Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Journal of Number Theory 121 (2006) 324–354
Journal of Number Theory
www.elsevier.com/locate/jnt

Strong characterizing sequences for subgroups of compact groups ☆

András Biró

A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13-15, 1053 Budapest, Hungary

Received 1 June 2004; revised 5 August 2005

Available online 9 June 2006

Communicated by David Gross

Abstract

In [A. Biró, V.T. Sós, Strong characterizing sequences in simultaneous Diophantine approximation, J. Number Theory 99 (2003) 405–414] we proved that if Γ is a subgroup of the torus $\mathbf{R}/\mathbf{Z}$ generated by finitely many independent irrationals, then there is an infinite subset $A \subseteq \mathbf{Z}$ which characterizes Γ in the sense that for $\gamma \in \mathbf{R}/\mathbf{Z}$ we have $\sum_{a\in A} \|a\gamma\| < \infty$ if and only if $\gamma \in \Gamma$. Here we consider a general compact metrizable Abelian group G instead of $\mathbf{R}/\mathbf{Z}$, and we characterize its finitely generated free subgroups Γ by subsets $A \subseteq G^*$, where G^* is the Pontriagin dual of G. For this case we prove stronger forms of the analogue of the theorem of the above mentioned work, and we find necessary and sufficient conditions for a kind of strengthening of this statement to be true.

MSC: 11J99; 22B99

Keywords: Strong characterizing sequences; Compact Abelian groups

0. Introduction

Let $T = \mathbf{R}/\mathbf{Z}$, where $\mathbf{R}$ denotes the additive group of the real numbers, $\mathbf{Z}$ is the subgroup of the integers. If $x \in \mathbf{R}$, then $\|x\|$ denotes its distance to the nearest integer, it is well defined also on T.

☆ Research partially supported by the Hungarian National Foundation for Scientific Research (OTKA) Grants Nos. T032236, T 042750, T043623 and T049693.

E-mail address: biroand@renyi.hu.

0022-314X/$ – see front matter
doi:10.1016/j.jnt.2006.03.008

Denote by $\chi_{[a,b]}$ the characteristic function of a real interval $[a, b]$. If v is a nonnegative function on $[0, \frac{1}{2}]$, and $A \subseteq \mathbf{Z}$, write

$$C_{T,A}(v) = \left\{ \gamma \in T\colon \sum_{a\in A} v\big(\|a\gamma\|\big) < \infty \right\}.$$

We now state more precisely the result of [B-S] mentioned in the abstract. We proved the following theorem with different notations. In fact we stated a somewhat weaker theorem there, but the same proof gives this statement.

Theorem. (*See* [B-S].) *Assume that $\Gamma \leqslant T$ is a finitely generated free Abelian group. Then there is an infinite subset $A \subseteq \mathbf{Z}$ such that for $\gamma \in T$ we have $\sum_{a\in A} \|a\gamma\| < \infty$ if and only if $\gamma \in \Gamma$, moreover, if $\|a\gamma\| < 1/10$ for all but finitely many $a \in A$, then we already know $\gamma \in \Gamma$. In other words, writing $v_1(x) = x$ and $v_2(x) = \chi_{[1/10,1/2]}(x)$ for $0 \leqslant x \leqslant \frac{1}{2}$, we have*

$$C_{T,A}(v_1) = C_{T,A}(v_2) = \Gamma. \tag{0.1}$$

Note that the statement of the theorem in [B-S] contains a misprint: lim inf should be replaced by lim sup there.

Such a set A was called a strong characterizing sequence of the subgroup Γ. The existence of strong characterizing sequences for any countable $\Gamma \leqslant T$ was proved in [Be].

We started to study this subject in [B-D-S]. We proved there that for any countable $\Gamma \leqslant T$ there is an infinite $A \subseteq \mathbf{Z}$ such that for $\gamma \in T$ we have $\|a\gamma\| \to 0$ if and only if $\gamma \in \Gamma$, moreover, if $\|a\gamma\| < 1/10$ for all but finitely many $a \in A$, then we already know $\gamma \in \Gamma$. Such an A was called a characterizing sequence, and one can easily see that to be a strong characterizing sequence is indeed a stronger property.

In the present paper, we deal with generalizations of *strong* characterizing sequences for compact metrizable Abelian groups. Our results here are not only generalizations, but also strengthening of the quoted theorem of [B-S], since, under some assumptions, we give necessary and sufficient conditions for more general pairs (v_1, v_2) of nonnegative increasing functions defined on $[0, \frac{1}{2}]$ (in place of the specific functions used in the [B-S] theorem) for which an analogous theorem is true.

We now describe briefly the results of the paper, but for simplicity, only for the case of T.

The cited theorem of [B-S] essentially means that the elements of the sequence $\|a\gamma\|_{a\in A}$ are asymptotically small (as $|a| \to \infty$) for $\gamma \in \Gamma$, but this is false for $\gamma \in T \setminus \Gamma$. Hence, the goal is to seek such an A for which the behavior is radically different in Γ and outside Γ. In this paper, we use the two functions v_1 and v_2 to measure the order of magnitude of $\|a\gamma\|_{a\in A}$ in the two parts Γ and $T \setminus \Gamma$, respectively.

Let V be the set of those real-valued, strictly increasing, continuous functions v on $[0, \frac{1}{2}]$ satisfying $v(0) = 0$ for which we have $v(2x) \ll v(x)$. We impose the last condition to ensure that $C_{T,A}(v)$ is always a subgroup of T.

Our Theorem 1 contains a characterization of functions $v_1 \in V$ for which there is an infinite $A \subseteq \mathbf{Z}$ satisfying (0.1) with this v_1, but maintaining $\chi_{[1/10,1/2]}$ as v_2.

If there is such a set A, this means that $C_{T,A}(v_1) = \Gamma$ with the stronger property that for $\gamma \in T \setminus \Gamma$ we even have that $\|a\gamma\|$ does not tend to 0. It turns out in Theorem 1 that if $v_1(x)$ tends to 0 sufficiently slowly as $x \to 0$ (e.g., $v_1(x) = \frac{1}{\log \frac{1}{x}}$), then there is no such A. However, it still may happen that for another function $v_2 \in V$, which tends to 0 faster than $v_1(x)$ as $x \to 0$, there is an A satisfying (0.1). If this is the case, then $C_{T,A}(v_1) = \Gamma$ with the stronger property that for $\gamma \in T \setminus \Gamma$ we even have $\sum_{a\in A} v_2(\|a\gamma\|) = \infty$. Such possibilities are analyzed in Theorem 2. See Example 1 in Section 1 for the case of $v_1(x) = \frac{1}{\log \frac{1}{x}}$.

In the extremal case when $v_2 = v_1$, we simply ask about the possibility of $C_{T,A}(v_1) = \Gamma$, without any stronger, additional property. Our results contain an interesting necessary and sufficient condition for this case, see Example 2 in Section 1.

We will see in Example 2 that $C_{T,A}(v) = \Gamma$ is impossible for some $v \in V$. However, in Theorem 3 we show that for any $v \in V$, it is possible to characterize Γ in a certain new sense with a set A satisfying $\Gamma \subseteq C_{T,A}(v)$.

It is remarkable that while our main interest lies in the case of T, we could not prove our statements directly for T. It turned out that it is easier to deal with the case of $\mathbf{Z_2}$ (the additive group of the 2-adic integers). In Section 2, we prove the theorems for this special group and its infinite cyclic subgroup $\Gamma = \mathbf{Z}$. Then, we will show in Section 3 that it is possible to extend the theorems from this seemingly very special case, using a certain transfer principle, to any compact metrizable Abelian group and its any finitely generated free dense subgroup.

The result from [Be] on countable subgroups of T raises the question whether the results proved here for finitely generated free dense subgroups could be extended for countable subgroups.

On the other hand, it would be nice to characterize the "good" pairs (v_1, v_2) in more general classes of functions. In particular, it would be interesting to prove Theorem 2 without the assumption $v_1(x^2) \gg v_1(x)$, especially in the case $v_1 = v_2$.

For generalizations of characterizing sequences for subgroups of more general topological groups, see [D-M-T,D-K,B-S-W]. For characterizations of subgroups of T in a different sense (with filters on the positive integers instead of subsets of $\mathbf{Z}$) see [W, Theorem 1].

1. Notations and statements of the results

In all of the theorems, G is a compact metrizable Abelian group, G^* is its character group (or Pontriagin dual, see, e.g., [R]), i.e., the group of continuous homomorphisms from G to T. It is well known that the property that G is metrizable is equivalent to the condition that G^* is countable (see again [R]). If $a \in G^*$, $\gamma \in G$, we write $a\gamma$ for the value of a at γ.

If $A \subseteq G^*$ is an infinite subset, v is a nonnegative function on $[0, \frac{1}{2}]$, and $\alpha_1, \alpha_2, \ldots, \alpha_t \in G$ generate a dense subgroup of G, let

$$C_{G,A}(v) = \left\{\gamma \in G\colon \sum_{a\in A} v\bigl(\|a\gamma\|\bigr) < \infty\right\}, \qquad L_{G,A} = \left\{\gamma \in G\colon \lim_{a\in A} \|a\gamma\| = 0\right\}$$

(here $\lim_{a\in A} \|a\gamma\| = 0$ means that for every $\epsilon > 0$ there are only finitely many $a \in A$ with $\|a\gamma\| \geqslant \epsilon$), and

$$B_{G,A}(\alpha_1, \alpha_2, \ldots, \alpha_t) = \left\{\gamma \in G\colon \sup_{a\in A,\, a\neq 0} \frac{\|a\gamma\|}{\max(\|a\alpha_1\|, \|a\alpha_2\|, \ldots, \|a\alpha_t\|)} < \infty\right\}$$

(here the denominator is obviously nonzero for $a \neq 0$ by the conditions).

To be precise, we repeat that V is the set of those real-valued, strictly increasing, continuous functions v on the interval $[0, \frac{1}{2}]$ satisfying $v(0) = 0$ for which there is a positive number

$K(v) > 0$ such that for every $x \geqslant 0$ we have (writing, for the sake of convenience, $v(y) = v(\frac{1}{2})$, if $y > \frac{1}{2}$)

$$v(2x) \leqslant K(v)v(x). \tag{1.1}$$

Condition (1.1) ensures that $C_{G,A}(v)$ is a subgroup of G. We can now state the results.

Theorem 1.

(i) *Assume that $\Gamma \leqslant G$ is a finitely generated free Abelian group, Γ is dense in G, and suppose that $v \in V$ satisfies*

$$\sum_{n=1}^{\infty} v\left(2^{-n}\right) < \infty. \tag{1.2}$$

Then there is an infinite subset $A \subseteq G^*$ such that $C_{G,A}(v) = C_{G,A}(\chi_{[\frac{1}{4},\frac{1}{2}]}) = \Gamma$.

(ii) *If Γ is an infinite cyclic dense subgroup of G, $v \in V$ satisfies*

$$\sum_{n=1}^{\infty} v\left(2^{-n}\right) = \infty, \tag{1.3}$$

and $A \subseteq G^*$ is such an infinite subset that $\Gamma \subseteq C_{G,A}(v)$, then $|L_{G,A}| = 2^{\aleph_0}$.

Theorem 1(i) in the case of $G = T$ is a more precise form of [B-S, Theorem]. We mention that analyzing the proof in [B-S] (which used the Freiman–Ruzsa theorem) we can see that Theorem 1(i) could be also proved by the method used there. However, Theorems 1(ii), and 2, 3 below are new results, even in the case of $G = T$.

If $v \in V$, then its inverse function, v^{-1} is defined on the interval $[0, v(\frac{1}{2})]$. We will write $v^{-1}(x) = \frac{1}{2}$ for $x > v(\frac{1}{2})$ (it is not important, but it will be convenient).

Theorem 2. *Let $v_1, v_2 \in V$ be such that*

$$v_2(x) \leqslant E v_1(x) \quad \text{for } 0 \leqslant x \leqslant \frac{1}{2} \tag{1.4}$$

with some constant $E > 0$, and there is a constant $q_1 > 0$ for which we have for every $0 \leqslant x \leqslant 1/2$ that

$$v_1\left(x^2\right) \geqslant q_1 v_1(x). \tag{1.5}$$

(i) *Assume that $\Gamma \leqslant G$ is a finitely generated free Abelian group, and Γ is dense in G. Assume also that v_1 and v_2 satisfy*

$$\sum_{n=1}^{\infty} v_1\left(\left(v_2^{-1}\left(\frac{1}{n}\right)\right)^n\right) < \infty. \tag{1.6}$$

Then there is an infinite subset $A \subseteq G^*$ such that $C_{G,A}(v_1) = C_{G,A}(v_2) = \Gamma$.

