The Isoperimetric inequality, the Brunn-Minkowski theory and Minkowski type Monge-Ampère equations on the sphere

Károly J. Böröczky, João P.G. Ramos, Alessio Figalli

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For Abel

Preface

The main topic of the book is how Geometric Isoperimetric-type inequalities intervene with functional inequalities such as the Prekopa-Leindler inequality, the Sobolev inequality, Poincaré inequality, on the one hand, and also with the uniqueness of the solution of certain Minkowski-type Monge-Ampère equations on the sphere, on the other hand. Various related ideas are discussed. Among others, we discuss proofs of the Brunn-Minkowski inequality using combinatorial ideas (à la Hadwiger-Ohmann), optimal transport, and spectral theory (à la Hilbert and Aleksandrov).

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0.1 Basic notation

$C_c(X)$ denotes the space of continuous real functions of compact support (zero outside of a compact set) for a topological space X. In addition, $C_c^k(\mathbb{R}^n)$, $k = 1, 2, \ldots$, denotes the space of real C^k functions on $\mathbb{R}^n$ with compact support, allowing $k = \infty$.

Minkowski sum:
$X + Y = \{x + y : x \in X \ \&\ y \in Y\}$ for $X, Y \subset \mathbb{R}^n$.
$\alpha X + \beta Y = \{\alpha x + \beta y : x \in X \ \&\ y \in Y\}$ for $\alpha, \beta \in \mathbb{R}$.

Origin in $\mathbb{R}^n$:
$o = o_{\mathbb{R}^n}$.

Scalar product and ℓ_2 norm in $\mathbb{R}^n$:
$\langle \cdot, \cdot \rangle$ and $\|x\| = \sqrt{\langle x, x \rangle}$.

Orthogonal complement in $\mathbb{R}^n$:
$L^\perp = \{x \in \mathbb{R}^n : \langle x, y \rangle = 0\ \forall y \in L\}$ for a linear subspace $L \subset \mathbb{R}^n$.
$u^\perp = \{x \in \mathbb{R}^n : \langle x, u \rangle = 0\ \forall y \in L\}$ for $u \in \mathbb{R}^n \backslash \{o\}$.

Orthogonal projection in $\mathbb{R}^n$:
$X|L = \Pi_L(X) = L \cap (X + L^\perp)$ for $X \subset \mathbb{R}^n$ and a linear subspace $L \subset \mathbb{R}^n$.

Topology in $\mathbb{R}^n$:
$\operatorname{int} X$, ∂X, and $\operatorname{cl} X$ denote, respectively, the interior, the boundary, and the closure of a set $X \subset \mathbb{R}^n$.

Relative topology in $\mathbb{R}^n$:
$\operatorname{relint} C$ and $\operatorname{relbd} C$ denote, respectively, the relative interior and relative boundary of a closed convex set $C \subset \mathbb{R}^n$ with respect to the affine subspace $\operatorname{aff} C$ generated by C (cf. Lemma 1.1.9).

Lebesgue measure, integration in $\mathbb{R}^n$:
$|X|$ is the Lebesgue measure for a measurable $X \subset \mathbb{R}^n$.
$\int_X f = \int_X f(x)\, dx$ denotes the Lebesgue integral for measurable $f : X \to \mathbb{R}$ (if the integral exists).

Unit (Euclidean or ℓ_2) ball in $\mathbb{R}^n$:
$B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ and $\omega_n = |B^n|$.

Hausdorff measure:
$\mathcal{H}^d$, normalized so that $\mathcal{H}^d(X) = |X|$ for Lebesgue measurable $X \subset \mathbb{R}^d$ (see Sections 1.B and 10.4).

Unit sphere S^{n-1} in $\mathbb{R}^n$, and integration on S^{n-1}:
$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$.
$\int_X f(x)\, dx = \int_X f\, d\mathcal{H}^{n-1}$ for $\mathcal{H}^{n-1}$ measurable $X \subset S^{n-1}$ and $f : X \to \mathbb{R}$ (if the integral exists).

ε-net:

Given $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, an ε-net $\Xi \subset X$ is a discrete set such that, for any $x \in X$, there exists $y \in \Xi$ with $\|x - y\| \leq \varepsilon$. Here, Ξ is finite if X is bounded.

Support function:

$h_K(u) = \max\{\langle x, u\rangle : x \in K\}$ for a compact convex $K \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n$ (see Section 1.6).

When speaking about compact convex sets, we always assume that the sets are non-empty.

Convex body:

$K \subset \mathbb{R}^n$ is convex body if it is compact, convex, and $\operatorname{int} K \neq \emptyset$.

- $\mathcal{K}^n_o$ is the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in K$.
- $\mathcal{K}^n_{(o)}$ is the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in \operatorname{int} K$.

Radial function:

For $K \in \mathcal{K}^n_o$ and $u \in S^{n-1}$, $\varrho_K(u) = \max\{t \geq 0 : tu \in K\}$ is the radial function (see Section 1.9 if $o \in \operatorname{int} K$, and Section 2.6 in general).

Polar (dual) body K^*:

$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \ \forall\, y \in K\}$ for $K \in \mathcal{K}^n_{(o)}$ (see Section 1.9).

Norm $\|x\|_K$:

$\|x\|_K = \max\{t \in \mathbb{R} : tx \in K\}$ if $x \in \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ for $K \in \mathcal{K}^n_{(o)}$ (see Section 1.9). In this case, $\|x\|_K = h_{K^*}(x)$, and $\|u\|_K = 1/\varrho_K(u)$ for $u \in S^{n-1}$.

Regular boundary points, $\partial' X$, $\nu_X(x)$:

For a closed set $X \subset \mathbb{R}^n$ with non-empty interior and locally Lipschitz boundary, $x \in \partial X$ is a regular boundary point (equivalently, $x \in \partial' X$) if ∂X is differentiable at x and therefore there exists a unique exterior unit normal $\nu_X(x)$ at x (see Section 1.5 if X is convex, and Chapter 4 if ∂X is locally Lipschitz).

Surface area measure S_K:

For a compact convex $K \subset \mathbb{R}^n$, the surface area measure S_K is a Borel measure on S^{n-1} such that $S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right)$ if K is a convex body (see Section 2.5).

Cone volume measure V_K:

For $K \in \mathcal{K}^n_o$, the cone volume measure V_K is a Borel measure on S^{n-1} with $dV_K = \frac{1}{n} h_K \, dS_K$ (see Section 2.6).

L_p surface area measure $S_{K,p}$:

For $K \in \mathcal{K}^n_o$, the L_p surface area measure $S_{K,p}$ is a Borel measure on S^{n-1} with $dS_{K,p} = h_K^{1-p} \, dS_K$ (see Section 9.3).

Derivative of Lipschitz functions on $\mathbb{R}^n$:

$D\varphi(x)$ is the $n \times m$ matrix derivative of a locally Lipschitz function $\varphi : \Omega \to \mathbb{R}^m$, for open $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$ where the derivative exists (so, for $\mathcal{H}^n$ a.e. $x \in \Omega$).

Hessian of C^2 or convex functions on $\mathbb{R}^n$:

$D^2\varphi(x)$ is the Hessian of the function $\varphi : \Omega \to \mathbb{R}^m$ for open $\Omega \subset \mathbb{R}^n$ and $x \in \Omega$ where φ is C^2, or φ and Ω are convex and $D^2\varphi(x)$ exists (so, for $\mathcal{H}^n$ a.e. $x \in \Omega$, see Section 10.6).

Spherical gradient and Hessian:

For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t > 0$ and $u \in S^{n-1}$; then (see Definition 8.1.6)

$$\nabla h(u) = D\bar{h}(u)|_{u^\perp} = D\tilde{h}(u)|_{u^\perp}, \qquad \nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp},$$
$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp} = \nabla^2 h(u) + h(u)I_{n-1}.$$

Subdifferential of a convex function

$\partial\varphi(z) = \{u \in \mathbb{R}^n : \varphi(x) - \varphi(z) \geq \langle u, x - z\rangle \ \forall x \in \Omega\}$ for $z \in \Omega$, for a convex and open set $\Omega \subset \mathbb{R}^n$ and a convex function $\varphi : \Omega \to \mathbb{R}$. In particular, $\partial\varphi(z)$ is nonempty, convex, and compact.

Log-concave functions and measures:

For a convex set $C \subset \mathbb{R}^n$, a function $f : C \to [0, \infty)$ is log-concave if $f((1 - \lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$ holds for $x, y \in C$ and $\lambda \in (0, 1)$; or in other words, $f = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. If $\operatorname{int} C \neq \emptyset$, then $d\mu = f\, d\mathcal{H}^n$ is a log-concave measure (cf. Section 10.9).

Chapter 1

Closed convex sets in $\mathbb{R}^n$, Support function

In this chapter we introduce the basic notions in convexity - bounded and unbounded closed convex sets, supporting hyperplanes, closest point map, convex polytopes and polyhedra, centroid, polarity, almost everywhere differentiability of the boundary of a convex body, convex functions, and encoding a compact convex set with the help of the convex and 1-homogeneous support function. Via Steiner symetrisation, we also prove two funcdamental geometric inequalities, namely, the Brunn-Minkowski inequality and the Isodiametric inequality (the latter also for non-convex sets).

1.1 Affine hull, Convex hull

In this section we introduce the basic notions of affine geometry.

Definition 1.1.1 (Affine subspace). $A \subset \mathbb{R}^n$ is an affine subspace of dimension d if there exists $w \in A$ such that $A - w$ is a d-dimensional linear subspace.

If $A \subset \mathbb{R}^n$ is a d-dimensional affine subspace, then $A - x$ is a d-dimensional linear subspace for any $x \in A$ where points, lines and hyperplanes are the 0-dimensional, 1-dimensional and $(n-1)$-dimensional affine subspaces, respectively. In addition, a non-empty intersection of affine subspaces is an affine subspace.

Definition 1.1.2 (Affine hull). For $X \subset \mathbb{R}^n$, aff X is the smallest affine subspace containing X.

We observe that if $x_1, \ldots, x_k \in \mathbb{R}^n$, then

$$\text{aff}\,\{x_1, \ldots, x_k\} = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Definition 1.1.3. We say $x_0, \ldots, x_d \in \mathbb{R}^n$ are affinely independent if $\sum_{i=0}^{d} \lambda_i x_i = o$ and $\sum_{i=0}^{d} \lambda_i = 0$ for $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ imply that each $\lambda_i = 0$.

Remark. When $d \geq 1$, it is equivalent to saying that the vectors $x_1 - x_0, \ldots, x_d - x_0$ are independent. In this case, aff $\{x_0, \ldots, x_d\}$ is of dimension d, and for any $z \in$ aff $\{x_0, \ldots, x_d\}$ there exist unique $\lambda_0, \ldots, \lambda_d \in \mathbb{R}$ with $z = \sum_{i=1}^{k} \lambda_i x_i$ and $\sum_{i=1}^{k} \lambda_i = 1$.

Definition 1.1.4 (Convex sets). We say that $X \subset \mathbb{R}^n$ is convex if $(1-\lambda)x + \lambda y \in X$ for any $x, y \in X$ and $\lambda \in [0, 1]$.

As the intersection of convex sets is convex, the following definition makes sense.

Definition 1.1.5 (Convex hull). For $X \subset \mathbb{R}^n$ the smallest convex set containing X is its convex hull $\operatorname{conv} X$; namely,

$$\operatorname{conv} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \forall\, x_i \in X \text{ and } \forall\, \lambda_i \geq 0 \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}. \tag{1.1}$$

Proposition 1.1.6 (Carathédory). *If $X \subset \mathbb{R}^n$ and $x \in \operatorname{conv} X$, then there exist an integer $1 \leq k \leq n+1$ and $x_1, \ldots, x_k \in X$ such that $x \in \operatorname{conv}\{x_1, \ldots, x_k\}$.*

Proof. The Proposition follows from (1.1) and by applying iteratively the following *Claim:* If $x = \sum_{i=1}^m \lambda_i x_i$ for $x_1, \ldots, x_m \in X$ and $\lambda_i > 0$ with $m \geq n+2$ and $\sum_{i=1}^m \lambda_i = 1$, then there exist $j \in \{1, \ldots, m\}$ and $\lambda'_i \geq 0$ for $i \in I_j$, where $I_j = \{1, \ldots, m\} \backslash \{j\}$, such that $x = \sum_{i \in I_j} \lambda'_i x_i$.

So, to conclude the proof, it suffices to prove the claim. To this aim, we observe that there exist $\mu_1, \ldots, \mu_m \in \mathbb{R}$ not all zero with $\sum_{i=1}^m \mu_i x_i = o$ and $\sum_{i=1}^m \mu_i = 0$ (since $m \geq n+2$). In particular, up to changing the sign of all μ_i's, there exists at least one index $i \in \{1, \ldots, m\}$ such that $\mu_i > 0$. Let $t = \min\left\{\frac{\lambda_i}{\mu_i} : \mu_i > 0\right\} > 0$, and choose $j \in \{1, \ldots, m\}$ such that $\mu_j > 0$ and $t = \frac{\lambda_j}{\mu_j}$. Then, for $i = 1, \ldots, m$, it follows that $\lambda'_i = \lambda_i - t\mu_i \geq 0$, $\sum_{i=1}^m \lambda'_i = 1$, and $x = \sum_{i=1}^m \lambda'_i x_i$. As $\lambda'_j = 0$, this proves the claim. ∎

For compact $X \subset \mathbb{R}^n$, its diameter is

$$\operatorname{diam} X = \max\{\|y - z\| : y, z \in X\}.$$

Proposition 1.1.7. *If $X \subset \mathbb{R}^n$ is compact, then* $\operatorname{conv} X$ *is compact, and* $\operatorname{diam} \operatorname{conv} X = \operatorname{diam} X$.

Remark. If $X \subset \mathbb{R}^n$ is closed, then $\operatorname{conv} X$ may not be closed; for example, if $X = \left\{\left(t, \frac{\pm 1}{t}\right) : t > 0\right\} \subset \mathbb{R}^2$, then $\operatorname{conv} X = \{(t, s) : t > 0 \;\&\; s \in \mathbb{R}\}$.

Proof. $\widetilde{X} = \operatorname{conv} X$ is compact as a consequence of Proposition 1.1.6. Now, let $D = \operatorname{diam} X$, and let $y, z \in \widetilde{X}$. We have $y \in X + DB^n$ for $x \in X$ because $\widetilde{X}$ is the intersection of all convex sets containing X, and hence $X \subset y + DB^n$, which in turn yields $\widetilde{X} \subset y + D\,B^n$. Therefore, $\|y - z\| \leq D$. ∎

The following statement is essentially equivalent to the Carathédory's result (Proposition 1.1.6).

Proposition 1.1.8 (Radon). *If $x_1, \ldots, x_k \in \mathbb{R}^n$ with $k \geq n+2$, then there exists $I \subset \{1, \ldots, k\}$ with $1 \leq \#I < k$ such that* $\operatorname{conv}\{x_i\}_{i \in I} \cap \operatorname{conv}\{x_i\}_{i \notin I} \neq \emptyset$.

Proof. For each i, let $y_i = (x_i, 1) \in \mathbb{R}^{n+1}$. Since $k \geq n+2$, there exist $\alpha_1, \ldots, \alpha_k$ not all zero with $\sum_{i=1}^k \alpha_i y_i = o_{\mathbb{R}^{n+1}}$; or equivalently, $\sum_{i=1}^k \alpha_i x_i = o$ and $\sum_{i=1}^k \alpha_i = 0$. It follows that $1 \leq \#I < k$ for $I = \{i \in \{1, \ldots, k\} : \alpha_i \geq 0\}$. In addition, we have $A = \sum_{i \in I} \alpha_i = \sum_{i \notin I} (-\alpha_i) > 0$ and $z \in \operatorname{conv}\{x_i\}_{i \in I} \cap \operatorname{conv}\{x_i\}_{i \notin I}$ for $z = \sum_{i \in I} \frac{\alpha_i}{A} x_i = \sum_{i \notin I} \frac{-\alpha_i}{A} x_i$. ∎

Lemma 1.1.9. *If $X \subset \mathbb{R}^n$ is a closed convex set and $d = \dim A$ for $A = \operatorname{aff} X$, then its relative interior* relint X *(with respect to the topology of A) is non-empty and convex and X is its closure.*

Remark. We write $d = \dim X$ where $|X| > 0$ if and only if $d = n$.

Proof. We may assume that $A = \mathbb{R}^n$, hence there exist affinely independent $x_0, \ldots, x_n \in X$. For $i = 0, \ldots, n$, there exists (unique) $u_i \in S^{n-1}$ such that $\langle u_i, x_i \rangle > \langle u_i, x_j \rangle$ for $j \neq i$, and $\langle u_i, x_k \rangle = \langle u_i, x_j \rangle$ for $k, j \neq i$. Choosing $s > 0$ such that $\langle u_i, x_i - x_j \rangle > s$ for all $j \neq i$, and setting $r = \frac{s}{n+1}$, we have $z_0 + r B^n \subset \operatorname{conv}\{x_0, \ldots, x_n\} \subset X$ for $z_0 = \frac{1}{n+1}(x_0 + \ldots + x_n)$. In particular, the convex set int$X$ is non-empty.

Finally, for any $y \in \partial X$ and $\lambda \in (0, 1]$, we have that $(1 - \lambda)y + \lambda z_0 + \lambda r B^n \subset X$; therefore, $X = \operatorname{cl} \operatorname{int} X$. ∎

1.2 Closed convex sets in $\mathbb{R}^n$

While one of the main topics of this book is to understand properties of compact convex sets in $\mathbb{R}^n$, we will need many properties of possibly unbounded closed convex sets.

Lemma 1.2.1 (Unbounded closed convex sets). *Let $X \subset \mathbb{R}^n$ be a convex closed set. Then X is unbounded if and only if there exists $u \in S^{n-1}$ such that $x + [0, \infty)u \subset X$ for any $x \in X$; this, in turn, is equivalent to saying that X contains a half line.*

Proof. The only non-trivial implication is the fact that if X is unbounded, then there exists $u \in S^{n-1}$ such that $x + [0, \infty)u \subset X$ for any $x \in X$.

Since X is unbounded, there exists a sequence $\{x_k\} \subset X$ with $\|x_k\| \to \infty$. Fix $y \in X$, and up to a subsequence assume that $u_k = \frac{x_k - y}{\|x_k - y\|} \in S^{n-1}$ converges to $u \in S^{n-1}$. Since

$$\|x_k - y\| - \|x - y\| \leq \|x_k - x\| \leq \|x_k - y\| + \|x - y\|$$

for any $x \in X$, this also implies that $\lim_{k\to\infty} \frac{x_k - x}{\|x_k - x\|} = u$. Then, for any $t > 0$, it follows from the convexity of X that $x + t u_k \in X$ whenever $\|x_k - x\| > t$. Therefore, since X is closed, letting $k \to \infty$ we deduce that $x + tu \in X$, as desired. ∎

Lemma 1.2.2. *Let $X \subset \mathbb{R}^n$ be convex closed, and let $z \notin X$. Then there exists a unique closest point $y \in X$ to z, and $X \subset H^+ = \{x \in \mathbb{R}^n : \langle x, z - y \rangle \leq \langle y, z - y \rangle\}$.*

Proof. The existence of a unique closest point $y \in X$ to z follows from the fact that the function $X \ni y \mapsto \|y - z\|^2$ has a minimum on X, as X is a closed.

Now, given $x \in X$, it follows that $y + t(x - y) = (1 - t)y + tx \in X$ for $t \in [0, 1]$ as X is convex, therefore $\|z - y - t(x - y)\|^2 \geq \|z - y\|^2$ for $t \in (0, 1)$; this is equivalent to saying that

$$\langle z - y, z - y \rangle - 2t\langle z - y, x - y \rangle + t^2\langle x - y, x - y \rangle \geq \langle z - y, z - y \rangle.$$

Letting $t > 0$ tend to zero, we deduce that $\langle z - y, x - y\rangle \le 0$, hence $x \in H^+$. Finally, if $x \in X$ with $x \ne y$, then $\langle z - y, x - y\rangle \le 0$ yields that $\|x - z\|^2 = \|(x - y) + (y - z)\|^2 > \|y - z\|^2$, showing the uniqueness of y as closest point. ■

The rest of the section collects various consequences of Lemma 1.2.2.

Lemma 1.2.3 (Exterior normal). *Let $X \subset \mathbb{R}^n$ be convex closed and $y \in \partial X$. Then there exists $u \in \mathbb{R}^n \backslash \{o\}$ such that $\langle x, u\rangle \le \langle y, u\rangle$ for $x \in X$.*

Remark. Any such u is called an exterior normal at y, and $y + u^\perp$ is called a supporting hyperplane, which then exists at any boundary point. It follows that X is the intersection of "supporting half spaces".

Proof. Let $z_k \in \mathbb{R}^n \backslash X$ a sequence tending to y, and let y_k be the closest point of X to z_k (cf. Lemma 1.2.2). As $\|y_k - z_k\| \le \|y - z_k\|$, also the sequence y_k tends to y. Since S^{n-1} is compact, we may assume that $u_k = \frac{z_k - y_k}{\|z_k - y_k\|}$ tends to $u \in S^{n-1}$. Using Lemma 1.2.2, we deduce that if $x \in X$ then $\langle u, x - y\rangle = \lim_{k\to\infty}\langle u_k, x - y_k\rangle \le 0$. ■

Definition 1.2.4 (Face). For a convex closed $X \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n \backslash \{o\}$, let us assume that there exists a $y \in \partial X$ where u is an exterior normal. Then $F_X(u) = \{x \in X : \langle x, u\rangle = \langle y, u\rangle\}$ is the face of X with exterior normal u.

Remarks.

