GEOMETRY OF SPLICE-QUOTIENT SINGULARITIES

GÁBOR BRAUN

Abstract. We obtain a new important basic result on splice-quotient singularities in an elegant combinatorial-geometric way: every level of the divisorial filtration of the ring of functions is generated by monomials of the defining coordinate functions. The elegant way is the language of line bundles based on Okuma’s description of the function ring of the universal abelian cover. As an easy application, we obtain a new proof of the End Curve Theorem of Neumann and Wahl.

1. Introduction

1.1. Results and applications. In this article we consider isolated complex normal surface singularities whose link is a rational homology sphere, i.e., the resolution graph is a tree of projective lines. A nice subclass of these are formed by splice-quotient singularities, which are generalizations of weighted-homogeneous singularities by Neumann and Wahl [9, Definition 7.1 & Theorem 7.2]. This class includes rational singularities and minimally elliptic singularities by [13, Theorem 5.1.]. For splice-quotient singularities, the resolution graph determines the leading terms of the equations for the universal abelian cover. Therefore one may expect that the resolution graph determines many analytical properties of these singularities. Our aim is to provide a basis for such results. On the other hand, we point out in Section 9 that the Hilbert–Samuel function is not determined by the resolution graph alone.

For every singularity, vanishing order on the exceptional curves produces the divisorial filtration on the ring of functions of the universal abelian cover. Our main result (Theorem 4.1) is that the divisorial filtration for splice-quotient singularities is algebraic i.e., coming from the equations: every level is generated by monomials. As a special case, this contains Okuma’s result [11, (3.3)] about the filtration defined by the vanishing order of only one exceptional curve.

This has many interesting consequences.

First of all, it leads to an easy proof of the geometric characterization of splice-quotient singularities, namely, the End Curve Theorem 4.2 [10, page 2].

Second, Némethi [7, Proposition 3.1.4.(1) with Theorem 4.1.1.] has used our result to explicitly compute the dimension of factors of different levels of the divisorial filtration and to determine the multiplicity of the singularity among others. This also gives formulas for the dimension of the cohomology of an important class of line bundles, which we shall call natural line bundles. These complement the additivity of the geometric genus by Okuma [11, Theorem 4.5] and the Seiberg–Witten Invariant Conjecture for splice-quotients (special case: [8, Theorem on page 1], general case: [11, Corollary 2.2.4]).

2000 Mathematics Subject Classification. Primary 14F05, 14J25; Secondary 14C17, 32S05, 32S50.

Key words and phrases. splice-quotient singularity, End Curve Theorem, divisorial filtration.
1.2. Method of proof. We employ a purely geometric approach to splice-quotient singularities, exploiting only their geometric characterization: the End Curve Condition (Definition 2.3.1). We shall consequently use the language of line bundles demonstrating that it is well-suited to the study of splice-quotient singularities. Therefore we present the definitions from the literature rephrased in this language. (The only exception is the counterexample in Section 9 which involves more delicate features.)

Thanks to Okuma’s description of the function ring of the universal abelian cover via natural line bundles, we can also rephrase our main result on the divisorial filtration in the language of line bundles: the essential part is that factors of sections of natural line bundles are generated by monomials of the defining coordinate functions. (Lemma 7.1(1)).

To ease the inductive proof, the lemma contains another result claiming that natural line bundles have many global sections compared to local ones. It is interesting that this second result has become the core of Némethi’s cohomology formulas [7, Theorem 4.1.1. & Corollary 4.1.3.].


2. Notation and setup

2.1. Resolutions. Let $(X, o)$ be a complex normal surface singularity whose link is a rational homology sphere. Let $\tilde{\pi}: \tilde{X} \to X$ be a good resolution with dual graph $\Gamma$. Recall that the link being a rational homology sphere means that $\Gamma$ is a tree and all the irreducible exceptional divisors have genus $0$.

We will use the same notation for a graph and its set of vertices, so $v \in \Gamma$ means $v$ is a vertex of $\Gamma$.

Let $L := H_2\left(\tilde{X}, \mathbb{Z}\right)$. It is freely generated by the classes of the irreducible exceptional curves $\{E_v\}_{v \in \Gamma}$, hence $L$ is also the group of integral divisors supported on the exceptional curves. Let $L'$ denote $H^2\left(\tilde{X}, \mathbb{Z}\right)$. Via Poincaré duality, $L'$ is the dual of $L$, so it is freely generated by the duals $E^*_v$ of the $E_v$, determined by $(E_v^*, E_w) = 1$ and $(E^*_v, E_w) = 0$ for $v \neq w$.

The intersection form $(\cdot, \cdot)$ on $L$ provides an embedding $L \hookrightarrow L'$ with factor the first homology group $H$ of the link $\partial \tilde{X}$. We denote by $[l']$ the image of $l' \in L'$ in $H$. The intersection form $(\cdot, \cdot)$ extends to $L'$.

The above embedding $L \hookrightarrow L'$ induces an isomorphism $L \otimes \mathbb{Q} \simeq L' \otimes \mathbb{Q}$ and hence realizes $L'$ as a subgroup of $L \otimes \mathbb{Q}$. The $\mathbb{Q}$ vector space $L \otimes \mathbb{Q}$ has a natural partial ordering: the elements greater than or equal to $0$ are the elements with non-negative coefficients in the base of the $E_v$. By restriction to subgroups, this provides a partial ordering on $L$ and $L'$. Let $L_{\geq 0}$ denote the semigroup of elements greater than or equal to $0$ of $L$. In other words, $L_{\geq 0}$ is the set of effective divisors supported on the exceptional curves.

Let $\theta: H \to \hat{H}$ be the isomorphism $[l'] \mapsto e^{2\pi i (l', \cdot)}$ of $H$ with its Pontrjagin dual $\hat{H}$.

2.2. Natural line bundles and the divisorial filtration.

2.2.1. Natural line bundles. It is well-known, see [5] 3.1 or [12] Lemma 2.2, that the first Chern class mapping from the Picard group to the second cohomology group is onto and has a group section $\mathcal{O}$ whose image contains the line bundles associated to divisors supported on the exceptional curves. The image is actually
unique. (Here is an easy, totally algebraic proof of the existence and uniqueness of the section. The claim is equivalent to the unique splitting of the induced short exact sequence $H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow \text{Pic}(\tilde{X})/L \rightarrow L'/L$. This sequence really splits uniquely as it is an extension of a torsion-group by a torsion-free divisible group.)

$$
\begin{array}{ccc}
0 & \rightarrow & H^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \rightarrow \\
& \rightarrow & \text{Pic}(\tilde{X})/L \rightarrow L'/L \rightarrow 0
\end{array}
$$

We shall call the line bundles of this subgroup as natural line bundles. They are important eg in the study of the universal abelian cover but have so far no name in the literature.

2.2.2. Eigen-decomposition of the universal abelian cover. Natural line bundles appear in the eigen-decomposition of the function ring of the universal abelian cover. Let $c: (Y, o) \rightarrow (X, o)$ be the universal abelian cover of $(X, o)$. Let $\pi_Y: \tilde{Y} \rightarrow Y$ be the (normalized) pullback of $\pi$ by $c$, and let $\tilde{c}: \tilde{Y} \rightarrow \tilde{X}$ be the morphism covering $c$.