(ii) *If Γ is an infinite cyclic dense subgroup of G, v_1 and v_2 satisfy*

$$\sum_{n=1}^{\infty} v_1\left(\left(v_2^{-1}\left(\frac{1}{n}\right)\right)^n\right) = \infty, \tag{1.7}$$

and $A \subseteq G^*$ is such an infinite subset that $\Gamma \subseteq C_{G,A}(v_1)$, then $|C_{G,A}(v_2)| = 2^{\aleph_0}$.

Example 1. We write $\log x$ for $\log_2 x$. We see that (1.2) is true, if

$$v(x) = \frac{1}{\log^{1+\epsilon} \frac{1}{x}} \tag{1.8}$$

with an $\epsilon > 0$. However, if we put $\epsilon = 0$ in (1.8), then (1.3) will be true. So Theorem 1(i) cannot be applied for $v(x) = \frac{1}{\log \frac{1}{x}}$. But we can analyze on the basis of Theorem 2 how strong statement can be proved for this function. We find that condition (1.6) is true, if $0 < A < 1$ and

$$v_1(x) = \frac{1}{\log \frac{1}{x}}, \qquad v_2(x) = 2^{-\log^A(\frac{1}{x})}, \tag{1.9}$$

but (1.7) is valid, if $B > 0$ and

$$v_1(x) = \frac{1}{\log \frac{1}{x}}, \qquad v_2(x) = x^B. \tag{1.10}$$

Hence (since the other conditions are obviously true) Theorem 2(i) is applicable for the pair in (1.9), but Theorem 2(ii) is applicable for the pair in (1.10).

Example 2. We get another very interesting case if in Theorem 2 we take $v_1 = v_2 = v$ with a $v \in V$ for which

$$v(x^2) \geqslant q_1 v(x)$$

for $0 \leqslant x \leqslant \frac{1}{2}$ with some constant $q_1 > 0$. For such functions Theorem 2 gives a necessary and sufficient condition for the existence of an infinite $A \subseteq G^*$ with $C_{G,A}(v) = \Gamma$: we see that if

$$\sum_{n=1}^{\infty} v\left(\left(v^{-1}\left(\frac{1}{n}\right)\right)^n\right) < \infty, \tag{1.11}$$

then we can always find an A with $C_{G,A}(v) = \Gamma$, but if the series in (1.11) is divergent, then this is false. Let us consider the concrete example

$$v(x) = 2^{-(\log\log \frac{2}{x})^C}$$

with a constant $0 < C \leqslant 1$. It is not hard to see that (1.11) is true, if $\frac{1}{2} < C \leqslant 1$, but (1.11) is false, if $0 < C \leqslant \frac{1}{2}$.

Theorem 3. *Assume that $\Gamma \leqslant G$ is a finitely generated free Abelian group, Γ is dense in G, and let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be any system of free generators of Γ. Let $v \in V$ be arbitrary, then there is an infinite subset $A \subseteq G^*$ such that on the one hand we have $\Gamma \subseteq C_{G,A}(v)$, on the other hand, $B_{G,A}(\alpha_1, \alpha_2, \ldots, \alpha_t) = \Gamma$.*

Remark. If $G = T$, and Γ is generated by an irrational $\alpha \in T$, then it follows from [K-L] that taking $A = \{q_n\colon\ n \geqslant 1\}$, where q_n are the continued fraction denominators of α, we have $B_{G,A}(\alpha) = \Gamma$. However, if, $v(x)$ tends very slowly to 0 as x tends to 0, then $\Gamma \subseteq C_{G,A}(v)$ will be false. Therefore, even in this special case the theorem tells something interesting, but it is of course more interesting for $t > 1$.

2. A special case

2.1. Introductory remarks and preliminaries

During this whole section, we consider the special case $G = \mathbf{Z_2}$, $\Gamma = \mathbf{Z}$, where $\mathbf{Z_2}$ is the additive group of the 2-adic integers; for an introduction to p-adic numbers, see, e.g., [K]. Hence, if we speak about Theorems 1–3 in Section 2 (e.g., in the title of Sections 2.2 and 2.3), then we always mean this special case.

We introduce some notations. Let $T^{(2)}$ be the subgroup of T defined by

$$T^{(2)} = \left\{ \frac{a}{2^N} \in T\colon\ N \geqslant 0,\ 1 \leqslant a \leqslant 2^N,\ \left(a, 2^N\right) = 1 \right\}.$$

If $k = \sum_{j=0}^{\infty} b_j 2^j \in \mathbf{Z_2}$, where b_j is 0 or 1 for every j, and $r = \frac{a}{2^N} \in T^{(2)}$, then their product is given by

$$kr = \sum_{j=0}^{\infty} a b_j 2^{j-N} \in T.$$

It is meaningful, since 2^{j-N} is 0 in T for $j \geqslant N$. It is well known that $T^{(2)} = (\mathbf{Z_2})^*$, i.e., every continuous homomorphism $f\colon \mathbf{Z_2} \to T$ has the form $f(k) = kr$ with an element $r \in T^{(2)}$ for every $k \in \mathbf{Z_2}$; this statement is easy to prove, since continuity implies that there is an integer $N \geqslant 0$ such that $f(2^N k) = 0$ for every $k \in \mathbf{Z_2}$.

If $A \subseteq T^{(2)}$, v is a nonnegative function on $[0, \frac{1}{2}]$, then $C_{\mathbf{Z_2},A}(v)$ is abbreviated to $C_A(v)$ in Section 2.

We now prove a basic lemma. Part (i) characterizes the ordinary integers in $\mathbf{Z_2}$, through their expansions.

Lemma 2.1.

(i) *Let $k = \sum_{j=0}^{\infty} b_j 2^j \in \mathbf{Z_2}$, where b_j is 0 or 1 for every j. If $k \notin \mathbf{Z}$, then there are infinitely many j such that $b_j \neq b_{j+1}$.*

(ii) *Let J be a positive integer, and let $x \in \mathbf{R}$ be given by*

$$x = \sum_{0 \leqslant j \leqslant J-1} b_j 2^{j-J},$$

where b_j is 0 *or* 1 *for every* $0 \leqslant j \leqslant J-1$. *If there is a*

$$0 \leqslant t \leqslant J-2$$

such that $b_t \neq b_{t+1}$, *then*

$$\|x\| \geqslant 2^{t-J}.$$

Proof. Statement (i) follows easily from $\sum_{j=0}^{\infty} 2^j = -1$, which is true in $\mathbf{Z_2}$. To prove (ii), we note that since $b_t = 1$ or $b_{t+1} = 1$, so $x \geqslant 2^{t-J}$, and since $b_t = 0$ or $b_{t+1} = 0$, so

$$x \leqslant 1 - \sum_{0 \leqslant j \leqslant J-1} (1-b_j)2^{j-J} \leqslant 1 - 2^{t-J}.$$

This proves the lemma. □

In Section 2.2 we prove all the theorems except Theorem 2(i), which is the hardest statement. We present its proof in Section 2.3.

2.2. Proofs of Theorems 1, 3 and 2(ii)

We begin with proving the easiest statements, i.e., Theorems 1(i) and 3.

Proof of Theorem 1(i). Define $A \subseteq T^{(2)}$ by

$$A = \left\{ \frac{1}{2^J} \colon J \geqslant 1 \right\}.$$

Then (1.2) and $v \in V$ easily imply $\mathbf{Z} \subseteq C_A(v)$. If $k = \sum_{j=0}^{\infty} b_j 2^j \in \mathbf{Z_2}$, where b_j is 0 or 1 for every j, and $t \geqslant 0$ is such that $b_t \neq b_{t+1}$, then by Lemma 2.1(ii), taking $J = t+2$ we have

$$\left\| \frac{k}{2^J} \right\| \geqslant \frac{1}{4}.$$

In view of Lemma 2.1(i), this proves that if $k \notin \mathbf{Z}$, then there are infinitely many $r \in A$ with $\|kr\| \geqslant \frac{1}{4}$. This completes the proof. □

Proof of Theorem 3. Let H be an infinite subset of the positive integers with the property

$$\sum_{J \in H} v\left(\frac{1}{2^J}\right) < \infty. \tag{2.1}$$

It is clear that there is such a subset H. Define

$$A = \left\{ \frac{1}{2^J} \colon J \in H \right\}.$$

Then (2.1) and $v \in V$ easily imply $\mathbf{Z} \subseteq C_A(v)$. If $k = \sum_{j=0}^{\infty} b_j 2^j \in \mathbf{Z_2}$, where b_j is 0 or 1 for every j, and $t \geqslant 0$ is such that $b_t \neq b_{t+1}$, then by Lemma 2.1(ii), taking a $J \in H$ with $J \geqslant t+2$ we have

$$\left\| \frac{k}{2^J} \right\| \geqslant 2^t \left\| \frac{1}{2^J} \right\|.$$

In view of Lemma 2.1(i) this proves the theorem. □

For the proof of Theorems 1(ii) and 2(ii) we need two lemmas.

Lemma 2.2. *Let $v \in V$, assume that g is a positive, monotonically increasing continuous function on the real interval $[n_0, \infty)$ with some positive integer n_0, and*

$$\sum_{n=n_0}^{\infty} v\bigl(2^{-ng(n)}\bigr) = \infty. \tag{2.2}$$

Assume also that

$$A = \{r_1, r_2, \ldots, r_n, \ldots\} \subseteq T^{(2)}$$

is such that

$$\|r_1\| \geqslant \|r_2\| \geqslant \cdots \geqslant \|r_n\| \geqslant \cdots \quad \text{and} \tag{2.3}$$

$$\sum_{r \in A} v\bigl(\|r\|\bigr) < \infty. \tag{2.4}$$

Then, for any positive constant K there are infinitely many positive integers n such that with a suitable positive integer T we have

$$\|Tr_1\|, \|Tr_2\|, \ldots, \|Tr_n\| \leqslant 2^{-Kg(Kn)} \quad \text{and} \tag{2.5}$$

$$\|r_{n+1}\| \leqslant \frac{1}{T^2}. \tag{2.6}$$

Proof. Remark first that it is enough to prove the lemma for large enough K. Condition (2.2) is equivalent to

$$\int_{n_0}^{\infty} v\bigl(2^{-xg(x)}\bigr)\,\mathrm{d}x = \infty,$$

which is equivalent to

$$\sum_{n=n_0}^{\infty} v\bigl(2^{-4Kng(4Kn)}\bigr) = \infty, \tag{2.7}$$

therefore (2.7) is true. Then (2.4) and (2.7) imply that there are infinitely many integers n such that

$$\|r_n\| \leqslant 2^{-4Kng(4Kn)}. \tag{2.8}$$

We use Dirichlet's approximation theorem (pigeon-hole principle) in the form that if M is a positive integer, we can take an integer $1 \leqslant T \leqslant M^n$ such that

$$\|Tr_1\|, \|Tr_2\|, \ldots, \|Tr_n\| \leqslant \frac{1}{M}. \tag{2.9}$$

We apply it with

$$\mathrm{M} = \left[2^{2Kg(4Kn)}\right] \geqslant 2^{Kg(4Kn)}$$

(integer part), the inequality is true for every $n \geqslant n_0$ if K is large enough, and we obtain an integer

$$1 \leqslant \mathrm{T} \leqslant 2^{2Kng(4Kn)} \tag{2.10}$$

such that

$$\|Tr_1\|, \|Tr_2\|, \ldots, \|Tr_n\| \leqslant 2^{-Kg(4Kn)} \leqslant 2^{-Kg(Kn)}.$$

This, together with (2.3), (2.8) and (2.10), proves the lemma. □

Lemma 2.3. *Let ϕ be a positive valued function on the positive integers such that*

(i) *there is a constant $K > 0$ with the property that*

$$\phi(2m) \leqslant K\phi(m) \tag{2.11}$$

for every positive integer m;

(ii) *for any $\epsilon > 0$, there is a positive integer m such that*

$$\phi(m) < \epsilon. \tag{2.12}$$

Then, for any sequence of positive numbers ϵ_i, there is a sequence m_i of positive integers such that

$$\phi(m_i) < \epsilon_i \tag{2.13}$$

for every $i \geqslant 1$, and for every $i \geqslant 1$ there is an integer $t_i \geqslant 1$ satisfying

$$m_i < 2^{t_i}, \quad 2^{t_i} | m_{i+1}. \tag{2.14}$$

Moreover, if the integers m_i and t_i satisfy these conditions, then

$$\sum_{i=1}^{\infty} b_i m_i \tag{2.15}$$

is convergent in $\mathbf{Z_2}$ for any sequence b_i, where b_i is 0 or 1 for every i, and if

$$\sum_{i=1}^{\infty} b_i m_i = \sum_{i=1}^{\infty} b_i^* m_i \tag{2.16}$$

in $\mathbf{Z_2}$, where every b_i and b_i^* is 0 or 1, then $b_i = b_i^*$ for every $i \geqslant 1$.