- It follows from Lemma 1.2.3 that $F_X(u) \subset \partial X$, and the condition on u is equivalent to saying that X contains no half line in the direction of u.
- An equivalent definition of a face of X is that it is the intersection with a supporting hyperplane.
- What is called face here, following for example, Grünbaum [277], is frequently called *exposed face in the literature (see, for example, Rockafellar* [498]*).*

Lemma 1.2.5 (Separation of a closed convex set and a compact set). *Let $X \subset \mathbb{R}^n$ be convex closed, and $K \subset \mathbb{R}^n$ compact convex with $X \cap K = \emptyset$. Then there exist $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that $\langle v, x\rangle < \alpha$ for $x \in X$ and $\langle v, y\rangle > \alpha$ for $y \in K$.*

Proof. Since K is compact and X is closed, there exist $x_0 \in X$ and $y_0 \in K$ such that $\|x_0 - y_0\|$ is the minimum of the distances of points of X and K. Therefore, the hyperplane normal to $x_0 - y_0$ and passing through $\frac{x_0 + y_0}{2}$ strictly separates X and K (thanks to Lemma 1.2.2). ■

Lemma 1.2.6 (Separation of closed convex sets). *If $X, Y \subset \mathbb{R}^n$ are convex closed, $\operatorname{int} Y \ne \emptyset$, and $X \cap \operatorname{int} Y = \emptyset$, then there exist $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that $\langle v, x\rangle \le \alpha$ for $x \in X$ and $\langle v, y\rangle \ge \alpha$ for $y \in Y$.*

Proof. We may assume that $o \in \operatorname{int} Y$, and chose a $x_0 \in X$. Using Lemma 1.2.5, for $k > \|x_0\|$ the disjoint compact sets $Y_k = kB^n \cap (1 - \frac{1}{k})Y \subset \operatorname{int} Y$ and $X_k = kB^n \cap X$ can be separated by a hyperplane H_k intersecting $\operatorname{conv}\{0, x_0\}$ in a point z_k. Then, it suffices to consider a subsequence $\{H_{k'}\}$ such that $z_{k'}$ tends to $z \in \operatorname{conv}\{0, x_0\}$ and some exterior unit vector $u_{k'}$ of $H_{k'}$ tends to a $v \in S^{n-1}$, and choose $\alpha = \langle v, z\rangle$. ■

Lemma 1.2.3 ensures that the normal cone below is not empty.

Definition 1.2.7 (Cone and Normal Cone).

(a) $C \subset \mathbb{R}^n$ is a convex cone if $\alpha x + \beta y \in C$ for any $x, y \in C$ and $\alpha, \beta \geq 0$. We say that C is non-trivial if $C \neq \{o\}$ and $C \neq \mathbb{R}^n$.

(b) If $X \subset \mathbb{R}^n$ is closed convex and $y \in \partial X$, then the normal cone at y is

$$N_X(y) = \{u \in \mathbb{R}^n : \langle u, x - y\rangle \leq 0 \ \forall x \in X\}$$

which is either a non-trivial closed convex cone, or $N_X(y) = \mathbb{R}^n$ in the case $X = \{y\}$.

Remarks.

- $v \in N_X(y)\backslash\{o\}$ for $y \in \partial X$ if and only if v is an exterior normal at y.
- If $C \subset \mathbb{R}^n$ is a closed convex cone with $C \neq \mathbb{R}^n$, then $\mathcal{U} = S^{n-1} \cap N_C(o)$ satisfies $C = \{x \in \mathbb{R}^n : \langle u, x\rangle \leq 0 \ \forall\, u \in \mathcal{U}\}$.

Lemma 1.2.8. *If $X \subset \mathbb{R}^n$ is closed and convex, $y_m, y \in \partial X$ with $y_m \to y$ and $u_m \in N_X(y_m) \cap S^{n-1}$, then $u \in N_X(y)$ for any accumulation point u of $\{u_m\}$.*

Proof. We may assume that $\lim_{m\to\infty} u_m = u$. For $x \in X$, we have $\langle u_m, x - y_m\rangle \leq 0$ for any m; therefore, $\langle u, x - y\rangle \leq 0$ also in the limit. ■

Next, we show that there always exists an exterior unit normal vector such that the opposite vector points inward.

Lemma 1.2.9. *Let $X \subset \mathbb{R}^n$ be closed, convex, with* $\operatorname{int} X \neq \emptyset$, *and let $x \in \partial X$. Then there exist $u \in S^{n-1} \cap N_X(x)$ and $t > 0$ such that $x - tu \in \operatorname{int} X$.*

Proof. We may assume that $x = o$. Let $C = \operatorname{cl}\{[0, \infty)z : z \in \operatorname{int} X\}$, that is a closed convex cone. Then the statement of the lemma is equivalent to proving that $N_X(o) \cap \operatorname{int}(-C) \neq \emptyset$.

By contradiction, we suppose that $N_X(o) \cap \operatorname{int}(-C) = \emptyset$. Then Lemma 1.2.6 provides a $v \in S^{n-1}$ such that $\langle v, y\rangle \geq 0$ for $y \in -C$ and $\langle v, z\rangle \leq 0$ for $z \in N_X(o)$. Hence, the first property of v yields that $v \in N_X(o)$, which combined with the second property of v implies that $1 = \langle v, v\rangle \leq 0$. This contradiction proves the result. ■

Lemma 1.2.2 makes it possible to consider the closest point map.

Definition 1.2.10 (Closest point map). If $X \subset \mathbb{R}^n$ convex closed and $z \in \mathbb{R}^n$, then the unique point $y \in X$ closest to z given by Lemma 1.2.2 is $\Pi_X(z) = y$.

Remarks:

(a) For $z \in \mathbb{R}^n \backslash X$, we have $\Pi_X(z) = y \in \partial X$ if and only if $z \in y + N_X(y)$;

(b) For a linear subspace $L \subset \mathbb{R}^n$, Π_L is the orthogonal projection.

The closest point map is actually a contraction, as we show now.

Lemma 1.2.11. *If $X \subset \mathbb{R}^n$ is convex and closed and $z_1, z_2 \in \mathbb{R}^n$, then*

$$\|\Pi_X(z_1) - \Pi_X(z_2)\| \leq \|z_1 - z_2\|.$$

Proof. We set $y_i = \Pi_X(z_i)$ and $u_i = z_i - y_i$. Then $\langle u_i, y_i - x\rangle \geq 0$ for $x \in X$ by Lemma 1.2.2 for $i = 1, 2$. Hence, the substitution $z_i = y_i + u_i$ implies

$$\|z_1 - z_2\|^2 = \|y_1 - y_2\|^2 + \|u_1 - u_2\|^2 + 2\langle u_1, y_1 - y_2\rangle + 2\langle u_2, y_2 - y_1\rangle \geq \|y_1 - y_2\|^2.$$

∎

1.3 Additional properties of convex cones

As the normal cones at a boundary point of a closed convex set is a closed convex cone, we need some better understanding of the properties of cones.

Definition 1.3.1 (Pointed convex cones). A closed convex cone $C \subset \mathbb{R}^n$ is pointed if it contains no line.

Remark. Pointed convex cones are also frequently called *strongly convex cones*.

As the convex hull of two linear subspaces is their sum, we have the following statement.

Lemma 1.3.2. *Any closed convex cone $C \subset \mathbb{R}^n$ of dimension at least 1 can be written as $L + C'$ where L is a linear subspace and $C' \subset L^\perp$ is a pointed convex cone.*

Remark. If $X \subset \mathbb{R}^n$ is a closed convex set with $\operatorname{int} X \neq \emptyset$, then the normal cone $N_X(x)$ is pointed convex at any $x \in \partial X$.

Definition 1.3.3 (Dual cone). For a closed convex cone $C \subset \mathbb{R}^n$, its dual cone is

$$C^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 0 \,\forall\, y \notin C\}.$$

Lemma 1.3.4. *If $C \subset \mathbb{R}^n$ is a closed convex cone, then C^* is a closed convex cone, and $C^{**} = C$.*

Proof. It is immediate to check that C^* is a closed convex cone and $C \subset C^{**}$.

If $z \notin C$, then Lemma 1.2.5 provides $v \in S^{n-1}$ and $\alpha \in \mathbb{R}$ such that

$$\langle v, z \rangle > \alpha \quad \text{and} \quad \langle v, x \rangle < \alpha \; \forall \, x \in C.$$

As C is a cone, choosing $x = o$ we deduce that $\alpha > 0$. Also, replacing x with tx and letting $t \to \infty$, we get $\langle v, x \rangle \leq 0$ for all $x \in C$. It follows that $v \in C^*$ and $\langle v, z \rangle > 0$, therefore $z \notin C^{**}$. ■

Now we establish various ways to characterize pointed convex cones.

Lemma 1.3.5 (Pointed convex cones). *For a closed convex cone $C \subset \mathbb{R}^n$, the following statements are equivalent:*

(i) $C \cap (-C) = \{o\}$;

(ii) C contains no line (and hence it is a pointed convex cone);

(iii) $\dim C^* = n$;

(iv) $C \neq \mathbb{R}^n$ and for any $z \notin C$ there exists $w \in \mathbb{R}^n$ such that

$$\langle w, z \rangle > 0 \quad \textit{and} \; \langle w, x \rangle < 0 \; \forall \, x \in C \backslash \{o\}.$$

Remark. It follows that a closed convex cone C is pointed convex if and only if there exists a supporting hyperplane H such that $H \cap C = \{o\}$. Because of this property, o is sometimes called an apex of the pointed convex cone.

Proof. (i) $\Longrightarrow$ (ii): If $\ell \subset C$ is a line, then Lemma 1.2.1 implies that $\ell' \subset C$ for the parallel linear 1-subspace by Lemma 1.2.1. Therefore, $\ell' \subset C \cap (-C)$.

(ii) $\Longrightarrow$ (iii): If $C^* \subset u^\perp$ for $u \in S^{n-1}$, then $\mathbb{R}u \subset C^{**} = C$.

(iii) $\Longrightarrow$ (iv): As $z \notin C$, Lemma 1.2.5 implies that there exists $w_0 \in \mathbb{R}^n$ such that

$$\langle w_0, z \rangle > 0 \quad \text{and} \; \langle w_0, x \rangle \leq 0 \; \forall \, x \in C.$$

Hence $w_0 \in C^*$. By continuity, it follows that there exists a $w \in \operatorname{int} C^*$ such that $\langle w, z \rangle > 0$. Now, for $x \in C \backslash \{o\}$ it follows that $w + \varepsilon x \in C^*$ for some small $\varepsilon > 0$. Thus $\langle x, w + \varepsilon x \rangle \leq 0$, which implies that $\langle x, w \rangle < 0$.

(iv) $\Longrightarrow$ (i): Since there exists $w \in \mathbb{R}^n$ $\langle w, x \rangle < 0$ for all $x \in C \backslash \{o\}$, we deduce that $C \cap (-C) = \{o\}$.

■

Finally, we consider a natural method to construct convex cones.

Definition 1.3.6 (Positive hull). If $X \subset \mathbb{R}^n$, then its positive hull is

$$\operatorname{pos} X = \left\{ \sum_{i=1}^{k} \lambda_i x_i : \forall \, x_i \in X \text{ and } \forall \, \lambda_i \geq 0 \right\},$$

which is a convex cone.

Since any finite set of vectors in $\mathbb{R}^n$ of cardinatlity at least $n+1$ is dependent, we have the following analogue of the Carathédory's Proposition 1.1.6.

Lemma 1.3.7. *If $X \subset \mathbb{R}^n$ and $z \in \operatorname{pos} X$, then there exist $x_1, \dots, x_n \in X$ and $\lambda_1, \dots, \lambda_n \geq 0$ such that $z = \sum_{i=1}^{d} \lambda_i x_i$.*

Lemma 1.3.8. *If $X \subset \mathbb{R}^n$ is compact and $o \notin \operatorname{conv} X$, then the convex cone $\operatorname{pos} X$ is a closed set.*

Remark. As X is compact, $o \notin \operatorname{conv} X$ is equivalent with the property that there exist $\alpha > 0$ and $v \in \mathbb{R}^n$ such that $\langle v, x\rangle \geq \alpha$ for $x \in X$.

Proof. Let $\alpha > 0$ and $v \in \mathbb{R}^n$ such that $\langle v, x\rangle \geq \alpha$ for $x \in X$. We consider $z = \lim_{k\to\infty} z_k$ for $z_k \in \operatorname{pos} X$, and hence $z_k = \sum_{i=1}^{n} \lambda_{ki} x_{ki}$ for $\lambda_{ki} \geq 0$ and $x_{ki} \in X$ by Lemma 1.3.7. There exists a $\gamma > 0$ such that $\langle v, z_k\rangle \leq \gamma$ for each k, and we may assume that $\lim_{k\to\infty} x_{ki} = y_i \in X$ for $i = 1, \dots, n$ by the compactness of X. The first property and $\langle v, x_{ki}\rangle \geq \alpha$ imply that each $\lambda_{ki} \leq \gamma/\alpha$, which in turn yields $z \in \operatorname{pos}\{y_1, \dots, y_n\} \subset \operatorname{pos} X$. ■

We note that if $X \subset \mathbb{R}^n$ is compact and $o \notin X$, then $\operatorname{pos} X$ may not be closed.

Example 1.3.9. If $X = \{(x, y, z) \in \mathbb{R}^3 : |x| = 1 \text{ and } y^2 + (z-1)^2 = 1\}$, then X is compact and $o \notin X$, but $\operatorname{pos} X$ is the union of the open half space $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ and the x axis; therefore, it is not closed.

Lemma 1.3.10. *If $X \subset \mathbb{R}^n$ is finite, then $\operatorname{pos} X$ is a closed convex cone.*

Proof. Since $\operatorname{pos} X$ is a convex cone, the only issue is whether it is closed. We prooceed by induction on $n \geq 1$ where the case $n = 1$ trivially holds as a positive hull in $\mathbb{R}$ is either a point, or a half line, or $\mathbb{R}$.

In the case $n \geq 2$, let $P = \operatorname{conv} X$. If $o \notin P$, then Lemma 1.3.8 verifies Lemma 1.3.10. If $o \in \operatorname{int} P$, then $\operatorname{pos} X = \mathbb{R}^n$, and hence closed.

Therefore the only case left open is when $n \geq 2$ and $o \in \partial P$, and let $u \in S^{n-1}$ be an exterior normal at o to P. For $X_0 = X \cap u^\perp$, $C_0 = \operatorname{pos} X_0$ is a closed convex cone by induction, thus we may assume that $X_+ = X \backslash u^\perp$ is non-empty. Here $\langle -u, x\rangle > 0$ for $x \in X_+$, and hence the finiteness of X_+ yields that $o \notin \operatorname{conv} X_+$, and there exists a $\beta > 0$ such that $\langle x, -u\rangle \geq \beta\|x\|$ for $x \in X_+$. In particular, $C_+ = \operatorname{pos} X_+$ is a closed convex cone by Lemma 1.3.8, and

$$\operatorname{pos} X = C_0 + C_+. \tag{1.2}$$

It also follows by the triangle inequality for $\|\cdot\|$ that $\langle y, -u\rangle \geq \beta\|y\|$ for $y \in C_+$.

Finally, to prove that $\operatorname{pos} X$ is closed, let $z_k \in \operatorname{pos} X$ tend to a $z \in \mathbb{R}^n$; in particular, $\langle -u, z_k\rangle \leq \gamma$ for some constant $\gamma > 0$. According to (1.2), we have $z_k = w_{k0} + w_{k+}$ for

$w_{k0} \in C_0 \subset u^\perp$ and $w_{k+} \in C_+$ where

$$\gamma \geq \langle -u, z_k \rangle = \langle -u, w_{k+} \rangle \geq \beta \|w_{k+}\|.$$

Therefore, we may assume that w_{k+} tends to w_+, and hence $w_{k0} = z_k - w_{k+}$ tends to some w_0. We have $w_+ \in C_+$ and $w_0 \in C_0$ as these cones are closed sets, and hence $z \in \operatorname{pos} X$ by (1.2). ∎

We note that the sum of two closed convex cones, which is always a convex cone, may not be closed.

Example 1.3.11. For the closed convex cones $C = \{(-t, -t, 0) : t \geq 0\}$ and $\widetilde{C} = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0 \ \&\ y^2 + z^2 \leq x^2\}$, we have that $p = (0, 0, 1) \notin C + \widetilde{C}$, but $p = \lim_{t\to\infty}(q_t + \tilde{q}_t)$ where $q_t = (-t, -t, 0) \in C$ and $\tilde{q}_t = \left(\sqrt{t^2 + (1 + \frac{1}{t})^2}, t, 1 + \frac{1}{t}\right) \in \widetilde{C}$.

1.4 Polyhedra, polytopes

Definition 1.4.1 (Polyhedron). A polyhedron $P \subset \mathbb{R}^n$ is the non-empty intersection of finitely many closed halfspaces in $\mathbb{R}^n$, allowing $P = \mathbb{R}^n$ (empty family of closed half spaces). A 0-dimensional and a 1-dimensional face are called vertex and edge, respectively. If $\dim P = n$ and $P \neq \mathbb{R}^n$, then the $(n-1)$-dimensional faces are called facets.

The following properties (including Lemma 1.4.2) are immediate from the definition:

- Any linear subspace is a polyhedron, and
- non-empty intersection of a finite family of polyhedra is a polyhedron.

Lemma 1.4.2. *If $P = H_1^+ \cap \ldots \cap H_k^+$ is a polyhedron for closed half spaces $H_i^+ \subset \mathbb{R}^n$, and $L \subset \mathbb{R}^n$ is a proper linear subspace, then the orthogonal projection $\Pi_L P$ is a polyhedron. In the case $\Pi_L P \neq L$, P is the intersection of all $H_i^+ \cap L$ such that H_i^+ contains a translate of $L^\perp$.*

Next we consider the "face lattice" of a polyhedron P; namely, the partially order set whose elements are the faces of P, and the ordering is by inclusion.

Proposition 1.4.3. *Let $P \subset \mathbb{R}^n$ be a polyhedron.*

(a) Any face of P is a polyhedron, and the number of faces is finite.

(b) Any face of a face of P is a face of P.

(c) If $\dim P = n$ and $P \neq \mathbb{R}^n$, then ∂P is the union of the facets, any face F of P of dimension at most $n-2$ is the face of a facet, and F is the intersection of the facets containing it.

(d) If $\dim P = n$ *and* $P \neq \mathbb{R}^n$*, then any* $(n-2)$*-dimensional face is contained in exactly two facets.*

Proof. We prove Proposition 1.4.3 by induction on n where the cases $n = 1, 2$ trivially hold. We may assume that $n \geq 3$, $o \in \operatorname{int} P$, and $P = \cap_{i=1}^{k} H_i^+$ where $H_1^+, \ldots, H_k^+$ is a minimal family of closed halfspaces whose intersection is P. Let $H_i = \partial H_i^+$.

For the first half of (a), any supporting hyperplane H of P, $H \cap P = \cap_{i=1}^{k}(H \cap H_i^+)$, and hence the face $H \cap P$ is a polyhedron.

Turning to (c), for any $z \notin P$, there exists an H_i^+ such that $z \notin H_i^+$, and hence the segment $\operatorname{conv}\{o, z\}$ intersects H_i. It follows that $\partial P = \cup_{i=1}^{k} F_i$ where $F_i = P \cap H_i$. The minimality of $H_1^+, \ldots, H_k^+$ yields that for any H_j^+, there exists a $z_j \in \cap_{i \neq j}^{k} H_i^+$ with $z_j \notin P$. Then $\operatorname{conv}\{o, z_j\}$ intersects F_j in a w_j that is the center of a $(n-1)$-ball contained in F_j as $o \in \operatorname{int} P$, thus each F_j is a facet.

For the second statement in (c), if $F = H \cap P$ is a face for a supporting hyperplane H, then let $z \in \operatorname{relint} F$. Here $z \in F_j$ for a facet F_j, and hence $F \subset H_j$, which in turn yields that $F = (H \cap H_j \cap P) \cap \cap_{i \neq j}^{k}(H \cap H_j \cap H_i^+) = H \cap F_j$, that is a face of F_j.

For the third statement in (c), we observe that $F_j = \cap_{i \neq j}(H_j \cap H_i^+)$, and let $I_j \subset \{1, \ldots, n\} \backslash \{j\}$ be minimal such that $F_j = \cap_{i \in I_j}(H_j \cap H_i^+)$, and hence $G_{ji} = H_i \cap F_j$, $i \in I_j$, are the $(n-2)$ faces of F_j. Here $G_{ji} = P \cap H_i \cap F_j = F_i \cap F_j$, and hence the induction hypothesis implies that

$$F = \cap_{F \subset G_{ji}} G_{ji} = \cap_{F \subset F_j \cap F_i} F_i \cap F_j = \cap_{F \subset F_i} F_i.$$

This completes the proof of (c), and turn the proof of (a) by induction.

For (d), let F be an $(n-2)$-dimensional face of P, and let $L \subset \mathbb{R}^n$ be the two-dimensional linear subspace orthogonal to $\operatorname{aff} F$, and hence $\Pi_L P$ is a polyhedron according to Lemma 1.4.2. The vertex $\Pi_L F$ of $\Pi_L P$ is contained in two edges f_1 and f_2 of Π_L, thus F is contained in exatly two facets of P by Lemma 1.4.2; namely, $f_1 + \operatorname{aff} F$ and $f_2 + \operatorname{aff} F$.

For (b), let G be a face of F for a face F of P. According (c), there exists a facet F_i such that F is either a face of F_i or $F = F_i$, and hence G is a face of F_i by induction. In particular, $A \cap F_i = G$ for an $(n-2)$-dimensional affine subspace $A \subset H_i$. If $L \subset \mathbb{R}^n$ is a two-dimensional linear subspace orthogonal to A, then $v = \Pi_L G$ is an endpoint of the edge $\Pi_L F_i$ of the polyhedron $\Pi_L P$. It follows that v is a vertex of $\Pi_L P$ by induction, and $v = \ell \cap \Pi_L P$ for a line $\ell \subset L$; therefore, $G = P \cap (\ell + A)$. ■

Definition 1.4.4 (Polytopes). A polytope is a bounded polyhedron.

In other words, a polytope is a bounded intersection of finitely many closed half-spaces. Now we show that this property is equivalent with the dual definition; namely, the convex hull of finitely many points.

