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An improved estimate in a power sum problem of Turán*

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ABSTRACT

Let $z_1, z_2, \ldots, z_n$ be complex numbers, and for $j \geq 1$ define

$$S_j = z_1^j + \ldots + z_1^j.$$

Let

$$R_n = \min_{z_1, z_2, \ldots, z_n} \max_{1 \leq j < n} |S_j|$$

under the condition that

$$\max_{1 \leq t \leq n} |z_t| = 1.$$

Improving our earlier result (see [B1]) we prove here that there is a constant $q > \frac{1}{2}$ such that $R_n > q$ for every n.

1. INTRODUCTION

To find lower bounds for the quantity R_n defined in the abstract is a classical problem of the power sum theory of Turán (see [T]; an account can be found in [M]). The minimum R_n exists by compactness, and it is trivial that the condition can be replaced by $z_1 = 1$.

The history of the problem is discussed in [B1]. We just mention here that the

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1942 conjecture of Turán that $R_n > c$ for some $c > 0$ independent of n was proved by F.V. Atkinson (see [A]) in 1961. In his book ([T], Problem 12) Turán posed the problem of finding the best possible constant c for which $R_n > c$. This problem is still unsolved. The best known lower bound so far was $R_n > \frac{1}{2}$ (see [B1]). In the present paper we improve this estimate, see our Theorem below.

It is trivial that $R_n \leq 1$. The best known upper bound so far was that of Komlós, Sárközy and Szemerédi ([K-S-Sz]). We will improve their bound in our forthcoming paper [B2] to

$$R_n < 1 - (1 - \varepsilon)\frac{\log\log n}{\log n}$$

for large n, with arbitrary $\varepsilon > 0$. It is very likely that in fact an estimate of the form $R_n < 1 - c$ with some positive constant independent of n is true. The paper [C-G] also supports this conjecture, where numerical evidence seems to show that the sequence R_n is decreasing and has a limit about 0.7.

Theorem. *There is an effectively computable absolute constant $q > \frac{1}{2}$ such that if $z_1, z_2, \ldots, z_n$ are complex numbers and $z_1 = 1$, then*

$$\max_{1 \leq j \leq n} |S_j| > q.$$

So $R_n > q$ for every n.

We do not compute a concrete value of q, but it would be possible following the steps of our proof. It would be interesting (and it seems to be rather complicated) to determine the best constant obtainable by the ideas of this proof.

A few words about the proof. We will assume that the numbers $z_1 = 1, z_2, \ldots, z_n$ are such that

$$|S_j| \leq q \tag{1}$$

for $1 \leq j \leq n$, here q is a number satisfying

$$\frac{1}{2} < q < q_0 < \frac{1}{\sqrt{2}}, \tag{2}$$

where $\frac{1}{2} < q_0 < \frac{1}{\sqrt{2}}$ is a fixed constant (we will need such an auxiliary upper bound for q). Eventually we will choose q very close to $\frac{1}{2}$, and we will get a contradiction with (1), this contradiction will prove the theorem.

The present proof is in fact the investigation of the possibility of 'asymptotic equality' in the proof of [B1], and we will find that asymptotic equality is impossible there. We will use the basic formulas of [B1]:

$$\begin{aligned} &S_t + b_1 S_{t-1} + \ldots + b_{t-1} S_1 = 1 + b_1 + \ldots + b_{t-1} - t b_t \\ &(t = 1, 2, \ldots, n-1); \end{aligned} \tag{3}$$

$$S_n + b_1 S_{n-1} + \ldots + b_{n-1} S_1 = 1 + b_1 + \ldots + b_{n-1}, \tag{4}$$

where the numbers $b_1, b_2, \ldots, b_{n-1}$ are defined by

$$(Z - z_2)(Z - z_3)\dots(Z - z_n) = Z^{n-1} + b_1 Z^{n-2} + \dots + b_{n-1}.$$

Formulas (3) and (4) follow by the Newton-Girard formulas for this polynomial and by $z_1 = 1$. The essential new ingredient in the present proof is the introduction of some new formulas (see (14) below) by summing the formulas in (3). These new formulas involve the numbers

$$c_t = 1 + b_1 + \dots + b_t$$

(we set $c_0 = b_0 = 1$), and we will investigate the geometric properties of the numbers c_t. Without using new formulas, i.e. just estimating in absolute value on the basis of formulas (3) and (4), it would be impossible to improve $\frac{1}{2}$. We prove a precise theorem confirming this heuristic statement in [B2].

If $z \neq 0$ is a complex number, we set its argument in $\arg(z)$ in $(-\pi, \pi]$. By $K_1, K_2, \dots$ we will denote positive absolute constants, and the constants involved in the O-symbols are also absolute. By $m_1, m_2, \dots$ we will denote 'absolute functions' in the sense that these functions are independent of the system $z_1, z_2, \dots, z_n$, i.e. if a statement involves a function m_r, then it is meant that the statement is true with a fixed function m_r for any system $z_1 = 1, z_2, \dots, z_n$ satisfying (1).