Proof. We define recursively the sequence m_i. We fix m_1 such that (2.13) is true with $i = 1$, which is possible by (ii). If $m_1, m_2, \ldots, m_i$ are given, then we take an integer t_i such that $2^{t_i} > m_i$, and we take $m_{i+1} = 2^{t_i} R$ with a positive integer R with

$$\phi(R) < \frac{\epsilon_{i+1}}{K^{t_i}}.$$

This is possible by (ii). Then (2.14) is true, and (2.13) with $i + 1$ in place of i follows by (2.11). Now, (2.14) easily implies that t_i is strictly increasing, so (2.15) is convergent indeed. If (2.16) is true, and j is the least integer for which $b_j \neq b_j^*$, then

$$\bigl(b_j - b_j^*\bigr) m_j = \sum_{i=j+1}^{\infty} \bigl(b_i^* - b_i\bigr) m_i. \tag{2.17}$$

The left-hand side here is an integer, its absolute value is m_j, so by (2.14) we see that it is not divisible by 2^{t_j}. However, every term on the right-hand side of (2.17) is divisible by 2^{t_j}, this follows from (2.14), since the sequence t_i is increasing. This is a contradiction, so the lemma is proved. □

Proof of Theorems 1(ii) and 2(ii). We will apply Lemma 2.3. For positive integers m we put

$$\phi_1(m) = \sup_{r \in A} \|mr\| \tag{2.18}$$

in the case of Theorem 1(ii), and

$$\phi_2(m) = \sum_{r \in A} v_2\bigl(\|mr\|\bigr) \tag{2.19}$$

in the case of Theorem 2(ii). Then for every $m \geqslant 1$ we have $0 < \phi_1(m) < \infty$ (obviously), and $0 < \phi_2(m) < \infty$ by (1.4) and $\mathbf{Z} \subseteq C_A(v_1)$, using also that $v_1 \in V$. Moreover, condition (i) of Lemma 2.3 is obviously satisfied, writing ϕ_1 or ϕ_2 in place of ϕ in (2.11) (we use $v_2 \in V$). Condition (ii) of Lemma 2.3 is also true for ϕ_1 and for ϕ_2, it will follow from Lemma 2.2. To prove this, remark first that (2.3) and (2.6) imply

$$\|Tr_l\| \leqslant \|r_l\|^{1/2} \quad \text{for } l > n. \tag{2.20}$$

Now, to prove (ii) of Lemma 2.3 for ϕ_1, we apply Lemma 2.2 with g identically 1 (it is possible, since (2.2) is true by (1.3), (2.4) is true by $\mathbf{Z} \subseteq C_A(v)$), and using (2.3), (2.20) and (2.5), we get

$$\sup_{r \in A} \|Tr\| \leqslant \max\bigl(2^{-K}, \|r_{n+1}\|^{1/2}\bigr).$$

Since we can take K and n to be arbitrarily large, and $\|r_{n+1}\| \to \infty$ as $n \to \infty$ by (2.3) and (2.4), so Lemma 2.3(ii) is proved for ϕ_1. In the case of ϕ_2, we apply Lemma 2.2 with $g(n) = \log \frac{1}{v_2^{-1}(1/n)}$, and v_1 in place of v. It is possible, since (2.2) is true by (1.7), and (2.4) is true by $\mathbf{Z} \subseteq C_A(v_1)$. Then by (2.5) and (2.20) we have

$$\sum_{r \in A} v_2\bigl(\|Tr\|\bigr) \leqslant n v_2\bigl(2^{-Kg(Kn)}\bigr) + \sum_{l=n+1}^{\infty} v_2\bigl(\|r_l\|^{1/2}\bigr). \tag{2.21}$$

Now, if $K \geqslant 1$, then

$$2^{-Kg(Kn)} \leqslant 2^{-g(Kn)} = v_2^{-1}\left(\frac{1}{Kn}\right), \tag{2.22}$$

so applying (1.4) and (1.5), we get from (2.21) and (2.22) that

$$\sum_{r \in A} v_2\bigl(\|Tr\|\bigr) \leqslant \frac{1}{K} + \frac{E}{q_1} \sum_{l=n+1}^{\infty} v_1\bigl(\|r_l\|\bigr).$$

Since we can take K and n to be arbitrarily large, so by $\mathbf{Z} \subseteq C_A(v_1)$, Lemma 2.3(ii) is proved for ϕ_2.

Hence we can apply Lemma 2.3 for ϕ_1 and for ϕ_2 in place of ϕ. If b_i is 0 or 1 for every i, and if $r \in A$ is given, then

$$\left\| \left(\sum_{i=1}^{\infty} b_i m_i \right) r \right\| \leqslant \sum_{i=1}^{\infty} \|m_i r\|, \tag{2.23}$$

and here the sum on the right-hand side is actually finite (since $r \in T^{(2)}$, (2.14) is true, and (2.14) implies $t_i \to \infty$).

We first consider the case of ϕ_1, i.e., Theorem 1(ii). By (2.23), (2.13) (for ϕ_1) and (2.18) we get

$$\left\| \left(\sum_{i=1}^{\infty} b_i m_i \right) r \right\| \leqslant \left(\sum_{i=1}^{I} m_i \right) \|r\| + \sum_{i=I+1}^{\infty} \epsilon_i$$

for any $r \in A$, any 0-1 sequence b_i and for any integer $I \geqslant 1$. If we take ϵ_i such that

$$\sum_{i=1}^{\infty} \epsilon_i < \infty,$$

then, fixing I to be an arbitrarily large constant, using that

$$\lim_{r\in A}\|r\|=0$$

by $\mathbf{Z}\subseteq C_A(v)$, we get

$$\lim_{r\in A}\left\|\left(\sum_{i=1}^{\infty}b_im_i\right)r\right\|=0$$

for any 0-1 sequence b_i, so by Lemma 2.3, Theorem 1(ii) is proved.

Consider now the case of ϕ_2, hence Theorem 2(ii). Note that if $0\leqslant x\leqslant\frac{1}{2}$, $I\geqslant 2$ is an integer, $0\leqslant x_i\leqslant\frac{1}{2}$ for $1\leqslant i\leqslant I$ and $x\leqslant\sum_{i=1}^{I}x_i$, then (1.1) gives

$$v_2(x)\leqslant\left(\sum_{i=1}^{I-1}K(v_2)^iv_2(x_i)\right)+K(v_2)^{I-1}v_2(x_I). \tag{2.24}$$

Indeed, for $I=2$ this follows directly from (1.1), and then we can prove the statement by induction for $I>2$. Then, since $K(v_2)\geqslant 1$, and we saw that the right-hand side of (2.23) is a finite sum, by (2.23), (2.24) we see for any $r\in A$ and for any 0-1 sequence b_i that

$$v_2\left(\left\|\left(\sum_{i=1}^{\infty}b_im_i\right)r\right\|\right)\leqslant\sum_{i=1}^{\infty}K(v_2)^iv_2\big(\|m_ir\|\big).$$

By (2.13) (for ϕ_2) and (2.19) we get

$$\sum_{r\in A}v_2\left(\left\|\left(\sum_{i=1}^{\infty}b_im_i\right)r\right\|\right)\leqslant\sum_{i=1}^{\infty}K(v_2)^i\epsilon_i.$$

Taking ϵ_i such that this last series is convergent, and using Lemma 2.3, Theorem 2(ii) is proved. □

2.3. *Proof of Theorem 2(i)*

In this subsection, the notations and assumptions of Theorem 2(i) are valid. Let $f(n)$ and $g(n)$ be positive integers for every $n\geqslant 1$, and assume that

$$f(n+1)\geqslant f(n),\qquad g(n+1)>g(n) \tag{2.25}$$

for large n. Let $0<N_1<N_2<\cdots<N_i<\cdots$ be a strictly increasing sequence of positive integers. For large i, define

$$A_{N_i}=\left\{\frac{v_n}{2^{g(n)+f(n)}}:\ N_i\leqslant n<N_{i+1}\right\}\cup\left\{\frac{p_n}{2^{g(n)+f(n)}}:\ N_i\leqslant n<N_{i+1}\right\},$$

where v_n are integers satisfying

$$1 \leqslant v_n \leqslant 2^{g(n)}, \quad \text{and} \quad v_n \quad \text{is odd for every } n,\ N_i \leqslant n < N_{i+1}, \tag{2.26}$$

p_n are primes satisfying

$$2^{\frac{g(n)}{2}} \leqslant p_n \leqslant 2^{1+\frac{g(n)}{2}}, \quad p_n \neq v_n \text{ for every } n,\ N_i \leqslant n < N_{i+1}. \tag{2.27}$$

We will determine the numbers v_n later. If the integers v_n are already fixed, then we can choose the primes p_n satisfying (2.27). By (2.25), since v_n and p_n are odd and $p_n \neq v_n$, for large enough i we have $|A_{N_i}| = 2(N_{i+1} - N_i)$, in other words, the elements written in the definition of A_{N_i} are indeed different. Similarly, we can see that for large i the sets A_{N_i} are pairwise disjoint.

We now choose the numbers v_n. They will be chosen randomly, satisfying (2.26). More precisely, we will need later the condition for v_n stated in the next lemma, and the proof of the lemma will show that choosing v_n randomly, this condition will be true. To state the lemma, we introduce some notations. For every $i \geqslant 1$, let $\epsilon_i < \frac{1}{10}$ be a positive number to be determined later, and for positive integers n, S and i, write

$$H_{n,S,i} = \left\{ 1 \leqslant v \leqslant 2^{g(n)}\colon \left\| \frac{vS}{2^{g(n)+f(n)}} \right\| < \epsilon_i \right\}. \tag{2.28}$$

Lemma 2.4. *Let i be large but fixed, and assume that*

$$\epsilon_i < 2^{-25-6\frac{g(N_{i+1})+f(N_{i+1})}{N_{i+1}-N_i}}. \tag{2.29}$$

If i is large enough, there are integers v_n for $N_i \leqslant n < N_{i+1}$ such that (2.26) *is true, and if for an integer S, $1 \leqslant S \leqslant 2^{1+g(N_{i+1})+f(N_{i+1})}$, the inequality*

$$\left| \left\{ N_i \leqslant n < N_{i+1}\colon |H_{n,S,i}| \leqslant 100\epsilon_i 2^{g(n)} \right\} \right| \geqslant \frac{N_{i+1} - N_i}{2} \tag{2.30}$$

holds, then

$$\left| \{ N_i \leqslant n < N_{i+1}\colon v_n \notin H_{n,S,i} \} \right| \geqslant \frac{N_{i+1} - N_i}{3} \tag{2.31}$$

also holds.