Then the action of $H$ on $(Y, o)$ lifts to $\tilde{Y}$, and one has the following eigenspace decomposition ([11], Lemma 3.5, [5] (3.7)). The eigenspaces are parametrized by

$(2.2.2.1) \quad Q := \left\{ \sum l'_i E_i \in L' \mid 0 \leq l'_i < 1 \right\}$.

The actual decomposition is

$(2.2.2.2) \quad \tilde{c}_* \mathcal{O}_{\tilde{Y}} = \bigoplus_{l' \in Q} \mathcal{O}(-l')$.

Here and below a line bundle $\mathcal{L}$ on the right-hand side is the $\theta(-[c_1(\mathcal{L})])$-eigenspace of the left-hand side. More generally, one has

$(2.2.2.3) \quad \tilde{c}_* \mathcal{O}_{\tilde{Y}} = \bigoplus_{l' \in Q} \mathcal{L} \otimes \mathcal{O}(-l') \quad \text{for every line bundle } \mathcal{L} \text{ over } \tilde{X}$.

The class $Q$ is a representative set of $L'/L$. For any $l' \in L'$, let $D_{l'} \in Q$ be the unique element of $Q$ representing $l' + L$. Thus all the eigenspaces of the ring $\tilde{c}_* \mathcal{O}_{\tilde{Y}}$ have the form $\mathcal{O}(-D_{l'})$. The multiplication in the ring is given by inclusion of line bundles $\mathcal{O}(-l'_{1}) \otimes \mathcal{O}(-l'_{2}) \simeq \mathcal{O}(-l'_{1} - l'_{2}) \hookrightarrow \mathcal{O}(-D_{l'_{1} + l'_{2}})$.

Remark 2.2.2.4. We have deliberately defined the line bundles $\mathcal{O}(-l')$ and the homomorphisms appearing in the eigen-decomposition and multiplication only up to isomorphism, ie up to a multiplication by an invertible function. This will suffice for our purposes, since we will be interested only in divisors of sections.

It is possible to define these exactly, but this involves only arbitrary choices and no new real relation.

Remark 2.2.2.5. The action of $H$ on the universal abelian cover differs by a sign between the sources [11] and [9]. As we make more direct use of formulas from the former, we have adopted its sign convention.
2.2.3. Divisorial filtration. For a resolution $\pi$ of an isolated local surface singularity, the divisorial filtration of the function ring $O_{Y,o}$ of the universal abelian cover is defined as in [4, (4.1.1)]; the levels of the filtration are indexed by $l' \in L'$ with $l' \geq 0$ and the level at $l'$ is

$$\mathcal{F}(l') := \{ f \in O_{Y,o} \mid \text{div}(f \circ \pi_Y) \geq \tilde{c}^*(l') \} = H^0\left(\tilde{Y}, O(\tilde{c}^*(l'))\right)$$

(2.2.3.1)

Recall eg from [3] (3.3) that for any $l' \in L'$, the pullback $\tilde{c}^*(l')$ is an integral cycle, and hence is uniquely represented by a divisor supported on $\pi_Y^{-1}(o)$. The notation $\tilde{c}^*(l')$ means this divisor.

We see that for $l_1' \geq l_2'$, the eigenspaces of the factor $\mathcal{F}(l_1')/\mathcal{F}(l_2')$ have the form $H^0(L)/H^0(L(-l))$ for some natural line bundle $L$ and effective divisor $l \in L_{\geq 0}$.

2.3. End curves and end curve sections. We define the fundamental geometric tool of splice-quotient singularities: end curves and their counterpart in the language of line bundles: end curve sections.

**Definition 2.3.1** (End Curve Condition). Let a good resolution of an isolated normal surface singularity be given. Let us consider an irreducible curve on it which intersects exactly one of the exceptional curves, and which intersects the exceptional curve transversally and in exactly one point (these are called transversal cuts). The curve is an end curve of the exceptional curve it intersects if it is a divisor of a natural line bundle. An end curve section is a section of a natural line bundle having the end curve as a divisor. (The natural line bundle is obviously unique up to isomorphism.)

The End Curve Condition for the resolution is that every exceptional curve is intersected by exceptional curves and end curves in at least two different points.

For end curves of a vertex $v$, their end curve sections are sections of $O(-E_v^*)$. Using (2.2.2), via the inclusion $O(-E_v^*) \subseteq O(-D_{c,E}) \subseteq O_{Y,o}$, we shall regard the end curve sections as eigen-vectors of the function ring of the universal abelian cover $(Y,o)$.

Recall that every splice-quotient singularity satisfies the End Curve Condition by design ([4, Theorem 7.2(6)]). We recall the construction in the next section.

**Remark 2.3.2.** The End Curve Condition is usually formulated as every exceptional curve of degree 1 has at least one end curve intersecting it in a point different from that of its neighbour exceptional curve. Our formulation is equivalent to this one, and is better suited to the spirit of the present article. Note that graphs with only 1 vertex are rational, and hence they have the required end curves.

**Remark 2.3.3.** Instead of being a divisor of a natural line bundle, the original definition for an end curve required the equivalent condition stating the existence of a non-constant function whose divisor is supported on the exceptional divisor and the curve. Such a function is called an end curve function for the curve.

So far in the literature, end curves were defined only for ends, ie exceptional curves having degree one in the resolution graph. Since we shall use end curves of other exceptional curves also (in the proof of Proposition 3.3.1), we consider less confusing to extend the notion of end curve than to introduce a new name. End curves at other exceptional curves appear naturally anyway, for example when passing to the restriction to a subgraph, see Lemma 6.1 [2].

**Remark 2.3.4.** The End Curve Condition is really a property of the resolution. There are singularities which have a resolution satisfying the End Curve Condition.
and also a resolution not satisfying it. The two resolutions can even have the same graph.

However, blowdowns preserve the End Curve Condition so if a resolution of a singularity satisfies the End Curve Condition, then so does the minimal resolution.

3. Definition of splice-quotient singularities

Splice-quotient singularities are defined as the result of the following construction. The spirit of the construction is that we want to construct a singularity with a given resolution graph together with a given collection of end curves demonstrating the End Curve Condition using end curve sections as coordinates for the universal abelian cover.

The construction is recalled from [9, Definition 7.1] with some reformulations from [11, Section 2.2] and a slight generalization to encompass our extended notion of end curves.

For the convenience of the reader, we shall include the geometric meaning of every notion in parenthesis below. Of course, these parenthesised expressions are not part of the formal definitions.

The construction starts with the following data: a tree $\Gamma$ with at least 3 vertices (the tree of all exceptional curves and some end curves) together with integers at vertices of degree at least 2, such that the subtree of vertices of degree at least 2 is negative definite. (Vertices with degree 1 are end curves, the other vertices are exceptional curves.)