Proposition 1.4.5. *A $P \subset \mathbb{R}^n$ is a polytope if and only if it is the convex hull of finitely many points.*

Remark. The polytope is actually the convex hull of its vertices.

Proof. We use induction on $n \geq 1$ where the case $n = 1$ is trivial.

To show that a polytope $P \subset \mathbb{R}^n$ is the convex hull of its vertices follows, we may assume that $\dim P = n$. For any $x \in P$, a line ℓ through x intersects ∂P in $y_1 \in F_1$ and $y_2 \in F_2$ for some facets F_1, F_2 of P by Proposition 1.4.3 (c) (possibly $F_1 = F_2$), and hence $x \in \operatorname{conv}\{y_1, y_2\}$. Now y_i is the convex hull of the finitely many vertices of F_i by induction and Proposition 1.4.3 (a), which are vertices of P, as well, by Proposition 1.4.3 (b).

Nex let $P = \operatorname{conv} X$ for $X = \{x_1, \dots, x_k\} \subset \mathbb{R}^n$. We may assume that $\dim P = n$ and $o \in \operatorname{int} P$. For any supporting hyperplane $H = \{x \in \mathbb{R}^n : \langle x, u \rangle = \alpha$ for $u \in S^{n-1}$ and $\alpha \in \mathbb{R}$ where $\langle x_i, u \rangle \geq \alpha$ for $x_i \in X$, any $z \in H \cap P$ is of the form $z = \sum_{i=1}^k \lambda_i x_i$ where $\sum_{i=1}^k \lambda_i = 1$ and each $\lambda_i \geq 0$. Since $\langle z, u \rangle = \alpha$ and each $\langle x_i, u \rangle \geq \alpha$, we deduce that $\lambda_i = 0$ if $x_i \notin H$; or in otherwords, the face $H \cap P$ is the convex hull of $H \cap X$. It follows that P has only finitely many faces.

Let Ξ be the union of $\operatorname{lin} F$ over all faces F of P with $\dim F \leq n-2$, and hence $|\Xi| = 0$. For any $z \in P \backslash \Xi$, the half line $\operatorname{pos} z$ intersects ∂P in a point y, which is contained in an $(n-1)$-dimensional face of P. Writing $F_1, \dots, F_m$ to denote the $(n-1)$-dimensional faces of P, and H_i^+ to denote the "supporting" half space containing P with $P \cap \partial H_i^+ = F_i$, we conclude that $P = \cap_{i=1}^m H_i^+$. ∎

These two dual representations of a polytope; namely, intersection of finitely many half spaces or convex hull finitely many points, are used in different setups. For example, given two polytopes $P, Q \subset \mathbb{R}^n$,

$$P \cap Q \text{ is a polytope provided } P \cap Q \neq \emptyset \tag{1.3}$$

because of the representation as a bounded intersection of finitely many half spaces, and the Minkowski sum

$$P + Q \text{ is a polytope} \tag{1.4}$$

because of the representation as a convex hull of finitely many points.

Let us consider some basic examples of polytopes that will be used in this book.

Example 1.4.6 (Polytopes). Let $e_1, \dots, e_n$ form an orthonormal basis of $\mathbb{R}^n$.

Cube, Parallopiped: Cube is congruent to $[-a, a]^n$ for $a > 0$, with $2n$ facets (each is an $(n-1)$-cube) and 2^n vertices. In particular, $[-1, 1]^n = B_\infty^n$ is the unit ball of the l_∞-norm.

A *parallopiped* is of the form $x + \Phi[-1, 1]^n$ for $\Phi \in GL(n)$ and $x \in \mathbb{R}^n$.

Crosspolytope: $\operatorname{conv}\{\pm e_1, \dots, \pm e_n\}$ is the unit ball of the l_1-norm.

Simplex: Convex hull of $(n+1)$ affinely independent points in $\mathbb{R}^n$.

Regular simplex: Simplex whose edges have the same length, for example, $\mathrm{conv}\{e_1,\dots,e_n\}$ is the regular $(n-1)$-simplex of edge length $\sqrt{2}$
If $v_1,\dots,v_{n+1}\in S^{n-1}$ satisfy $\langle v_i,v_j\rangle=\frac{-1}{n}$ for $i\neq j$, then $\mathrm{conv}\{v_1,\dots,v_{n+1}\}$ is the regular simplex insribed into B^n, and $\{x\in\mathbb{R}^n:\langle x,v_i\rangle\leq 1,\ i=1,\dots,n+1\}$ is the regular simplex circumscribed around B^n

Pyramid or "Cone": Congruent to $C=\mathrm{conv}\{o,F\}$ where F is an $(n-1)$-dimensional polytope with $o\notin\mathrm{aff}\,F$ with exterior unit normal u, and hence $|C|=\frac{1}{n}\,h\cdot\mathcal{H}^{n-1}(F)$ where h is the distance of aff F from o.

Definition 1.4.7 (Polyhedral Cone). A $C=\cap_{i=1}^m H_i^+$ where each H_i^+ is a half space of $\mathbb{R}^n$ with $o\in\partial H_i^+$, is called a polyhedral cone. Writing $d=\dim C$, the "normalized angle" of C is

$$\beta(C)=\int_C e^{-\pi\|x\|^2}\,d\mathcal{H}^d(x)=\frac{\mathcal{H}^d(C\cap B^n)}{\omega_d}=\frac{\mathcal{H}^{d-1}(C\cap S^{n-1})}{d\omega_d},\tag{1.5}$$

and hence $\beta(\mathbb{R}^n)=1$.

The dual representation of polyhedral cone is the positive hull of finitely many vectors. We observe that if $L\subset\mathbb{R}^n$ is a lnear subspace of dimension $\delta\geq 1$, then $L=\mathrm{pos}\{\pm x_1,\dots,\pm x_\delta\}$ for any basis $x_1,\dots,x_\delta$ of L.

Proposition 1.4.8. *$C\subset\mathbb{R}^n$ polyhedral cone if and only if $C=\mathrm{pos}\{x_1,\dots,x_k\}$ for $x_1,\dots,x_k\in\mathbb{R}^n$.*

Proof. Both in the cases if C is a polyhedral cone or if it is the positive hull of a finite set, C is a closed convex cone (cf. Lemma 1.3.10), thus Lemma 1.3.2 yields that $C=L+C'$ where L is a linear subspace and $C'\subset L^\perp$ is a pointed convex cone. We may assume that $d=\dim C'\geq 1$, and let $A\subset L^\perp$ be a $(d-1)$-dimensional linear subspace such that $A\cap C=\{o\}$ (cf. Lemma 1.3.5). It follows that if $z\in\mathrm{relint}\,C'$, then $P'=(A+z)\cap C'$ is bounded, thus it is a compact convex set satisfying that $\mathrm{pos}\,P'=C'$.

Now if C is a polyherdral cone, then P is a polyhedron, and hence polytope, that is the convex hull of its vertices $v_1,\dots,v_m$ by Proposition 1.4.5. Therefore C is the positive hull of $v_1,\dots,v_m$ and a finite set of vectors whose positive hull is L.

Next we assume that $C=\mathrm{pos}\{x_1,\dots,x_k\}$. We may assume that for some m with $1\leq m\leq k, x_i\in L$ if and only if $i>m$. It follows that $C'=\mathrm{pos}\{x'_1,\dots,x'_k\}$ for $x'_i=\Pi_{L^\perp}x_i$. Let $y_i=t_ix'_i\in A+z$ for suitable $t_i>0$. It follows that $P=\mathrm{conv}\{y_1,\dots,y_m\}$, and hence P is a polyhedron by Proposition 1.4.5, which in turn yields that $C'=\cap_{i=1}^p H_i'^+$ where each $H_i'^+$ is a closed half space of $L^\perp$ and o lies on the relative boundary of $H_i'^+$. Therefore $C=\cap_{i=1}^p H_i^+$ where $H_i^+=H_i'^++L$. ■

Next we show that the dual of a polyhedral cone is also a polyhedral cone.

Lemma 1.4.9. *For* $u_1, \ldots, u_k \in \mathbb{R}^n \backslash \{o\}$, $C = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le 0,\ i = 1, \ldots, k\}$ *satisfies* $C^* = \mathrm{pos}\{u_1, \ldots, u_k\}$.

Proof. We have $\mathrm{pos}\{u_1, \ldots, u_k\} \subset C^*$ by the definition of a dual cone. If $z \notin \mathrm{pos}\{u_1, \ldots, u_k\}$, then Lemma 1.2.5 yields the existence of a $v \in \mathbb{R}^n$ such that $\langle v, z \rangle > 0$ and $\langle v, u_i \rangle \le 0$ for $i = 1, \ldots, k$. It follows that $v \in C$, and hence $z \notin C^*$. ■

It follows from the definition of the normal cone and the dual cone that for any compact convex set $K \subset \mathbb{R}^n$ and $z \in \partial K$,

$$N_K(z) = C^* \quad \text{for the closed convex cone } C = \mathrm{cl}\,\mathrm{pos}(K - z). \tag{1.6}$$

We deduce (i) of Lemma 1.4.10 below from (1.6), Proposition 1.4.3 and Lemma 1.4.9, (i) yields (ii), (ii) yields (1.7) and in turn (1.7) yields (1.8) where possibly $\dim P < n$ in (ii) and (iii).

Lemma 1.4.10 (Normal cone of at a face). *Let* $P \subset \mathbb{R}^n$ *be a polytope.*

(i) If $\dim P = n$, $F_1, \ldots, F_k$ *are the facets of* P *and* u_i *is the exterior unit normal to* F_i, *then for any face* F *of* P *and* $z \in \mathrm{relint}\, F$, *we have*

$$N_P(z) = \mathrm{pos}\{u_i : F \subset F_i\}.$$

(ii) If F *is a face of* P, *or* P *is lower dimensional and* $F = P$, *then we can set* $N_P(F) = N_P(z)$ *for any* $z \in \mathrm{relint}\, F$, *and* $y \in (\mathrm{relint}\, F) + N_P(F)$ *if and only if* $\Pi_P(y) \in \mathrm{relint}\, F$.

(iii) If $\varrho > 0$, *then using* $\sqcup$ *for disjoint union, we have*

$$P + \varrho B^n = \mathrm{relint}\, P \sqcup \bigsqcup_{F \text{ face}} ((N_F \cap \varrho B^n) + \mathrm{relint}\, F)\,; \tag{1.7}$$

$$|(N_F \cap \varrho B^n) + \mathrm{relint}\, F| = \varrho^{n-d} \omega_{n-d} \beta(N_F) \mathcal{H}^d(F) \quad \text{if } d = \dim F. \tag{1.8}$$

1.5 Regular boundary points of closed convex sets and Convex functions

At a boundary point of a closed convex set $X \subset \mathbb{R}^n$, there might be various supporting hyperplanes; or in other words, various exterior unit normals (think about the vertices of a polytope). However, as we will soon see, there is a unique supporting hyperplane at a typical boundary point provided $\mathrm{int}\, X \neq \emptyset$ (like at the points in the relative interior of a facet of a polytope).

Definition 1.5.1 (Regular boundary points). For a closed convex set $X \subset \mathbb{R}^n$ with $\mathrm{int}\, X \neq \emptyset$, we say that $x \in \partial X$ is a regular boundary point if there exists a unique supporting hyperplane at x; or equivalently, there exists a unique exterior unit normal $\nu_X(x) \in N_X(x) \cap S^{n-1}$. Let $\partial' X \subset \partial X$ be the set of regular boundary points.

Actually $\mathcal{H}^{n-1}$ a.e. boundary points are regular.

Theorem 1.5.2. *For a closed convex set $X \subset \mathbb{R}^n$ with* $\operatorname{int} X \neq \emptyset$, *$\partial X \backslash \partial' X$ is σ-compact with σ-finite $\mathcal{H}^{n-2}$-measure, thus $\partial' X$ is Borel and $\mathcal{H}^{n-1}(\partial X \backslash \partial' X) = 0$.*

Proof. We may assume that X is bounded (compact) and $o \in X$.

Let $u_1, \ldots, u_n$ be an orthonormal basis of $\mathbb{R}^n$, and let $u_{i+n} = -u_i$ for $i = 1, \ldots, n$. For an integer $a > \operatorname{diam} X + 2$, let $W = [-a, a]^n$, and let F_i be the facet of W with exterior unit normal u_i, $i = 1, \ldots, 2n$. For $k \geq 2$, let $\Gamma_k \subset \partial W$ be the subset such that

$$\Gamma_k \cap F_i = \left\{x \in F_i : \exists\, u_j \in u_i^\perp \text{ such that } \langle x, u_j \rangle \in \tfrac{1}{kn}\mathbb{Z}\right\} \quad \text{for } i = 1, \ldots, 2n.$$

It follows that $\mathcal{H}^{n-2}(\Gamma_k) < \infty$, relbd $F_i \subset \Gamma_k$ for $i = 1, \ldots, 2n$ and if $s \subset F_i$ for a segment of length at least $\frac{1}{k}$, then $s \subset \Gamma_k \neq \emptyset$.

Let $X_k \subset \partial X$ be the set of points of $x \in \partial X$ such that there exist $u, v \in S^{n-1} \cap N_X(x)$ with $\angle(u, v) \geq \frac{1}{k}$. Then X_k is compact (cf. Lemma 1.2.8), and as any point of ∂W is of distance at least 2 from X, the properties of Γ_k yield that $N_X(x) \cap \Gamma_k$ for $x \in X_k$. In particulr, $X_k \subset \Pi_X(\Gamma_k)$, and hence $\mathcal{H}^{n-2}(X_k) < \infty$ as Π_X is a contraction by Lemma 1.2.11. We conclude Theorem 1.5.2 from $\partial X \backslash \partial' X = \cup_k X_k$. ■

Lemma 1.2.8 yields that ν_X is continuous on regular boundary points of X.

Lemma 1.5.3. *If $X \subset \mathbb{R}^n$ is a closed convex set with* $\operatorname{int} X \neq \emptyset$, *then $\nu_X : \partial' X \to S^{n-1}$ is continuous.*

Definition 1.5.4 (Subdifferential of a convex function). If $\varphi : \Omega \to \mathbb{R}$ convex for a convex and open $\Omega \subset \mathbb{R}^d$ and $x \in \Omega$, then

$$\partial\varphi(x) = \{z \in \mathbb{R}^d : \varphi(y) \geq \varphi(x) + \langle z, y - x\rangle \text{ for any } y \in \Omega\}.$$

Remarks.

- $\partial\varphi(x) \neq \emptyset$ and is compact for $x \in \Omega$.
- For the convex epi-graph $X = \{(x, t) \in \mathbb{R}^{d+1} : x \in \Omega \text{ and } t \geq \varphi(x)\}$, of φ, $x \in \Omega$ and $y = (x, \varphi(x)) \in \partial X$, we have $z \in \partial\varphi(x)$ if and only if $(z, -1) \in N_X(y) \subset \mathbb{R}^{d+1}$.
- $\partial\varphi(x)$ is one point if and only if $D\varphi(x)$ exists and $\partial\varphi(x) = \{D\varphi(x)\}$.

Theorem 1.5.2 yields a stronger form of the Rademacher theorem about differentiability of convex functions.

Corollary 1.5.5. *For a convex $\varphi : \Omega \to \mathbb{R}$ on an open convex $\Omega \subset \mathbb{R}^d$, φ is differentiable at every $x \in \Omega$ but at a σ-compact subset of Ω with σ-finite $\mathcal{H}^{d-1}$-measure.*

Proof. Write Ω as the countably union of the sets $z_j + r_j B^n \subset \Omega$, $j \in \mathbb{N}$, where $z_j \in \Omega$ and $r_j > 0$, and apply Theorem 1.5.2 to the sets $X_j = \{(x, t) \in \mathbb{R}^{d+1} : x \in z_j + r_j B^n \text{ and } t \geq \varphi(x)\}$. ■

The Remarks after Definition 1.5.4 of the subdifferential yield the following correspondence between regular boundary points and differentiability of convex functions.

Lemma 1.5.6. *For closed convex $X \subset \mathbb{R}^n$ with* $\mathrm{int}X \neq \emptyset$ *and* $X \neq \mathbb{R}^n$, *∂X is locally Lipschitz, and if a relatively open $U \subset \partial X$ is the graph of a convex function φ on $\Omega \subset v^\perp$ for $v \in S^{n-1}$, then $y = x + \varphi(x))v \in U$ is a regular point if and only if $D\varphi(x)$ exists.*

Lemma 1.2.8 yields that an everywhere differentiability convex function is C^1.

Proposition 1.5.7 (Differentiable and convex is C^1).

(i) If $\varphi : \Omega \to \mathbb{R}$ convex and differentiable for open convex $\Omega \subset \mathbb{R}^d$, then φ is C^1.

(ii) If every $x \in \partial X$ is regular for a closed convex $X \subset \mathbb{R}^n$ with $\mathrm{int}X \neq \emptyset$, *then ∂X is C^1 a manifold.*

1.6 Support function of a compact convex set

As we are going to see throughout this monograph, the support function encodes a compact convex set in a rather natural manner, and builds an very signicant bridge between Geometry and Analysis.

Definition 1.6.1 (Support function, Faces). Let $K \subset \mathbb{R}^n$ be compact convex and $u \in \mathbb{R}^n$.

- $h_K(u) = \max\{\langle x, u\rangle : x \in K\}$ is the "support function".
- For $u \neq o$, $F_K(u) = \{x \in K : \langle x, u\rangle = h_K(u)\}$ is the face with exterior normal u.

The following properties directly follow from the definition and the fact that any point not contained in a compact convex set is cut off by a supporting hyperplane (cf. Lemma 1.2.2).

Lemma 1.6.2. *Let $K, C \subset \mathbb{R}^n$ be compact convex and $u, v \in \mathbb{R}^n$.*

- *h_K is convex and one-homogeneous; namely, $h_K(\gamma u) = \gamma h_K(u)$ for $\gamma \geq 0$, and for any $\alpha, \beta \geq 0$, we have*

$$h_K(\alpha u + \beta v) \leq \alpha h_K(u) + \beta h_K(v).$$

- *For $x \in \partial K$, $u \in N_K(x)$ if and only if $h_K(u) = \langle u, x\rangle$.*
- *$K = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)\ \forall u \in \mathbb{R}^n\} = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)\ \forall u \in S^{n-1}\}$.*
- *$h_{\Phi K}(u) = h_K(\Phi^t u)$ for $\Phi \in \mathrm{GL}(n, \mathbb{R})$.*
- *$h_{\alpha K + \beta C} = \alpha\, h_K + \beta\, h_C$ and $F_{\alpha K + \beta C}(u) = \alpha\, F_K(u) + \beta\, F_C(u)$ for $\alpha, \beta \geq 0$.*
- *$h_K(u) = h_{K|L}(u)$ for a proper linear subspace $L \subset \mathbb{R}^n$ and $u \in L$.*

Let us consider some fundamental examples of a the support function.

Example 1.6.3. In these example, we always have $u \in \mathbb{R}^n$.

- *Polytopes:* $h_P(u) = \max\{\langle u, v_i\rangle : i = 1, \dots, m\}$ for a polytope $P = \mathrm{conv}\{v_1, \dots, v_m\}$; for example, if $v_1, \dots, v_m$ are the vertices of P.
- *Euclidean ball:* $h_{rB^n}(u) = r \cdot \|u\|$ for $r > 0$.
- *Unit ball of l_p norm:* Let $1 \leq p \leq \infty$.
  $B_p^n = \{x \in \mathbb{R}^n : \|x\|_p \leq 1\} = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_1|^p + \dots + |x_n|^p \leq 1\}$ if $p \in [1, \infty)$, and $B_\infty^n = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\} = [-1, 1]^n$.
  For $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (e.g. $q = \infty$ if $p = 1$), we claim that
  $$h_{B_p^n}(u) = \|u\|_q. \tag{1.9}$$
  On the one hand, Hölder inequality yields that $\langle x, u\rangle \leq \|x\|_p\|u\|_q \leq \|u\|_q$ for $x \in B_p^n$, and hence $h_{B_p^n}(u) \leq \|u\|_q$. On the other hand, there exists $x \in \partial B_p^n$ with $\langle x, u\rangle = \|x\|_p\|u\|_q = \|u\|_q$, which in turn yields (1.9).
- *Zonotopes, Zonoids:* Zonotopes are Minkowski sums of finitely many segments, and zonoids are limits of zonotopes (limits are discussed in the next Section 1.7). Zonoids are always centrally symmetric, and an o-symmetric convex body $K \subset \mathbb{R}^n$ is a zonoid if and only if there exists a finite even Borel measure μ on S^{n-1} not concentrated on any great subsphere such that
  $$h_K(x) = \int_{S^{n-1}} |\langle x, v\rangle|\, d\mu(v).$$
  Here K is a zonotope if and only if μ is discrete.
  Any planar centrally symmetric convex body is a zonoid, but this property fails in higher dimensions. A typical example for zonoids is the projection body, see Section 2.B. For "classical" properties of zonoids, see Schneider, Weil [525].

Extreme points form a small subset of a convex compact set K such that for any $u \in \mathbb{R}^n$, the maximum of the linear function $x \mapsto \langle u, x\rangle$ over $x \in K$; namely, the value of the support function h_K at u, is realized at an extreme point.

Definition 1.6.4 (Extreme point). If $K \subset \mathbb{R}^n$ is convex, compact, then an $x \in K$ is called an extreme point if $x = (1 - \lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0, 1)$ implies $y = z = x$ (and hence $x \in \partial K$). The set of extreme points is $\mathrm{ext}\, K$.

Lemma 1.6.5. *Let $K \subset \mathbb{R}^n$ be convex, compact.*

(i) If $F \subset K$ is a face, then $\mathrm{ext}\, F \subset \mathrm{ext}\, K$.

(ii) The set $\mathrm{ext}\, K$ *is non-empty, and is smallest set such that* $K = \mathrm{conv}\,\mathrm{ext}\, K$.