Proof. Let us fix an $1 \leqslant S \leqslant 2^{1+g(N_{i+1})+f(N_{i+1})}$ such that (2.30) is true for S. If for an integer vector

$$(\ldots, v_n, \ldots)_{N_i \leqslant n < N_{i+1}}, \quad 1 \leqslant v_n \leqslant 2^{g(n)} \text{ for every } n,\ N_i \leqslant n < N_{i+1}, \tag{2.32}$$

formula (2.31) is not true, then

$$\left| \left\{ N_i \leqslant n < N_{i+1}\colon |H_{n,S,i}| \leqslant 100\epsilon_i 2^{g(n)},\ v_n \in H_{n,S,i} \right\} \right| \geqslant \frac{N_{i+1} - N_i}{6}.$$

Hence for this fixed S, the number of integer vectors (2.32) for which (2.31) is not true, is at most

$$\left(\prod_{N_i \leqslant n < N_{i+1}} 2^{g(n)}\right)(100\epsilon_i)^{(N_{i+1}-N_i)/6}.$$

Then the number of vectors (2.32) for which there is an $1 \leqslant S \leqslant 2^{1+g(N_{i+1})+f(N_{i+1})}$ such that (2.30) is true but (2.31) is not, is at most

$$\left(\prod_{N_i \leqslant n < N_{i+1}} 2^{g(n)}\right)(100\epsilon_i)^{(N_{i+1}-N_i)/6} 2^{1+g(N_{i+1})+f(N_{i+1})}.$$

And the number of vectors (2.32) with v_n odd for every n, $N_i \leqslant n < N_{i+1}$, is

$$\left(\prod_{N_i \leqslant n < N_{i+1}} 2^{g(n)}\right)2^{-(N_{i+1}-N_i)}.$$

So the lemma will be proved, if we can show that

$$(100\epsilon_i)^{(N_{i+1}-N_i)/6} 2^{1+g(N_{i+1})+f(N_{i+1})} < 2^{-(N_{i+1}-N_i)-1}. \tag{2.33}$$

It is not hard to see, using (2.29), that (2.33) will be true for large i. The lemma is proved. □

From now on, we assume that (2.29) is true for large i, and in A_{N_i} we always take numbers v_n with the properties stated in Lemma 2.4.

For the proof of the next lemma we will need the following well-known general lemma. For the sake of completeness, we present a proof.

Lemma 2.5. *Let $\theta \in \mathbf{R}$, $0 < \epsilon < \frac{1}{10}$, and let $N \geqslant 1$ be an integer. Then at least one of the following two conditions is satisfied*:

(i) $|\{1 \leqslant n \leqslant N\colon \|n\theta\| < \epsilon\}| \leqslant 100\epsilon N$;
(ii) $|\theta - \frac{a}{q}| \leqslant \frac{1}{N}$ *for an* $1 \leqslant q \leqslant \frac{1}{\epsilon}$ *and* $(a, q) = 1$.

Proof. Let Q_1 and Q_2 be positive integers satisfying

$$Q_2 = 2N, \quad \frac{1}{3\epsilon} \leqslant Q_1 + 1 \leqslant \frac{1}{2\epsilon}, \tag{2.34}$$

and for real t consider the well-known Fourier series

$$f(t) = \sum_{m=-Q_1}^{Q_1} \left(1 - \frac{|m|}{Q_1+1}\right) e^{2\pi i m t} = \frac{\sin^2((Q_1+1)\pi t)}{(Q_1+1)\sin^2(\pi t)},$$

$$F(t) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n t} = \begin{cases} 1 - Q_2\|t\|, & \text{if } \|t\| \leqslant \frac{1}{Q_2}, \\ 0, & \text{if } \frac{1}{Q_2} \leqslant \|t\| \leqslant \frac{1}{2}, \end{cases}$$

where $c_n = Q_2 \frac{\sin^2(\pi n/Q_2)}{\pi^2 n^2}$. Remark that the formulas mean $f(t) = Q_1 + 1$ for $t \in \mathbf{Z}$, and $c_0 = \frac{1}{Q_2}$. Since $\frac{2}{\pi}x \leqslant \sin x \leqslant x$ is valid for $0 \leqslant x \leqslant \frac{\pi}{2}$, so

$$c_n f(t) \geqslant \left(\frac{2}{\pi}\right)^4 \frac{Q_1+1}{Q_2}, \quad \text{if } |n| \leqslant \frac{Q_2}{2} \text{ and } |t| \leqslant \frac{1}{2(Q_1+1)}.$$

Hence, since $c_n, f(t) \geqslant 0$ for every n and t, so if (i) is false, by (2.34) we get

$$\sum_{n=-\infty}^{\infty} c_n f(n\theta) \geqslant \left(\frac{2}{\pi}\right)^4 \frac{Q_1+1}{Q_2} \bigl|\{1 \leqslant n \leqslant N\colon \|n\theta\| < \epsilon\}\bigr| \geqslant \left(\frac{2}{\pi}\right)^4 \frac{1}{6N\epsilon} 100N\epsilon > 1.$$

On the other hand, we have

$$\sum_{n=-\infty}^{\infty} c_n f(n\theta) = \sum_{m=-Q_1}^{Q_1} \left(1 - \frac{|m|}{Q_1+1}\right) \sum_{n=-\infty}^{\infty} c_n e^{2\pi i m n\theta} = \sum_{m=-Q_1}^{Q_1} \left(1 - \frac{|m|}{Q_1+1}\right) F(m\theta).$$

Therefore, using $F(0) = 1$ and $F(t) = F(-t)$, we conclude that there is an integer m with $1 \leqslant m \leqslant Q_1$ such that $F(m\theta) \neq 0$, which implies $\|m\theta\| \leqslant \frac{1}{Q_2}$. Hence, using (2.34), (ii) is true. □

Lemma 2.6. *Let i be large but fixed, and assume that*

$$2^{2-\frac{g(N_i)}{2}} < \epsilon_i. \tag{2.35}$$

If i is large, then for every $1 \leqslant S \leqslant 2^{1+g(N_{i+1})+f(N_{i+1})}$ at least one of the following two conditions is satisfied:

(i) $|\{r \in A_{N_i}\colon \|rS\| \geqslant \frac{\epsilon_i}{2}\}| \geqslant \frac{N_{i+1}-N_i}{3}$;
(ii) *there is an $N_i \leqslant n < N_{i+1}$ for which $\|\frac{S}{2^{f(n)+g(n)}}\| \leqslant \frac{1}{2^{g(n)}}$.*

Proof. Let $1 \leqslant S \leqslant 2^{1+g(N_{i+1})+f(N_{i+1})}$ be fixed. Let $N_i \leqslant n < N_{i+1}$, and

$$\theta = \frac{S}{2^{g(n)+f(n)}}.$$

Then by Lemma 2.5, either

$$\left|\left\{1 \leqslant v \leqslant 2^{g(n)}\colon \left\|\frac{vS}{2^{g(n)+f(n)}}\right\| < \epsilon_i\right\}\right| \leqslant 100\epsilon_i 2^{g(n)}, \quad \text{or} \tag{2.36}$$

$$\left|\frac{S}{2^{g(n)+f(n)}} - \frac{a}{q}\right| \leqslant \frac{1}{2^{g(n)}} \quad \text{for an } q,\ 1 \leqslant q \leqslant \frac{1}{\epsilon_i}, \text{ and } (a,q) = 1 \tag{2.37}$$

is true. If for our fix S there is at least one $N_i \leqslant n < N_{i+1}$ for which (2.37) is true with $q = 1$, then we get (ii) of the lemma. So we may assume that (2.37) is never true with $q = 1$. If (2.37) holds for an n, $N_i \leqslant n < N_{i+1}$, with some $q > 1$, then if i is large enough, using (2.27), (2.35),

$n \geqslant N_i$ and the monotonicity of g, we get

$$\left|\frac{p_n S}{2^{g(n)+f(n)}} - \frac{p_n a}{q}\right| \leqslant 2^{1-\frac{g(n)}{2}} < \frac{\epsilon_i}{2} \leqslant \frac{1}{2q} \quad \text{and} \tag{2.38}$$

$$p_n \geqslant 2^{\frac{g(n)}{2}} > \frac{1}{\epsilon_i} \geqslant q. \tag{2.39}$$

Since p_n is a prime, so $(q, p_n a) = 1$ by (2.37) and (2.39), hence (2.38) and $q > 1$ gives

$$\left\|\frac{p_n S}{2^{g(n)+f(n)}}\right\| \geqslant \left\|\frac{p_n a}{q}\right\| - \frac{1}{2q} \geqslant \frac{1}{2q} \geqslant \frac{\epsilon_i}{2}.$$

This shows that if for our fix S (2.37) holds for at least $\frac{N_{i+1}-N_i}{3}$ integers n with $N_i \leqslant n < N_{i+1}$, then (i) of the lemma is true. Hence we may assume that this is not the case, but then (2.36) is valid for at least $\frac{N_{i+1}-N_i}{2}$ integers n with $N_i \leqslant n < N_{i+1}$. The definition of $H_{n,S,i}$ in (2.28) shows that this means that (2.30) is true for our fixed S, and then Lemma 2.4 (and (2.28) again) shows that (i) of the present lemma is true. This completes the proof of the lemma. □

From now on we assume also that (2.35) is true for large i. Let i_0 be large and

$$A = \bigcup_{i \geqslant i_0} A_{N_i}.$$

Lemma 2.7. *Assume that*

$$\liminf_{i\to\infty} (N_{i+1} - N_i) v_2(\epsilon_i) > 0, \tag{2.40}$$

$$\sum_{n=1}^{\infty} v_1\bigl(2^{-f(n)}\bigr) < \infty, \tag{2.41}$$

and for large i we have

$$g(N_i) \geqslant \max_{N_{i+1} \leqslant n < N_{i+2}} f(n). \tag{2.42}$$

Then we have

$$\sum_{r\in A} v_1\bigl(\|r\|\bigr) < \infty, \tag{2.43}$$

and if $k \in \mathbf{Z_2}$, but $k \notin \mathbf{Z}$, then

$$\sum_{r\in A} v_2\bigl(\|kr\|\bigr) = \infty. \tag{2.44}$$

Proof. Since $v_n, p_n \leqslant 2^{g(n)}$ for large n by (2.26) and (2.27), so

$$\sum_{r\in A} v_1(\|r\|) \leqslant 2\sum_{i\geqslant i_0}\sum_{N_i\leqslant n<N_{i+1}} v_1(2^{-f(n)}),$$

which, in view of (2.41), proves (2.43). Let $k=\sum_{j=0}^{\infty} b_j 2^j \in \mathbf{Z_2}$, where b_j is 0 or 1 for every j, and assume that $k \notin \mathbf{Z}$. Let i be large, and

$$S_i = \sum_{0\leqslant j\leqslant f(N_{i+1})+g(N_{i+1})} b_j 2^j. \tag{2.45}$$

Then $1 \leqslant S_i \leqslant 2^{1+f(N_{i+1})+g(N_{i+1})}$, and for every $r \in A_{N_i}$ we have $\|kr\| = \|S_i r\|$ by (2.25) and the definition of A_{N_i}. If Lemma 2.6(i) is true for this i and $S = S_i$, then

$$\sum_{r\in A_{N_i}} v_2(\|kr\|) = \sum_{r\in A_{N_i}} v_2(\|S_i r\|) \geqslant \frac{N_{i+1}-N_i}{3} v_2\left(\frac{\epsilon_i}{2}\right) \geqslant \frac{(N_{i+1}-N_i)v_2(\epsilon_i)}{3K(v_2)}$$

by (1.1). If this would happen for infinitely many i, then, in view of (2.40), we would get (2.44). Therefore, we may assume that if i is large enough, then Lemma 2.6(ii) is valid for i and for $S = S_i$ defined in (2.45). Hence for some $N_i \leqslant n < N_{i+1}$, writing $\theta = \frac{S_i}{2^{g(n)+f(n)}}$, we have

$$\|\theta\| \leqslant \frac{1}{2^{g(n)}}.$$

On the other hand, writing

$$x = \sum_{0\leqslant j\leqslant f(n)+g(n)-1} b_j 2^{j-f(n)-g(n)},$$

we have

$$\|\theta\| = \|x\|.$$

In view of Lemma 2.1(ii), we then see that if

$$b_t \neq b_{t+1} \tag{2.46}$$

for an integer

$$0 \leqslant t \leqslant f(n)+g(n)-2,$$

then

$$2^{t-f(n)-g(n)} \leqslant \|x\| \leqslant \frac{1}{2^{g(n)}},$$

hence

$$t \leqslant f(n).$$

Therefore, since $N_i \leqslant n < N_{i+1}$, using (2.25) we see that if we can choose i and t in such a way that they are large, (2.46) holds for t, and

$$\max_{N_i \leqslant n < N_{i+1}} f(n) + 1 \leqslant t \leqslant f(N_i) + g(N_i) - 2, \tag{2.47}$$

then we will get a contradiction, which will prove (2.44).