Let $L := \bigoplus_{u, \delta_u \geq 2} E_u$ be the group generated by the vertices of degree at least 2 together with the negative definite symmetric bilinear map $(-, -)$ induced by the numbers at these vertices. Let $L'$ be the dual of $L$, which is freely generated by the duals $E^*_v$ of the $E_v$, defined via $(E^*_v, E_u) = -1$ and $(E^*_v, E_u) = 0$ for $v \neq u$. (The groups $L'$ and $L$ are the second homology group and second cohomology group of the resolution, respectively. The elements $E_v$ and $E^*_v$ are the canonical generators, as introduced in Subsection 2.1.) Let $H := L'/L$ (the first homology group of the link).

For every vertex $w$ of degree 1, let $z_w$ be a variable (an end curve section of $w$). Furthermore, let $E^*_w := E^*_u$, where $u$ is the unique neighbour of $w$.

In Subsection 3.1, we make some definitions, which will be used in the construction in Subsection 3.2.

3.1. Degrees and Monomial Condition. We introduce a rudimentary form of notions related to divisors of functions, i.e., power series in the $z_w$.

Definition 3.1.1. For every vertex $v$, the $v$-degree of a monomial $\prod_i z_i^{\alpha_i}$ is

$$- \sum_i \alpha_i (E^*_v, E^*_i).$$

The $v$-degree (vanishing order on $E_v$) of a power series is the minimum of the $v$-degrees of the monomials having non-zero coefficient.

Remark 3.1.3. In general, the $v$-degree is a non-negative rational number. In the literature, there is an additional constant factor in the definition to make it an integer. We prefer our choice, because it will make the $v$-degree of a function to be its vanishing order on $E_v$.

Remark 3.1.4. In the literature, the term $v$-degree, $v$-order and $v$-weight are used in the same meaning. For clarity, we use only $v$-degree.

Definition 3.1.5. For any tree $\Gamma$ we define the following.
(1) The branches of a vertex $v$ are the components of $\Gamma \setminus v$. The variables of a branch are the variables $z_w$ of the vertices $w$ of the component having degree 1 in $\Gamma$.

(2) **Monomial Condition** [13] Condition 3.3, [11] Definition 2.4: The Monomial Condition for a branch of a vertex $v$ is the existence of a non-negative integer $\alpha_w$ for every vertex of the branch such that

\[
\sum_{w: \delta_w=1} \alpha_w E_w^* - \sum_{w: \delta_w \geq 2} \alpha_w E_w = E_v^*,
\]

where $\delta_w$ is the degree of $w$ in $\Gamma$. For such $\alpha_w$, the monomial $\prod_{w: \delta_w=1} z_w^{\alpha_w}$ is admissible for the branch. (In other words, $\prod_{w: \delta_w=1} z_w^{\alpha_w}$ is admissible if it is a holomorphic section of $\mathcal{O}(-E_v^*)$ with divisor supported on the branch via the inclusions $H^0(\mathcal{O}(-E_v^*)) \subseteq H^0(\mathcal{O}(-D_{E_v^*})) \subseteq \mathcal{O}_{Y,o}$. The divisor of $\prod_{w: \delta_w=1} z_w^{\alpha_w}$ in $\mathcal{O}(E_v^*)$ is $\sum_{w: \delta_w=1} \alpha_w w - \sum_{w: \delta_w \geq 2} \alpha_w E_w$ where in the first sum $w$ also stands for the end curve it represents.)

The Monomial Condition for a negative definite tree is that for every vertex having at least 3 branches, all of its branches satisfy the Monomial Condition.

The $v$-degree of all admissible monomials of the branches of $v$ is $-(E_v^*, E_v^*)$ as easily seen from (3.1.6).

### 3.2. Construction of splice-quotient singularities

We are ready to recall the construction of splice-quotient singularities. For the reader’s convenience, we recall the given data and some notation from the beginning of this section.

**Definition 3.2.1** (Splice diagram equations [8] Definition 7.1). Let $\Gamma$ be a tree with at least 3 vertices such that the subtree of vertices of degree at least 2 is negative definite. Let $\Gamma$ satisfy the Monomial Condition. We shall define equations in the ring of convergent power series in the variables $z_w$ using the notations introduced in this section.

First, we define the action of the group $H$ of $\Gamma$ on the ring via

\[
[E_v^*] \cdot z_w := e^{-2\pi i \langle [E_v^*], E_v^* \rangle} z_w,
\]

ie $z_w$ is a $\theta([E_v^*])$-eigenvector.

We make some arbitrary choices. We select an admissible monomial $M_{v,C}$ for every branch $C$ of every vertex $v$ with at least 3 branches. We select complex numbers $a_{v,i,C}$ for $1 \leq i \leq \delta_v - 2$ such that for every $v$, all the maximal minors of the matrix $(a_{v,i,C})$ have full rank (ie are non-degenerate). Finally, we choose convergent power series $H_{v,i}$ for $1 \leq i \leq \delta_v - 2$, which are $\theta([E_v^*])$-eigenvectors of the $H$-action and have $v$-degree greater than $-(E_v^*, E_v^*)$, ie greater than the $v$-degree of the $M_{v,C}$. The splice diagram equations are the equations for every vertex $v$ with at least 3 branches:

\[
\sum_C a_{v,i,C} M_{v,C} + H_{v,i} = 0, \quad i = 1, \ldots, \delta_v - 2,
\]

where $C$ runs over the branches of $v$.

The splice diagram equations define a singularity with an $H$-action. The factor of the singularity by the $H$-action is the result of the construction. Singularities arising this way are called splice-quotient singularities.

**Theorem 3.2.4** ([8] Theorem 7.2). Let $\Gamma$ be a tree with at least 3 vertices such that the subtree of vertices of degree at least 2 is negative definite. Let $\Gamma$ satisfy the Monomial Condition.

Then the splice diagram equations of $\Gamma$ define an isolated complete surface singularity with $H$ acting freely on it outside the origin. It is the universal abelian
cover of its factor by $H$, i.e., of the splice-quotient singularity resulting from the construction. The factor is an isolated surface singularity with a good resolution whose resolution graph is the subtree of $\Gamma$ consisting of vertices of degree at least 2. Every 1-degree vertex of $\Gamma$ is an end curve whose variable is an end curve section of it.

Remark 3.2.5. Actually, [9, Theorem 7.2] claims the above only for quasi-minimal resolution graphs (a technical modification of minimal resolution graphs) with one end curve at every vertex of degree 1. However, the proof can be easily extended to this general case.

The weak equisingularity type of the constructed splice-quotient singularity depends only on the graph. Independence on the choice of the $H_{v,i}$ and $a_{v,i,C}$ are easy, see eg [13, Theorem 1.1]. Independence on the choice of admissible monomials is stated in [9, Theorem 10.1].

4. Main results

We formulate our main results on splice-quotient singularities.

**Theorem 4.1.** For splice-quotient singularities, every level of the divisorial filtration is generated by monomials of the defining coordinate functions (the $z_w$ in Definition 3.2.1).

This has many applications, for example the End Curve Theorem, which states that every singularity satisfying the End Curve Condition is splice-quotient.