Finally: besides q, we will have another parameter, $0 < \alpha < \frac{\pi}{4}$, and eventually we will fix α close enough to $\frac{\pi}{4}$, and then we fix q close enough to $\frac{1}{2}$. The functions $f_1(q, \alpha), f_2(q, \alpha), \dots$ will also denote absolute functions in the above sense, and if a function is denoted by $f_r(q, \alpha)$ (with an $r \geq 1$), then we mean that

$$f_r\colon \left(\frac{1}{2}, 1\right) \times \left(0, \frac{\pi}{4}\right) \to (0, \infty),$$

and that this function satisfies the condition

$$\lim_{\alpha \to \frac{\pi}{4}} \left(\limsup_{q \to \frac{1}{2}} f_r(q, \alpha) \right) = 0.$$

The notations introduced above will be freely used in the sequel.

2. AUXILIARY LEMMAS

We will repeatedly use the following relations concerning complex numbers. Observe that each statement is actually an elementary geometric fact.

Lemma 1. *If $w_1 \neq 0$ and $w_2 \neq 0$ are complex numbers, and $\arg \frac{w_1}{w_2} = \varphi$, then*

(i) $|w_1 - w_2|^2 = |w_1|^2 \sin^2 \varphi + (|w_2| - |w_1| \cos \varphi)^2$;

(ii) $|w_1 - w_2|^2 = |w_1|^2 + |w_2|^2 - 2|w_1||w_2| \cos \varphi$;

(iii) $|w_1 - w_2| \geq |w_1| \sin \varphi$;

(iv) $|w_1 + w_2| \geq |w_1| + |w_2| \cos \varphi$.

Furthermore, if w is any complex number, then

(v) $||w| - w|^2 = 2(|w| - \mathrm{Re}\, w)|w|$.

Proof. For the proof of statements (i)-(iv) we may assume that $w_1 = 1$, and let $w_2 = r\cos\varphi + ir\sin\varphi$, where $r > 0$. Then (i) and (ii) follow by direct computation, (iii) is a consequence of (i), and we get (iv) from the fact that $\mathrm{Re}(1 + w_2) = 1 + r\cos\varphi$.

Statement (v) is also easy, because its left hand side is $(|w| - w)(|w| - \overline{w})$, so the lemma is proved.

If and $1 \leq t \leq n - 1$ and $1 + b_1 + \ldots + b_{t-1} \neq 0$, $b_t \neq 0$, then define

$$\alpha_t = \left|\arg\frac{b_t}{1 + b_1 + \ldots + b_{t-1}}\right|,$$

otherwise we put $\alpha_t = \pi$ (say).

If $\alpha_t \leq \pi/4$ for $1 \leq t \leq n - 1$, then we get by repeated application of Lemma 1 (iv) (since $\cos\pi/4 = 1/\sqrt{2}$) that

$$|1 + b_1 + \ldots + b_{n-1}| \geq \frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{n-1}|),$$

which is impossible in view of (1), (2) and (4).

This means that the following definition is meaningful. Let $1 \leq k \leq n - 1$ be such that

(5) $$\alpha_1, \alpha_2, \ldots, \alpha_{k-1} \leq \frac{\pi}{4};$$

but

(6) $$\alpha_k > \frac{\pi}{4}.$$

Lemma 2. *We have the inequalities*

(i) $|1+b_1+\ldots+b_{t-1}-tb_t| \leq q(1+|b_1|+\ldots+|b_{t-1}|)$ $(t=1,2,\ldots$

(ii) $|1+b_1+\ldots+b_{t-1}| \geq 1+|b_1|\cos\alpha_1+\ldots+|b_{t-1}|\cos\alpha_{t-1}$ $(t=1,2,\ldots$

(iii) $|1+b_1+\ldots+b_{t-1}-tb_t| \geq |1+b_1+\ldots+b_{t-1}|\sin\alpha_t$ $(t=1,2,\ldots$

and $\alpha_k < \pi/2$;

(iv) $|1+b_1+\ldots+b_{k-1}| \leq q\sqrt{2}(1+|b_1|+\ldots+|b_{k-1}|)$.

Proof. Statement (i) follows from (3) and (1), and (ii) follows from Lemma 1 (iv). In case $\alpha_k \geq \pi/2$ we would have

$$|1 + b_1 + \ldots + b_{k-1} - kb_k| \geq |1 + b_1 + \ldots + b_{k-1}|$$

e.g. by Lemma 1 (iv), and this would contradict to (i), (ii), (5) and (2). So $\alpha_k < \pi/2$, and (iii) is a consequence of Lemma 1 (iii). We obtain (iv) from (i), (iii) and (6).