Since $k \notin \mathbf{Z}$, there are infinitely many t satisfying (2.46), in view of Lemma 2.1(i). Since $f(N_i)$ tends to infinity as i tends to infinity by (2.41), we see that

$$f(N_i) + g(N_i) - 2 \geqslant \max_{N_{i+1} \leqslant n < N_{i+2}} f(n) + 1$$

for large i. This shows that if t is large enough, we can choose i in such a way that (2.47) holds. This proves the lemma. □

To conclude the proof of Theorem 2(i), we have to prove that we can choose the positive integers $f(n)$ and $g(n)$ for every $n \geqslant 1$, the integer sequence $0 < N_1 < N_2 < \cdots < N_i < \cdots$, and the numbers $0 < \epsilon_i < \frac{1}{10}$ for every $i \geqslant 1$ in such a way that (2.25) holds for large n, (2.29), (2.35) and (2.42) are true for large i, and (2.40) and (2.41) are also true.

Introduce the notation $D_i = N_{i+1} - N_i$, and write

$$t_i = v_1\left(v_2^{-1}\left(\frac{1}{D_i}\right)^{D_i}\right). \tag{2.48}$$

Assume first that the integer sequence $0 < N_1 < N_2 < \cdots < N_i < \cdots$, and for every $i \geqslant 1$ positive numbers ϵ_i and positive integers $f(N_i)$, $g(N_i)$ are given in such a way that

$$\lim_{i \to \infty} D_i t_{i-1} = 0, \tag{2.49}$$

$$\sum_{i=3}^{\infty} D_i t_{i-2} < \infty, \tag{2.50}$$

and for large enough i we have the following conditions:

$$\epsilon_i = 2^{-30-6\frac{g(N_{i+1})+f(N_{i+1})}{D_i}}, \tag{2.51}$$

$$g(N_i) = f(N_{i+1}), \tag{2.52}$$

$$v_2^{-1}\left(\frac{1}{D_i}\right)^{D_i} \leqslant 2^{-12g(N_{i+1})} < v_2^{-1}\left(\frac{1}{D_i}\right)^{D_i/4}, \tag{2.53}$$

$$g(N_{i+1}) \geqslant 2g(N_i). \tag{2.54}$$

We show that we can then choose the positive integers $f(n)$ and $g(n)$ for every $n \geqslant 1$ in such a way that the required conditions are true. Indeed, since $g(N_{i+1}) \to \infty$ by (2.54), it is clear from the right inequality of (2.53) that

$$\lim_{i \to \infty} \frac{g(N_{i+1})}{D_i} = \infty. \tag{2.55}$$

For every $i \geqslant 1$ we take

$$f(n) = f(N_i), \tag{2.56}$$

if $N_i < n < N_{i+1}$. On the other hand, if i is large enough, we choose the integers $g(n)$ for $N_i < n < N_{i+1}$ in such a way that

$$g(N_i) < g(N_i + 1) < g(N_i + 2) < \cdots < g(N_{i+1} - 1) < g(N_{i+1}),$$

which is possible by (2.54) and (2.55). Then (2.25) is true for large n. It is trivial from (2.51) that (2.29) is true for large i (and we see from (2.55) and (2.51) that $\epsilon_i < \frac{1}{10}$ is true for large i). It is also clear by (2.56) and (2.52) that (2.42) is true for large i. Formulas (2.52) and (2.54) imply that $g(N_{i+1}) \geqslant f(N_{i+1})$ for large i. Using this, (2.51) and the left inequality of (2.53), we see for large i that

$$\epsilon_i \geqslant 2^{-30} v_2^{-1}\left(\frac{1}{D_i}\right). \tag{2.57}$$

This shows at once by (1.1) that (2.40) is true. Observe that by (2.53), using (1.5), we have

$$t_i \leqslant v_1\left(2^{-12g(N_{i+1})}\right) \leqslant \frac{1}{q_1^2} t_i \tag{2.58}$$

for large i. For the validity of (2.35), by (2.57) and the monotonicity of v_1, it is enough to prove that

$$v_1\left(2^{32-\frac{g(N_i)}{2}}\right) < v_1\left(v_2^{-1}\left(\frac{1}{D_i}\right)\right). \tag{2.59}$$

Here the right-hand side is at least $\frac{1}{ED_i}$ by (1.4), so by (1.1), (1.5), and (2.58) we see that (2.59) (and so (2.35) for large i) follows by (2.49). To prove (2.41), we remark that by (2.56) and (2.52), for large i we have

$$\sum_{n=N_i}^{N_{i+1}-1} v_1\left(2^{-f(n)}\right) = D_i v_1\left(2^{-f(N_i)}\right) = D_i v_1\left(2^{-g(N_{i-1})}\right).$$

Using again (1.5) and (2.58), we see that (2.41) follows from (2.50).

Hence we have proved that it is enough to achieve that (2.49), (2.50) are true, and (2.51)–(2.54) hold for large i. At the end of Section 2, we will prove the following lemma.

Lemma 2.8. *Let $\{a(n)\}_{n=1}^{\infty}$ be a sequence of positive real numbers, and assume that*

$$\sum_{n=1}^{\infty} a(n) < \infty. \tag{2.60}$$

Let $0 < c < 1$ be a fixed constant. Then there is a strictly increasing sequence $\{n_i\}_{i=1}^{\infty}$ of positive integers such that

$$\sum_{i=2}^{\infty} n_i a(n_{i-1}) < \infty, \tag{2.61}$$

and for every $i \geqslant 1$ we have

$$\frac{a(n_{i+1})}{a(n_i)} < c. \tag{2.62}$$

Using this statement, we now complete the proof of Theorem 2(i). Apply Lemma 2.8 with some $0 < c < 1$ to be determined later, and with

$$a(n) = v_1\left(v_2^{-1}\left(\frac{1}{n}\right)^n\right). \tag{2.63}$$

Then (2.60) follows from (1.6). Let $\{n_i\}_{i=1}^{\infty}$ be the sequence given by Lemma 2.8, and define the positive integers D_i for $i \geqslant 1$ in such a way that

$$D_{2j} = n_j, \qquad D_{2j+1} = D_{2j} \tag{2.64}$$

for $j \geqslant 1$. Then define the sequence N_i such that $D_i = N_{i+1} - N_i$ for large i, and define the integers $g(N_i)$ such that for large j we have

$$2^{-12} v_2^{-1}\left(\frac{1}{D_{2j}}\right)^{D_{2j}/4} \leqslant 2^{-12g(N_{2j+1})} < v_2^{-1}\left(\frac{1}{D_{2j}}\right)^{D_{2j}/4}, \tag{2.65}$$

$$v_2^{-1}\left(\frac{1}{D_{2j-1}}\right)^{D_{2j-1}} \leqslant 2^{-12g(N_{2j})} < 2^{12} v_2^{-1}\left(\frac{1}{D_{2j-1}}\right)^{D_{2j-1}}, \tag{2.66}$$

finally, define $f(N_i)$ and ϵ_i in such a way that (2.52) and (2.51) hold for large i. Then (2.50) follows from (2.64), (2.63) and (2.61) (see (2.48)). Condition (2.49) is a consequence of (2.50), since t_i is a decreasing sequence, because n_i is increasing, and so by (2.64), D_i is also increasing. Since (2.53) follows at once from (2.65) and (2.66) for large i, it is enough to prove that (2.54) holds for large i.

Now, it is clear from (2.65), (2.66) and (2.64) that $g(N_{2j+2}) > 2g(N_{2j+1})$ for large j. Assume that

$$g(N_{2j+1}) \leqslant 2g(N_j) \tag{2.67}$$

for a large j. Remember that (2.58) follows from (2.53) for large i. Since we have already proved (2.53), we can use (2.58). Then, by (1.5), (2.58) and (2.67) we have

$$\frac{1}{q_1^2} t_{2j} \geqslant v_1\left(2^{-12g(N_{2j+1})}\right) \geqslant q_1 v_1\left(2^{-12g(N_{2j})}\right) \geqslant q_1 t_{2j-1}. \tag{2.68}$$

Now, by (2.48), (2.63) and (2.64) we have $t_{2j} = a(n_j)$ and $t_{2j-1} = a(n_{j-1})$. Therefore, (2.68) and (2.62) imply

$$q_1^3 a(n_{j-1}) \leqslant a(n_j) < c a(n_{j-1}).$$

Hence, if we choose $c < q_1^3$, then this is a contradiction, so (2.67) is false. So we have proved (2.54) for large i.

So, the proof of Theorem 2(i) will be complete, if we prove Lemma 2.8.

Proof of Lemma 2.8. Let $n_1 = 1$, and if $n_1 < n_2 < \cdots < n_i$ are given, let $n_{i+1} > n_i$ be the least integer satisfying (2.62). We have to prove (2.61). If $n_i \leqslant n < n_{i+1}$, then

$$\frac{a(n)}{a(n_i)} \geqslant c.$$

Therefore,

$$\frac{1}{c} \sum_{n=n_i}^{n_{i+1}-1} a(n) \geqslant (n_{i+1} - n_i) a(n_i).$$

Then (2.60) implies

$$\sum_{i=1}^{\infty} (n_{i+1} - n_i) a(n_i) < \infty. \tag{2.69}$$

But for any integer $I \geqslant 1$, we have

$$\sum_{i=1}^{I} (n_{i+1} - n_i) a(n_i) = -n_1 a(n_1) + n_{I+1} a(n_I) + \sum_{i=2}^{I} n_i \big(a(n_{i-1}) - a(n_i)\big). \tag{2.70}$$

Since

$$a(n_{i-1}) - a(n_i) \geqslant (1-c) a(n_{i-1})$$

by (2.62), so (2.69) and (2.70) imply (2.61). The lemma is proved. □

3. The general case

3.1. Conditional proof of the theorems

In Section 2 we proved that Theorems 1–3 are true if we write $G = \mathbf{Z_2}$, $\Gamma = \mathbf{Z}$. We will prove the theorems for the general case from this special case, using Lemma 3.1 below. In this subsection we assume Lemma 3.1 and prove the theorems, and then we prove Lemma 3.1 in the rest of the paper.

If $\alpha_1, \alpha_2, \ldots, \alpha_t$ generate a group Γ, we say that they freely generate Γ, if Γ is a free Abelian group, and $\alpha_1, \alpha_2, \ldots, \alpha_t$ is a system of free generators of Γ.

Lemma 3.1. *Let G_1 and G_2 be compact metrizable Abelian groups. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_t \in G_1$ freely generate a dense subgroup Γ_1 of G_1, and $\beta \in G_2$ freely generates a dense subgroup Γ_2 of G_2. Let $v_1, v_2 \in V$. Assume that $A \subseteq G_2^*$ is such an infinite subset that $\Gamma_2 \subseteq C_{G_2,A}(v_1)$. Then there is an infinite subset $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v_1)$, and*

(i) *if $C_{G_2,A}(\chi_{[1/4,1/2]}) \subseteq \Gamma_2$, then $C_{G_1,H}(\chi_{[1/10,1/2]}) \subseteq \Gamma_1$,*
(ii) *if $C_{G_2,A}(v_2) \subseteq \Gamma_2$, then $C_{G_1,H}(v_2) \subseteq \Gamma_1$,*
(iii) *if $|C_{G_2,A}(v_2)| < 2^{\aleph_0}$, then $|C_{G_1,H}(v_2)| < 2^{\aleph_0}$,*
(iv) *if $|L_{G_2,A}| < 2^{\aleph_0}$, then $|L_{G_1,H}| < 2^{\aleph_0}$,*
(v) *if $B_{G_2,A}(\beta) \subseteq \Gamma_2$, then $B_{G_1,H}(\alpha_1, \alpha_2, \ldots, \alpha_t) \subseteq \Gamma_1$.*

We now prove the theorems assuming this lemma, and using the special cases of the theorems proved in Section 2. The most complicated proof is that of Theorem 1(i), so we start with the other proofs.