**Theorem 4.2** (End Curve Theorem [10, page 2]). If a good resolution of an isolated normal surface singularity with rational homology sphere satisfies the End Curve Condition, then it is splice-quotient. Moreover, the resolution arises as a result of the splice-quotient construction in Definition 3.2.1 with the coordinate functions being arbitrarily chosen end curve sections for an arbitrarily fixed set of end curves demonstrating the End Curve Condition, i.e. every exceptional curve is intersected by at least two exceptional curves and end curves altogether with no three curves having a common point.

These results will be proved in Section 8.

Némethi has applied our Lemma 7.1 (2) for refining Theorem 4.1 to explicit formulas for the cohomology of natural line bundles. Without proof, here we mention only a power series formula, a direct generalization of [2, Theorem 1] and its successor [4, Theorem 5.1.5] from rational singularities and minimal elliptic singularities to splice-quotient singularities.

**Corollary 4.3.** [7, Theorem 4.1.1. in its (4.4.1) form] For every resolution of a splice-quotient singularity satisfying the End Curve Condition, the following hold.

\[
\sum_{k_v \geq 0} \sum_{I \subseteq \Gamma} (-1)^{|I|+1} \dim \frac{H^0(\mathcal{O}(-\sum_v k_v E_v^*))}{H^0(\mathcal{O}(-\sum_v k_v E_v^* - \sum_{w \in I} E_w))} \prod_{v \in I} x_v^{k_v} = \prod_{v \in \Gamma} (1 - x_v)^{\delta_v - 2}.
\]

Here $|I|$ denotes the number of elements of the set $I$.

5. Numerically effective cycles

In this section, we recall some combinatorial results regarding the homological cycles of isolated surface singularities with the link being a rational homology sphere.

The results are mostly about numerically effectiveness, which we recall.
A cycle \( l' \in L \otimes \mathbb{Q} \) is numerically effective if for every exceptional curve \( E_v \), we have \( (E_v, l') \geq 0 \). A line bundle is numerically effective if its first Chern class is so.

The basic property of numerically effective cycles is their relationship with global sections with line bundles.

**Lemma 5.2.** \[ (4.2)(a) \& (c) \] For every line bundle \( \mathcal{L} \), there is a least effective cycle \( x \in L \) such that \( \mathcal{L}(-x) \) is numerically effective. Moreover, for every cycle \( 0 \leq y \leq x \) the inclusion \( H^0(\mathcal{L}(-y)) \hookrightarrow H^0(\mathcal{L}) \) is an isomorphism.

For every vertex \( v \) with \( (c_1(\mathcal{L}), E_v) < 0 \), we have \( x \geq E_v \).

**Remark 5.3.** Actually, the reference \[ (5) \ (4.2)(c) \] is about first cohomology. Our reformulation for the zeroth cohomology can be obtained as follows. Via the cohomology exact sequence of the short exact sequence \( L(-x) \to \mathcal{L} \to L_x \), the reference is equivalent to that the inclusion \( H^0(\mathcal{L}(-x)) \hookrightarrow H^0(\mathcal{L}) \) is an isomorphism. Applying this to \( \mathcal{L}(-y) \) instead of \( \mathcal{L} \), we obtain that the inclusion \( H^0(\mathcal{L}(-x)) \hookrightarrow H^0(\mathcal{L}(-y)) \) is also an isomorphism. Hence the claimed inclusion \( H^0(\mathcal{L}(-y)) \hookrightarrow H^0(\mathcal{L}) \) is an isomorphism, as well.

Another well-known result is about the sparse distribution of numerically effective cycles.

**Lemma 5.4.** For every \( l' \in L \otimes \mathbb{Q} \), all but finitely many numerically effective \( l'_0 \in L' \) satisfies \( l' \geq l'_0 \). For example, all numerical effective cycles are less than or equal to 0.

The final result allows some simplification in the definition of splice-quotient singularities.

**Lemma 5.5.** \[ (\text{Cf } [9] \text{ Lemma 3.2}) \] For every numerically effective cycle \( l' \in L \otimes \mathbb{Q} \) and every vertex \( v \) of the resolution graph:

\[
l' \leq \frac{(l', E_v^*)}{(E_v^*, E_v^*)} E_v^*.
\]

**Proof.** Let us express the difference \( l' - (l', E_v^*)/(E_v^*, E_v^*) \) \( E_v^* \) in the basis of the \( E_w \). The coefficient of \( E_v \) is the scalar product with \(-E_v^*\), which is 0. Therefore the difference can be considered as a rational cycle over the subgraph obtained by removing the vertex \( v \). Over this subgraph, the scalar product with the \( E_w \) for \( w \neq v \) remains the same as over the whole graph, which is the same as the scalar product of \( l' \) with the \( E_w \). Hence the difference \( l' - (l', E_v^*)/(E_v^*, E_v^*) \) \( E_v^* \) is numerically effective over the subgraph. Thus, by Lemma 5.4, the difference is less than or equal to 0.

### 6. Restrictions

In this subsection, we show that if a singularity satisfies the **End Curve Condition** then so do the singularities determined by its subgraphs. Moreover, the restrictions of natural line bundles to these singularities are also natural. These are mostly known results, but we present the proof in the language of line bundles.

**Lemma 6.1.** \[ (\text{Cf } [11] \text{ Proposition 2.16}) \] For a singularity satisfying the **End Curve Condition**, the following hold.

1. Every divisor supported on exceptional curves and end curves is a divisor of a natural line bundle.
2. Every subtree satisfies the **End Curve Condition**. In particular, the subtree has the following end curves: the exceptional curves and end curves of the original graph intersecting the subtree.
(3) Restrictions of natural line bundles to subtrees are natural. Moreover, for every divisor on the subtree representing the first Chern class of the restriction of a natural bundle and which is supported on the subtree and the exceptional curves and end curves of the big graph intersecting the subtree, the following extensibility property holds. The divisor extends to a divisor of the original bundle supported on the exceptional curves and end curves of the original graph. In particular, the divisor is really a divisor of the restriction of the line bundle.

Proof. Since a natural line bundle is uniquely determined by its Chern class, statement (1) follows.

We show the extensibility property of divisors from (3). This is merely a combinatorial claim: we have to extend a divisor to represent a given cohomology class. This can be done step by step: each step adds a new vertex to the subtree and extends the divisor to the larger subtree. At every step all we have to do is find coefficients for the end curves and exceptional curves intersecting the added vertex which do not have a coefficient yet, ie all of them but one.

The requirement to represent the given cohomology class is that the intersection numbers of the extended divisor with the exceptional curves of the extended subtree is the same as that of the cohomology class. This is automatic for all vertices except the newly added one. For the newly added vertex, the requirement is that the sum of the new coefficients be a prescribed value, which can be fulfilled as there is at least one undetermined coefficient by the hypothesis that the added vertex (as every exceptional curve) is intersected by at least two end curves and exceptional curves of the original graph altogether. Hence the extension of the divisor is possible.

It also follows that the original divisor is a divisor of the restriction. This proves the extensibility property from (3).

Restriction of natural line bundles form a subgroup of the Picard group of the subtree, which we shall call the restriction group. We shall show that this is the group of natural line bundles.