Lemma 3. *There are positive absolute constants K_1 and K_2 such that*

(i) $$K_1 \leq \frac{t|b_t|}{1 + |b_1| + \ldots + |b_{t-1}|} \leq K_2 \quad (t = 1, 2, \ldots, k);$$

(ii) $K_1 \leq \dfrac{|1 + b_1 + \ldots + b_t|}{t|b_t|} \leq K_2 \qquad (t = 1, 2, \ldots, k-1),$

and there are functions $m_1, m_2\colon \{1, 2, \ldots\} \to (1, \infty)$ *such that*

$$\lim_{k \to \infty} m_1(k) = \infty,$$

and

$$m_1(k) \leq 1 + |b_1| + \ldots + |b_{k-1}| \leq m_2(k)$$

for any system satisfying (1).

Proof. We get (i) by (5), (2), and Lemma 2 (i), (ii). Statement (ii) also follows (perhaps changing the constants $K_1$and K_2) from these relations and from (i), which is already proved. The existence of such functions m_1 and m_2 can be easily proved using (i), the choice

$$m_j(k) \prod_{t=1}^{k-1} \left(1 + \frac{K_j}{t}\right)$$

$(j = 1, 2)$ will be suitable.

Corollary 1. *There is a function* $m_3\colon (1/2, 1) \to (1, \infty)$ *such that*

$$\lim_{q \to 1/2} m_3(q) = \infty,$$

and

$$m_3(q) < k, \quad m_3(q) < 1 + |b_1| + \ldots + |b_{k-1}|$$

for any system satisfying (1).

Proof. We know by Lemma 2 (iv), (ii) and (5) that

$$1 + \frac{1}{\sqrt{2}}(|b_1| + \ldots + |b_{k-1}|) \leq \sqrt{2}q(1 + |b_1| + \ldots + |b_{k-1}|),$$

and so

$$1 - \frac{1}{\sqrt{2}} \leq \left(\sqrt{2}q - \frac{1}{\sqrt{2}}\right)(1 + |b_1| + \ldots + |b_{k-1}|) \leq \sqrt{2}\left(q - \frac{1}{2}\right)m_2(k).$$

This proves the statement concerning k, and then, taking into account the existence of m_1, we also get the statement (perhaps changing the function m_3) for $1 + |b_1| + \ldots + |b_{k-1}|$.

Remember the notation $c_t = 1 + b_1 + \ldots + b_t$ and the convention $b_0 = c_0 = 1$.

Corollary 2. *There is a function* $m_4 : (0, 1] \to (0, 1]$ *such that*

$$\lim_{y \to 0} m_4(y) = 0$$

and the following assertion holds for any system satisfying (1): if $H \subseteq \{0, 1, \ldots, k-1\}$,

and

$$\sum_{t \in H} |c_t| \le y(1 + |c_1| + \ldots + |c_{k-1}|), \tag{7}$$

then

$$\sum_{t \in H} |b_t| \le m_4(y)(1 + |b_1| + \ldots + |b_{k-1}|). \tag{8}$$

Proof. It is easy to see from Lemma 3 that if y is small enough, and (7) is true with some nonempty H, then k is large. So we may assume that k is arbitrarily large. Let $\varepsilon > 0$ be fixed, then by Lemma 3, we have

$$\left(\sum_{t=0}^{[\varepsilon k]} |b_t|\right)\left(\prod_{t=[\varepsilon k]+1}^{k-1} \left(1 + \frac{K_1}{t}\right)\right) \le \sum_{t=0}^{k-1} |b_t|.$$

For a fixed (but small enough) ε one has for large k that

$$\prod_{t=[\varepsilon k]+1}^{k-1} \left(1 + \frac{K_1}{t}\right) \ge \left(\frac{1}{\varepsilon}\right)^{K_3},$$

and so (using also Lemma 3 (ii))

$$\sum_{t \in H} |b_1| \le \left(\sum_{t=0}^{[\varepsilon k]} |b_t|\right) + \sum_{t \in H, t > \varepsilon k} |b_1| \le \varepsilon^{K_3} \sum_{t=0}^{k-1} |b_t| + \frac{1}{\varepsilon k K_1} \sum_{t \in H, t > \varepsilon k} |c_t|.$$

Since $1 + |c_1| + \ldots + |c_{k-1}| \le k(1 + |b_1| + \ldots + |b_{k-1}|)$, we have by (7) that

$$\sum_{t \in H} |b_1| \le \left(\sum_{t=0}^{k-1} |b_t|\right)\left(\varepsilon^{K_3} + \frac{y}{\varepsilon K_1}\right),$$

and this proves the corollary.

Lemma 4. *If* $1 \le t \le k$, *then*

$$|1 + b_1 + \ldots + b_{t-1}| = \frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{t-1}|) + O\left(\left(q - \frac{1}{2}\right)(1 + |b_1| + \ldots + |b_{k-1}|)\right).$$

Proof. On the one hand, by Lemma 2 (ii) and (5),

$$|1 + b_1 + \ldots + b_{t-1}| \ge \frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{t-1}|),$$

on the other hand, by Lemma 1 (iv) and (5) we have

$$|1 + b_1 + \ldots + b_{k-1}| \geq |1 + b_1 + \ldots + b_{t-1}| + \frac{1}{\sqrt{2}}(|b_t| + \ldots + |b_{k-1}|).$$

This inequality and Lemma 2 (iv) give

$$|1 + b_1 + \ldots + b_{t-1}| \leq q\sqrt{2}(1 + |b_1| + \ldots + |b_{t-1}|) + \left(q\sqrt{2} - \frac{1}{\sqrt{2}}\right)(|b_t| + \ldots + |b_{k-1}|),$$

and this proves the lemma.