Proof. For (i), there exists a halfspace $H^+ \subset \mathbb{R}^n$ such that $K \subset H^+$ and $F = \partial H^+ \cap K$. If $x \in \mathrm{ext}\, F$ and $x = (1 - \lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0, 1)$, then $y, z \in \partial H^+$, and hence $y = z = x$.

For (ii), if $K = \operatorname{conv} X$, then $X \subset \operatorname{ext} K$, as $\operatorname{conv} X$ is the family of all convex linear combinations of the points of X.

Finally we show that $K = \operatorname{conv} \operatorname{ext} K$ by induction on $\dim K$ where the statement readily holds if $\dim K = 0$. If $\dim K \geq 1$ and x is a point of the relative boundary of K, then $x \in F$ for a face $F \subset K$; therefore, $x \in \operatorname{conv} \operatorname{ext} F \subset \operatorname{conv} \operatorname{ext} K$ by induction and (i).

■

Remark. $K = \operatorname{conv} \operatorname{ext} K$ can be proved applying induction to the faces of K, and for $Z \subset K$, $K = \operatorname{conv} Z$ if and only if $\operatorname{ext} K \subset Z$. In particular, for any $u \in \mathbb{R}^n$, $h_K(u) = \langle u, x \rangle$ for an $x \in \operatorname{ext} K$.

Next we show how the support function encodes the facial structure of a compact convex set.

Lemma 1.6.6 (Faces are subdifferentials of the support function). *If $K \subset \mathbb{R}^n$ is compact convex and $u \in \mathbb{R}^n \backslash \{o\}$, then $\partial h_K(u) = F_K(u)$.*

Proof. If $z \in F_K(u)$, then $\langle z, u \rangle = h_K(u)$, and hence $h_K(v) - h_K(u) \geq \langle z, v \rangle - \langle z, u \rangle = \langle z, v - u \rangle$ for any $v \in \mathbb{R}^n$. Therefore, $z \in \partial h_K(u)$.

On the other hand, if $z \in \partial h_K(u)$, then for any $t \in (0, 1)$, we have

- $t \cdot h_K(u) = h_K((1 + t)u) - h_K(u) \geq \langle z, (1 + t)u - u \rangle = t\langle z, u \rangle$ and
- $-t \cdot h_K(u) = h_K((1 - t)u) - h_K(u) \geq \langle z, (1 - t)u - u \rangle = -t\langle z, u \rangle$,

thus $\langle z, u \rangle = h_K(u)$. In addition, for any $v \in \mathbb{R}^n$, $z \in \partial h_K(u)$ yields that

$$h_K(v) = h_K(u) + (h_K(v) - h_K(u)) \geq \langle z, u \rangle + \langle z, v - u \rangle = \langle z, v \rangle,$$

and hence $z \in F_K(u)$. ■

h_K is typically not diferentiable at o for a compact convex set K (unless K is a point), and next we characterize when h_K is differentiable on $\mathbb{R}^n \backslash \{o\}$.

Lemma 1.6.7. *Let $K \subset \mathbb{R}^n$ be compact convex. Then h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ if and only if* $\operatorname{int} K \neq \emptyset$ *and ∂K contains no segment, that is in turn equivalent to saying that $\|u + v\|_{K-z} > \|u\|_{K-z} + \|v\|_{K-z}$ for any $z \in$* $\operatorname{int} K$ *and independent $u, v \in \mathbb{R}^n \backslash \{o\}$.*

Remark. Such convex bodies are called *strictly convex*. In this case, $Dh_K(u) = x$ if $x \in \partial K$ and $u \neq o$ is an exterior normal at x by Lemma 1.6.6.

Proof. As h_K is convex, h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ if and ly if $\partial h_K(u) = F_K(u)$ is a unique point for any $u \in \mathbb{R}^n \backslash \{o\}$ (cf. Lemma 1.6.6), that is equivalent to saying that ∂K contains no segment. This in turn yields that $\| \cdot \|_{K-z}$ is "strictly convex" for $z \in \operatorname{int} K$. ■

Next we prove that support functions are characterized by the properties convex and one-homogeneous.

Proposition 1.6.8. *Let $h : \mathbb{R}^n \to \mathbb{R}$. There exists compact convex set $K \subset \mathbb{R}^n$ satisfying $h = h_K$ if and only if h is convex and $h(\gamma u) = \gamma h(u)$ holds for $u \in \mathbb{R}^n$ and $\gamma \geq 0$.*

In addition, the compact convex set $K \subset \mathbb{R}^n$ satisfying $h = h_K$ is unique.

Proof. If $h = h_K$ for a compact convex set $K \subset \mathbb{R}^n$, then h_K is convex and one-homogeneous by Lemma 1.6.2, which also yields the uniqueness of K.

Therefore let h be convex and one-homogeneous. If $z \in \partial h(u)$ for $u \in \mathbb{R}^n \backslash o$, then any $v \in \mathbb{R}^n$ satisfies

$$h(v) - h(u) \geq \langle z, v - u \rangle. \tag{1.10}$$

Applying this with $v = 2u$, we deduce $h(u) = h(2u) - h(u) \geq \langle z, u \rangle$. On the other hand, using $v = \frac{1}{2}u$, it follows that $\frac{-1}{2}h(u) = h(\frac{1}{2}u) - h(u) \geq \langle z, \frac{-1}{2}u \rangle$, and hence $h(u) \leq \langle z, u \rangle$. In particular, if $z \in \partial h(u)$ for $u \in \mathbb{R}^n \backslash o$, then

$$\langle z, u \rangle = h(u) \text{ and } h(v) \geq \langle z, v \rangle \text{ for } v \in \mathbb{R}^n \tag{1.11}$$

We deduce that $z \in K = \{x \in \mathbb{R}^n : \langle x, w \rangle \leq h(w)\ \forall w \in \mathbb{R}^n\}$, and hence $\partial h(u) \subset K$ for any $u \in \mathbb{R}^n$.

Therefore, K is non-empty, and its definition shows that $h_K(u) \leq h(u)$ for any $u \in \mathbb{R}^n$. In addition, if $u \in \mathbb{R}^n$ and $z \in \partial h(u) \subset K$, then $h_K(u) \geq \langle z, u \rangle = h(u)$, proving $h_K(u) = h(u)$ for any $u \in \mathbb{R}^n$. ∎

Next we describe the normal cones of the Minkowski sums of polytopes (see Proposition 1.4.3 for face structure of a polytope).

Lemma 1.6.9 (Normal cones of at a Minkowski sum). *If $P, Q \subset \mathbb{R}^n$ are polytopes, then the family of normal cones of $P + Q$ is the family of all $N_P(F) \cap N_Q(G)$ with $N_P(F) \cap N_Q(G) \neq \{o\}$ where F (G) is a face of P (Q), or P is lower dimensional and $F = P$ (or Q is lower dimensional and $G = Q$).*

Proof. If F (G) is a face of P (Q), or P is lower dimensional and $F = P$ (or Q is lower dimensional and $G = Q$), and $N_P(F) \cap N_Q(G) \neq \{o\}$, then $F = F_P(u)$ and $G = F_Q(u)$ for some $u \in N_P(F) \cap N_Q(G) \cap S^{n-1}$, and hence $F + G = F_{P+Q}(u)$ is a face of $P + Q$ with $N_{P+Q}(F + G) = N_P(F) \cap N_Q(G)$.

On the other hand, for any face M of $P + Q$, we have $M = F_{P+Q}(u)$ for a $u \in S^{n-1}$; therefore, $M = F + G$ and $N_{P+Q}M = N_P(F) \cap N_Q(G)$ for $F = F_P(u)$ and $G = F_Q(u)$. ∎

1.7 Hausdorff distance of compact convex sets

In this section, we equip the space of compact convex subsets of $\mathbb{R}^n$ with a natural topology. The easiest way to do this is to provide a very useful metric.

Definition 1.7.1 (Hausdorff distance). The Hausdorff distance between non-empty compact convex sets $K, C \subset \mathbb{R}^n$ is

$$\delta_H(K, C) = \min\{\varrho \geq 0 : K \subset C + \varrho B^n \text{ and } C \subset K + \varrho B^n\}.$$

$\delta_H(K, C)$ is a metric on the space of compact convex subsets of $\mathbb{R}^n$ (see Section 3.7 for the version on the space of compact subsets). We always consider the space of compact convex sets with the topology induced by δ_H (where we always assume that the compact convex sets are non-empty), and write $K_m \to K$ or $\lim_{m\to\infty} K_m = K$ to signal that $\lim_{m\to\infty} \delta_H(K_m, K) = 0$.

Remark. Abusing the notation, even if h_K, h_C are functions on $\mathbb{R}^n$ for convex compact sets $K, C \subset \mathbb{R}^n$, we write

$$\|h_K - h_C\|_\infty = \max\left\{|h_K(u) - h_C(u)| : u \in S^{n-1}\right\} = \delta_H(K, C). \tag{1.12}$$

Lemma 1.7.2. *If $K \subset R\, B^n$ compact convex for $R > 0$, then*

$$|h_K(u) - h_K(v)| \leq R\|u - v\| \quad \text{for } u, v \in \mathbb{R}^n.$$

Proof. If $y, z \in K$ satisfy $\langle y, u\rangle = h_K(u)$ and $\langle z, v\rangle = h_K(v)$, then $\langle z, v\rangle \geq \langle y, v\rangle$, and

$$h_K(u) - h_K(v) = \langle y, u\rangle - \langle z, v\rangle = \langle y, u - v\rangle + \langle y - z, v\rangle \leq \langle y, u - v\rangle \leq R\|u - v\|.$$

Similar argument shows that $h_K(v) - h_K(u) \leq R\|v - u\|$. ■

Theorem 1.7.3 (Blaschke Selection Theorem). *Any bounded sequence $\{K_m\}$ of compact convex sets in $\mathbb{R}^n$ has a subsequence that tends to a compact convex set.*

Proof. Assume that each $K_m \subset RB^n$ for $R > 0$, set $\tilde{h}_{K_m} = h_{K_m}|_{S^{n-1}}$, and choose a countable dense set $X \subset S^{n-1}$. As $\tilde{h}_{K_m} \leq R$, there exists subsequence $\{\tilde{h}_{K_{m'}}\}$ such that $\tilde{h}_{K_{m'}}(x)$ is convergent for any $x \in X$. It follows from Lemma 1.7.2 that $\{\tilde{h}_{K_{m'}}\}$ is a Cauchy sequence with respect to $\|\cdot\|_\infty$, and hence $\tilde{h}_{K_{m'}}$ tends to a continuous $\tilde{h}$ on S^{n-1} uniformly.

We define $h(tu) = t\tilde{h}(u)$ for $u \in S^{n-1}$ and $t \geq 0$. As $h_{K_{m'}}$ tends pointwise to h on $\mathbb{R}^n$, we deduce that h is convex and one-homogeneous; therefore, $h = h_C$ for a compact convex set C. In turn, $K_{m'}$ tends to C by (1.12). ■

Lemma 1.7.4. *The volume (Lebesgue measure) is continuous as a function of compact convex subsets on $\mathbb{R}^n$.*

Remark. As we will see in Section 3.7, Hausdorff distance can be naturally extended to compact subets of $\mathbb{R}^n$; however, the Lebesgue measure is readily not continuous as a function of compact subsets.

Proof. Let $K_m \to K$ for compact convex sets $K, K_m \subset \mathbb{R}^n$. We may assume that $o \in \operatorname{relint} K$. If $\dim K = n$, then $rB^n \subset K$ for some $r > 0$, and hence $\varrho B^n \subset \frac{\varrho}{r} K$ for any $\varrho > 0$. If $\delta_H(K_m, K) < \varrho$ for $\varrho \in (0, r)$, then $(1 - \frac{\varrho}{r}) K \subset K_m \subset (1 + \frac{\varrho}{r}) K$; therefore, $|K_m| \to |K|$.

On the other hand, if $\dim K < n$, then $K \subset u^\perp$ for some $u \in S^{n-1}$. For an $R > 0$ such that $K \subset RB^n$, if $\delta_H(K_m, K) < \varrho$ for $\varrho \in (0, 1)$, then $K_m \subset (R+1)(B^n \cap u^\perp) + [-\varrho, \varrho]u$, and hence $|K_m| \leq (R+1)^{n-1}\omega_{n-1} \cdot 2\varrho$. In particular, $|K_m| \to 0 = |K|$. ■

Any compact convex set $K \subset \mathbb{R}^n$ can be arbitrarily well approximated by polytopes (see (1.13) and (1.14) below and Section 1.13 for more precise estimates), and by convex bodies of C^∞_+ boundary (see Theorem 8.1.10) in terms of the Hausdorff distance. ε-nets (cf. Section 0.1) are simple tools to handle polytopal approximation; namely, if $\varepsilon > 0$, then

$$\delta_H(K, P) \leq \varepsilon \text{ for any } \varepsilon\text{-net } \Xi \subset K \text{ and } P = \operatorname{conv} \Xi. \tag{1.13}$$

Sometimes we need approximation by full dimensional polytopes containing the compact convex set $K \subset \mathbb{R}^n$. If $\varepsilon > 0$, P is a polytope with $\delta_H(K, P) \leq \frac{\varepsilon}{4n}$, then

$$\delta_H(K, Q) < \varepsilon \text{ and } P \subset \operatorname{int} Q \text{ for the polytope } Q = P + \left[-\tfrac{\varepsilon}{2n}, \tfrac{\varepsilon}{2n}\right]^n \tag{1.14}$$

where both the P in (1.13) and the Q in (1.14) can be assumed o-symmetric if K is o-symmetric.

1.8 Simple polytopes and Strongly Isomorphic polytopes

The main goal of this section is to introduce the families of strongly isomorphic simple polytopes. We will need the following special case of Lemma 1.6.9:

Lemma 1.8.1. *If $P, Q \subset \mathbb{R}^n$ are n-polytopes, then the family of normal cones of $P + Q$ is the family of all $N_P(F) \cap N_Q(G)$ with $N_P(F) \cap N_Q(G) \neq \{o\}$ where F (G) is a face of P (Q).*

According to Proposition 1.4.3, it $P \subset \mathbb{R}^n$ is an n-dimensional, then any vertex v is the interesection the facets containing v; therefore, v is contained in at least n facets. It follows by induction on the dimension n that v is also contained in at least n edges. It also follows that exactly n facets meet at v if and only if exactly n edges meet at v, and in this case, any $(n-1)$ of these edges are edges of one of the facets containing v, as well.

Definition 1.8.2 (Simple polytopes). We say that a d-dimensional polytope $P \subset \mathbb{R}^n$ is simple, $1 \le d \le n$, if exactly d edges meet at any vertex of P.

Remark. The considerations above and Lemma 1.4.10 show the following properties:

- Any at least one-dimensional face of a simple polytope is simple.
- For an n-dimensional polytope $P \subset \mathbb{R}^n$, P is simple if and only if the normal cone at any vertex is simplicial; namely, it is the positive hull of n independent vectors.

We note that Lemma 1.4.10 yields the property that if $u_1, \ldots, u_k$ are the facet exterior unit normas of an n-polytope Q with corresponding facets $F_1, \ldots, F_k$, and F is a face of Q, then

$$N_Q F = \mathrm{pos}\{u_i : F \subset F_i\}. \tag{1.15}$$

Next, we show that any n-polytope can be approximated by simple polytopes with the same the facet unit exterior normals:

Lemma 1.8.3. *If $P \subset \mathbb{R}^n$ is an n-polytope and $\varepsilon > 0$, then there exists a simple n-polytope P' such that $\delta_H(P, P') < \varepsilon$ for the Hausdorff distance, the families of unit exterior normals to the facets of P and P' coincide, and the normal cone at any vertex of P' is contained in the normal cone of some vertex of P.*

Proof. Let $u_1, \ldots, u_k$ be the facet exterior unit normas of P, and for $\tau = (\tau_1, \ldots, \tau_k) \in \mathbb{R}^k$, let $P_\tau = \{x \in \mathbb{R}^n : \langle x, u_i\rangle \le \tau_i \;\; i = 1, \ldots, k\}$. Thus there exists a $\varepsilon_0 > 0$ such that if $\delta_H(P, P_\tau) < \varepsilon_0$, then P_τ is an n-polytope and $u_1, \ldots, u_k$ are the exterior normals of the facets of P_τ.

If $\delta_H(P, P_\tau) < \varepsilon_0$ and P is not simple, then there exist a subset $I \subset \{1, \ldots, k\}$ of cardinality $n + 1$ and a vertex v of P_τ such that $\langle v, u_i\rangle = \tau_i$ for $i \in I$; therefore, the determinant of of the $(n + 1) \times (n + 1)$ matrix made up from the vectors (u_i, τ_i), $i \in I$, is 0. This shows that except for the $\tau \in \mathbb{R}^k$ lying in the union of certain $\binom{k}{n+1}$ proper linear subspaces of $\mathbb{R}^k$, any $\tau \in \mathbb{R}^k$ such that $\delta_H(P, P_\tau) < \varepsilon_0$ gives us a simple polytope P_τ.

Finally, indirectly, we suppose that there exist $\tau^{(m)} = (\tau_1^{(m)}, \ldots, \tau_k^{(m)}) \in \mathbb{R}^k$ such that $\lim_{m\to\infty} \tau_i^{(m)} = h_P(u_i)$ for $i = 1, \ldots, k$, and each $P_{\tau^{(m)}}$ has a vertex $v^{(m)}$ where the normal cone is not contained in the in the normal cone of any vertex of P. As $P_{\tau^{(m)}}$ tends to P, we may assume that $v^{(m)}$ tends to an $x_0 \in \partial P$, and hence $x_0 \in \mathrm{relint}\, F$ for a face F of P. According to (1.15), $N_P(x_0) = N_F$ is the positive hull of a subset of $\mathcal{U} \subset \{u_1, \ldots, u_k\}$. It also follows from (1.15) (with $Q = P_{\tau^{(m)}}$) and $N_{P_{\tau^{(m)}}}(v^{(m)}) \not\subset N_P F$ that we may assume that there exists a $u_j \notin \mathcal{U}$ such that $u_j \notin N_F$ but $u_j \in N_{P_{\tau^{(m)}}}(v^{(m)})$ for each m. However, the last property yields $u_j \in N_P(x_0) = N_F$, which is a contradiction, completing the proof of Lemma 1.8.3. ■

Strongly isomorphic have similar face structures, which is very useful in certain calculations (see for example the proof the Aleksandrov Fenchel inequality in Section 7.A).

Definition 1.8.4 (Strongly Isomorphic Polytopes). The polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ are strongly isomorphic if for any $1 \le i < j \le n$, P_i and P_j have exactly the same set of normal cones; or equivalently, $\dim F_{P_i}(u) = \dim F_{P_j}(u)$ for any $u \in S^{n-1}$.

It follows that if the polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ are strongly isomorphic, then they have the same dimension, if the common dimension is less than n, then their affine hulls are parallel.

Proposition 1.8.5. *For compact convex sets $C_1, \ldots, C_k \subset \mathbb{R}^n$ and $\varepsilon > 0$, there exist strongly isomorphic simple n-polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ such that $\delta_H(C_i, P_i) < \varepsilon$ for $i = 1, \ldots, k$.*

Proof. Choose $R > 0$ such that $C_1, \ldots, C_k \subset \operatorname{int} RB^n$. According to (1.14), there exist n polytopes $Q_1, \ldots, Q_k \subset RB^n$ such that $\delta_H(C_i, Q_i) < \varepsilon/3$ for $i = 1, \ldots, k$. It follows from Lemma 1.8.1 that $Q, \widetilde{Q}_1, \ldots, \widetilde{Q}_k$ are strongly isomorphic for $Q = Q_1 + \ldots + Q_k \subset \operatorname{int}(kRB^n)$ and

$$\widetilde{Q}_i = Q_i + \frac{\varepsilon}{3kR} \sum_{j \ne i} Q_j,$$

and hence $\delta_H(Q_i, \widetilde{Q}_i) < \varepsilon/3$ and $\delta_H(C_i, \widetilde{Q}_i) < 2\varepsilon/3$ for $i = 1, \ldots, k$.

Finally, Lemma 1.8.3 provides a a simple n-polytope $Q' \subset kRB^n$ such that the families of unit exterior normals to the facets of Q and Q' coincide, and the normal cone at any vertex of Q' is contained in the normal cone of some vertex of Q. Readily,

$$P_i = \widetilde{Q}_i + \frac{\varepsilon}{3kR} \cdot Q'$$

satisfies that $\delta_H(C_i, P_i) < \varepsilon$ for $i = 1, \ldots, k$. As Q is strongly isomorphic to $\widetilde{Q}_i$, $i = 1, \ldots, k$, we deduce from that the normal cone at any vertex of Q' is contained in the normal cone of some vertex of $\widetilde{Q}_i$, and hence Lemma 1.6.9 for the Minkowski sum of polytopes yields that P_i is strongly isomorphic to the simple polytope Q'. ■

Simple polytopes have the advantage that local deformations keeping the exterior unit normals of the facets do not change the face structure (see Lemma 1.8.6). For independent $u_1, \ldots, u_n \in \mathbb{R}^n$, let $u_1^*, \ldots, u_n ** \in \mathbb{R}^n$ be the dual basis; namely, $\langle u_i^*, u_j \rangle = 0$ if $i \ne j$, and $\langle u_i^*, u_i \rangle = 1$, and let $R = \max_{i=1}^n \|u_i^*\}$. It follows that if $p, q \in \mathbb{R}^n$ satisfy that $\langle p, u_i \rangle = t_i$ and $\langle q, u_i \rangle = s_i$ for $i = 1, \ldots, n$ (and hence $p = \sum_{i=1}^n t_i u_i^*$ and $q = \sum_{i=1}^n s_i u_i^*$), then

$$\|p - q\| \le nR \max_{i=,\ldots,n} |t_i - s_i|. \tag{1.16}$$

Since any vertex of a simple polytope is contained in n facets, (1.16) yields the folowing:

Lemma 1.8.6. *For a simple n-polytope $P \subset \mathbb{R}^n$, there exists $\varepsilon_0 > 0$, such that if $u_1, \ldots, u_n$ are the unit exterior normals of the facets of P, and $|t_i - h_P(u_i)| < \varepsilon_0$ for $i = 1, \ldots, k$, then $P' = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq t_i\}$ is a simple n-polytope strongly isomorphic to P.*

1.9 The polar (dual) of a convex body

We call a compact convex set $K \subset \mathbb{R}^n$ with non-empty interior a *convex body*. Whenever $o \in \operatorname{int} K$, then we can consider it as the unit ball of a norm type function.