First, the restriction group consists of the line bundles associated to divisors supported on the subtree and the end curve candidates of (2) ie the exceptional curves and end curves of the original graph intersecting the subtree. In particular, the restriction group contains all line bundles associated to divisors supported on the exceptional curves.

Obviously, the End Curve Condition implies that every exceptional curve of the subtree has at least two end curve candidates, and thus the end curve candidates together with the exceptional curves generate the second cohomology group. Here we use that the second cohomology group \( L' \) is generated by the \( E^*_v \) where \( v \) runs through all but one vertices of degree 1, see eg [9, Proosition 5.1]. Hence the Chern class restricted to the restriction group is onto.

Second, the kernel of the Chern class on the restriction group consists of restrictions having Chern class 0. By the part of (3) already proven, these restrictions have divisor 0, ie all of them are trivial. Hence the Chern class is injective on the restriction group finishing the proof that the restriction group is the group of natural line bundles.

As a consequence, the end curve candidates of (2) are really end curves as they are divisors of natural line bundles.
Generators for global sections

Now we are ready to formulate our main lemma. It has two parts, which are of different nature. Nevertheless, as we have already mentioned in the introduction, we present them together since the proof is a simultaneous induction of both statements.

Lemma 7.1 (Main Lemma). Let \( \pi : \tilde{X} \to X \) be a good resolution of a singularity satisfying the End Curve Condition. Let \( \Gamma \) denote the resolution graph. Let us have a fixed collection of end curves demonstrating the End Curve Condition, ie every exceptional curve is intersected by at least two other exceptional curves and end curves altogether, and no three curves have a common point. Let \( \mathcal{L} \) be a natural line bundle on \( \tilde{X} \).

Then the following hold.

1. Let \( l \in \mathcal{L} \geq 0 \) be an effective cycle on the exceptional curves. Then every subset of \( H^0(\mathcal{L})/H^0(\mathcal{L}(-l)) \) satisfying the following condition generates it as a vector space:
   - For every effective divisor of \( \mathcal{L} \) supported on the exceptional curves and the fixed end curves which is not greater than or equal to \( l \), the set contains the class of a section of \( \mathcal{L} \) with this divisor.

2. Let \( v \) be a vertex of degree 1 of the graph \( \Gamma \) such that \( (c_1(\mathcal{L}), E_v) \geq 0 \). Let \( w \) be the unique neighbour of \( v \). Then the restriction map

\[
\frac{H^0(\mathcal{L})}{H^0(\mathcal{L}(-E_w))} \to \frac{H^0(\mathcal{L}|_{\Gamma \setminus \{v\}})}{H^0(\mathcal{L}|_{\Gamma \setminus \{w\}}(-E_w))},
\]

is an isomorphism. Here \( \mathcal{L}|_{\Delta} \) denotes the restriction of \( \mathcal{L} \) to the neighbourhood of the subgraph \( \Delta \).

**Proof.** We prove both statements by a simultaneous induction on the number of vertices of the resolution graph.

First we prove (2) together with (1) for the factors \( H^0(\mathcal{L})/H^0(\mathcal{L}(-E_w)) \) appearing in (2). We shall use the notations of (2). For brevity, let us call subsets satisfying the condition of (1) as generating subset candidates.

Obviously, the restriction map is injective, so we need to prove surjectivity and that the generating subset candidates are really generating subsets. These two statements together are equivalent to a single statement: the image of generating subset candidates are generating subsets.

As the reader may expect, we use the induction hypothesis to show that the images are generating subsets, so we shall prove that the image of generating subset candidates are again generating subset candidates.

Recall that being a generating subset candidate means that certain divisors appear as divisors of some sections in the subset. So we will prove the best we can hope for divisors, which is sufficient for the claim of the previous paragraph by Lemma 6.1: every effective divisor of the restriction of \( \mathcal{L} \) supported on the exceptional curves and the fixed set of end curves with the vanishing order on \( E_w \) being 0 extends to an effective divisor of \( \mathcal{L} \) supported on the exceptional curves and the fixed set of end curves also. (This is not a special case of Lemma 6.1: as it does not guarantee an effective extension to divisors. But we shall extend its argument to the present case.)

Indeed, let \( k \geq 0 \) be the vanishing order on \( E_w \) of such a divisor of the restricted bundle. To extend it, we only need to choose non-negative orders for the end curves of \( E_v \) such that their sum is \( (E_v, c_1(\mathcal{L})) - kE_v^2 \), which is non-negative since \( (E_v, c_1(\mathcal{L})) \geq 0 \) and \( E_v^2 < 0 \). Hence the extension is possible.
Now we turn to statement (1). We start with a special case of a one-vertex graph. Let \( v \) be the single vertex. At the moment, we prove (1) only for factors \( H^0(L)/H^0(L(E_v)) \) with \( c_1(L) \) numerically effective ie \( L = -kE_v^* \) for some \( k \geq 0 \).

We have an embedding

\[
\frac{H^0(O(-kE_v^*))}{H^0(O(-kE_v^* - E_v^*))} \rightarrow H^0(O_{E_v}(-kE_v^*)).
\]

The vector space on the right-hand side has dimension \( k + 1 \).

Let us have a generating subset candidate of the factor, ie a subset satisfying the condition of (1). Let \( H_1 \) and \( H_2 \) be two of the fixed end curves. Let us choose sections in the given set with divisors \( iH_1 + (k - i)H_2 \) representing \( kE_v^* \) for \( i = 0, \ldots, k \), which is possible by hypothesis. Clearly, the images of these \( k + 1 \) sections are linearly independent in the right-hand side, since they have different vanishing order on the intersection of \( H_1 \) and \( E_v^* \). So they form a basis of the right-hand side, hence they also form a basis of the left-hand side and the inclusion is an isomorphism.

Now we turn to statement (1) in general. We construct recursively an increasing sequence \( x_n \in L_{\geq 0} \), such that \( x_0 = 0 \) and the following are satisfied.

(i) The line bundle \( L(-x_{2n+1}) \) is numerically effective.

(ii) The statement of (1) holds for the factor \( H^0(L(-x_n))/H^0(L(-x_{n+1})) \).

(iii) The sequence is strictly increasing at even steps: \( x_{2n+1} < x_{2n} \).

Given \( x_{2n} \), we can find an \( O_{2n+1} \) by Lemma 5.2 which even satisfies a condition stronger than (iii) \( H^0(L(-x_{2n}))/H^0(L(-x_{2n+1})) = 0 \).

Given \( x_{2n-1} \), we can always choose a suitable \( x_{2n} \) by the special cases of (1) already proved. If the graph has only one vertex, then we must have \( x_{2n-1} = -kE_v^* \) for some \( k \geq 0 \) and we can choose \( x_{2n} := x_{2n-1} + E_v \) where \( v \) is the single vertex of the graph. If the graph has at least two vertices, then we can choose \( x_{2n} := x_{2n-1} + E_w \) where \( w \) is a neighbour of a vertex of degree 1.

The conditions imply that the \( c_1(L(-x_{2n+1})) \) are different numerically effective cycles, hence by Lemma 5.4 \( x_k \geq l \) for some odd \( k \).