Now let $0 < \alpha < \pi/4$ be fixed (but we will choose it close to $\pi/4$), and let

$$H_\alpha = \{1 \leq t \leq k - 1 : \alpha_t \leq \alpha\}.$$

Remember what was told about the functions f_r in the Introduction.

Lemma 5. *There are functions* $f_1(q, \alpha)$ *and* $f_2(q, \alpha)$ *such that*

(i) $\sum_{t \in H_\alpha} |b_t| \leq f_1(q, \alpha)(1 + |b_1| + \ldots + |b_{k-1}|)$;

(ii) *if* $1 \leq t \leq k - 1$ *and* $t \notin H_\alpha$, *then*

$$\left| t|b_t| - \frac{1}{2}(1 + |b_1| + \ldots + |b_{t-1}|)\right| \leq f_2(q, \alpha)(1 + |b_1| + \ldots + |b_{t-1}|),$$

and

$$\left|\frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{t-1}|) - |1 + b_1 + \ldots + b_{t-1}|\right| \leq f_2(q, \alpha)(1 + |b_1| + \ldots + |b_{t-1}|).$$

Proof. By Lemma 2 (ii) and (iv), (5), and the definition of H_α we get

$$q\sqrt{2}(1 + |b_1| + \ldots + |b_{k-1}|) \geq \frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{k-1}|) + \left(\cos\alpha - \frac{1}{\sqrt{2}}\right) \sum_{t \in H_\alpha} |b_t|,$$

and this proves (i).

For the proof of (ii) observe that Lemma 2 (i) and Lemma 1 (i) give for $1 \leq t \leq k - 1$ that

$$q^2(1 + |b_1| + \ldots + |b_{t-1}|)^2 \geq |1 + b_1 + \ldots + b_{t-1}|^2 \sin^2\alpha_t + |(t|b_t| - |1 + b_1 + \ldots + b_{t-1}| \cos\alpha_t)|^2, \tag{9}$$

and by (9), Lemma 2 (ii) and (5) we have

(10) $$\frac{q}{\sin\alpha_t}(1+|b_1|+\ldots+|b_{t-1}|) \geq |1+b_1+\ldots+b_{t-1}| \geq \frac{1}{\sqrt{2}}(1+|b_1|+\ldots+|b_{t-1}|).$$

This proves the second formula in (ii), because for $1 \leq t \leq k-1$, $t \notin H_\alpha$ one has $\alpha \leq \alpha_t \leq \pi/4$.
By (9) and the right-hand inequality of (10),

$$\left| t|b_t| - \frac{1}{\sqrt{2}}|1+b_1+\ldots+b_{t-1}| \right| \leq \left(\left|\cos\alpha_t - \frac{1}{\sqrt{2}}\right| + \sqrt{q^2 - \frac{\sin^2\alpha_t}{2}} \right)(1+|b_1|+\ldots+|b_{t-1}|).$$

This proves the first formula in (ii), using also the second formula of (ii) (and perhaps changing f_2), which is already proved.

Lemma 6. *For $1 \leq t \leq k$ we have*

$$\frac{t|c_t|}{1+|c_1|+\ldots+|c_{t-1}|} = \frac{3}{2} + O\left(f_5(q,\alpha)\frac{1+|b_1|+\ldots+|b_{k-1}|}{1+|b_1|+\ldots+|b_t|}\right).$$

Proof. First we assume that $1 \leq t \leq k-1$.
The relations

(11) $$\left| \sum_{j=1}^{t} j|b_j| - \frac{1}{2}\sum_{j=1}^{t}(1+|b_1|+\ldots+|b_{j-1}|) \right| \leq tf_3(q,\alpha)(1+|b_1|+\ldots+|b_{k-1}|)$$

and

(12) $$\left| \sum_{j=1}^{t} |1+b_1+\ldots+b_{j-1}| - \frac{1}{\sqrt{2}}\sum_{j=1}^{t}(1+|b_1|+\ldots+|b_{j-1}|) \right| \leq tf_3(q,\alpha)(1+|b_1|+\ldots+|b_{k-1}|)$$

are easily obtained by considering separately the cases $j \in H_\alpha$ and $j \notin H_\alpha$, and using Lemma 5 (i), (ii) and Lemma 3 (i). Since

(13) $$\sum_{j=1}^{t} j|b_j| + \sum_{j=1}^{t}(1+|b_1|+\ldots+|b_{j-1}|) = t(1+|b_1|+\ldots+|b_t|),$$

so from this, (11) and (12) we infer that for $1 \leq t \leq k-1$

$$\frac{3}{\sqrt{2}}\sum_{j=1}^{t}|1+b_1+\ldots+b_{j-1}| = t(1+|b_1|+\ldots+|b_t|) + O(tf_4(q,\alpha)(1+|b_1|+\ldots+|b_{k-1}|)).$$

Applying Lemma 4 (for $t+1$) and the fact that

$$\sum_{j=1}^{t} |1+b_1+\ldots+b_{j-1}| \geq K_4 t(1+|b_1|+\ldots+|b_t|)$$

(which follows by Lemma 3 and (13)) we obtain the desired estimate for $1 \leq t \leq k-1$.