Definition 1.9.1. For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, if $x \in \mathbb{R}^n$, then let

$$\|x\|_K = \min\{t \geq 0 : x \in t\,K\}.$$

Remarks. $x \mapsto \|x\|_K$ is convex and $\|\lambda\,x\|_K = \lambda\|x\|_K$ for $\lambda \geq 0$.

$\|\cdot\|_K$ is an actual norm if and only if $K = -K$ (K is origin symmetric).

Definition 1.9.2 (Polar (dual) of a convex body). If $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then its polar is

$$K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \ \forall\, y \in K\},$$

which is a convex body with $o \in \operatorname{int} K^*$, and is o-symmetric if K is o-symmetric.

In particular, if K is o-symmetric, then $\|\cdot\|_{K^*}$ is the norm of the space dual to the one with norm $\|\cdot\|_K$. In general, the following statement summarizes the fundamental properties of polarity of convex bodies.

Proposition 1.9.3. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$.*

(i) $z \in \partial K$ and $y \in \mathbb{R}^n$ is an exterior normal at z to K with $\langle y, z \rangle = 1$ if and only if $y \in \partial K^*$ and z is an exterior normal at y to K^*;

(ii) $(K^*)^* = K$;

(iii) $h_{K^*}(u) = \|u\|_K$ and $h_K(u) = \|u\|_{K^*}$ for $u \in \mathbb{R}^n$;

(iv) $(\Phi\, K)^* = \Phi^{-t} K^*$ for $\Phi \in \mathrm{GL}(n)$.

Remark. $h_K(y) = h_{K^*}(z) = 1$ in (i).

Proof. If $z \in \partial K$ and $y \in \mathbb{R}^n$ is an exterior normal at z to K with $\langle y, z \rangle = 1$ in (i), then $\langle y, x \rangle \leq \langle y, z \rangle = 1$ for $x \in K$, and hence $y \in K^*$, and $\langle w, z \rangle \leq 1 = \langle y, z \rangle$ for $w \in K^*$, which in turn yields that z is exterior normal at $y \in \partial K^*$, proving half of (i).

For (ii), readily $K \subset (K^*)^*$. If $p \notin K$, then $p = \lambda z$ for $\lambda > 1$ and $z \in \partial K$, and hence $\langle p, y \rangle = \lambda > 1$ where y is an exterior normal at z to K with $\langle y, z \rangle = 1$. Since $y \in K^*$, we have $p \notin (K^*)^*$. Having (ii), interchanging the role of K and K^* verifies the other implication in (i).

For (iii), $h_K(y) = 1$ is equivalent to $\|y\|_{K^*} = 1$ by (i).

For (iv), $\langle \Phi^{-t}x, \Phi y\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$. ■

We deduce the following examples for polars from Example 1.6.3 for support functions and Proposition 1.9.3.

Example 1.9.4. *l_p-balls:* $(B_p^n)^* = B_q^n$ if $\frac{1}{p} + \frac{1}{q} = 1$ for $p, q \in [1, \infty]$.

Polytopes: If $P = \{x \in \mathbb{R}^n : \langle x, v_i\rangle \leq 1,\ i = 1, \dots, k\}$ is bounded for $v_1, \dots, v_k \in \mathbb{R}^n \backslash \{o\}$, then $o \in \operatorname{int} P$ and $P^* = \operatorname{conv}\{v_1, \dots, v_k\}$.

Definition 1.9.5 (Radial function). For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, if $u \in S^{n-1}$, then $\varrho_K(u) = \max\{t \geq 0 : tu \in K\}$.

Remark. In particular, $\varrho_K(u)\, u \in \partial K$ for $u \in S^{n-1}$.

Proposition 1.9.3 yields that if $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ and $u \in S^{n-1}$, then

$$\varrho_K(u) = \frac{1}{\|u\|_K} = \frac{1}{h_{K^*}(u)}. \tag{1.17}$$

We observe that $u \mapsto \varrho_K(u) \cdot u = (\|u\|_K)^{-1} \cdot u$ parametrizes ∂K for $u \in S^{n-1}$ (see Section 2.2).

Polarity (cf. Proposition 1.9.3), the parametrization of the boundary by the radial function and the characterization of the differentiablity of the support function by Lemma 1.6.7 yields the following:

Lemma 1.9.6. *Let $K \subset \mathbb{R}^n$ be compact convex. Then* $\operatorname{int} K \neq \emptyset$ *and ∂K is a C^1 manifold if and only if $\|\cdot\|_{K-z}$ is C^1 on S^{n-1} for $z \in \operatorname{int} K$, that is equivalent to saying that $h_K(u+v) > h_K(u) + h_K(v)$ for independent $u, v \in \mathbb{R}^n$.*

The next statement follows from Proposition 1.9.3 via Lemma 1.6.7 and Lemma 1.9.6.

Corollary 1.9.7. *Given convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, its boundary ∂K is a C^1 manifold if and only if K^* is strictly convex.*

We note that the polar body satisfies various fundamental inequalities. The probably most widely used is the Blaschke-Santaló inequality and its reverse form (see Sections 6.5 and 6.6) stating that if $K \subset \mathbb{R}^n$ is a centered convex body, then

$$4^{-n}|B^n|^2 < |K| \cdot |K^*| \leq |B^n|^2$$

where equality holds in the upper bound (the Blaschke-Santaló inequality) if and only if K is a centered ellipsoid, and the lower bound actually holds whenever $o \in \operatorname{int} K$. For related inequalities for the so-called projection body, see Section 2.B.

1.10 Steiner and Schwarz symmetrizations of convex bodies in $\mathbb{R}^n$, and the Isodiametric and the Brunn-Minkowski inequalities

In this section, we discuss two classical symmetrization methods, the Steiner symmetrization (the "mother of all symmetrizations") and a variant of it, the Schwarz symmetrizations. The main use of a symmetrization is that when proving an inequality, the symmetrization reduces the problem to more symmetric objects. We demonstrate this method by proving the Isodiametric and the Brunn-Minkowski inequalities *via* Steiner symmetrization. We recall that $X|L$ is the orthogonal projection of an $X \subset \mathbb{R}^n$ into a proper linear subspace $L \subset \mathbb{R}^n$.

Definition 1.10.1 (Steiner symmetrization). Let $K \subset \mathbb{R}^n$ convex body, and let $u \in S^{n-1}$. The followings are equivalent definitions of the Steiner symmetrial $\Theta_{u^\perp}K$.

(a) $\Theta_{u^\perp}K = \bigcup \left\{x + [-q, q]u : x \in K|u^\perp \text{ and } \mathcal{H}^1\Big(K \cap (x + \mathbb{R}u)\Big) = 2q\right\}$.

(b) $\Theta_{u^\perp}K = \left\{x + \frac{t-s}{2}\, u : x \in K|u^\perp \text{ and } x + tu, x + su \in K\right\}$.

(c) If $f, g : K|u^\perp \to \mathbb{R}$ concave such that

$$K = \left\{x + t\,u : x \in K|u^\perp \text{ and } -g(x) \le t \le f(x)\right\},$$

then

$$\Theta_{u^\perp}K = \left\{x + t\,u : x \in K|u^\perp \text{ and } -\frac{f(x) + g(x)}{2} \le t \le \frac{f(x) + g(x)}{2}\right\}.$$

Definition 1.10.2 (Inradius and circumradius). For a convex body $K \subset \mathbb{R}^n$, $r(K)$ is the maximal radius of any ball contained in K, and $R(K)$ is the minimal radius of any ball containing K.

Remark. There exists a unique circumscribed ball of radius $R(K)$ containing K, but there might exist several balls of radius $r(K)$ contained in K. If K is o-symmetric, then $r(K)B^n \subset K \subset R(K)B^n$.

Proposition 1.10.3. *Let $K, C \subset \mathbb{R}^n$ be convex bodies, and let $u \in S^{n-1}$.*

(i) $\Theta_{u^\perp}K$ is symmetric through $u^\perp$, and $\Theta_{u^\perp}K = (w|u^\perp) + rB^n$ if $K = w + rB^n$ for $r > 0$ and $w \in \mathbb{R}^n$;

(ii) $\Theta_{u^\perp}K$ is a convex body;

(iii) $\Theta_{u^\perp}(\lambda K) = \lambda \cdot \Theta_{u^\perp}K$ if $\lambda \in \mathbb{R}\backslash\{0\}$;

(iv) $r(\Theta_{u^\perp}K) \ge r(K)$ and $R(\Theta_{u^\perp}K) \le R(K)$;

(v) $|\Theta_{u^\perp}K| = |K|$;

(vi) $\operatorname{diam}\Theta_{u^\perp}K \le \operatorname{diam}K$*;*

(vii) $|\alpha\Theta_{u^\perp}K + \beta\Theta_{u^\perp}C| \le |\alpha K + \beta C|$ for $\alpha, \beta > 0$.

Proof. (i), (ii) and (iii) follow by definition (b), (iv) by (i) and containment, and (v) by Fubini and definition (a).

For (vi), let $\operatorname{diam}\Theta_{u^\perp}K = \|z_1 - z_2\|$ where $z_i = x_i + \frac{1}{2}(t_i - s_i)$, $x_i \in K|u^\perp$, $y_i = x_i + t_i u \in K$ and $w_i = x_i + s_i u \in K$ for $t_i, s_i \in \mathbb{R}$. Since $z_1 - z_2 = \frac{1}{2}(y_1 - y_2 + w_2 - w_1)$, the triangle inequality yields that either $\|z_1 - z_2\| \le \|y_1 - y_2\|$ or $\|z_1 - z_2\| \le \|y_2 - y_1\|$.

For (vii), it is sufficient to prove that $\alpha\Theta_{u^\perp}K + \beta\Theta_{u^\perp}C \subset \Theta_{u^\perp}(\alpha K + \beta C)$ according to (v), which in turn follows from definition (b). ∎

We say that a convex body $K \subset \mathbb{R}^n$ is unconditional with respect to an orthonormal basis $u_1, \dots, u_n$ of $\mathbb{R}^n$ if it is symmetric through $u_1^\perp, \dots, u_n^\perp$.

Lemma 1.10.4. *If $u_1, \dots, u_n$ form an orthonormal basis of $\mathbb{R}^n$ and $K \subset \mathbb{R}^n$ is a convex body, then $\Theta_{u_n^\perp} \circ \dots \circ \Theta_{u_1^\perp}K$ is unconditional, and hence it is o-symmetric.*

Proof. We observe that if $C \subset \mathbb{R}^n$ convex body and $j = 2, \dots, n$, and C is symmetric through $u_i^\perp$ for $i = 1, \dots, j-1$, then $\Theta_{u_j^\perp}C$ is also symmetric through $u_i^\perp$ for $i = 1, \dots, j-1$. ∎

We now demonstrate how Lemma 1.10.4 leads to the Isodiametric Inequality.

Theorem 1.10.5 (Isodiametric Inequality for convex bodies). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, then* $\operatorname{diam}K \ge 2r$*; or in other works,* $\operatorname{diam}K \ge 2\omega_n^{-1/n}|K|^{1/n}$.

Proof. Let $u_1, \dots, u_n$ form an orthonormal basis of $\mathbb{R}^n$, and hence $K' = \Theta_{u_n^\perp} \circ \dots \circ \Theta_{u_1^\perp}K$ is an o-symmetric convex body by Lemma 1.10.4 with $\operatorname{diam}K' \le \operatorname{diam}K$ by Proposition 1.10.3 (vi). Since $|K'| = |rB^n|$ by Proposition 1.10.3 (v), it follows that $K'\backslash\operatorname{int}(rB^n) \ne \emptyset$, which in turn yields that $\operatorname{diam}K' \ge 2r$. ∎

For a bounded measurable set $X \subset \mathbb{R}^n$, its diameter is $\operatorname{diam} X = \sup\{\|x - y\| : x, y \in X\}$. Since $\operatorname{diam}\operatorname{conv}\operatorname{cl} X = \operatorname{diam} X$ by Proposition 1.1.7, Theorem 1.10.5 directly yields its extension to measurable sets.

Theorem 1.10.6 (Isodiametric Inequality). *If $X \subset \mathbb{R}^n$ is bounded and measurable, then* $\operatorname{diam} X \ge 2\omega_n^{-1/n}|X|^{1/n}$.

The Isodiametric Inequality indicates what the right normalization of the Hausdorff measure is in order to conincide with Lebesgue measure (see Section 1.B).

Typically, Steiner symmetrization is useful to prove inequalities where balls are extremizers because iterated Steiner symmetrizations can lead to a ball.

Theorem 1.10.7 (Iterated Steiner symmetrizations). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, and $\varepsilon \in (0, \frac{1}{2})$, then there exist finitely many Steiner symmetrizations starting with K producing a convex body K' with $(1-\varepsilon)rB^n \subset K' \subset (1+\varepsilon)rB^n$.*

Or equivalently, there exists a sequence $\{K_m\}$ *of convex bodies tending to* rB^n *where* $K_0 = K$ *and* $K_{m+1} = \Theta_{u_m^\perp} K_m$ *for some* $u_m \in S^{n-1}$.

Remark. Klartag [370] proves that $cn^4 |\log \varepsilon|^2$ Steiner symmetrizations are enough for an absolute constant $c > 1$.

Proof. According to Lemma 1.10.4, we may assume that K is o-symmetric and $K \neq rB^n$. Writing $\mathcal{F}_K$ to denote the family of convex bodies resulting from finitely many iterated Steiner symmetrizations starting from K, Proposition 1.10.3 (v) yields that Theorem 1.10.7 is equivalent proving that

$$\Xi = \sup\{|C \cap rB^n| : C \in \mathcal{F}_K\} = |rB^n|. \tag{1.18}$$

The argument is indirect, so we suppose that $\Xi < |rB^n|$, and seek a contradiction. For $k \geq 2$, let $C_k \in \mathcal{F}_K$ such that $|C_k| > \Xi - \frac{1}{k}$. As $C_k \subset R(K)B^n$, we may assume that $C_k \to C_0$ for an origin symmetric convex body C_0 by the Blaschke Selection Theorem 1.7.3 where $|C_0| = |rB^n|$ by the continuity of volume (cf. Lemma 1.7.4). As $|C_0 \cap rB^n| = \Xi < |rB^n|$, there exist balls $z_1 + \varrho B^n \subset \mathrm{int} B^n \backslash C_0$ and $z_2 + \varrho B^n \subset \mathrm{int} C_0 \backslash B^n$ for some $\varrho > 0$. There exist k large enough such that $\frac{1}{k} < |\varrho B^n|$, $z_1 + \varrho B^n \subset B^n \backslash C_k$ and $z_2 + \varrho B^n \subset C_k \backslash B^n$; therefore, if $u = \frac{z_1 - z_2}{\|z_1 - z_2\|}$ and $\|x - x_0\| < \varrho$ for $x \in u^\perp$ and $x_0 = z_i | u^\perp$, then $\ell_x = x + \mathbb{R}u$ satifies

$$\begin{aligned}\mathcal{H}^1\left(\ell_x \cap rB^n \cap \Theta_{u^\perp} C_k\right) &= \min\left\{\mathcal{H}^1\left(\ell_x \cap rB^n\right), \mathcal{H}^1\left(\ell_x \cap C_k\right)\right\}\\ &\geq \mathcal{H}^1\left(\ell_x \cap rB^n \cap C_k\right) + \mathcal{H}^1\left(\ell_x \cap \left(z_2 + \varrho B^n\right)\right),\end{aligned}$$

and hence

$$|rB^n \cap \Theta_{u^\perp} C_k| \geq |C_k| + |\varrho B^n| > \Xi,$$

which is a contradiction verifying (1.18). ■

Remark 1.10.8. According to Theorem 1.A.3, we can choose a sequence of Steiner symmetrizations whose results tend to a ball in a way such that the hyperplanes are chosen independently of the convex body. Let $v_1, \ldots, v_n \in S^{n-1}$ be independent such that $\angle(v_i, v_j)/\pi$ is irrational for $i \neq j$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$. Now if $K \subset \mathbb{R}^n$ is an o-symmetric convex body with $|K| = |rB^n|$ for $r > 0$, then K_m tends to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$.

As an example for how to use Steiner symmetrization to prove geometric inequalities, we provide the proof of the Brunn-Minkowski inequality.

Theorem 1.10.9 (Brunn-Minkowski inequality). *If* $K, C \subset \mathbb{R}^n$ *are convex bodies and* $\alpha, \beta \geq 0$, *then*

$$|\alpha K + \beta C|^{\frac{1}{n}} \geq \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}}. \tag{1.19}$$

Proof. We may assume that $\alpha = \beta = 1$. Let $|K| = |rB^n|$ and $|C| = |RB^n|$ for $r, R > 0$. We apply simulatanious iterated Steiner symmetrization to K and C, which does not increase the volume of the Minkowski sum by Proposition 1.10.3 (vii). We may assume that K and C are unconditional according to Lemma 1.10.4.

According to Proposition 1.10.3 (iv), Steiner symmetrization does not increase the Hausdorff distance from rB^n or RB^n. Therefore, when we apply iterated Steiner symmetrizations simultaniously to K and C in stages, and at each stage, we apply finitely many Steiner symmetrizations based on Theorem 1.10.7 to make sure that either the image of K, or the image of C (always just one of them), gets closer to the respective ball. Eventually, we construct a sequence $\{\Theta_{u_m^\perp}\}$ of Steiner symmetrizations such that writing $K_0 = K$, $C_0 = C$, $K_{m+1} = \Theta_{u_m^\perp} K_m$ and $C_{m+1} = \Theta_{u_m^\perp} C_m$, the sequences $\{K_m\}$ and $\{C_m\}$ tend to rB^n and RB^n, respectively. It follows from Proposition 1.10.3 (vii) and the continuity of volume Lemma 1.7.4 that

$$|K + C|^{\frac{1}{n}} \geq \lim_{m\to\infty} |K_m + C_m|^{\frac{1}{n}} = |rB^n + RB^n|^{\frac{1}{n}} = |rB^n|^{\frac{1}{n}} + |RB^n|^{\frac{1}{n}} = |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}.$$

■

It is not hard to see (using triangulations similarly as in Remark 1.10.10) that Steiner symmetrization produces polytope from a polytope; however, the result has typically more vertices. Lemma 1.10.11 exhibit some special cases when the number of vertices does not grow during Steiner symmetrization.

Remark 1.10.10. Given a polytope $P \subset \mathbb{R}^n$ and some facets $F_1, \ldots, F_q$ of P, a triangulation $\mathcal{F}$ of $\{F_1, \ldots, F_q\}$ is a finite family $\mathcal{F}$ of $(n-1)$-simplices whose union is $F_1 \cup \ldots \cup F_q$, each vertex of simplex in $\mathcal{F}$ is a vertex of some F_i, and the non-empty intersection of any two simplices in $\mathcal{F}$ is a common face. $\mathcal{F}$ can constructed by constructing a triangulation $\mathcal{F}_m$ of the family of m-dimensional polytopes that are faces of some F_i into m-dimensional simplices by induction on $m = 0, \ldots, n-1$ in a way such that if F is a face of some F_i, and $G \in \mathcal{F}_m$ intersects F in a k-dimensional set, $0 \leq k \leq m$, then $F \cap G \in \mathcal{F}_k$.

Lemma 1.10.11. *For an n-dimensional polytope $P \subset \mathbb{R}^n$, assume that there exists $u \in S^{n-1}$ such that if v is a vertex of P, then either $(v + \mathbb{R}u) \cap P = \{v\}$, or $(v + \mathbb{R}u) \cap P$ is a segment whose other endpoint is also a vertex of P. Then the Steiner symmetrial $\Theta_{u^\perp} P$ is a polytope with the same number of vertices as P.*

Proof. We use the notations as in Definition 1.10.1 (c) of the Steiner symmetrization, only with $K = P$. Let $\mathcal{F}$ be a triangulation of the faces of P lying in the graph of f, and let $\mathcal{F}'$ be the family of the projections of the elements of $\mathcal{F}$ into $u^\perp$. Now elements of $\mathcal{F}'$ partition $P|u^\perp$, and the condition in Lemma 1.10.11 yields that for any $S \in \mathcal{F}'$ with vertices $x_1, \ldots, x_n$, $x_i + f(x_i)u$ and $x_i + g(x_i)u$ are vertices of P, and both f and g are linear on S. In turn, $\frac{1}{2}(f + g)$ is linear on S for any $S \in \mathcal{F}'$, and $x_i + \frac{f(x_i)+g(x_i)}{2}\,u$ is a vertex of $\Theta_{u^\perp} P$, and any vertices of P and $\Theta_{u^\perp} P$ can be obtained this way. ■

A close relative of the Steiner symmetrization is the Schwarz symmetrization. We recall that $\omega_n = |B^n|$.

Definition 1.10.12 (Schwarz symmetrization). If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then the Schwarz symmetrial is

$$\begin{aligned}\Theta_{\mathbb{R}u} K &= \bigcup \Big\{ tu + (u^\perp \cap r_t B^n) : t \in \mathbb{R},\ (tu + u^\perp) \cap K \neq \emptyset, \\ &\qquad \mathcal{H}^{n-1}\Big(K \cap (tu + u^\perp)\Big) = r_t^{n-1}\omega_{n-1},\ r_t \geq 0 \Big\}.\end{aligned}$$

Lemma 1.10.13. *For a convex body $K \subset \mathbb{R}^n$ and $u \in S^{n-1}$, there exists a sequence $\{\Theta_{v_m^\perp}\}$ of Steiner symmetrizations for $v_m \in u^\perp \cap S^{n-1}$ such that if $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$, then $\{K_m\}$ tends to $\Theta_{\mathbb{R}u} K$.*

Proof. Let $K|\mathbb{R}u = [a, b]u$ for $a < b$. According to Lemma 1.10.4, we may assume that K is unconditional with respect to a basis involving u, and hence $K \cap (tu + u^\perp)$ is symmetric through tu for $t \in (a, b)$.