Next we show by induction on \( n \) that \( H^0(L)/H^0(L(-x_n)) \) satisfies (1). This is obvious for \( n = 0 \).

The inductive step follows from the short exact sequence

\[
0 \rightarrow \frac{H^0(L(-x_n))}{H^0(L(-x_{n+1}))} \rightarrow \frac{H^0(L)}{H^0(L(-x_{n+1}))} \rightarrow \frac{H^0(L)}{H^0(L(-x_n))} \rightarrow 0.
\]

The right-hand side satisfies (1) by the inductive hypothesis. The left-hand side satisfies (1) too, by (ii). Since the left-hand side and the right-hand side of the exact sequence satisfy (1) so does the middle term.

Finally, \( H^0(L)/H^0(L(-l)) \) is a factor of \( H^0(L)/H^0(L(-x_k)) \), and hence it also satisfies (1).

8. Geometric characterization of splice-quotient singularities

In this section, we prove the End Curve Theorem 4.2 and Theorem 4.1 based on Lemma 7.1. We will prove Theorem 4.1 for every singularity satisfying the End Curve Condition because we want to use it in the proof of the End Curve Theorem 4.2.

In Subsection 8.1, we determine the place of monomials of end curve sections in the divisorial filtration, as a preparation for proving Theorem 4.1 in Subsection 8.2, i.e that levels of the divisorial filtration are generated by monomials. In particular,
this will show that the function ring of the universal abelian cover consists of convergent power series in end curve sections, which is a major step for the End Curve Theorem 4.2.

Our next step is to derive the Monomial Condition, which is required by the splice-quotient construction, from the End Curve Condition in Subsection 8.3. Finally, we finish the proof in Subsection 8.3 by finding splice-diagram equations.

8.1. Divisors of monomials. We describe the levels of the divisorial filtration of the function ring of the universal cover, which contain any given product of end curve sections.

Lemma 8.1.1. Let the \( z_i \) be end curve sections in the function ring of the universal abelian cover. Let \( E_i^* \) be the inverse of the second cohomology class represented by the end curve of \( z_i \). Then for every non-negative second cohomology class \( l' \) of a resolution of the singularity and every non-negative integers \( \alpha_i \):

\[
\prod_i z_i^{\alpha_i} \in \mathcal{F}(l') \iff \sum_i \alpha_i E_i^* \geq l'.
\]

Proof. Via the identifications of sections of natural line bundles with eigen-functions on the universal abelian cover, every end curve section \( z_i \) is a section of \( \mathcal{O}(-E_i^*) \) with divisor its end curve. Therefore the product \( \prod_i z_i^{\alpha_i} \) is a section of \( \mathcal{O}(-\sum_i \alpha_i E_i^*) \) with divisor supported on end curves. The product is also an eigen-function. By Theorem 4.2.3.1, the corresponding eigen-space of \( \mathcal{F}(l') \) is \( H^0(\mathcal{O}(-l' - l)) \) for the \( l \in \mathbb{Q} \) making the line bundles \( \mathcal{O}(-\sum_i \alpha_i E_i^*) \) and \( \mathcal{O}(-l' - l) \) differ by a divisor supported on the exceptional curves. In other words, \( l \) is the rational part of \( \sum_i \alpha_i E_i^* - l' \) by the definition of \( \mathcal{F}(l') \) of \( \mathbb{Q} \).

Since the divisor of \( \prod_i z_i^{\alpha_i} \) in \( \mathcal{O}(-\sum_i \alpha_i E_i^*) \) does not contain exceptional curves, the product is a section of \( H^0(\mathcal{O}(-l' - l)) \) if and only if \( -\sum_i \alpha_i E_i^* \geq -l' - l \) \iff \( \sum_i \alpha_i E_i^* - l' \geq l \). Since \( l \) is the rational part of the left-hand side, the last inequality is equivalent to \( \sum_i \alpha_i E_i^* - l' \geq 0 \). \( \square \)

8.2. The divisorial filtration is generated by monomials. We prove Theorem 4.2.4 that levels of the divisorial filtration are generated by monomials of end curve sections for singularities satisfying the End Curve Condition. First we reformulate it to be suitable for use in the proof of the End Curve Theorem 4.2.

Proposition 8.2.1. For every resolution of a singularity satisfying the End Curve Condition and every collection of end curve sections belonging to end curves such that every exceptional curve is intersected by at least two end curves and exceptional curves altogether, the following holds. Every element of every level of the divisorial filtration is a convergent power series in the monomials of the end curve sections lying in the same level of the divisorial filtration.

This is a simple consequence of Lemma 7.4.1 via standard arguments on complete rings, as the proof below shows.

Proof of Proposition 8.2.1. Let \( m \) denote the maximal ideal of the function ring of the universal abelian cover. The topology of the ring is given by the \( m \)-adic filtration, ie the powers of \( m \) as a neighbourhood basis of \( 0 \). As a preliminary, we show that the divisorial filtration gives the same topology and thus the same completion, ie every ideal of either filtration contains an ideal from the other filtration. Clearly, every ideal of the divisorial filtration contains a large power of the maximal ideal since all functions in the latter ideal have large vanishing orders on all the exceptional curves.
We now prove the other direction. By \(3\), (3.2), there exists a non-zero effective cycle \(a \in L_{\geq 0}\) such that for \(k \geq 2\)
\[
H^0\left(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-\tilde{e}^*(a))\right)^{\otimes k} \longrightarrow H^0\left(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(-k \cdot \tilde{e}^*(a))\right)
\]
is onto. Hence \(F(ka) \subseteq F(a)^k\) lies in the \(k\)th power of the maximal ideal.

We now turn the proof of the proposition. Recall that the levels of the divisorial filtration are isomorphic to natural line bundles. Hence applying \(\text{Lemma 7.1(1)}\) to the eigenspaces of the levels show that that the eigenspaces of factors of two comparable levels are generated by monomials. In particular, the factor of any two comparable levels is also generated by monomials. Here we call two levels comparable if one is contained in the other, as usual for partial orders.

We prove that the ring homomorphism is surjective. By standard completion arguments, it is enough to prove that the maximal ideal is mapped surjectively onto \(m/m^2\). By extension of the constant functions, this is equivalent to the homomorphism being surjective onto \(F(0)/m^2\), where the level \(F(0)\) is the whole function ring. As we have proved, there is a level \(F(l')\) of the divisorial filtration contained in \(m^2\), hence it is enough to prove that the homomorphism is surjective onto \(F(0)/F(l')\). This is indeed the case, as this factor is a factor of two comparable levels of the divisorial filtration.

Finally, we show that every level \(F(l')\) of the divisorial filtration as an ideal is generated by monomials, which obviously shows that every element of the level is a convergent power series in the monomials lying in the level. The level is a finitely generated ideal, hence by Nakayama’s lemma it is enough to show that \(F(l')/m F(l')\) is generated by monomials. We prove this by repeating the argument in the previous paragraph: there is a level \(F(a)\) of the divisorial filtration contained in \(m F(l')\), and the factor of two levels \(F(l')/F(a)\) is generated by monomials, hence so is \(F(l')/m F(l')\).