But Lemma 3 and its Corollary 1 show (since $|c_k| = |c_{k-1}| + O(|b_k|)$) that this proves the lemma (perhaps changing f_5), including the case $t = k$.

3. NEW FORMULAS

Now we introduce the important new formulas

(14) $$S_t + c_1 S_{t-1} + \ldots + c_{t-1} S_1 = 2(1 + c_1 + \ldots + c_{t-1}) - t c_t \quad (t = 1, 2, \ldots, k).$$

These formulas are obtained from (3) by induction ($k \leq n-1$): (14) for $t = 1$ is just (3) for $t = 1$, and the difference of (14) for t and $t-1$ is exactly (3) for t, because for $t \geq 2$ one has

$$(2(1+c_1+\ldots+c_{t-1}) - tc_t) - (2(1+c_1+\ldots+c_{t-2}) - (t-1)c_{t-1}) = 1 + b_1 + \ldots + b_{t-1} - tb_t\,.$$

We define the angles β_t similarly to α_t. If $1 \leq t \leq k$, and $1 + c_1 + \ldots + c_{t-1} \neq 0$, $c_t \neq 0$, then define

$$\beta_t = \left| \arg \frac{c_t}{1 + c_1 + \ldots + c_{t-1}} \right|,$$

otherwise we put $\beta_t = \pi$ (say).

We will need a crude auxiliary bound.

Lemma 7. *We have*

$$\beta_t \leq \frac{\pi}{8}$$

for $1 \leq t \leq k$.

Proof. Let $\beta = \pi/8$ and assume that $1 \leq t \leq k$ is such that $\beta_t > \beta$, but $\beta_1, \beta_2, \ldots, \beta_{t-1} \leq \beta$.

Then, by repeated application of Lemma 1 (iv) we get

(15) $$|1 + c_1 + \ldots + c_{t-1}| \geq (1 + |c_1| + \ldots + |c_{t-1}|) \cos\beta.$$

On the other hand, $\beta_t > \beta$. If $\beta_t \leq \pi/2$, then by Lemma 1 (iii) one has

(16) $$|2(1 + c_1 + \ldots + c_{t-1}) - tc_t| \geq 2|1 + c_1 + \ldots + c_{t-1}| \sin\beta,$$

but this is also true in the case $\beta_t > \pi/2$, moreover, then

$$|2(1 + c_1 + \ldots + c_{t-1}) - tc_t| \geq 2|1 + c_1 + \ldots + c_{t-1}|,$$

e.g. by Lemma 1 (iv). So in any case, (15) and (16) imply that

$$|2(1 + c_1 + \ldots + c_{t-1}) - tc_t| \geq (1 + |c_1| + \ldots + |c_{t-1}|)(2\cos\beta \sin\beta),$$

and this means (by (14) and (1)) that

$$2\cos\beta\sin\beta \le q.$$

But this contradicts to (2), since

$$2\cos\beta\sin\beta = \sin 2\beta = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

so the lemma is proved.

The following lemma is very important in the proof of the theorem.

Lemma 8. *We have*

(i) $\dfrac{|1 + c_1 + \ldots + c_{k-1}|}{1 + |c_1| + \ldots + |c_{k-1}|} = 1 + O(f_9(q, \alpha)),$

and

(ii) $\cos\beta_k = 1 + O(f_{10}(q, \alpha))$

Proof. Introduce the notation

$$h_t = \frac{|1 + c_1 + \ldots + c_{t-1}|}{1 + |c_1| + \ldots + |c_{t-1}|}.$$

Using Lemma 7 and Lemma 1 (iv) it is clear that

$$(17) \qquad h_t \ge \cos\frac{\pi}{8}$$

for $1 \le t \le k$. It is also clear by Lemma 1 (iv) that

$$h_{t+1} \ge \frac{|1+c_1+\ldots+c_{t-1}|+|c_t|\cos\beta_t}{1+|c_1|+\ldots+|c_{t-1}|+|c_t|} = h_t+(\cos\beta_t-h_t)\frac{|c_t|}{1+|c_1|+\ldots+|c_t|}.$$

Hence, since for $1 \le t \le k - 1$ by Lemma 3

$$\frac{|c_t|}{1 + |c_1| + \ldots + |c_t|} \ge \frac{K_5}{t},$$

so for $1 \le t \le k - 1$ we have that

$$(18) \qquad h_{t+1} \ge h_t + K_5\frac{\cos\beta_t - h_t}{t}, \text{ if } \cos\beta_t \ge h_t.$$