Proof of Theorems 2(i) and 3. We take $G_1 = G$, $\Gamma_1 = \Gamma$, let $\alpha_1, \alpha_2, \ldots, \alpha_t$ be a system of free generators of Γ, and let $G_2 = \mathbf{Z_2}$, $\Gamma_2 = \mathbf{Z}$, $\beta = 1$.

In the case of Theorem 2(i), by the special case already proved, there is an infinite $A \subseteq G_2^*$ such that $C_{G_2,A}(v_1) = C_{G_2,A}(v_2) = \Gamma_2$. Then by Lemma 3.1(ii) there is an infinite $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v_1)$ and $C_{G_1,H}(v_2) \subseteq \Gamma_1$. By the definitions and (1.4), $C_{G_1,H}(v_1) = C_{G_1,H}(v_2) = \Gamma_1$ follows, so Theorem 2(i) is proved.

In the case of Theorem 3 we take $v_1 = v$. Then by the special case already proved, there is an infinite $A \subseteq G_2^*$ such that $\Gamma_2 \subseteq C_{G_2,A}(v)$ and $B_{G_2,A}(\beta) = \Gamma_2$. By Lemma 3.1(v) there is an infinite $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v)$ and $B_{G_1,H}(\alpha_1, \alpha_2, \ldots, \alpha_t) \subseteq \Gamma_1$. Since $\Gamma_1 \subseteq B_{G_1,H}(\alpha_1, \alpha_2, \ldots, \alpha_t)$ is trivial, Theorem 3 follows. □

Proof of Theorems 1(ii) and 2(ii). We take $G_2 = G$, $\Gamma_2 = \Gamma$, $G_1 = \mathbf{Z_2}$, $\Gamma_1 = \mathbf{Z}$.

In the case of Theorem 1(ii) we take $v_1 = v$. Let $A \subseteq G_2^*$ be such an infinite subset that $\Gamma_2 \subseteq C_{G_2,A}(v)$, and assume that $|L_{G_2,A}| < 2^{\aleph_0}$. Then by Lemma 3.1(iv), there is an infinite subset $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v)$, and $|L_{G_1,H}| < 2^{\aleph_0}$. This contradicts the special case of Theorem 1(ii) already proved. Therefore $|L_{G_2,A}| \geqslant 2^{\aleph_0}$, and since $|G_2| \leqslant 2^{\aleph_0}$ is trivial (because $G_2 = (G_2^*)^*$, and G_2^* is countable), so $|L_{G_2,A}| = 2^{\aleph_0}$, Theorem 1(ii) follows.

In the case of Theorem 2(ii), let $A \subseteq G_2^*$ be such an infinite subset that $\Gamma_2 \subseteq C_{G_2,A}(v_1)$, and assume that $|C_{G_2,A}(v_2)| < 2^{\aleph_0}$. Then by Lemma 3.1(iii), there is an infinite $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v_1)$, and $|C_{G_1,H}(v_2)| < 2^{\aleph_0}$. This contradicts the special case of Theorem 2(ii) already proved. Then, using again that $|G_2| \leqslant 2^{\aleph_0}$, we get $|C_{G_2,A}(v_2)| = 2^{\aleph_0}$, so Theorem 2(ii) is proved. □

Proof of Theorem 1(i). As in the proof of Theorems 2(i) and 3, let $G_1 = G$, $\Gamma_1 = \Gamma$, $G_2 = \mathbf{Z_2}$, $\Gamma_2 = \mathbf{Z}$. Define $v_1 = v$. By the special case of Theorem 1(i) already proved there is an infinite $A \subseteq G_2^*$ such that $C_{G_2,A}(v) = C_{G_2,A}(\chi_{[1/4,1/2]}) = \Gamma_2$. Then by Lemma 3.1(i) there is an infinite $H \subseteq G_1^*$ such that $\Gamma_1 \subseteq C_{G_1,H}(v)$ and $C_{G_1,H}(\chi_{[1/10,1/2]}) \subseteq \Gamma_1$. Now, let $\hat{H} \subseteq G_1^*$ be defined by

$$\hat{H} = \{2^r h\colon\ h \in H,\ 0 \leqslant r \leqslant 2\}.$$

We claim that $C_{G_1,\hat{H}}(\chi_{[1/4,1/2]}) \subseteq C_{G_1,H}(\chi_{[1/10,1/2]})$. Indeed, assume that $\gamma \in C_{G_1,\hat{H}}(\chi_{[1/4,1/2]})$ is such that $\|h\gamma\| \geqslant \frac{1}{10}$ for every $h \in H_0$, where $H_0 \subseteq H$ is an infinite subset. Let $h \in H_0$. If $\|h\gamma\| \in [\frac{1}{8}, \frac{1}{4}]$, then $\|(2h)\gamma\| \geqslant \frac{1}{4}$, and if $\|h\gamma\| \in [\frac{1}{10}, \frac{1}{8}]$, then $\|(4h)\gamma\| \geqslant \frac{1}{4}$. This means that there is an integer $0 \leqslant r_0 \leqslant 2$ and an infinite subset $H_1 \subseteq H_0$ such that $\|(2^{r_0}h)\gamma\| \geqslant \frac{1}{4}$ for every $h \in H_1$. But then $\gamma \in C_{G_1,\hat{H}}(\chi_{[1/4,1/2]})$ implies that

$$\left\{2^{r_0}h\colon\ h \in H_1\right\}$$

is a finite set. Since $H_1 \subseteq G_1^*$ is infinite, this shows that

$$\left\{h \in G_1^*\colon\ 2^{r_0}h = 0\right\}$$

is an infinite set. But this is false. Indeed, on the one hand, G_1 is topologically generated by a system of free generators $\alpha_1, \alpha_2, \ldots, \alpha_t$ of Γ, so every $h \in G_1^*$ is determined by its values on $\alpha_1, \alpha_2, \ldots, \alpha_t$; on the other hand, if $2^{r_0}h = 0$, then there are only finitely many possibilities for the t-tuple $(h\alpha_1, h\alpha_2, \ldots, h\alpha_t)$. This is a contradiction, so we proved $C_{G_1,\widehat{H}}(\chi_{[1/4,1/2]}) \subseteq C_{G_1,H}(\chi_{[1/10,1/2]})$.

Consequently, we have $C_{G_1,\widehat{H}}(\chi_{[1/4,1/2]}) \subseteq \Gamma_1$. On the other hand, it is clear by $v \in V$ that $C_{G_1,H}(v) \subseteq C_{G_1,\widehat{H}}(v)$, hence (since $\Gamma_1 \subseteq C_{G_1,H}(v)$) we have $\Gamma_1 \subseteq C_{G_1,\widehat{H}}(v)$, so $C_{G_1,\widehat{H}}(v) = C_{G_1,\widehat{H}}(\chi_{[1/4,1/2]}) = \Gamma_1$ follows, and this completes the proof of Theorem 1(i). □

In the rest of the paper, our aim is to prove Lemma 3.1.

3.2. Preliminary lemmas

In this short subsection we present two easy, but important lemmas.

Lemma 3.2. *If $\omega \in T$, $k \geqslant 1$ is an integer, and*

$$\|\omega\|, \|2\omega\|, \|4\omega\|, \ldots, \left\|2^k\omega\right\| \leqslant \delta < \frac{1}{10},$$

then $\|\omega\| \leqslant \frac{\delta}{2^k}$.

Proof. This is proved as in [B-S, Lemma 3]. □

Lemma 3.3. *Let G be a compact Abelian group. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_t \in G$ freely generate a dense subgroup of G. Then*

(i) *the subgroup*

$$\left\{(\mu\alpha_1, \mu\alpha_2, \ldots, \mu\alpha_t)\colon\ \mu \in G^*\right\}$$

is dense in T^t;

(ii) *if $\mu \in G^*$ is such that*

$$\mu\alpha_1 = \mu\alpha_2 = \cdots = \mu\alpha_t = 0,$$

then $\mu = 0$;

(iii) *if $\beta \in G$ is such that $\mu\beta = 0$ for every $\mu \in G^*$, then $\beta = 0$.*

Proof. Since $\alpha_1, \alpha_2, \ldots, \alpha_t$ are independent, part (i) is a particular case of the well-known fact that discontinuous characters of G can be approximated by continuous ones (cf. [H-R]). Part (ii) is trivial from the fact that $\alpha_1, \alpha_2, \ldots, \alpha_t$ generate a dense subgroup of G. Part (iii) follows from Pontriagin's duality theorem (see [R]). □

3.3. *Proof of Lemma 3.1*

In this subsection, we use the notations of Lemma 3.1. The set $H \subseteq G_1^*$ we produce will have the form $H = \bigcup_{k=1}^4 H_k$, we define the subsets H_k successively, and we show the impact of each step on the final proof. The first three subsets, H_1, H_2 and H_3 will not depend on the set $A \subseteq G_2^*$, but H must depend on A, so we will use A in the definition of the last subset H_4.

At each step, on the one hand, we need $\Gamma_1 \subseteq C_{G_1,H_k}(v_1)$ (which will finally imply $\Gamma_1 \subseteq C_{G_1,H}(v_1)$). This means essentially that $\|h\alpha_i\|$ is small, when $h \in H_k$ and $1 \leqslant i \leqslant t$. On the other hand, we need restrictions on γ, if $\gamma \in G_1$ and $\|h\gamma\|$ is small for $h \in H_k$ (because in each case we have to force such elements γ into a countable set).

We now start the formal proof. Our basic tools will be Lemmas 3.4 and 3.5 below. We first introduce a notation: by identification of the groups $(G_2^*)^t$ and $(G_2^t)^*$, for a general element $\lambda \in (G_2^t)^*$ we write

$$\lambda = \big(\lambda(1), \lambda(2), \ldots, \lambda(t)\big),$$

where $\lambda(i) \in G_2^*$ for $1 \leqslant i \leqslant t$.

Lemma 3.4. *For every integer $j \geqslant 1$ and for every $\lambda \in (G_2^t)^*$ let $\epsilon(j, \lambda)$ be a given positive number. Then for every such pair (j, λ) we can choose a character $\mu_j^\lambda \in G_1^*$ (in other words, for every λ we can choose a sequence μ_j^λ) with the following properties*:

$$\big\|\lambda(i)\beta - \mu_j^\lambda \alpha_i\big\| < \epsilon(j, \lambda) \tag{3.1}$$

for every $1 \leqslant i \leqslant t$, and

$$\mu_1^\lambda \neq \mu_1^{\lambda^\star}, \quad \textit{if } \lambda \neq \lambda^\star. \tag{3.2}$$

Proof. Since $(G_2^t)^*$ is countable, by Lemma 3.3(i) we can choose recursively the characters μ_j^λ in such a way that (3.1) and (3.2) will be true. □

Lemma 3.5. *For every integer $n \geqslant 1$ and for every $\tau \in G_1^*$ let $\delta(n, \tau)$ be a given positive number. Then for every such pair (n, τ) we can choose a character $\kappa_n^\tau \in (G_2^t)^*$ (in other words, for every τ we can choose a sequence κ_n^τ) with the following properties*:

$$\big\|\kappa_n^\tau(i)\beta - \tau\alpha_i\big\| < \delta(n, \tau) \tag{3.3}$$

for every $1 \leqslant i \leqslant t$, and

$$\kappa_n^\tau \neq \kappa_{n^\star}^{\tau^\star}, \quad \textit{if } (n, \tau) \neq (n^\star, \tau^\star). \tag{3.4}$$

If the numbers $\delta(n,\tau)$ are small enough, then for every $1 \leqslant i \leqslant t$ and $\tau \in G_1^*$ we have

$$\lim_{n\to\infty} \kappa_n^\tau(i)\beta = \tau\alpha_i. \tag{3.5}$$

Proof. Since G_1^* is countable, by Lemma 3.3(i) (applied for G_2 and β) we can choose recursively the elements κ_n^τ in such a way that (3.3) and (3.4) will be true. The last statement is trivial from (3.3). □

We apply these lemmas with some positive numbers $\epsilon(j,\lambda)$ ($j \geqslant 1$, $\lambda \in (G_2^t)^*$) and $\delta(n,\tau)$ ($n \geqslant 1$, $\tau \in G_1^*$) to be determined later, and we choose characters $\mu_j^\lambda \in G_1^*$, $\kappa_n^\tau \in (G_2^t)^*$ satisfying the conditions of Lemmas 3.4 and 3.5.