### 8.3. End Curve Condition implies Monomial Condition.

In all known proofs that a singularity is splice-quotient, it is an important step to show that the resolution graph satisfies the Monomial Condition. In this subsection, we do this step for the End Curve Theorem 4.2.

**Proposition 8.3.1.** If a good resolution of an isolated normal surface singularity with rational homology sphere link satisfies the End Curve Condition then it also satisfies the Monomial Condition for every finite set of end curves intersecting the exceptional divisor in pairwise disjoint points and every exceptional curve being intersected by at least two exceptional curves and end curves altogether.

To better understand the proof, first we present the idea behind it. First, the Monomial Condition is local, ie it holds for a branch of a vertex if and only if it holds in the subgraph spanned by the branch and the vertex. Restricting to this subgraph has the advantage that the other branches of the vertex become just end curves, hence we have admissible monomials for them. The splice diagram equations suggest that an admissible monomial for the branch is a linear combination of admissible monomials of two other branches up to higher degree terms. So we will take a suitable linear combination of admissible monomials (ie end curve sections) of two other branches and cut out the higher degree terms to obtain an admissible monomial.

This is done precisely as follows.

**Proof of Proposition 8.3.1.** We verify the Monomial Condition for every branch of every vertex \(v\) having at least 3 branches.
The Monomial Condition for a branch in the whole graph is equivalent to the Monomial Condition for the branch in the subgraph spanned by v and the branch, since the required equation \(3.1.6\) means the same condition for both graphs. (The easiest way to see this is that an equivalent form of \(3.1.6\) is that both of its side has the same intersection number with the exceptional curves of the graph. The intersection numbers with exceptional curves outside the subgraph is always 0 for both sides, hence only the subgraph matters.)

Hence we shall work with this subgraph, and our notations will be implicitly used for this subgraph.

The advantage of using this subgraph is that v inherits end curves cut out by the other branches of v, see Lemma 6.1. Since v has at least 3 branches, it has at least 2 end curves intersecting it in different points in the subgraph.

We select two end curve sections \(s_1\) and \(s_2\) of \(O(-E_v^*)\) whose end curves intersect \(E_v\) in different points. Obviously, these sections lie outside of \(H^0(O(-E_v^* - E_w))\) where \(w\) is the vertex of the branch adjacent to v.

In particular, their images under the embedding

\[
\frac{H^0(O(-E_v^*))}{H^0(O(-E_v^* - E_w))} \hookrightarrow H^0(O_{E_{w}}(-E_v^*)) \simeq \mathbb{C}
\]

are non-zero. Since the codomain is 1 dimensional, there exists a non-zero complex number \(\alpha\) such that

\[
(8.3.2) \quad s_1 - \alpha s_2 \in H^0(O(-E_v^* - E_w)).
\]

(This is supposed to be an admissible monomial of the branch plus higher degree terms.) Since the end curves (ie the divisors) of \(s_1\) and \(s_2\) intersect \(E_v\) in different points, the linear combination \(s_1 - \alpha s_2\) is not zero on \(E_v\). Thus, \(s_1 - \alpha s_2\) is a non-zero element of the factor \(H^0(O(-E_v^* - E_w)) / H^0(O(-E_v^* - E_w - E_v))\). Hence by Lemma 7.1, the line bundle \(H^0(O(-E_v^* - E_w))\) has a non-zero section whose divisor is supported by the end curves and exceptional curves other than \(E_v\). Since \((E_v, c_1(O(-E_v^* - E_w))) = 0\), the divisor cannot contain any curves intersecting \(E_v\), so the divisor is supported by the branch and its end curves (other than \(E_v\)). In the larger bundle \(O(-E_v^*)\), the section is still supported on the branch and its end curves, hence the coefficients of the divisor are non-negative numbers \(\alpha_s\) for the end curves and exceptional curves of branch, respectively, satisfying \(3.1.6\).

8.4. Proof of the End Curve Theorem. In this subsection, we prove the End Curve Theorem. Our arguments are taken from [33] Section 5], and adapted to the use of [Theorem 4.1] instead of the properties of special singularities.

Proof of Theorem 4.2 We are given a ring homomorphism from the ring of convergent power series to the ring of functions of the universal abelian cover of a singularity satisfying the End Curve Condition. The variables are mapped to end curve sections. What we have to show is that this homomorphism is surjective with kernel generated by splice diagram equations.

The surjectivity is part of Proposition 8.2.1 which is already proven. It follows at once that the kernel is a prime ideal of dimension 2. Since any ideal generated by splice diagram equations is also prime of dimension 2 by Theorem 3.2.3, it is enough to show that the kernel contains a system of splice diagram equations.

We fix a vertex \(v\) with at least 3 branches. We are going to find splice diagram equations \((3.2.3)\) for this vertex as follows.

First of all, we choose admissible monomials \(M_{v,C}\) for the branches arbitrarily. The existence of admissible monomials was shown in Proposition 8.3.1.

By Lemma 5.5 via \(8.1.2\), the monomials having \(v\)-degree at least \(- (E_v^*, E_v^*)\) are exactly the monomials in \(\mathcal{F}(E_v^*)\). The admissible monomials lie in the subspace
of the $\theta([E^*_v])$-eigenspace of $\mathcal{F}(E^*_v)$, which is isomorphic to $H^0(\mathcal{O}(-E^*_v))$. Under such an isomorphism, the vector space of power series with higher $v$-degree and in the same eigenspace of the group action is the subspace $H^0(\mathcal{O}(-E^*_v) - E_v)$. So we have to show that the admissible monomials of $v$ satisfy $\delta_v - 2$ linear equations in the factor $H^0(\mathcal{O}(-E^*_v))/H^0(\mathcal{O}(-E^*_v) - E_v))$ whose matrix of coefficients has all its maximal minors non-degenerate. The latter condition means that the subspace generated by all the admissible monomials in the factor is generated by any two of the admissible monomials.

To see that this condition really holds, we consider the embedding

$$H^0(\mathcal{O}(-E^*_v))/H^0(\mathcal{O}(-E^*_v) - E_v)) \hookrightarrow H^0(\mathcal{O}_{E_v}(-E^*_v)) \cong \mathbb{C}^2.$$  

The space on the right-hand side has dimension 2. Any non-zero section of it has exactly one zero point on $E_v$. Any two non-zero sections having different zero points are linearly independent, and therefore form a basis of it. Hence any two of the admissible monomials form a basis of it, thus any two of the admissible monomials generate the subspace generated by all of the admissible monomials, as claimed.

9. Counterexample for Hilbert–Samuel function

In this section we give an example that the Hilbert–Samuel function of a splice-quotient singularity is not determined by the resolution graph. Let $m$ denote the maximal ideal of an isolated singularity. Recall that the Hilbert–Samuel function of the singularity is the power series

$$\sum_{k=0}^{\infty} \dim \frac{m^k}{m^{k+1}} \cdot t^k \in \mathbb{Z}[[t]].$$

9.1. The singularities. In this subsection, we present the example and state the Hilbert–Samuel functions. The computation of the functions are left to Subsection 9.2.