We would like to estimate $\cos\beta_t - h_t$ from below. For this we apply Lemma 1 (ii) with the choice $w_1 = (2(1 + c_1 + \ldots + c_{t-1}))/(1 + |c_1| + \ldots + |c_{t-1}|)$ and $w_2 = (tc_t)/(1 + |c_1| + \ldots + |c_{t-1}|)$ with any $1 \le t \le k$. Using Lemma 6 and the fact that $|w_1 - w_2| \le q$ by (14) and (1), we obtain by Lemma 1 (ii) that

$$4h_t^2 + \frac{9}{4} - 6h_t\cos\beta_t + O\left(f_6(q, \alpha)\left(\frac{1 + |b_1| + \ldots + |b_{k-1}|}{1 + |b_1| + \ldots + |b_t|}\right)^2\right) \le q^2$$

whence (since $q^2 - \frac{1}{4} = f_7(q, \alpha)$, and using also (17))

$$(19)\qquad \cos\beta_t \geq \frac{2h_t^2+1}{3h_t} + O\left(f_8(q,\alpha)\left(\frac{1+|b_1|+\ldots+|b_{k-1}|}{1+|b_1|+\ldots+|b_t|}\right)^2\right)$$

for $1 \leq t \leq k$.

The case $t = k$ of (19) shows that (ii) is a consequence of (i), so it is enough to prove (i). We would like to define the function $f_9(q,\alpha)$. If $f_8(q,\alpha) \geq 1$ (trivial case), let $f_9(q,\alpha) = 1$. So we may assume that $f_8(q,\alpha) < 1$.

Assume that we have

$$h_t \geq 1 - f_9(q,\alpha)$$

for some t (we will determine the function $f_9(q,\alpha)$ later), and let $H = 1 - f_9(q,\alpha)$.

Since the function $F(h) = (2h^2+1)/(3h)$ is increasing for $1/\sqrt{2} \leq h \leq 1$, so if $H \geq 1/\sqrt{2}$, then

$$(20)\qquad F(h_t) \geq F(H) = H + \frac{1-H^2}{3H} \geq H + \frac{1-H}{3},$$

since $H \leq 1$. This means (taking into account (19)) that for $1 \leq t \leq k-1$, if $h_t \geq 1 - f_9(q,\alpha) \geq 1/\sqrt{2}$, then

$$(21)\qquad \cos\beta_t \geq 1 - f_9(q,\alpha) + \frac{f_9(q,\alpha)}{3} + O\left(f_8(q,\alpha)\left(\frac{1+|b_1|+\ldots+|b_{k-1}|}{1+|b_1|+\ldots+|b_t|}\right)^2\right).$$

On the other hand, since the function $F(h) - h = (1-h^2)/(3h)$ is decreasing for $0 < h \leq 1$, so from (19), using again the right-hand side inequality in (20), we get for $1 \leq t \leq k-1$, if $h_t \leq 1 - f_9(q,\alpha)$, the inequality

$$(22)\qquad \cos\beta_t - h_t \geq \frac{f_9(q,\alpha)}{3} + O\left(f_8(q,\alpha)\left(\frac{1+|b_1|+\ldots+|b_{k-1}|}{1+|b_1|+\ldots+|b_t|}\right)^2\right).$$

We will choose an integer $1 \leq t_0 \leq k-1$, and let

$$(23)\qquad Q = \frac{1+|b_1|+\ldots+|b_{k-1}|}{1+|b_1|+\ldots+|b_{t_0}|}.$$

If we assume that

$$(24)\qquad f_9(q,\alpha) \geq K_6 f_8(q,\alpha) Q^2$$

with a large enough constant K_6, then by (21), (22) and (23) we infer that if $t_0 \leq t \leq k-1$, and $h_t \geq 1 - f_9(q,\alpha)$, then

$$(25)\qquad \cos\beta_t \geq 1 - f_9(q,\alpha)$$

(of course this is also true in the case $1 - f_9(q,\alpha) \leq 1/\sqrt{2}$, by Lemma 7), and if $t_0 \leq t \leq k-1$, and $h_t \leq 1 - f_9(q,\alpha)$, then

$$(26)\qquad \cos\beta_t - h_t \geq K_7 f_9(q,\alpha).$$

It follows from (25) and Lemma 1 (iv) that if $h_t \geq 1 - f_9(q,\alpha)$ for some $t_0 \leq t \leq k-1$, then $h_{t+1} \geq 1 - f_9(q,\alpha)$, so $h_k \geq 1 - f_9(q,\alpha)$ by induction, and we are done.