For every $\lambda \in (G_2^t)^*$ let $R(\lambda)$ be a positive integer, to be determined later. If $j \geqslant 1$ is an integer, $\lambda \in (G_2^t)^*$, let $R(j,\lambda) = j + R(\lambda)$. Define

$$H_1 = \left\{2^r\left(\mu_{j+1}^\lambda - \mu_j^\lambda\right):\ \lambda \in \left(G_2^t\right)^*,\ j \geqslant 1,\ 0 \leqslant r \leqslant R(j,\lambda)\right\}.$$

Lemma 3.6. *Let $R(\lambda)$ be given for every $\lambda \in (G_2^t)^*$. If the numbers $\epsilon(j,\lambda)$ are small enough, then the set $H_1 \subseteq G_1^*$ defined above satisfies the following conditions*:

(i) *We have*

$$\Gamma_1 \subseteq C_{G_1,H_1}(v_1). \tag{3.6}$$

(ii) *If $\gamma \in G_1$ is such that $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1$, then*

$$F_\gamma(\lambda) := \lim_{j\to\infty} \mu_j^\lambda \gamma \tag{3.7}$$

exists for every $\lambda \in (G_2^t)^*$, and

$$\left\|F_\gamma(\lambda) - \mu_1^\lambda\gamma\right\| < 2^{-R(\lambda)} \tag{3.8}$$

holds for all but finitely many $\lambda \in (G_2^t)^*$.

Proof. If $1 \leqslant i \leqslant t$, $\lambda \in (G_2^t)^*$, $j \geqslant 1$, then by (3.1) we have

$$\left\|\left(\mu_{j+1}^\lambda - \mu_j^\lambda\right)\alpha_i\right\| \leqslant \epsilon(j,\lambda) + \epsilon(j+1,\lambda).$$

It is clear by $v_1 \in V$ that for arbitrary fixed positive integers $R(j,\lambda)$ we can choose the numbers $\epsilon(j,\lambda)$ so small that

$$\sum_{\lambda\in(G_2^t)^*}\sum_{j\geqslant 1}\sum_{0\leqslant r\leqslant R(j,\lambda)} v_1\left(2^r\left(\epsilon(j,\lambda)+\epsilon(j+1,\lambda)\right)\right) < \infty$$

will hold. Therefore,

$$\sum_{\lambda\in(G_2^t)^*}\sum_{j\geqslant 1}\sum_{0\leqslant r\leqslant R(j,\lambda)} v_1\left(\left\|2^r\left(\mu_{j+1}^\lambda - \mu_j^\lambda\right)\alpha_i\right\|\right) < \infty \tag{3.9}$$

will be true for every $1 \leqslant i \leqslant t$. This gives (3.6) at once.

Let $\gamma \in G_1$ be such that $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1$. Let $0 \neq h \in H_1$ be fixed. We see by (3.9) and Lemma 3.3(ii) that there may be only finitely many pairs $\lambda \in (G_2^t)^*$, $j \geqslant 1$ such that there is a $0 \leqslant r \leqslant R(j,\lambda)$ satisfying

$$2^r\left(\mu_{j+1}^{\lambda} - \mu_j^{\lambda}\right) = h.$$

Hence, by the property of γ, using Lemma 3.2 and $R(j,\lambda) = j + R(\lambda)$, we obtain

$$\left\|\left(\mu_{j+1}^{\lambda} - \mu_j^{\lambda}\right)\gamma\right\| < \frac{1/10}{2^{j+R(\lambda)}} \tag{3.10}$$

for all but finitely many pairs $\lambda \in (G_2^t)^*$, $j \geqslant 1$. Then (3.10) shows that $\mu_j^{\lambda}\gamma$ is a Cauchy sequence in T for every $\lambda \in (G_2^t)^*$, so (3.7) exists, and (3.10) shows also (3.8). Lemma 3.6 is proved. □

From now on, we assume that the numbers $\epsilon(j,\lambda)$ are as small as Lemma 3.6 requires. Hence, we will fix first the parameters $R(\lambda)$, and then we will choose the numbers $\epsilon(j,\lambda)$ small enough. So (i) and (ii) of Lemma 3.6 will be true, in particular, we may use the notation $F_\gamma(\lambda)$.

To motivate the next lemma, observe that if $\epsilon(j,\lambda)$ are small enough, then $F_{\alpha_i}(\lambda) = \lambda(i)\beta$ by (3.1), so the map $F_{\alpha_i} : (G_2^t)^* \to T$ is a group homomorphism. By taking a new set H_2, we extend this property for general γ.

Lemma 3.7. *If the numbers $\epsilon(j,\lambda)$ are small enough, then there is a subset $H_2 \subseteq G_1^*$ such that*

$$\Gamma_1 \subseteq C_{G_1,H_2}(v_1), \tag{3.11}$$

and if $\gamma \in G_1$ satisfies $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1 \cup H_2$, then for $1 \leqslant i \leqslant t$ there are elements $\phi_i(\gamma) \in G_2$ such that

$$F_\gamma(\lambda) = \sum_{i=1}^{t} \lambda(i)\phi_i(\gamma) \tag{3.12}$$

for every $\lambda \in (G_2^t)^*$.

Proof. Let us choose for every pair $\lambda_1, \lambda_2 \in (G_2^t)^*$ an infinite subset (to be determined later) J_{λ_1,λ_2} of the positive integers. Then define

$$H_2 = \left\{2^r\left(\mu_j^{\lambda_1} + \mu_j^{\lambda_2} - \mu_j^{\lambda_1+\lambda_2}\right):\ \lambda_1, \lambda_2 \in \left(G_2^t\right)^*,\ j \in J_{\lambda_1,\lambda_2},\ 0 \leqslant r \leqslant j\right\}.$$

If $1 \leqslant i \leqslant t$, $\lambda_1, \lambda_2 \in (G_2^t)^*$, $j \in J_{\lambda_1,\lambda_2}$, then by (3.1) we have

$$\left\|\left(\mu_j^{\lambda_1} + \mu_j^{\lambda_2} - \mu_j^{\lambda_1+\lambda_2}\right)\alpha_i\right\| \leqslant \epsilon(j,\lambda_1) + \epsilon(j,\lambda_2) + \epsilon(j,\lambda_1+\lambda_2). \tag{3.13}$$

It is clear that we can choose the numbers $\epsilon(j,\lambda)$ so small that

$$\lim_{j\to\infty} \sum_{0\leqslant r\leqslant j} v_1\left(2^r\left(\epsilon(j,\lambda_1) + \epsilon(j,\lambda_2) + \epsilon(j,\lambda_1+\lambda_2)\right)\right) = 0$$

will be true for every fixed pair $\lambda_1, \lambda_2 \in (G_2^t)^*$. If this is true, then we can choose the infinite subsets J_{λ_1,λ_2} of the positive integers in such a way that (the first summation below is over every pair from $(G_2^t)^*$)

$$\sum_{\lambda_1,\lambda_2} \sum_{j\in J_{\lambda_1,\lambda_2}} \sum_{0\leqslant r\leqslant j} v_1\bigl(2^r\bigl(\epsilon(j,\lambda_1)+\epsilon(j,\lambda_2)+\epsilon(j,\lambda_1+\lambda_2)\bigr)\bigr) < \infty$$

will hold. Therefore, using (3.13), we get (the first summation is again over every pair from $(G_2^t)^*$)

$$\sum_{\lambda_1,\lambda_2} \sum_{j\in J_{\lambda_1,\lambda_2}} \sum_{0\leqslant r\leqslant j} v_1\bigl(\bigl\|2^r\bigl(\mu_j^{\lambda_1}+\mu_j^{\lambda_2}-\mu_j^{\lambda_1+\lambda_2}\bigr)\alpha_i\bigr\|\bigr) < \infty \tag{3.14}$$

will be true for every $1 \leqslant i \leqslant t$. This gives (3.11) at once.

Let $\gamma \in G_1$ be such that $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1 \cup H_2$. Let $0 \neq h \in H_2$ be fixed. We see by (3.14) and Lemma 3.3(ii) that for any fixed pair $\lambda_1, \lambda_2 \in (G_2^t)^*$ there may be only finitely many $j \in J_{\lambda_1,\lambda_2}$ such that there is a $0 \leqslant r \leqslant j$ satisfying

$$2^r\bigl(\mu_j^{\lambda_1}+\mu_j^{\lambda_2}-\mu_j^{\lambda_1+\lambda_2}\bigr) = h. \tag{3.15}$$

Using (3.15) and the property of γ, by Lemma 3.2 we obtain

$$\bigl\|\bigl(\mu_j^{\lambda_1}+\mu_j^{\lambda_2}-\mu_j^{\lambda_1+\lambda_2}\bigr)\gamma\bigr\| < \frac{1/10}{2^j} \tag{3.16}$$

for every pair $\lambda_1, \lambda_2 \in (G_2^t)^*$ and for large enough $j \in J_{\lambda_1,\lambda_2}$. Since every J_{λ_1,λ_2} is an infinite set, so $F_\gamma : (G_2^t)^* \to T$ is a group homomorphism by (3.7) and (3.16). Therefore, there are elements $\phi_1(\gamma), \phi_2(\gamma), \ldots, \phi_t(\gamma) \in G_2$ such that (3.12) holds for every $\lambda \in (G_2^t)^*$. Indeed, it is implied by basic facts from the duality theory of locally compact Abelian groups (see [R]): the dual group of a compact group is discrete, therefore, using Pontriagin's duality theorem, any algebraic homomorphism $F : (G_2^t)^* \to T$ has the form (3.12). Lemma 3.7 is proved. □

We assume in the sequel that the $\epsilon(j,\lambda)$ are small enough, so Lemma 3.7 is true, and we may use the notations H_2 and $\phi_i(\gamma)$.

The role of the next part H_3 is that we will be able to reconstruct γ from the elements $\phi_i(\gamma)$. For this purpose we start to use the characters κ_n^τ.

Lemma 3.8. *If the numbers $R(\lambda)$ are large enough, but $\epsilon(j,\lambda)$ and $\delta(n,\tau)$ are small enough, then there is a subset $H_3 \subseteq G_1^*$ with the following conditions*:

$$\Gamma_1 \subseteq C_{G_1,H_3}(v_1), \tag{3.17}$$

and if $\gamma \in G_1$ satisfies $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1 \cup H_2 \cup H_3$, then

$$\lim_{n\to\infty} \sum_{i=1}^{t} \kappa_n^\tau(i)\phi_i(\gamma) = \tau\gamma \tag{3.18}$$

for every $\tau \in G_1^*$.

Proof. Let

$$H_3 = \{2^r(\mu_1^{\kappa_n^\tau} - \tau)\colon\ n \geqslant 1,\ \tau \in G_1^*,\ 0 \leqslant r \leqslant n\}.$$

If $1 \leqslant i \leqslant t$, $n \geqslant 1$ and $\tau \in G_1^*$, then by (3.1) and (3.3) we have

$$\|(\mu_1^{\kappa_n^\tau} - \tau)\alpha_i\| \leqslant \|\mu_1^{\kappa_n^\tau}\alpha_i - \kappa_n^\tau(i)\beta\| + \|\kappa_n^\tau(i)\beta - \tau\alpha_i\| \leqslant \epsilon(1, \kappa_n^\tau) + \delta(n, \tau).$$

It is clear (using (3.4)) that we can choose the numbers $\delta(n, \tau)$ and $\epsilon(1, \kappa_n^\tau)$ to be small enough that

$$\sum_{n\geqslant 1}\sum_{\tau\in G_1^*}\sum_{0\leqslant r\leqslant n} v_1\big(2^r\big(\delta(n,\tau) + \epsilon(1, \kappa_n^\tau)\big)\big) < \infty$$

will hold. Therefore,

$$\sum_{n\geqslant 1}\sum_{\tau\in G_1^*}\sum_{0\leqslant r\leqslant n} v_1\big(\|2^r(\mu_1^{\kappa_n^\tau} - \tau)\alpha_i\|\big) < \infty \tag{3.19}$$

will be true for every $1 \leqslant i \leqslant t$. This gives (3.17) at once.

