We iterate this step many times, and then we apply Cauchy–Scwarz inequality and use
$$\frac{1}{2\pi}\int_0^{\infty}\frac{\Gamma\left(\frac{1}{2}\pm i\tau\right)\Gamma\left(\frac{1}{4}\pm i\tau\pm it\right)}{\Gamma\left(\pm 2i\tau\right)}S_l^2\left(\tau^2\right)d\tau=\Gamma\left(l+\frac{1}{2}\right)\left|\Gamma\left(l+\frac{3}{4}+it\right)\right|^2 l!;$$
see, e.g., [Ko-S]. We can prove in this way that for any $A>0$ we have a $B>0$ such that
$$M_l\ll e^{-\frac{\pi}{2}T}\left(1+T\right)^B\left(1+l\right)^{-A}.$$
Therefore, it follows from (A2.9) that for the proof of (A2.3) and (A2.4) it is enough to see that there is a C such that for $T\le t_j, r\le T+1$ we have
$$\int_{D_4}\left|B_l(w)\frac{\left(K_{l-1+\frac{1}{4}}\cdots K_{\frac{5}{4}}K_{\frac{1}{4}}f\right)(w)}{\Gamma\left(\frac{1}{2}+l\right)}u_j(w)\right|d\mu_w\ll\left(1+l+T\right)^C,$$
$$\int_{D_4}\left|B_l(w)\frac{\left(K_{l-1+\frac{1}{4}}\cdots K_{\frac{5}{4}}K_{\frac{1}{4}}f\right)(w)}{\Gamma\left(\frac{1}{2}+l\right)}E_a\left(w,\frac{1}{2}+ir\right)\right|d\mu_w\ll\left(1+l+T\right)^C.$$

We write every $w \in D_4$ as $w = \gamma_j z$, $z \in D_1$ (see the beginning of Section 5), where D_1 is the standard fundamental domain of the full modular group. It is easy to see that there is a C such that

$$u_j(w) \ll (1+T)^C$$

for every $w \in D_4$, and

$$E_a\left(w, \frac{1}{2}+ir\right) \ll (1+l+T)^C$$

for $w \in D_4$, Im$z \leq l^{10}$ (say). Then the required estimates on these sets follow by Cauchy's inequality and [F], p. 146, formula (11). It remains to estimate the second integral for Im$z \geq l^{10}$. If Im$z \geq l^{10}$, then we have

$$\frac{\left(K_{l-1+\frac{1}{4}} \cdots K_{\frac{5}{4}} K_{\frac{1}{4}} f\right)(w)}{\Gamma\left(\frac{1}{2}+l\right)} E_a\left(w, \frac{1}{2}+ir\right) \ll (1+l+T)^C$$

with a C. In fact, we have a much better estimate, since the first factor is very small there, as one can see by a polynomial upper bound for the Fourier coefficients of f (which can be proved for positive Fourier coefficients by formula (83) of [P], using there the function ϕ from [D], Section 5; the negative Fourier coefficients can be estimated in the same way, considering $\overline{Lf}$ in place of f), by the description of the action on Fourier coefficients of the Maass operators on p. 78 of [D] (use also $K_{-k}\overline{f}=\overline{L_k f}$), and by a very small upper bound (obtained trivially from [G-R], p. 1015, 9.223) for the involved W-functions. The required estimate follows then again by [F], p. 146, formula (11).

The lemma is proved. ∎

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