Hence we may assume that $h_t \leq 1 - f_9(q, \alpha)$ for $t_0 \leq t \leq k-1$, so that (26) is true for every $t_0 \leq t \leq k-1$. But then we get from (18) that

$$h_k \geq h_{t_0} + K_8 f_9(q, \alpha) \sum_{t=t_0}^{k-1} \frac{1}{t},$$

so (since $h_k \leq 1$ and $h_{t_0} \geq 0$)

$$(27) \qquad f_9(q, \alpha) \sum_{t=t_0}^{k-1} \frac{1}{t} \leq K_9.$$

On the other hand, Lemma 3 (i) gives us

$$\sum_{t=0}^{k-1} |b_t| \leq \left(\sum_{t=0}^{t_0} |b_t| \right) \left(\prod_{t=t_0+1}^{k-1} \left(1 + \frac{K_2}{t} \right) \right),$$

whence

$$\sum_{t=t_0+1}^{k-1} \frac{1}{t} \geq K_{10} \log Q,$$

and, combining with (27),

$$(28) \qquad f_9(q, \alpha) \log Q \leq K_{11}.$$

Remember that (28) holds if the assumption (24) is satisfied.

We would like to define the function $f_9(q, \alpha)$. If $f_8(q, \alpha) \geq 1$ (trivial case), let $f_9(q, \alpha) = 1$. So we may assume that $f_8(q, \alpha) < 1$.

In preparation to the definition of $f_9(q, \alpha)$, we firstly define the number t_0. If

$$(29) \qquad f_8(q, \alpha)^{1/3}(1 + |b_1| + \ldots + |b_{k-1}|) \leq 1,$$

let $t_0 = 1$, and if

$$(30) \qquad f_8(q, \alpha)^{1/3}(1 + |b_1| + \ldots + |b_{k-1}|) > 1$$

let $1 \leq t_0 \leq k-1$ be the least integer for which

$$(31) \qquad \frac{1 + |b_1| + \ldots + |b_{k-1}|}{1 + |b_1| + \ldots + |b_{t_0}|} f_8(q, \alpha)^{1/3} < 1,$$

i.e. we have

$$(32) \qquad \frac{1 + |b_1| + \ldots + |b_{k-1}|}{1 + |b_1| + \ldots + |b_{t_0-1}|} f_8(q, \alpha)^{1/3} \geq 1.$$

Such a t_0 exists because $f_8(q, \alpha) < 1$, and (30) is true.
In view of (23), (29) and (31), if the inequality

$$(33) \qquad f_9(q, \alpha) \geq K_6 f_8(q, \alpha)^{1/3}$$

holds, then (24) will be true. On the other hand, observe that in the case (29)

$$(34) \qquad Q \geq m_5(q),$$

where $m_5: (1/2, 1) \to (1, \infty)$ is a function with

$$(35) \qquad \lim_{q \to 1/2} m_5(q) = \infty$$

(we use Lemma 3 and its Corollary 1 for this); and in the case (30),

$$(36) \qquad Q \geq K_{12} f_8(q, \alpha)^{-1/3}$$

by (31), (32) and Lemma 3 (i).

We now define $f_9(q, \alpha)$ for the case $f_8(q, \alpha) < 1$. Let K_{13} be a large enough constant, and

$$(37) \qquad f_9(q, \alpha) = K_{13} \max \left(\frac{1}{\log \frac{1}{f_8(q, \alpha)}}, \frac{1}{\log m_5(q)} \right).$$

It is clear by (35) that $f_9(q, \alpha)$ has the desired property (see the Introduction) of the functions $f_r(q, \alpha)$.

We may assume that $f_8(q, \alpha) < K_{14}$ with an arbitrarily small constant $K_{14} > 0$, since otherwise $f_9(q, \alpha) > K_{15}$, and the assertion of the lemma is trivial. But then (33) is true, and so (24) is satisfied, hence (28) also holds. This is a contradiction if K_{13} is large enough, in view of (34), (36) and (37). Remember that this contradiction means that (26) can not be true for every $t_0 \leq t \leq k - 1$, but then (25) is true with some $t_0 \leq t \leq k - 1$, and as we have seen, this proves the lemma.

Remark that our proof above actually depends on the fact that 1 is an attractive fixed point of the transformation $h \to (2h^2 + 1)/3h$.

4. PROOF OF THE THEOREM

Using Lemma 8 (i) and (ii), Lemma 6 and (14) for $t = k$ we have

$$(38) \qquad S_k + c_1 S_{k-1} + \ldots + c_{k-1} S_1 = \frac{1}{2}(1 + c_1 + \ldots + c_{k-1}) + O(f_{11}(q, \alpha)(1 + |c_1| + \ldots + |c_{k-1}|)).$$

Define

$$(39) \qquad t_j = \frac{c_j}{1 + c_1 + \ldots + c_{k-1}} \qquad (j = 0, 1, \ldots, k-1)$$

(Remember that $c_0 = 1$, and $1 + c_1 + \ldots + c_{k-1} \neq 0$, e.g. by Lemma 7.) Then we know that

$$(40) \qquad t_0 + t_1 + \ldots + t_{k-1} = 1, \text{ and } |t_0| + |t_1| + \ldots + |t_{k-1}| = 1 + O(f_{12}(q, \alpha)).$$