For $k \geq 2$ and $i = 1, \dots, 2^k - 1$, consider the hyperplanes $H_{k,i} = u^\perp + (a + i \cdot \frac{b-a}{2^k})u$, and apply finitely many Steiner symmetrizations based on Theorem 1.10.7 using vectors from $u^\perp$ to ensure that the ratio of the relative circumradius and inradius of $H_{k,i} \cap K$ becomes less than $1 + 2^{-k}$ for each $i = 1, \dots, 2^k - 1$, and then let k tend to infinity to complete the proof of Lemma 1.10.13.

Alternatively, one may apply Theorem 1.A.3 instead of Theorem 1.10.7 in the following way. We deduce from Theorem 1.A.3 (cf. Remark 1.10.8) that choosing independent $v_1, \dots, v_{n-1} \in S^{n-1} \cap u^\perp$ such that $\angle(v_i, v_j)/\pi$ is irrational for $i \neq j$, the iterated Steiner symmetrizations through $v_1^\perp, \dots, v_{n_1}^\perp$ lead to a convex body K' such that $K' \cap (tu + u^\perp)$ is an $(n-1)$-ball centered at tu for $t \in (a, b)$, and hence $K' = \Theta_{\mathbb{R}u} K$. ■

Corollary 1.10.14. *$K, C \subset \mathbb{R}^n$ convex body, $u \in S^{n-1}$.*

(i) $\Theta_{\mathbb{R}u} K$ has axial rotational symmetry through $\mathbb{R}u$, and $\Theta_{\mathbb{R}u}(rB^n) = rB^n$ for $r > 0$;

(ii) $\Theta_{\mathbb{R}u} K$ is a convex body;

(iii) $\Theta_{\mathbb{R}u}(\lambda K) = \lambda \Theta_{\mathbb{R}u} K$ if $\lambda \in \mathbb{R} \backslash \{0\}$;

(iv) $r(\Theta_{\mathbb{R}u} K) \geq r(K)$ and $R(\Theta_{\mathbb{R}u} K) \leq R(K)$;

(v) $|\Theta_{\mathbb{R}u} K| = |K|$;

(vi) $\operatorname{diam} \Theta_{\mathbb{R}u} K \leq \operatorname{diam} K$;

(vii) $|\alpha\Theta_{\mathbb{R}u} K + \beta\Theta_{\mathbb{R}u} C| \leq |\alpha K + \beta C|$ for $\alpha, \beta > 0$.

Similarly argument to the one leading to Theorem 1.10.7 yields

Theorem 1.10.15 (Iterated Schwarz symmetrizations). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, then there exists sequence $\{K_m\}$ of convex bodies tending to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{\mathbb{R}u_m} K_m$ for some $u_m \in S^{n-1}$.*

1.11 Centroid

In physics, in many circumstances, we can replace a body by a single point; namely, by the body's centroid having assigned the same mass. In mathematics, centroids prove to be the most natural "centers" a convex body, even if there are some other useful candidates. In this section, we prove the elementary properties of the centroid that are sufficient for most of the applications, and point to references for some advanced properties.

Definition 1.11.1 (Centroid). For a convex body $K \subset \mathbb{R}^n$, its centroid is

$$\sigma_K = \frac{1}{|K|} \int_K x\, dx,$$

and hence if $u \in \mathbb{R}^n$, then

$$\langle u, \sigma_K \rangle = \frac{1}{|K|} \int_K \langle u, x \rangle\, dx = \frac{1}{|K|} \int_{\mathbb{R}} t \cdot \mathcal{H}^{n-1}\Big(K \cap (u^\perp + tu)\Big)\, dt. \tag{1.20}$$

The convex body K is called centered if $\sigma_K = 0$.

Lemma 1.11.2 (Affine equivariance of the centroid). *If $K \subset \mathbb{R}^n$ convex body, then $\sigma_{\Phi K} = \Phi \sigma_K$ where $\Phi x = Ax + b$ for $A \in \mathrm{GL}(n)$ and $b \in \mathbb{R}^n$.*

Examples

- If $K \subset \mathbb{R}^n$ o-symmetric convex body, then $\sigma_K = o$ by affine equivariance.
- If $C = \mathrm{conv}\{o, F\}$ where $F \subset u^\perp + hu$ is an $(n-1)$-dimensional compact convex set for $u \in S^{n-1}$ and $h > 0$, then for $t \in [0, h]$, we have $\mathcal{H}^{n-1}(C \cap (u^\perp + tu)) = (t/h)^{n-1}\mathcal{H}^{n-1}(F)$, and hence (1.20) yields

$$\langle u, \sigma_C \rangle = \tfrac{n}{n+1}\, h \tag{1.21}$$
$$|\{x \in C : \langle u, x \rangle \le \langle u, \sigma_C \rangle\}| = \left(\tfrac{n}{n+1}\right)^n |C|. \tag{1.22}$$

The centroid is also equivariant under Steiner and Schwarz symmetrizations.

Lemma 1.11.3. *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then*

$$\sigma_{\Theta_{u^\perp} K} = \sigma_K | u^\perp \quad \textit{and} \quad \sigma_{\Theta_{\mathbb{R}u} K} = \sigma_K | \mathbb{R}u$$

Proof. Affine equivariance (cf. Lemma 1.11.2) yields that $\sigma_{\Theta_{u^\perp} K} \in u^\perp$ and $\sigma_{\Theta_{\mathbb{R}u} K} \in \mathbb{R}u$. On the other hand, (1.20) and Definition 1.10.1 (a) of the Steiner symmetrization imply $\langle v, \sigma_{\Theta_{u^\perp} K} \rangle = \langle v, \sigma_K \rangle$ for $v \in u^\perp$, and (1.20) also yields $\langle u, \sigma_{\Theta_{\mathbb{R}u} K} \rangle = \langle u, \sigma_K \rangle$. ■

Now we show that the centroid is sitting in the "middle" of a convex body.

Lemma 1.11.4. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) If $\sigma_K = o$, then $-K \subset nK$;

(ii) If H^+ is a closed halfspace with $\sigma_K \in H^+$, then $|H^+ \cap K| \geq (\frac{n}{n+1})^n |K| > \frac{1}{e}\,|K|$.

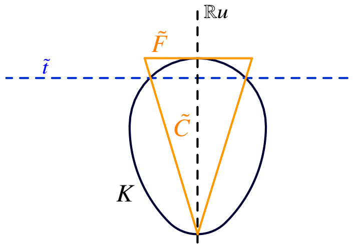
Remark. Simplices are optimal for (i) and (ii) (cf. (1.21) and (1.22)).

Proof. According to Lemma 1.11.3, we may assume that $\Theta_{\mathbb{R}u} K = K$; namely, K has axial rotational symmetry through $\mathbb{R}u$.

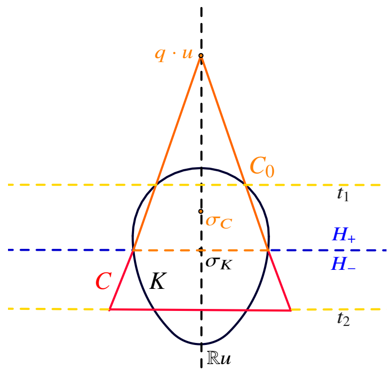
Concerning (i), it is equivalent to saying that if $u \in S^{n-1}$, then $h_{K-\sigma_K}(-u) \leq n h_{K-\sigma_K}(u)$; or in other words, using that $h_{K-\sigma_K}(-u) = h_K(-u) + \langle \sigma_K, u\rangle$ and $h_{K-\sigma_K}(u) = h_K(u) - \langle \sigma_K, u\rangle$, for any convex body $K \subset \mathbb{R}^n$ and $u \in S^{n-1}$, we have

$$\text{if } h_K(-u) = 0, \text{ then } \langle \sigma_K, u\rangle \leq \tfrac{n}{n+1}\, h_K(u). \tag{1.23}$$

Let $\widetilde{C} = \operatorname{conv}\{o, \widetilde{F}\}$ be the "bounded cone" with axial rotational symmetry through $\mathbb{R}u$ with $|\widetilde{C}| = |K|$ where $\widetilde{F} \subset u^\perp + h_K(u)u$ is an $(n-1)$-dimensional ball centered at $h_K(u)u$. It follows that $h_{\widetilde{C}}(u) = h_K(u)$ and $h_{\widetilde{C}}(-u) = 0$, $|\widetilde{C}\backslash K| = |K\backslash\widetilde{C}|$, and using axial rotational symmetry through $\mathbb{R}u$ and the convexity of K and C, we deduce the existence of a $\tilde{t} \in (0, h_K(u))$ such that $\langle x, u\rangle \geq \tilde{t}$ if $x \in \widetilde{C}\backslash K$, and $\langle x, u\rangle \leq \tilde{t}$ if $x \in K\backslash\widetilde{C}$. In particular, $\langle \sigma_K, u\rangle \leq \langle \sigma_{\widetilde{C}}, u\rangle = \frac{n}{n+1}\, h_K(u)$ (cf. (1.21)), which in turn yields (1.23).



$\langle \sigma_K, u\rangle + h_K(-u)$
$\leq \frac{n}{n+1}\,(h_K(u) + h_K(-u))\,.$



$|K \cap H_+| = |C \cap H_+| = |C_0|.$

For (ii), we may assume that $\sigma_K = o$, and $H^+ = \{x : \langle x, u\rangle \geq 0\}$ for a $u \in S^{n-1}$. Let $H^- = \{x : \langle x, u\rangle \leq 0\}$ where $\sigma_K = o \in \operatorname{int} K$ for example by (i).

Let $q > 0$ such that $|C_0| = |K \cap H^+|$ for $C_0 = \operatorname{conv}\{qu, K \cap u^\perp\}$, and hence $|C_0\backslash(K \cap H^+)| = |(K \cap H^+)\backslash C_0|$, and using axial rotational symmetry through $\mathbb{R}u$ and the convexity of K, there exists a $t_1 \in (0, h_K(u))$ such that

$$\langle x, u\rangle \geq t_1 \text{ if } x \in C_0\backslash(K \cap H^+), \text{ and } \langle x, u\rangle \leq t_1 \text{ if } x \in (K \cap H^+)\backslash C_0. \tag{1.24}$$

Next let $C = qu + \frac{|K|^{\frac{1}{n}}}{|K \cap H^+|^{\frac{1}{n}}}(C_0 - qu)$ be the bounded cone such that qu is the apex of C and $|C| = |K|$. It follows that $|(C \cap H^-)\backslash(K \cap H^-)| = |(K \cap H^-)\backslash(C \cap H^-)|$ and for $t_2 = -h_C(-u)$, we have

$$\begin{aligned} \langle x,u\rangle \ge t_2 \quad &\text{if} \quad x \in (C\cap H^-)\backslash(K\cap H^-),\\ \langle x,u\rangle \le t_2 \quad &\text{if} \quad x \in (K\cap H^-)\backslash(C\cap H^-). \end{aligned} \tag{1.25}$$

We deduce from (1.24) and (1.25) that $\langle \sigma_C, u\rangle \ge \langle \sigma_K, u\rangle = 0$, and hence (1.22) implies

$$|K\cap H^+| = |C\cap H^+| \ge |C\cap(\sigma_C + H^+)| = (\tfrac{n}{n+1})^n|C| = (\tfrac{n}{n+1})^n|K|.$$

■

The following statement is proved in Kannan, Lovász, Simonovits [361] where the factor n is optimal for example for simpleces.

Lemma 1.11.5 (KLS ellipsoid). *If $K\subset\mathbb{R}^n$ is a convex body, then there exists an o-symmetric ellipsoid E with $\sigma_K + E \subset K \subset \sigma_K + nE$.*

Remark. See Proposition 6.4.9 for a more precise statement. In particular, E is a ball if the ellipsoid of inertia of K is a ball; namely, if $\int_{K-\sigma_K}\langle u,x\rangle^2\,dx$ is the same value for any $u\in S^{n-1}$.

The proof of Lemma 1.11.6 uses polar coordinates; namely, the formula that if $f\in L_1(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} f = \int_{S^{n-1}}\int_0^\infty f(ru)r^{n-1}\,dr\,d\mathcal{H}^{n-1}(u). \tag{1.26}$$

Lemma 1.11.6. *If $K\subset\mathbb{R}^n$ is a convex body with $o\in\operatorname{int}K$, then*

(i) $|K| = \dfrac{1}{n}\displaystyle\int_{S^{n-1}} \varrho_K(u)^n\,du;$

(ii) $\sigma_K = o$ *if and only if* $\displaystyle\int_{S^{n-1}} u\cdot\varrho_K(u)^{n+1}\,du = o.$

Proof. For (i), take $f = \mathbf{1}_K$ in (1.26).

For (ii), $\sigma_K = o$ if and only if $\int_{\mathbb{R}^n}\langle v,x\rangle\cdot\mathbf{1}_K(x)\,dx = 0$ for $v\in\mathbb{R}^n$, which is equivalent to (ii) by taking $f(x) = \langle v,x\rangle\cdot\mathbf{1}_K(x)$ in (1.26). ■

The following statements are proved by Milman, Pajor [453].

Theorem 1.11.7 (Milman, Pajor). *If $K,C\subset\mathbb{R}^n$ are centered convex bodies ($\sigma_K = \sigma_C = o$), then*

$$|K-C|\cdot|K\cap C| \ge |K|\cdot|C| \text{ and } |K\cap(-K)| \ge 2^{-n}|K|. \tag{1.27}$$

Remark. Here $|K-C|\cdot|K\cap C| \le \binom{2n}{n}|K|\cdot|C| < 4^n|K|\cdot|C|$ by (1.28) below.

While do not provide the proof of Theorem 1.11.7, we verify the following variant.

Theorem 1.11.8 (Rogers-Shepard). *If $K, C \subset \mathbb{R}^n$ re convex bodies, then*

$$|K|\,|C| \le |K - C| \cdot \max_{x \in \mathbb{R}^n} |K \cap (x + C)| \le \binom{2n}{n} |K|\,|C| < 4^n |K|\,|C|; \tag{1.28}$$

$$|K - K| \le \binom{2n}{n} |K| < 4^n |K|, \tag{1.29}$$

$$|\widetilde{K} \cap (-\widetilde{K})| \ge 2^{-n} |K| \ \textit{for some}\ \widetilde{K} = K - w,\ w \in K. \tag{1.30}$$

Remarks.

- $w = \sigma_K$ works in (1.30) by (1.27).
- $|K - K| = \binom{2n}{n} |K|$ if K is a simplex in (1.29).
- If K, C o-symmetric, then $|K \cap C| = \max_{x \in \mathbb{R}^n} |K \cap (x + C)|$ by the Brunn-Minkowski inequality (1.19).

Proof. Taking convolution, $\mathbf{1}_K * \mathbf{1}_{-C}(x) = \int_{\mathbb{R}^n} \mathbf{1}_K(y) \cdot \mathbf{1}_{-C}(x - y)\, dy = |K \cap (x + C)|$ for $x \in \mathbb{R}^n$ where $K \cap (x + C) \ne \emptyset$ if and only $x \in D = K - C$; therefore,

$$|K| \cdot |C| = \int_{\mathbb{R}^n} \mathbf{1}_K * \mathbf{1}_{-C} = \int_D |K \cap (x + C)|\, dx. \tag{1.31}$$

Now (1.31) yields the first inequality in (1.28).

For the second inequality in (1.28), assume that $|K \cap C| = \max_{x \in \mathbb{R}^n} |K \cap (x + C)|$, and hence $o \in \mathrm{int} D$. For $x \in D$, there exists $z \in \partial D$ with $x = \|x\|_D \cdot z$ where $z = a - b$ for $a \in K$, $b \in L$, thus $x = \|x\|_D (a - b)$. We deduce the containment relation

$$\|x\|_D \cdot a + (1 - \|x\|_D)(K \cap C) \subset K \cap (x + C)$$

implying that $|K \cap (x + C)| \ge (1 - \|x\|_D)^n |K \cap C|$, and in turn we conclude from (1.31) and $\frac{d}{dt} \int_{tD} (1 - \|x\|_D)^n\, dx = |D|(1 - t)^n t^{n-1}$ for $t \in (0, 1)$ that

$$|K| \cdot |C| \ge |K \cap C| \cdot |D| \int_0^1 (1 - t)^n t^{n-1} dt = \binom{2n}{n}^{-1} |K \cap C| \cdot |K - C|.$$

Here $\binom{2n}{n}^{-1} < 4^n$ follows by the Stirling formula, and finally (1.28) implies (1.29) and (1.30). ■

If $K \subset \mathbb{R}^n$ is convex body, then the Brunn-Minkowski inequality (1.19) and the Rogers-Shepard inequality (1.29) yield that

$$2^n |K| \le |K - K| \le \binom{2n}{n} |K| < 4^n |K|.$$

1.12 The Brunn-Minkowski and the Isodiametric inequalities with equality for convex bodies

We note that we have already proved the Brunn-Minkowski inequality

$$|\alpha K+\beta C|^{\frac{1}{n}}\geq \alpha|K|^{\frac{1}{n}}+\beta|C|^{\frac{1}{n}} \tag{1.32}$$

for convex bodies $K,C\subset\mathbb{R}^n$ and $\alpha,\beta\geq 0$ via Steiner symmetrization in Theorem 1.10.9. In this section, we provide Brunn's original ideas in [131] at the end of the 19th centrury that also leads to the characterization of the case of equality (see Theorem 1.12.3). While the Brunn-Minkowski inequality is one of the most widely used estimates in convexity, Theorem 1.11.7 shows that it is far from optimal if for example $|K|=|C|$ and $|K\cap C|$ is small for centered convex bodies $K,C\subset\mathbb{R}^n$.

Before proving the Brunn-Minkowski inequality, we state some useful equivalent forms of it.

Definition 1.12.1 (Homothetic convex bodies). Convex bodies $K,C\subset\mathbb{R}^n$ are homothetic if there exist $\gamma>0$ and $x\in\mathbb{R}^n$ such that $K=\gamma C+x$ (it is readily a symmetric relation).

Lemma 1.12.2 (Equivalent forms of Brunn-Minkowski). *The following are equivalent assuming that they hold for any convex bodies $K,C\subset\mathbb{R}^n$.*

(i) $|\alpha K+\beta C|^{\frac{1}{n}}\geq\alpha|K|^{\frac{1}{n}}+\beta|C|^{\frac{1}{n}}$ for $\alpha,\beta>0$;

(ii) $|K+C|^{\frac{1}{n}}\geq|K|^{\frac{1}{n}}+|C|^{\frac{1}{n}}$;

(iii) $|(1-\lambda)K+\lambda C|^{\frac{1}{n}}\geq(1-\lambda)|K|^{\frac{1}{n}}+\lambda|C|^{\frac{1}{n}}$ for $\lambda\in(0,1)$;

(iv) $|(1-\lambda)K+\lambda C|\geq|K|^{1-\lambda}|C|^{\lambda}$ for $\lambda\in(0,1)$;

(v) $f(t)=|C+tK|^{\frac{1}{n}}$ and $g(t)=|(1-t)C+tK|^{\frac{1}{n}}$ are concave for $t\in[0,1]$.

Equality holds in (i), (ii) or (iii) if and only if K and C are homothetic, equality holds in (iv) if and only if K and C are translates, and f or g in (v) is linear if and only if K and C are homothetic.

Proof. (i), (ii) and (iii) are equivalent as $|\alpha K|=\alpha^n|K|$ for $\alpha>0$, and (iii) yields (iv) by the AM-GM inequality.

To show that (iv) implies (i), set $\alpha_0=|X|^{\frac{1}{n}}$, $\beta_0=|Y|^{\frac{1}{n}}$, $X_0=X/\alpha_0$ and $Y_0=Y/\beta_0$, and hence $|X_0|=|Y_0|=1$. Setting $\lambda=\frac{\beta\beta_0}{\alpha\alpha_0+\beta\beta_0}$, (iv) yields $|(1-\lambda)X_0+\lambda Y_0|_*\geq 1$, thus $|\alpha X+\beta Y|_*^{\frac{1}{n}}=(\alpha\alpha_0+\beta\beta_0)\left|\frac{\alpha\alpha_0}{\alpha\alpha_0+\beta\beta_0}X+\frac{\beta\beta_0}{\alpha\alpha_0+\beta\beta_0}Y\right|_*^{\frac{1}{n}}\geq\alpha\alpha_0+\beta\beta_0$.

(v) yields either (ii) (if f is concave) or (iii) (if g is concave).

To show that (i) implies (v), we observe that the convexity of K and C yields that $\frac{1}{2}(C+sK)+\frac{1}{2}(C+rK)=C+tK$ and $\frac{1}{2}((1-s)C+sK)+\frac{1}{2}((1-r)C+rK)=(1-t)C+tK$ for $s,r\geq 0$ and $t=\frac{1}{2}(s+r)$.

Concerning equality, the conditions for (i), (ii), (iii) and (iv) are readily equivalent.

In (v), f is linear on $[0,1]$ if and only if $|C+\frac{1}{2}K|^{\frac{1}{n}} = \frac{1}{2}|C|^{\frac{1}{n}} + \frac{1}{2}|C+K|^{\frac{1}{n}}$, which in turn equivalent with $C+K=\gamma C+z$ for $\gamma>0$, $z\in\mathbb{R}^n$. In other words, $h_C+h_{K-z} = \gamma h_C$, and since there exists $u\in S^{n-1}$ with $h_C(u)>0$ and $h_{K-z}(u)>0$, we deduce that $\gamma>1$ and $K=(\gamma-1)C+z$.