Let $\gamma \in G_1$ be such that $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1 \cup H_2 \cup H_3$. Let $0 \neq h \in H_3$ be fixed. Formula (3.19) and Lemma 3.3(ii) give that for any fixed $\tau \in G_1^*$ there may be only finitely many $n \geqslant 1$ for which there is a $0 \leqslant r \leqslant n$ satisfying

$$2^r(\mu_1^{\kappa_n^\tau} - \tau) = h.$$

Then, using the property of γ, by Lemma 3.2 we obtain

$$\|(\mu_1^{\kappa_n^\tau} - \tau)\gamma\| < \frac{1/10}{2^n}$$

for every $\tau \in G_1^*$ and for large enough n. It follows that

$$\lim_{n\to\infty} \mu_1^{\kappa_n^\tau}\gamma = \tau\gamma \tag{3.20}$$

for every $\tau \in G_1^*$. If we have $R(\kappa_n^\tau) \to \infty$ as $n \to \infty$ for every $\tau \in G_1^*$, which we may assume in view of (3.4), then (3.8) and (3.20) imply

$$\lim_{n\to\infty} F_\gamma(\kappa_n^\tau) = \tau\gamma \tag{3.21}$$

for every $\tau \in G_1^*$. Formulas (3.12) and (3.21) give (3.18). Lemma 3.8 is proved. □

Again, we assume in the sequel that the parameters satisfy the conditions needed in Lemma 3.8.

Now, since $\gamma \in G_1$ is determined by the elements $\phi_i(\gamma) \in G_2$, it is enough to force $\phi_i(\gamma)$ into a small set. This is the point where we use the set $A \subseteq G_2^*$.

Lemma 3.9. *Assume that the numbers $R(\lambda)$ are large enough, but $\epsilon(j,\lambda)$ and $\delta(n,\tau)$ are small enough. There is an infinite subset $H_4 \subseteq G_1^*$ such that by taking $H=\bigcup_{k=1}^4 H_k$, on the one hand we have*

$$\Gamma_1 \subseteq C_{G_1,H}(v_1), \tag{3.22}$$

on the other hand, the following statements hold (*we mean in each case that the condition implies that γ satisfies $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H_1 \cup H_2$, so $\phi_i(\gamma)$ are defined*):

(i) *if $\gamma \in C_{G_1,H}(\chi_{[1/10,1/2]})$, then $\phi_i(\gamma) \in C_{G_2,A}(\chi_{[1/4,1/2]})$ for $1 \leqslant i \leqslant t$,*
(ii) *if $\gamma \in C_{G_1,H}(v_2)$, then $\phi_i(\gamma) \in C_{G_2,A}(v_2)$ for $1 \leqslant i \leqslant t$,*
(iii) *if $\gamma \in L_{G_1,H}$, then $\phi_i(\gamma) \in L_{G_2,A}$ for $1 \leqslant i \leqslant t$,*
(iv) *if $\gamma \in B_{G_1,H}(\alpha_1,\alpha_2,\ldots,\alpha_t)$, then $\phi_i(\gamma) \in B_{G_2,A}(\beta)$ for $1 \leqslant i \leqslant t$.*

Proof. If $a \in A$ and $1 \leqslant i \leqslant t$, let $\pi_{a.i} \in (G_2^t)^*$ be defined by

$$\pi_{a,i}(i) = a, \quad 1 \leqslant i \leqslant t, \quad \text{and} \tag{3.23}$$

$$\pi_{a,i_1}(i_2) = 0 \in G_2^* \quad 1 \leqslant i_1, i_2 \leqslant t,\ i_1 \neq i_2. \tag{3.24}$$

Now, for $a \in A$ and $1 \leqslant i \leqslant t$, define

$$f_i(a) = \mu_1^{\pi_{a,i}}. \tag{3.25}$$

It is clear by (3.2) that $f_i : A \to G_1^*$ is an injection for every i, $1 \leqslant i \leqslant t$. Let

$$H_4 = \bigcup_{i=1}^{t} f_i(A),$$

it is clearly an infinite set. We have by (3.1), (3.25), (3.23) and (3.24) for every $a \in A$ that

$$\bigl\| f_i(a)\alpha_i - a\beta \bigr\| < \epsilon(1,\pi_{a,i}), \quad 1 \leqslant i \leqslant t, \quad \text{and} \tag{3.26}$$

$$\bigl\| f_{i_1}(a)\alpha_i \bigr\| < \epsilon(1,\pi_{a,i_1}), \quad 1 \leqslant i_1, i \leqslant t,\ i_1 \neq i. \tag{3.27}$$

We get for any fixed $1 \leqslant i \leqslant t$ that

$$\sum_{h\in H_4} v_1\bigl(\|h\alpha_i\|\bigr) \leqslant \sum_{a\in A}\biggl(v_1\bigl(\|a\beta\| + \epsilon(1,\pi_{a,i})\bigr) + \sum_{1\leqslant i_1\leqslant t,\, i_1\neq i} v_1\bigl(\epsilon(1,\pi_{a,i_1})\bigr)\biggr).$$

We then see, using $v_1 \in V$ and $\beta \in C_{G_2,A}(v_1)$ that if we choose the numbers $\epsilon(1,\pi_{a,i})$ small enough, which is possible, then (3.22) follows (using also (3.6), (3.11), (3.17)).

We now prove statements (i)–(iv). Remark first that in each case it is clear that $\gamma \in G_1$ is such that $\|h\gamma\| < \frac{1}{10}$ for all but finitely many $h \in H$ (in case (iv) it follows from (3.22)), so $\phi_i(\gamma)$ are defined indeed. By (3.8) and (3.25), we see that

$$\bigl\| F_\gamma(\pi_{a.i}) - f_i(a)\gamma \bigr\| < 2^{-R(\pi_{a.i})} \tag{3.28}$$

for all but finitely many $a \in A$ and for every i, $1 \leqslant i \leqslant t$. And, by (3.12), (3.23) and (3.24) we have

$$F_\gamma(\pi_{a,i}) = a\phi_i(\gamma)$$

for every $a \in A$ and $1 \leqslant i \leqslant t$, so for all but finitely many $a \in A$ by (3.28) we obtain

$$\left\| a\phi_i(\gamma) \right\| < 2^{-R(\pi_{a.i})} + \left\| f_i(a)\gamma \right\|, \quad 1 \leqslant i \leqslant t. \tag{3.29}$$

Take the numbers $R(\pi_{a.i})$ so large that we have

$$\sum_{a \in A} v_2\big(2^{-R(\pi_{a.i})}\big) < \infty, \quad \lim_{a \in A} 2^{-R(\pi_{a.i})} = 0$$

for every i, $1 \leqslant i \leqslant t$ (in fact, the second formula follows from the first one here). Then, using (3.29), $v_2 \in V$ and that $f_i : A \to H$ is an injection for every i, statements (i)–(iii) follow.

For the proof of (iv), take $R(\pi_{a.i})$ so large and $\epsilon(1, \pi_{a,i})$ so small that

$$2^{-R(\pi_{a.i})} < \|a\beta\|, \qquad \epsilon(1, \pi_{a,i}) < \|a\beta\| \tag{3.30}$$

for every $0 \neq a \in A$, and for every i, $1 \leqslant i \leqslant t$. It is possible, since $a\beta \neq 0$ for $0 \neq a \in G_2^*$. We note that (3.30), (3.26) and (3.27) imply

$$\max\big(\left\| f_i(a)\alpha_1 \right\|, \left\| f_i(a)\alpha_2 \right\|, \ldots, \left\| f_i(a)\alpha_t \right\|\big) \leqslant 2\|a\beta\|$$

for any $0 \neq a \in A$ and $1 \leqslant i \leqslant t$, and then (3.29) and (3.30) give (iv), so Lemma 3.9 is proved. □

Conclusion of the proof of Lemma 3.1. We fix first the numbers $R(\lambda)$ large enough, then we fix $\epsilon(j, \lambda)$ small enough. We also choose $\delta(n, \tau)$ small enough. Then we can apply Lemmas 3.8, 3.9, and the last statement of Lemma 3.5 is also applicable.

We take the set H of Lemma 3.9. In view of (3.22), it is enough to prove statements (i)–(v) of Lemma 3.1. We take

$\gamma \in C_{G_1,H}(\chi_{[1/10,1/2]})$ to prove (i) of Lemma 3.1,

$\gamma \in C_{G_1,H}(v_2)$ to prove (ii) and (iii) of Lemma 3.1,

$\gamma \in L_{G_1,H}$ to prove (iv) of Lemma 3.1, and

$\gamma \in B_{G_1,H}(\alpha_1, \alpha_2, \ldots, \alpha_t)$ to prove (v) of Lemma 3.1.

In (i), (ii) and (v) of Lemma 3.1 by Lemma 3.9 and the conditions of Lemma 3.1 we have $\phi_i(\gamma) \in \Gamma_2$, so $\phi_i(\gamma) = k_i\beta$ with some integers k_i for $1 \leqslant i \leqslant t$. Hence, by (3.5) and (3.18), we have

$$\tau\left(\gamma - \sum_{i=1}^{t} k_i\alpha_i\right) = 0$$

for every $\tau \in G_1^*$, which implies by Lemma 3.3(iii) that

$$\gamma = \sum_{i=1}^{t} k_i \alpha_i \in \Gamma_1,$$

so (i), (ii) and (v) of Lemma 3.1 are proved.

In (iii) and (iv) of Lemma 3.1, by Lemma 3.9 and the conditions of Lemma 3.1 we have, writing $S = C_{G_1,H}(v_2)$ in (iii) and $S = L_{G_1,H}$ in (iv) that

$$\left|\left\{(\phi_1, \phi_2, \ldots, \phi_t)\colon \phi_i \in G_2 \text{ and } \phi_i = \phi_i(\gamma) \text{ with some } \gamma \in S \text{ for } 1 \leqslant i \leqslant t\right\}\right| < 2^{\aleph_0}.$$

By (3.18), this implies that the cardinality of the set of those functions $f : G_1^* \to T$ for which there is a $\gamma \in S$ such that $f(\tau) = \tau\gamma$ for every $\tau \in G_1^*$ is less than $2^{\aleph_0}$. (Indeed, since the characters κ_n^τ are fixed, (3.18) shows that $\tau\gamma$ depends only on the t-tuple $(\phi_1(\gamma), \phi_2(\gamma), \ldots, \phi_t(\gamma))$.) This implies by Lemma 3.3(iii) that $|S| < 2^{\aleph_0}$, and the lemma is proved. □

Acknowledgments

I thank Professors Gábor Halász and Vera T. Sós for several useful discussions on the subject. I am also grateful to the referee for many valuable remarks concerning the presentation of the paper.

References

[Be] M. Beiglbock, Strong characterizing sequences of countable groups, preprint, 2003.
[B-S-W] M. Beiglbock, C. Steineder, R. Winkler, Sequences and filters of characters characterizing subgroups of compact Abelian groups, Topology Appl. 153 (2006) 1682–1695.
[B-D-S] A. Biró, J.-M. Deshouillers, V.T. Sós, Good approximation and characterization of subgroups of R/Z, Studia Sci. Math. Hungar. 38 (2001) 97–113.
[B-S] A. Biró, V.T. Sós, Strong characterizing sequences in simultaneous Diophantine approximation, J. Number Theory 99 (2003) 405–414.
[D-K] D. Dikranjan, K. Kunen, Characterizing subgroups of compact Abelian groups, J. Pure Appl. Algebra, in press.
[D-M-T] D. Dikranjan, C. Milan, A. Tonolo, A characterization of the maximally almost periodic Abelian groups, J. Pure Appl. Algebra 197 (1–3) (2005) 23–41.
[H-R] E. Hewitt, K. Ross, Abstract Harmonic Analysis, vol. I, Springer, Berlin, 1979.
[K] N. Koblitz, P-Adic Numbers, p-Adic Analysis and Zetafunctions, second ed., Springer, New York, 1984.
[K-L] C. Kraaikamp, P. Liardet, Good approximations and continued fractions, Proc. Amer. Math. Soc. 112 (2) (1991) 303–309.
[R] W. Rudin, Fourier Analysis on Groups, Wiley–Interscience, 1990.
[W] R. Winkler, Ergodic group rotations, Hartman sets and Kronecker sequences, Monatsh. Math. 135 (2002) 333–343.