This means in particular that

$$\sum_{j=0}^{k-1} |t_j| - \sum_{j=0}^{k-1} t_j = O(f_{12}(q, \alpha)),$$

and so

$$\sum_{j=0}^{k-1} |t_j| - \sum_{j=0}^{k-1} \mathrm{Re}\, t_j = O(f_{12}(q,\alpha)).$$

By Lemma 1 (v) and the Cauchy-Schwarz inequality this implies that

$$\sum_{j=0}^{k-1} ||t_j| - t_j| \le \sqrt{2}\sqrt{\sum_{j=0}^{k-1} (|t_j| - \mathrm{Re}\, t_j)}\sqrt{\sum_{j=0}^{k-1} |t_j|} = O(f_{13}(q,\alpha)). \tag{41}$$

Then (38), (39), (40), (41) and (1) give

$$|t_0|S_k + |t_1|S_{k-1} + \ldots + |t_{k-1}|S_1 = \frac{1}{2} + O(f_{14}(q,\alpha)),$$

and taking real parts,

$$|t_0|\mathrm{Re}S_k + |t_1|\mathrm{Re}S_{k-1} + \ldots + |t_{k-1}|\mathrm{Re}S_1 = \frac{1}{2} + O(f_{14}(q,\alpha)). \tag{42}$$

Let $c < 1/2$ be an arbitrary number, and

$$H_c = \{0 \le j \le k-1 \colon \mathrm{Re}S_{k-j} \le c\}.$$

Then by (1) we have

$$|t_0|\mathrm{Re}S_k + |t_1|\mathrm{Re}S_{k-1} + \ldots + |t_{k-1}|\mathrm{Re}S_1 \le q(|t_0| + |t_1| + \ldots + |t_{k-1}|) - (q-c)\sum_{j \in H_c} |t_j|,$$

and so (40) and (42) show that

$$\left(\frac{1}{2} - c\right)\sum_{j \in H_c} |t_j| \le f_{15}(q,\alpha).$$

(We used also that $q > 1/2$.) Hence, if we choose

$$c = \frac{1}{2} - f_{16}(q,\alpha),$$

where $f_{16}(q,\alpha) = \sqrt{f_{15}(q,\alpha)}$, then

$$\sum_{j \in H_c} |t_j| \le f_{16}(q,\alpha)$$

In view of (39), this implies

$$\sum_{j \in H_c} |c_j| \le f_{16}(q,\alpha)(1 + |c_1| + \ldots + |c_{k-1}|),$$

and Corollary 2 of Lemma 3 then gives

$$\sum_{j \in H_c} |b_j| \le f_{17}(q,\alpha)(1 + |b_1| + \ldots + |b_{k-1}|). \tag{43}$$

If $j \notin H_c$, then (since $\mathrm{Re}S_{k-j} \ge 1/2 - f_{16}(q,\alpha)$ and $S_{k-j} \le q$) we have

$$S_{k-j} = \frac{1}{2} + O(f_{18}(q,\alpha)).$$

Therefore, from (43) we obtain (considering separately the summation over H_c and over the complement of H_c) that

$$S_k + b_1 S_{k-1} + \ldots + b_{k-1} S_1 =$$
$$\frac{1}{2}(1 + b_1 + \ldots + b_{k-1}) + O(f_{19}(q,\alpha)(1 + |b_1| + \ldots + |b_{k-1}|)).$$

Hence, applying the case $t = k$ of (3) (remember that $k \leq n - 1$)

$$\frac{1}{2}(1 + b_1 + \ldots + b_{k-1}) = \tag{44}$$
$$1 + b_1 + \ldots + b_{k-1} - kb_k + O(f_{19}(q,\alpha)(1 + |b_1| + \ldots + |b_{k-1}|))$$

holds. But (6) and Lemma 2 (iii) give

$$|1 + b_1 + \ldots + b_{k-1} - kb_k| \geq \frac{1}{\sqrt{2}} |1 + b_1 + \ldots + b_{k-1}|,$$

hence from (44) it follows that

$$|1 + b_1 + \ldots + b_{k-1}| \leq f_{20}(q,\alpha)(1 + |b_1| + \ldots + |b_{k-1}|). \tag{45}$$

On the other hand, because of (5) and Lemma 2 (ii),

$$|1 + b_1 + \ldots + b_{k-1}| \geq \frac{1}{\sqrt{2}}(1 + |b_1| + \ldots + |b_{k-1}|). \tag{46}$$

The inequalities (45) and (46) imply that

$$\frac{1}{\sqrt{2}} \leq f_{20}(q,\alpha). \tag{47}$$

But we know that

$$\lim_{\alpha \to \frac{\pi}{4}} \left(\limsup_{q \to \frac{1}{2}} f_{20}(q,\alpha) \right) = 0,$$

hence if we fix $\alpha < \pi/4$ close enough to $\pi/4$, and then fix $q > 1/2$ close enough to $1/2$, (47) will be a contradiction. This means that (1) can not hold, and so the theorem is proved.

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