In addition, g is linear on $[0,1]$ if and only if $|\frac{1}{2}C+\frac{1}{2}K|^{\frac{1}{n}} = \frac{1}{2}|C|^{\frac{1}{n}} + \frac{1}{2}|K|^{\frac{1}{n}}$, that is equivalent to saying that K, C are homothetic. ■

Theorem 1.12.3 (Brunn-Minkowski with equality). *If $K_1, K_2\subset\mathbb{R}^n$ are convex bodies and $\alpha_1,\alpha_2>0$, then $|\alpha_1K_1+\alpha_2K_2|^{\frac{1}{n}} \geq \alpha_1|K_1|^{\frac{1}{n}}+\alpha_2|K_2|^{\frac{1}{n}}$, and equality holds if and only if K_1 and K_2 are homothetic.*

Proof. The argument is by induction on $n\geq 1$ where the case $n=1$ trivial.

Let $n\geq 2$, and assume that $\alpha_1+\alpha_2=1$, moreover $|K_i|=1$ and $\sigma_{K_i}=\int_{K_i}x\,dx=o$ for $i=1,2$. Therefore writing $K=\alpha_1K_1+\alpha_2K_2$, the Brunn-Minkowski inequality with characterization of equality is equivalent to saying that

$$|K| = |\alpha_1K_1+\alpha_2K_2| \geq 1, \tag{1.33}$$

with equality if and only if $K_1=K_2$.

As a first step, for a fixed $u\in S^{n-1}$, we prove (1.33) using the sections of K_1 and K_2 orthogonal to u, and verify the claim that equality in (1.33) yields that $h_{K_1}(u)=h_{K_2}(u)$.

If $-h_{K_i}(-u)<r_i<h_{K_i}(u)$ for $i=1,2$ and $-h_K(-u)<s<h_K(u)$, then let

$$\begin{aligned} V_i(r_i) &= |\{x\in K_i : \langle x,u\rangle \leq r_i\}|;\\ \widetilde{C}_i(r_i) &= K_i\cap(r_iu+u^\perp);\\ C(s) &= K\cap(su+u^\perp). \end{aligned}$$

In particular, $\int_{-h_{K_i}(-u)}^{h_{K_i}(u)}\mathcal{H}^{n-1}(\widetilde{C}_i(t))\,dt=1$, $\frac{\partial}{\partial t}V_i(t)=\mathcal{H}^{n-1}(\widetilde{C}_i(t))$ and $\alpha_1\widetilde{C}_1(r_1)+\alpha_2\widetilde{C}_2(r_2)\subset C(\alpha_1r_1+\alpha_2r_2)$.

The key idea of the argument is that the level sets $\widetilde{C}_i(r_i)$ of K_i are parametrized by the volume $v\in(0,1)$ below the level set. Thus for $v\in(0,1)$, we define $t_i(v)\in\mathbb{R}$ by $V_i(t_i(v))=v$, and set $C_i(v)=\widetilde{C}_i(t_i(v))$, and hence

$$\frac{\partial}{\partial v}t_i(v) = \frac{1}{\mathcal{H}^{n-1}(C_i(v))}. \tag{1.34}$$

Using substitution $t=\alpha_1t_1(v)+\alpha_2t_2(v)$, the notation $A_i(v)=\mathcal{H}^{n-1}(C_i(v))$ and the fact that the Brunn-Minkowski inequality (1.32) is known in dimension $n-1$ by induction, we deduce that

$$\begin{aligned}|K| &= \int_{-\alpha_1 h_{K_1}(-u)-\alpha_2 h_{K_2}(-u)}^{\alpha_1 h_{K_1}(u)+\alpha_2 h_{K_2}(u)} \mathcal{H}^{n-1}(\widetilde{C}(t))\,dt \\ &= \int_0^1 \mathcal{H}^{n-1}(C(\alpha_1 t_1(v)+\alpha_2 t_2(v)))\left(\frac{\alpha_1}{A_1(v)}+\frac{\alpha_2}{A_2(v)}\right)dv \\ &\geq \int_0^1 \left(\alpha_1 A_1(v)^{\frac{1}{n-1}}+\alpha_2 A_2(v)^{\frac{1}{n-1}}\right)^{n-1}\left(\frac{\alpha_1}{A_1(v)}+\frac{\alpha_2}{A_2(v)}\right)dv.\end{aligned}$$

Since $\left(\alpha_1 A_1^{\frac{1}{n-1}}+\alpha_2 A_2^{\frac{1}{n-1}}\right)^{n-1} \geq A_1^{\alpha_1}A_2^{\alpha_2} \geq \left(\frac{\alpha_1}{A_1}+\frac{\alpha_2}{A_2}\right)^{-1}$ for $A_1, A_2 > 0$ by the AM-GM inequality where equality yields $A_1 = A_2$, we conclude that

$$|\alpha_1 K_1 + \alpha_2 K_2| \geq \int_0^1 1\,dv = 1,$$

proving the Brunn-Minkowski inequality (1.33).

Equality in Brunn-Minkowski in (1.33) implies that $A_1(v) = A_2(v)$ for $v \in (0,1)$, and hence (1.34) yields that the existence of a constant $c \in \mathbb{R}$ with $t_2(v) = t_1(v) + c$ for every $v \in (0,1)$. Using $\sigma_{K_1} = \sigma_{K_2} = o$ and $\mathcal{H}^{n-1}(\widetilde{C}_2(t_i(v))) = A_i(v)$, it follows from (1.20) that

$$\begin{aligned}0 &= \int_{K_2}\langle x,u\rangle\,dx = \int_{-h_{K_2}(-u)}^{h_{K_2}(u)} t\cdot\mathcal{H}^{n-1}(\widetilde{C}_2(t))\,dt = \int_0^1 t_2(v)\cdot A_2(v)\cdot\frac{1}{A_2(v)}\,dv \\ &= \int_0^1 t_1(v)+c\,dv = c + \int_0^1 t_1(v)\,dv = c + \int_{K_1}\langle x,u\rangle\,dx = c.\end{aligned}$$

We deduce that $t_2(v) = t_1(v)$ for $v \in (0,1)$, and hence $h_{K_2}(u) = \lim_{v\to 1^-} t_2(v) = \lim_{v\to 1^-} t_1(v) = h_{K_1}(u)$, completing the proof of the claim for a fixed $u \in S^{n-1}$.

Finally equality in the Brunn-Minkowski inequality (1.33) implies $h_{K_2}(u) = h_{K_1}(u)$ for every $u \in S^{n-1}$ by the claim; therefore, $K_1 = K_2$. ■

Finally we show how the Brunn-Minkowski inequality with equality yields the characterization of the equality case in the Isodiametric Inequality Theorem 1.10.5.

Theorem 1.12.4 (Isodiametric Inequality for convex bodies with equality). *If $K \subset \mathbb{R}^n$ is a convex body, then* $\mathrm{diam}K \geq 2\omega_n^{-1/n}|K|^{1/n}$*, with equality if and only if K is a ball.*

Proof. Let $|K| = |rB^n|$ for $r > 0$, and let us consider the o-symmetric convex body $K' = \frac{1}{2}(K-K)$. As $\mathrm{diam}\,C = \max_{u\in S^{n-1}} h_C(u) + h_C(-u)$ for a convex body $C \subset \mathbb{R}^n$, we deduce that $\mathrm{diam}K' = \mathrm{diam}K$. It follows from the Brunn-Minkowski inequality Theorem 1.12.3 that $|K'| \geq |K|$. Since $K' \backslash \mathrm{int}(rB^n) \neq \emptyset$, we deduce that $\mathrm{diam}K = \mathrm{diam}K' \geq 2r$.

If diam$K = 2r$, then $K' = rB^n$, and hence $|K| = |\frac{1}{2}K + \frac{1}{2}(-K)|$. It follows from the equality case of Brunn-Minkowski inequality Theorem 1.12.3 that K' is a translate of K; therefore, K is a ball. ■

Let us discuss the rather simple extension of the equality case of the Isodiametric Inequality to any bounded measurable set $X \subset \mathbb{R}^n$ where the diameter of X is

$$\mathrm{diam}\, X = \sup\{\|x - y\| : x, y \in X\}.$$

Theorem 1.12.5 (Isodiametric Inequality with equality). *If $X \subset \mathbb{R}^n$ is bounded, measurable and* diam $X > 0$*, then* diam $X \geq 2\omega_n^{-1/n}|X|^{1/n}$*, with equality if and only if there exists a Euclidean ball $B \supset K$ with $|B \backslash X| = 0$.*

Proof. We consider the following equivalent form: If diam $X = 2r > 0$, then $|X| \leq |rB^n|$. It follows from the definition that diam cl X = diam X, and $\widetilde{X}$ = conv cl X is a compact convex set with diam $\widetilde{X}$ = diam X by Proposition 1.1.7. We deduce from Theorem 1.12.4 that $|X| \leq |\widetilde{X}| \leq |rB^n|$.

If $|X| = |rB^n|$, then $|\widetilde{X}| = |rB^n|$, and hence Theorem 1.12.4 yields that $\widetilde{X}$ is a ball of radius r. As $|X| = |\widetilde{X}|$, we conclude that $|\widetilde{X} \backslash X| = 0$. ■

1.13 Hausdorff distance from a polytope

We have already seen (cf. (1.13)) that a compact convex set can be arbitraily well approximated by polytopes in terms of the Hausdorff distance. This section discusses approximation by low complexity polytopes where estimates on the the cardinality of ε-nets have a key role. We recall that according to Section 0.1, if $X \subset \mathbb{R}^n$ and $\varepsilon > 0$, then an ε-net $\Xi \subset X$ is a discrete set such that for any $x \in X$ there exists a $y \in \Xi$ with $\|x - y\| \leq \varepsilon$. We start with polytopal approximation of B^n by inscribed polytopes because in this case, we have the following direct correspondence with ε-nets in S^{n-1}.

Lemma 1.13.1. *If $\varepsilon \in (0, \sqrt{2})$ and $P = \mathrm{conv}\Xi$ for a discrete set $\Xi \subset S^{n-1}$, then the following three statements are equivalent.*

- *Ξ is an ε-net;*
- *$(1 - \frac{\varepsilon^2}{2})B^n \subset P$;*
- *$\delta_H(B^n, P) \leq \frac{\varepsilon^2}{2}$.*

Proof. It follows as $S^{n-1} \cap (y + \varepsilon B^n) = \{z \in S^{n-1} : \langle z, y\rangle \geq 1 - \frac{\varepsilon^2}{2}\}$ for $y \in S^{n-1}$. ■

For ε-nets in S^{n-1} of minimal cardinality, a simple argument gives estimates of the right order in terms of ε.

Lemma 1.13.2. *Let $\varepsilon \in (0, \sqrt{2})$.*

(i) $\#\Xi \le \sqrt{n\pi}\sqrt{2}^n \cdot \varepsilon^{-(n-1)}$ *for some* ε*-net* $\Xi \subset S^{n-1}$*;*

(ii) $\#\Xi \ge 2(1-\frac{\varepsilon^2}{2})\sqrt{n}\varepsilon^{-(n-1)}$ *for any* ε*-net* $\Xi \subset S^{n-1}$*; in particular,* $\#\Xi \ge \sqrt{n}\varepsilon^{-(n-1)}$ *if* $\varepsilon \le 1$.

Remark. See Theorem 1.13.6 for improvements.

Proof. For any $z \in S^{n-1}$, projecting the cap $B_z(\varepsilon) = S^{n-1} \cap (z + \varepsilon B^n)$ to $z^\perp$ results in an $(n-1)$-ball of radius between $\varepsilon/\sqrt{2}$ and ε, and hence

$$\frac{\omega_{n-1}\varepsilon^{n-1}}{\sqrt{2}^{n-1}} < \mathcal{H}^{n-1}(B_z(\varepsilon)) < \left(1-\frac{\varepsilon^2}{2}\right)^{-1} \omega_{n-1}\varepsilon^{n-1}. \tag{1.35}$$

For (i), let $\Xi = \{z_1, \dots, z_N\} \subset S^{n-1}$ be maximal with the property that $\|z_i - z_j\| \ge \varepsilon$ for $i \ne j$, and hence $\Xi \subset S^{n-1}$ is an ε-net, and the caps $B_{z_i}(\varepsilon)$, $z_i \in \Xi$ cover S^{n-1}. It follows from (1.35) and $\mathcal{H}^{n-1}(S^{n-1}) = n\omega_n$ that $N \le \frac{n\omega_n}{\omega_{n-1}}\sqrt{2}^{n-1}\varepsilon^{-(n-1)} < \sqrt{n\pi}\sqrt{2}^n \cdot \varepsilon^{-(n-1)}$ by $\frac{\omega_n}{\omega_{n-1}} < \sqrt{\frac{2\pi}{n}}$ (cf. (10.1)).

For (ii), if $\Xi \subset S^{n-1}$ ε-net, then $S^{n-1} = \cup_{z\in\Xi} B_z(\varepsilon)$, and hence (1.35) yields that $N \ge (1-\frac{\varepsilon^2}{2})\frac{n\omega_n}{\omega_{n-1}}\varepsilon^{-(n-1)} > 2(1-\frac{\varepsilon^2}{2})\sqrt{n}\varepsilon^{-(n-1)}$ by $\frac{\omega_n}{\omega_{n-1}} > \sqrt{\frac{2\pi}{n+1}}$ (cf. (10.1)). ■

Combining Lemmas 1.13.1 and 1.13.2, and using polarity for circumscribed polytopes, we deduce that following estimates:

Corollary 1.13.3. *For* $c = 90$ *and* $k \ge 2^n$*, we have the followings:*

(i) If $P_k \subset B^n$ *polytope with* k *vertices minimizing* $\delta_H(B^n, P_k)$*, then*

$$c^{-1}k^{\frac{-2}{n-1}} \le \delta_H(B^n, P_k) \le ck^{\frac{-2}{n-1}}.$$

(ii) If $k \ge 2^n$ *and* $P_{(k)} \supset B^n$ *polytope with* k *facets minimizing* $\delta_H(B^n, P_{(k)})$*, then*

$$c^{-1}k^{\frac{-2}{n-1}} \le \delta_H(B^n, P_{(k)}) \le ck^{\frac{-2}{n-1}}.$$

Given the estimates of Lemma 1.13.2 for ε-nets, we are ready to construct for any convex body a well approximating polytope of low complexity.

Theorem 1.13.4. *If* $C \subset \mathbb{R}^n$ *convex body and* $k \ge 4^n n^n$*, then there exists polytope* $P_k \subset C$ *with at most* k *vertices and* $P_{(k)} \supset C$ *with at most* k *facets with*

$$\delta_H(C, P_k),\ \delta_H(C, P_{(k)}) \le 600 \cdot \mathrm{diam} C \cdot k^{\frac{-2}{n-1}}$$

Remarks. Bronshtein, Ivanov [128] proved the existence of a polytope containg C, having at most k vertices and satisfying a similar estimate. The estimates of Theorem 1.13.4 are of the right order in general by Corollary 1.13.3. We note that if C is a convex body with C^2 boundary, then the limits $\lim_{k\to\infty} k^{\frac{2}{n-1}}\delta_H(C, P_k)$ and

$\lim_{k\to\infty} k^{\frac{2}{n-1}} \delta_H(C, P_{(k)})$ exist and are positive for the best approximating polytopes $P_k \subset C$ with at most k vertices and $P_{(k)} \supset C$ with at most k facets (see Section 8.10 and Böröczky [90]).

Proof. Assume that $\mathrm{diam} C = 1$ and $C \subset B^n$, and hence $C + B^n \subset 2B^n$.

Let $k = \left\lfloor \frac{3n4^{n-1}}{\varepsilon^{n-1}} \right\rfloor$ for $\varepsilon \in (0, \frac{1}{n}]$ where $\lfloor t \rfloor$ is the largest integer not larger than t, and hence Lemma 1.13.2 yields that the existence of an ε-net $\widetilde{\Xi} = \{z_1, \dots, z_N\} \subset 2S^{n-1}$ with $N \le k$. In turn Lemma 1.2.11 yields that $\Xi = \{y_1, \dots, y_N\} \subset \partial(C + B^n)$, $N \le k$, is a ε-net where $y_i = \Pi_{C+B^n} z_i$ the closest point.

Next let $x_i = \Pi_C y_i$, and hence $y_i = x_i + u_i$ for some $u_i \in S^{n-1}$ where $\langle u_i, x_i - p\rangle \ge 0$ for $p \in C$. It follows that $P_k = \mathrm{conv}\{x_1, \dots, x_N\} \subset C$, and for $\delta = \delta_H(C, P_k)$, there exists an $x \in \partial C$ such that $\|x - w\| = \delta$ for $w = \Pi_{P_k} x$ where

$$\langle u, x - p\rangle \ge \delta \text{ for } u = \frac{x-w}{\|x-w\|} \in S^{n-1} \text{ and } p \in P_k. \tag{1.36}$$

Since x is a farthest point of C from P_k, u is an exterior normal to C at x, thus $y = x + u \in \partial(C + B^n)$, which in turn yields that the existence of a $y_i \in \Xi$ with $\|x - y_i\| \le \varepsilon$. It follows using (1.36) at the end that

$$\begin{aligned} \varepsilon^2 &\ge \|y - y_i\|^2 = \|x - x_i\|^2 + \|u - u_i\|^2 + 2\langle u, x - x_i\rangle + 2\langle u_i, x_i - x\rangle \\ &\ge 2\langle u, x - x_i\rangle \ge 2\delta, \end{aligned} \tag{1.37}$$

and hence $\delta_H(C, P_k) \le \frac{\varepsilon^2}{2} \le \frac{(3n)^{\frac{2}{n-1}} 16}{2k^{\frac{2}{n-1}}} < \frac{300}{k^{\frac{2}{n-1}}}$.

Next let $P_{(k)} = \{p \in \mathbb{R}^n : \forall \langle p, u_i\rangle \le \langle x_i, u_i\rangle\}$, and hence the facets of $P_{(k)}$ touch C at $x_1, \dots, x_N$). For $z \in P_{(k)} \backslash C$, we have $z = x + \delta u$ for $\delta > 0$ where $x = \Pi_C z$ and $u = \frac{z-x}{\|z-x\|} \in S^{n-1}$, and our task is to provide a good upper bound on δ. Since $y = x + u \in \partial(C + B^n)$, there exists $y_i \in \Xi$ with $\|x - y_i\| \le \varepsilon$, and (1.37) yields that

$$\|u - u_i\|^2 + 2\langle u_i, x_i - x\rangle \le \varepsilon^2.$$

We deduce that $\langle u_i, u\rangle = 1 - \frac{1}{2}\|u - u_i\|^2 \ge 1 - \frac{\varepsilon^2}{2} > \frac{1}{2}$, thus $\langle u_i, x_i\rangle \ge \langle u_i, z\rangle = \langle u_i, x\rangle + \delta\langle u_i, u\rangle$ yields $\frac{\varepsilon^2}{2} \ge \langle u_i, x_i - x\rangle \ge \delta\langle u_i, u\rangle > \frac{\delta}{2}$. As $z \in P_{(k)} \backslash C$ was an arbitrary point, we conclude that $\delta_H(C, P_{(k)}) \le \varepsilon^2 < 600 k^{-\frac{2}{n-1}}$. ■

Next we show that any compact convex set can be reasonable well approximated by n-dimensional polytopes containing the set.

Remark 1.13.5 (Approximating by n-polytopes containing the compact convex set). Given a compact convex set $C \subset \mathbb{R}^n$, the simple construction in (1.13) and the more sophisticated Theorem 1.13.4 provides a polytope $Q \subset C$ with at most m vertices such that $\delta = \delta_H(Q, C)$ is small (and it is possible to ensure that Q is o-symmetric if C is o-symmetric), and hence

$$C \subset Q + M \text{ for the } n\text{-polytope } Q + M \text{ with at most } 2nm \text{ vertices} \quad (1.38)$$
$$\text{and satisfying } \delta_H(Q + M, C) \leq \sqrt{n}\,\delta$$

where for an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, $M = \text{conv}\{\pm\sqrt{n}\,\delta \cdot e_i\}_{i=1}^n$ is a cross-polytope with $2n$ vertices and $\delta\, B^n \subset M$.

In particular, if $\dim C = d \geq 2$ and $k \geq 2n4^d d^d$, then Theorem 1.13.4 and $(2d)^{\frac{2}{d-1}} \leq 8$ yield an n-polytope $\widetilde{P}_k \supset C$ (o-symmetric if C is o-symmetric) with at most k vertices and satisfying the estimates

$$\delta_H(C, \widetilde{P}_k) \leq 4800\sqrt{n} \cdot \text{diam} C \cdot k^{\frac{-2}{d-1}}. \quad (1.39)$$

In the second half of this section, we discuss approximation of B^n by polytopes of reasonable compplexity in detail because very good estimates and even exact solutions are known in many cases. We note that the inscribed regular crosspolytope $C_n = \text{conv}\,\{\pm e_1, \ldots, \pm e_n\} \subset B^n$ for an orthonormal basis $e_1, \ldots, e_n$ is a polytope has $2n$ vertices and $\frac{1}{\sqrt{n}}\,B^n \subset C_n$. Therefore, first we consider the case when the number of vertices is at least $2n$. The following estimates were proved by Böröczky, Wintsche [120]:

Theorem 1.13.6. *If $n^{-1/2} \leq r < 1$, then there exists a polytope P_k with k vertices satisfying $rB^n \subset P_k \subset B^n$ (and hence $\delta_H(P_k, B^n) \leq 1 - r$) and*

$$k \leq c \cdot r(1 - r^2)^{\frac{1-n}{2}} \cdot n^{\frac{3}{2}} \log(1 + r^2 n)$$

where $c > 1$ is an absolute constant. In particular, if $0 < \varepsilon < 1$, then the minimal cardinality N_ε of an ε-net in S^{n-1} satisfies

$$n^{1/2}\varepsilon^{-(n-1)} \leq N_\varepsilon \leq cn^{3/2} \log n \cdot \varepsilon^{-(n-1)},$$

and even $N_\varepsilon \geq c^{-1} n^{3/2} \varepsilon^{-(n-1)}$ if $0 < \varepsilon < \frac{1}{\sqrt{n}}$.