The example is a star-shaped graph: i.e. there is a central vertex, and the branches of the vertex are simply paths. In our example, there are four branches whose determinant we denote by $a_1, a_2, b$ and $c$. We require these four numbers to be pairwisely relative prime. For every such quadruple of numbers, there is a unique graph whose determinant is 1. Our example will be this unique graph, so the universal abelian cover will be the resolution itself (this is just for simplifying computations).

Now we write up a possible system of splice-quotient equations, which define the singularity:

$$x_1^{a_1} + y^b + z^c = 0,$$
$$x_2^{a_2} + y^b + \gamma z^c = 0,$$

where $\gamma$ is a complex number different from 0 and 1. Its Hilbert–Samuel function is

$$\frac{(1 - t^{a_1})(1 - t^{a_2})}{(1 - t)^4}.$$  

By adding higher degree terms to the equations, we obtain another singularity with the same resolution graph:

$$x_1^{a_1} + y^b + z^c = 0,$$
$$x_2^{a_2} + y^b + \gamma z^c + x_1^{a_1-1}x_2 = 0.$$
Here we impose the following restrictions on the numbers:

(9.1.4a) \( 2 \leq a_1 < a_2 < b < c, \)
(9.1.4b) \( ia_1 > a_2, \)
(9.1.4c) \( a_1 - 1 + i < a_2, \)
(9.1.4d) \( b + a_1 - 1 > 2a_2 - i. \)

The inequality of (9.1.4b) expresses that the added term is of higher degree. The other inequalities are of technical nature. In particular, the sole purpose of (9.1.4d) is to simplify computations.

The Hilbert–Samuel function of the latter singularity is

\[
\frac{t^{a_1-1} - t^{a_2} + t^{2a_2-i} - t^{a_1-1}t^i}{(1-t)^2}
\]

(9.1.5)

9.2. Computation of Hilbert–Samuel function. In this subsection, we verify the Hilbert–Samuel functions (9.1.2) and (9.1.3).

Recall that the codegree of a power series is the minimum of the degrees of its terms. Obviously, a function is in the \( k \)th power of the maximal ideal if and only if it is represented by a power series of codegree at least \( k \).

The idea is to find a good basis for the function rings: ie a collection of monomials such that every function can be uniquely written as a convergent power series having only these monomials as its terms. Furthermore, these unique representation should have the minimal possible codegree so that it reflects in which power of maximal ideal the function is contained.

Given such a collection of monomials, the monomials of degree \( k \) obviously form a basis of the factor \( m^k/m^{k+1} \), leading to a combinatorial formula for the Hilbert–Samuel function.

We claim that for the equations (9.1.1), a good collection of monomials are the ones not divisible by \( x_1^{a_1} \) or \( x_2^{a_2} \). For the equations (9.1.3), a good collection is formed by the monomials divisible by none of \( x_1x_2^2, x_1^{a_1}, x_1^{a_1-1}x_1^2 \) and \( x_2^{a_2-i} \).

The claimed form of the Hilbert–Samuel function follows easily from these claims.

We verify our claim only for the equations (9.1.3), since the verification for (9.1.1) is similar and easier. We call the monomials outside of the good collection to be forbidden.

The verification is based on expressing the forbidden monomials in terms of the allowed ones:

(9.2.1a) \( x_1x_2^{a_2} = (x_2^i - x_1)y^b + (x_1^2 - \gamma x_1)z^c - x_1^3(x_1^a + y^b + z^c) + x_1(x_2^{a_2} + y^b + \gamma z^c + x_1^{a_1-1}x_2^i) \)
(9.2.1b) \( x_1^{a_1} = -y^b - z^c + (x_1^{a_1} + y^b + z^c) \)
(9.2.1c) \( x_1^{a_1-1}x_2^i = -x_2^{a_2} - y^b - \gamma z^c + (x_2^{a_2} + y^b + \gamma z^c + x_1^{a_1-1}x_2^i) \)
(9.2.1d) \( x_2^{2a_2-i} = (x_1^{a_1-1} - x_2^{a_2-i} - x_1^{a_1-2}x_2^i)y^b + (\gamma x_1^{a_1-1} - \gamma x_2^{a_2-i} - x_1^{a_1-2}x_2^i)z^c + x_1^{a_1-2}x_2^i(x_1^a + y^b + z^c) + (x_2^{a_2-i} - x_1^{a_1-1})(x_2^{a_2} + y^b + \gamma z^c + x_1^{a_1-1}x_2^i) \)
In these equations, the first line contains the equivalent expression without the forbidden monomials, and of non-smaller codegree thanks to (9.1.4c) and (9.1.4d). Further lines contain a linear combination of the defining equations to make the equation hold, which is just for the reader’s convenience.

We now verify that every function is represented by a unique power series not containing a term divisible by the mentioned monomials. By the Weierstrass Preparation Theorem, the ring $\mathbb{C}\{x_1, x_2, y, z\}/(x_1^{a_1} + y^b + z^c)$ is a free module over $\mathbb{C}\{y, z\}$ with basis $1, x_1, \ldots, x_1^{a_1-1}$. Similarly, the ring $\mathbb{C}\{x_1, y, z\}/(x_1^{a_1} + y^b + z^c)$ is a free module over $\mathbb{C}\{y, z\}$ with basis $1, x_1, \ldots, x_1^{a_1-1}$. It follows that $\mathbb{C}\{x_1, x_2, y, z\}/(x_1^{a_1} + y^b + z^c, x_2^{b_2} + y^b + z^c + x_2^{a_2-1}x_2^1)$ is a free module over $\mathbb{C}\{y, z\}$ with basis $x_1^{a_1-1}x_2^j$ for $0 \leq j_1 < a_1 - 1$ and $0 \leq j_2 < a_2 - 1$. In this basis, the monomials $x_1^{a_1-1}x_2^j$ for $i < j < a_2 - 1$ can be replaced by $x_2^{2a_2+j-i-1}$ by (9.2.1a). This gives us the claimed basis of the function ring.

We still need to check that the unique representative is of minimal codegree. We do this by transforming any representative $f_0$ into the unique one omitting the forbidden monomials such that the codegree does not decrease. The main idea behind this is to replace forbidden monomials with another expressions of non-smaller codegree. Equations (9.2.1) provide these replacements.

In more details, we first eliminate the monomials divisible by $x_2^{2a_2-1}$ via (9.2.1d) and the Weierstrass Preparation Theorem, which does not decrease the codegree. Then we eliminate the terms divisible by $x_1x_2^{a_2}$ via (9.2.1a). The third step is to eliminate the terms divisible by $x_1^{a_1}$ via (9.2.1b) and the Weierstrass Preparation Theorem. The final step is the elimination of monomials divisible by $x_1^{a_1-1}x_2^j$ via (9.2.1c). The reader can easily verify that during these steps forbidden monomials claimed to be eliminated in previous steps cannot reappear, so finally we have obtained a representative without forbidden monomials and with codegree at least the original.

References


Rényi Institute of Mathematics, Budapest, Reáltanoda u. 13–15, 1053, Hungary
E-mail address: braung@renyi.hu