Remarks. Lemma 1.13.2 (ii) implies that $k \geq c^{-1} r(1 - r^2)^{\frac{1-n}{2}} \cdot n^{\frac{1}{2}}$.

Theorem 1.13.6 yields the lower bound in (1.40), and the volume estimates in Theorem 6.8.3 imply the bound in (1.40), and (1.41) follows by polarity.

Theorem 1.13.7. *For an absolute constant $c > 1$ and $2n \leq k \leq 2^n$, if ϱ_k is the maximal $\varrho > 0$ with the property that there exists a polytope $P_k \subset B^n$ with k vertices and $\varrho B^n \subset P_k$, and R_k is the minimal $R > 0$ with the property that there exists a polytope*

$P_{(k)} \supset B^n$ with k faces and $P_{(k)} \subset RB^n$, then

$$c^{-1}\sqrt{\frac{\log\frac{k}{n}}{n}} \leq \quad \varrho_k \quad \leq c\sqrt{\frac{\log\frac{k}{n}}{n}}; \tag{1.40}$$

$$c^{-1}\sqrt{\frac{n}{\log\frac{k}{n}}} \leq \quad R_k \quad \leq c\sqrt{\frac{n}{\log\frac{k}{n}}}. \tag{1.41}$$

Finally, let us discuss approximation by inscribed polytopes of at most $2n$ vertices.

Proposition 1.13.8 (Steiner). *If $P \subset B^n$ is a simplex and $rB^n \subset P$ for $r > 0$, then $r \leq \frac{1}{n}$, with equality if and only if P is an inscribed regular simplex.*

Remark. The proof of Lemma 1.13.8 yields $r(P) \leq \frac{1}{n}$.

Proof. If $Q \subset B^n$ is an inscribed regular simplex, then $r(Q) = \frac{1}{n}$, and the origin is the center of the inscribed ball. Therefore, it is sufficient to prove that $r(P) \leq \frac{1}{n}$ for a simplex $P \subset B^n$ with equality if and only if P is an inscribed regular simplex.

We may assume that P has maximal inradius, and hence each vertex of P lies on S^{n-1}. We suppose that there exist vertices x_0, x_1, x_2 of P such that $\|x_1 - x_0\| \neq \|x_2 - x_0\|$, and seek a contradiction. We apply Steiner symmetrization with respect to $u^\perp$ for of $u = \frac{x_2 - x_1}{\|x_2 - x_1\|}$. Then $P' = \Theta_u P \subset B^n$ is a simplex according to Lemma 1.10.11, $r(P') \geq r(P)$, and the vertex $x_0|u^\perp$ of P' lies in $\mathrm{int}B^n$. Therefore the simplex $P' \subset B^n$ does not have maximal inradius, which is absurd. ∎

Proposition 1.13.9. *If $P \subset B^n$ is a polytope with at most $n+2$ vertices and $rB^n \subset P$ for $r > 0$, then $r \leq (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$, with equality if and only if P is the convex hull of a $\lceil \frac{n}{2} \rceil$-dimensional and a $\lfloor \frac{n}{2} \rfloor$-dimensional centered regular simplex of circumradius one whose affine hulls are orthogonal.*

Remark. The proof of Proposition 1.13.9 yields $r(P) \leq (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$.

Proof. If $Q \subset B^n$ is an optimal polytope, then $r(Q) = (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$ that is larger than the inradius of any simplex in B^n according to Proposition 1.13.8, and the origin is the center of the inscribed ball of Q. Therefore, it is sufficient to prove that $r(P) \leq (\lceil \frac{n}{2} \rceil^2 + \lfloor \frac{n}{2} \rfloor^2)^{\frac{-1}{2}}$ for a $P \subset B^n$ a polytope with at most $n+2$ vertices with equality as in Proposition 1.13.9.

We may assume that P has maximal inradius, which is than larger the indarius of any simplex in B^n. Therefore, P has $n+2$ vertices, and for any ball of radius $r(P)$ in P, each vertex of P lies in a facet touching that ball, and hence each vertex of P lies on S^{n-1} by the maximality property of P. According to Radon's Theorem Proposition 1.1.8, there exist $m, k \geq 1$ with $m + k = n$ such that the vertices of P are $x_0, \ldots, x_m, y_0, \ldots, y_k$ and $S \cap S' \neq \emptyset$ for $S = \mathrm{conv}\{x_0, \ldots, x_m\}$ and $S' = \{y_0, \ldots, y_k\}$.

We suppose that there exist vertices x_i, x_j and y_ℓ such that $\|x_i - y_\ell\| \neq \|x_j - y_\ell\|$, and seek a contradiction. We apply Steiner symmetrization with respect to $u^\perp$ for of $u = \frac{x_i - x_j}{\|x_i - x_j\|}$. Then $P' = \Theta_u P \subset B^n$ is a polytope with $n+2$ vertices according to Lemma 1.10.11, $r(P') \geq r(P)$, and the vertex $y_\ell | u^\perp$ of P' lies in intB^n. Therefore the simplex $P' \subset B^n$ does not have maximal inradius, which is absurd.

Similarly, $\|y_i - x_\ell\| \neq \|y_j - x_\ell\|$ for any y_i, y_j and x_ℓ, and hence S is a centered regular m-simplex and S' is a centered regular k-simplex of circumradius one, and their affine hulls are orthogonal. As the relaive indarius of S is $\frac{1}{m}$, and the relative inradius of S' is $\frac{1}{k}$, the symmetries of P yield that $r(P) = (k^2 + m^2)^{\frac{-1}{2}}$, which is maximal when $\{k, m\} = \left\{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor\right\}$. ■

If the number of k vertices satisfy $n+1 \leq k \leq 2n$, then Tikhomirov [552] managed to find the essentially optimal estimates.

Theorem 1.13.10 (Tikhomirov). *For some absolute constant $c > 1$, if $n+1 \leq k \leq 2n$ and r_k is maximal with the property that there exists a polytope P_k with at most k vertices satisfying $r_k B^n \subset P_k \subset B^n$, then*

$$c^{-1} \cdot \frac{\sqrt{k-n}}{n} \leq r_k \leq c \cdot \frac{\sqrt{k-n}}{n}. \tag{1.42}$$

Remark. The lower bound follows from considering polytopes that are direct sums of lower dimensional centered regular simplices of circumradius one whose dimensions are either $\lceil \frac{n}{k-n} \rceil$ or $\lfloor \frac{n}{k-n} \rfloor$. This polytope might be extremal in (1.42).

1.14 Comments to Chapter 1

Concerning additional properties of the closed and compact convex sets and the support function, see the monographs Bonnesen, Fenchel [81], Gruber [276], Hug, Weil [343] and Schneider [522]. For properties of the polar body, see Artstein-Avidan, Giannopoulos, Milman [28, 29] and Schneider [522]. For more advanced properties of normal cones, polytopes and polyhedral sets, see Barvinok [55], Grünbaum [277] and Ziegler [582].

Families of strongly isomorphic polytopes have been already used by Aleksandrov [5, 7] in one of his proofs of the Aleksandrov-Fenchel inequality (see Section 7.A and Schneider [522]). The main idea is that mixed volumes of strongly isomorphic polytopes can be interperted as a multilinear forms of the values of the support functions at the common exterior unit vectors of the facets (see Theorem 7.A.7). This idea has been generalized into his polytope algebra by McMullen [445, 447]. Putterman [495] used strongly isomorphic polytopes in his argument about the Logarithmic (L_0) Brunn-Minkowski conjecture for $p \in [0, 1)$ origin symmeric convex bodies (see Section 8.7).

Strongly isomorphic polytopes can represent ample divisors on projective toric varieties (see Cox, Little, Schenck [179], Ewald [207], Fulton [250] and Oda [477]), or certain properties of compact hyperbolic manifolds (see Fillastre [230]),.

For in depth studies on Steiner symmetrization and Schwarz symmetrization, see Bianchi, Gardner, Gronchi [71, 72]. They provide a broader class of n hyperplanes for the iterated Steiner Symmetrization than Theorem 1.A.3; namely, $v_1, \ldots, v_n \in S^{n-1}$ are independent in a way such that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for irrational $\alpha \in (0, 1)$.

The Brunn-Minkowski inequality (see Section 1.12) for convex bodies was proved by Brunn [131] in dimensions $n = 2, 3$, and by Minkowski in any dimensions (see Section 3.B for their argument). It was Minkowski's work [465] where the importance of the inequality was recognized, and it has found its place within a whole, now called Brunn-Minkowski, theory as reviewed by the thorough monograph Schneider **??**. Various natural strengthened versions of the Brunn-Minkowski inequality for convex bodies are known or conjectured, like the stability version Theorem 8.6.4 essentially due to Figalli, Maggi, Pratelli [225], and the Logarithmic Brunn-Minkowski conjecture for o-symmetric convex bodies (see Section 8.7). Dar's conjectured strengthening of the Brunn-Minkowski inequality states in [186] that if K and C are convex bodies in $\mathbb{R}^n$, and $\Theta = \max_{x \in \mathbb{R}^n} V(K \cap (x + C))$, then

$$|K + C|^{\frac{1}{n}} \geq \Theta^{\frac{1}{n}} + \left(\frac{|K| \cdot |C|}{\Theta}\right)^{\frac{1}{n}}. \tag{1.43}$$

Dar's conjecture is only known to hold in the plane (see Xi, Leng [570]), and in some very specific cases in higher dimension (see Dar [186]). For extensions of the Brunn-Minkowski inequality to non-convex sets or to functions - that is, the Prékopa-Leindler inequality - see Chapter 3.

Concerning best approximation of a convex body K with C^2 boundary by polytopes of high complexity, if $P_m \subset K$ polytope with at most m vertices, and $P_{(m)} \supset K$ polytope with at most m facets minimizing $\delta_H(K, P_m)$ and $\delta_H(K, P_{(m)})$, then Böröczky [90] determine the finite positive limits $\lim_{m\to\infty} m^{\frac{2}{n-1}} \delta_H(K, P_m)$ and $\lim_{m\to\infty} m^{\frac{2}{n-1}} \delta_H(K, P_{(m)})$. For the earlier history of polytopal approximation of smooth convex bodies in terms of the Hausdorff metric, see Gruber [275, 276]. More recently, Naszódi, Nazarov, Ryabogin [471] prove an estimate for any centered convex body in terms of dilation distance that works for polytopal approximation using reasonably large number of vertices.

Concerning related polytopal approximation of a ball using smaller number of vertices, Bárány, Füredi [46] contains many estimates (see also Böröczky, Wintsche [120] and Galicer, Litvak, Merzbacher, Pinasco [261]). A related and even more intensively investigated topic is polytopal approximation in terms of volume difference that is discussed in Section 6.8 (see also Prochno, Schütt, Werner [494]).

1.A Supplement: Iterated Steiner symmetrizations with respect to *n* fixed hyperplanes

The main goal of this section is to prove Theorem 1.A.3. Lemma 1.A.1 and Lemma 1.A.2 are needed in the proof of Theorem 1.A.3.

Lemma 1.A.1. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $u \in S^{n-1}$.*

(i) If $\varrho > 0$, then $|\varrho B^n \cap \Theta_{u^\perp} K| \geq |\varrho B^n \cap K|$;

(ii) If K is not symmetric through $u^\perp$, then there exists $\varrho > 0$ such that $|\varrho B^n \cap \Theta_{u^\perp} K| > |\varrho B^n \cap K|$.

Proof. (i) follows from the fact that $(x + \mathbb{R}u) \cap rB^n$ is symmetric through $u^\perp$ for any $x \in K|u^\perp$.

If K is not symmetric through $u^\perp$, then there exists $x \in (\mathrm{int} K)|u^\perp$ such that $x + tu, x + su \in \partial K$ where $t > |s|$. Let $\|x + su\| < \varrho < \|x + su\|$. Thus for some $\eta > 0$, $\mathcal{H}^1\big((z + \mathbb{R}u) \cap \varrho B^n\big) < \mathcal{H}^1\big((z + \mathbb{R}u) \cap K\big)$ holds if $z \in u^\perp$ satisfies $\|z - x\| < \eta$, which in turn yields (ii). ■

Lemma 1.A.2. *If $v_1, \ldots, v_n \in S^{n-1}$, $n \geq 2$, are independent such that $\angle(v_i, v_j) = \alpha_{ij}\pi$ for $i \neq j$ and irrational $\alpha \in (0,1)$, and non-empty o-symmetric $X \subset S^{n-1}$ is compact and symmetric through $v_i^\perp$ for $i = 1, \ldots, n$, then $X = S^{n-1}$.*

Proof. The proof is by induction on $n \geq 2$. If $n = 2$, then the composition of the reflections through $v_1^\perp$ and $v_2^\perp$ is a rotation of angle $\alpha_{12}2\pi$; therefore, it is suffiicient to prove that numbers of the form $\{m\alpha_{12}\}$ for $m \in \mathbb{N}$ are dense in $(0,1)$ where $\{t\} = t - \lfloor t \rfloor$ is the fractional part of a $t \in \mathbb{R}$. In turn, this follows from diophantine approximation; namely, that there exists arbitrary large integer q such that $|\alpha_{12} - \frac{p}{q}| \leq \frac{1}{q^2}$ for some integer p.

Now let $n \geq 3$, and assume that Lemma 1.A.2 is known in $\mathbb{R}^{n-1}$. For $i = 1, \ldots, n$, let $w_i \in S^{n-1}$ be such that $\langle w_i, v_j \rangle = 0$ for $j \neq i$, and hence $w_1, \ldots, w_n$ re independent.

We suppose that there exists $z \in S^{n-1} \backslash X$, and seek a contradiction. Let $x_0 \in X$ such that $\alpha = \max\{\langle x, z \rangle : x \in X\} = \langle x_0, z \rangle$, and hence $\alpha \in [0,1)$ as X is o-symmetric. Choose $w_i \notin \mathrm{lin}\{z, x_0\}$. The set $S = \{y \in S^{n-1} : \langle y, w_i \rangle = \langle x_0, w_i \rangle\}$ is an $(n-2)$-dimensional sphere that is symmetric through each $v_j^\perp$ with $j \neq i$. As $x_0 \in X \cap S$, the induction hypothesis yields that $S \subset X$. On the other hand, the supporting affine $(n-2)$-space at x_0 to S is $x_0 + L$ for $L = x_0^\perp \cap w_i^\perp$, which satisfies $L \not\subset z^\perp$ as x_0, w_i, z are independent. It follows that there exists $y \in S \subset X$ with $\langle y, z \rangle = \langle x_0, z \rangle = \alpha$, which is a contradiction, proving Lemma 1.A.2. ■

Remark (Existence of basis in Lemma 1.A.2): The family of bases $v_1, \ldots, v_n \in S^{n-1}$ of $\mathbb{R}^n$ up to isometry can be parametrized by the family C of symmetric positive definite so-called Gram matrices $A = [a_{ij}]$ with $a_{ii} = 1$ on the diagonal. The correspondence is provided by $a_{ij} = \langle v_j, v_j \rangle$. We can consider $C \subset \mathbb{R}^N$ for $N = \frac{n(n-1)}{2}$ taking $a_{ij}, i < j$ as coordinates. Since any symmetric matrix $A = [a_{ij}]$ with $a_{ii} = 1$ and $|a_{ij}| < \frac{1}{n-1}$ for $i < j$ lies in C, it follows that C is a bounded open convex subset of $\mathbb{R}^n$. If $A = [a_{ij}] \in C$ does not satify the condition in Theorem 1.A.2, then there exists $i < j$ such that a_{ij} is an algebraic number; therefore, $\mathcal{H}^N$ a.e. matrices $A = [a_{ij}] \in C$ represents a basis of $\mathbb{R}^n$ satisfying the conditions in Lemma 1.A.2.

The following elegant way to produce iterated Steiner symmetrizations is due to Klain [368]:

Theorem 1.A.3 (Iterated Steiner symmetrizations, Klain). *Let $v_1, \ldots, v_n \in S^{n-1}$ be independent such that $\angle(v_i, v_j) = \alpha_{ij}\pi$ for $i \neq j$ and irrational $\alpha \in (0,1)$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.*

If $K \subset \mathbb{R}^n$ is an o-symmetric convex body with $|K| = |rB^n|$ for $r > 0$, then K_m tends to rB^n where $K_0 = K$ and $K_{m+1} = \Theta_{u_m^\perp} K_m$.

Proof. According to the Blaschke Selection Theorem 1.7.3 and $K_m \subset R(K)B^n$, it is equivalent to prove that if a subsequence $\{K_{m_j}\}$ of $\{K_m\}$ tends to an o-symmetric convex body C, then $C = rB^n$. We may assume that for some $\alpha \in \{1, \ldots, n\}$, n divides $m_j - \alpha$ for any m_j.

We claim that C is symmetric through each $v_i^\perp$, $i = 1, \ldots, n$. We suppose that the claim does not hold, and seek a contradiction. Let $\{w_1, \ldots, w_n\} = \{v_1, \ldots, v_n\}$ in a way such that $w_{i+\alpha-1} = v_j$ if n divides $i + \alpha - 1 - j$, and let $\ell \in \{1, \ldots, n\}$ be the smallest index such that C is not symmetric through $w_\ell^\perp$. According to Lemma 1.A.1, there exist some $\varrho, \eta > 0$ such that $\left|\varrho B^n \cap \Theta_{w_\ell^\perp} C\right| > |\varrho B^n \cap C| + 3\eta$. Choose $\lambda > 1$ such that $(\lambda^n - 1)|C| < \eta$, and hence $(1 - \lambda^{-n})|C| < \eta$, and a $J > 1$ such that $\lambda^{-1}C \subset K_{m_j} \subset \lambda C$ if $j \geq J$. It follows from Proposition 1.10.3 (iii) and from $\Theta_{w_i^\perp} C = C$ for $i < \ell$ that

$$
\begin{aligned}
\left|\varrho B^n \cap \Theta_{w_\ell^\perp} \ldots \Theta_{w_1^\perp} K_{m_J}\right| &\geq \left|\varrho B^n \cap \Theta_{w_\ell^\perp} \ldots \Theta_{w_1^\perp} \lambda^{-1} C\right| \\
&= \left|\varrho B^n \cap \lambda^{-1} \Theta_{w_\ell^\perp} C\right| \\
&\geq \left|\varrho\varrho B^n \cap \Theta_{w_\ell^\perp} C\right| - \left(\left|\Theta_{w_\ell^\perp} C\right| - \left|\lambda^{-1}\Theta_{w_\ell^\perp} C\right|\right) \\
&\geq \left|rB^n \cap \Theta_{w_\ell^\perp} C\right| - \eta \geq |\varrho B^n \cap C| + 2\eta.
\end{aligned}
$$

As $K_{m_{J+1}}$ is obtained by further Steiner symmetrizations, Lemma 1.A.1 implies that

$$
\left|\varrho B^n \cap K_{m_{J+1}}\right| \geq |\varrho B^n \cap C| + 2\eta. \tag{1.44}
$$

On the other hand, $K_{m_{J+1}} \subset \lambda C$ yields that

$$
\left|\varrho B^n \cap K_{m_{J+1}}\right| \leq |\varrho B^n \cap \lambda C| \leq |\varrho B^n \cap C| + |\lambda C| - |C| \leq |\varrho B^n \cap C| + \eta,
$$

contradicting (1.44). In turn, we conclude the claim that C is symmetric through each $v_i^\perp$.

Finally, applying Lemma 1.A.2 to the set $Z = C \cap R(C)S^{n-1}$ yields that $R(C)S^{n-1} \subset C$, and hence $C = R(C)B^n = rB^n$. ■

1.B Supplement: The Hausdorff Measure

In this section, we introduce the Hausdorff measure and establish its basic properties (see Section 10.4, Falconer [208] and Federer [212] for more advanced properties). Given $s \geq 0$, let $\omega_s := \frac{\pi^{s/2}}{\Gamma(1+s/2)}$ where $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}\,dt$ denotes the Γ function, and hence $\omega_n = |B^n|$.

Definition 1.B.1. Given $s \geq 0$ and $\delta > 0$, we define

$$\mathcal{H}^s_\delta(E) = \inf \sum_{F \in \mathcal{F}} \omega_s \left(\frac{\operatorname{diam} F}{2}\right)^s$$

for $E \subset \mathbb{R}^n$ where the infimum is taken over all countable coverings $\mathcal{F}$ of E such that $\operatorname{diam}(F) < \delta$ for every $F \in \mathcal{F}$. The *s-dimensional Hausdorff measure* $\mathcal{H}^s$ is defined by $\mathcal{H}^s(E) := \lim_{\delta \searrow 0} \mathcal{H}^s_\delta(E)$.

Remark 1.B.2. The following properties hold:

- $\mathcal{H}^s_\delta$ and $\mathcal{H}^s$ are outer measures.
- If E is a segment, then $\mathcal{H}^1(E) = \mathcal{H}^1_\delta(E) = \operatorname{length}(E)$ for every $\delta > 0$.
- If E is a curve (Lipschitz image of $[0, 1]$), then $\mathcal{H}^1_\delta(E) \leq \operatorname{diam}(E)$ provided $\delta \geq \operatorname{diam}(E)$, and as $\delta \searrow 0$, $\mathcal{H}^1_\delta(E) \to \mathcal{H}^1(E) = \operatorname{length}(E)$.
- If E is countable, then $\mathcal{H}^s(E) = 0$ for every $s > 0$.
- If E is finite, then $\mathcal{H}^0(E) = \#E$.
- $\mathcal{H}^s(E) = \mathcal{H}^s(x + E) = \mathcal{H}^s(\Phi E)$ for every $x \in \mathbb{R}^n$ and every $\Phi \in O(n)$.
- $\mathcal{H}^s(\lambda E) = \lambda^s \mathcal{H}^s(E)$ for $\lambda > 0$.

Proposition 1.B.3. *$\mathcal{H}^s$ is a Borel measure on $\mathbb{R}^n$ for $s \geq 0$.*

Remark 1.B.4. $\mathcal{H}^0$ is the counting measure.

The normalization of the Hausdorff measure is explained by the fact that $\mathcal{H}^n$ agrees with the Lebesgue measure $\mathcal{L}^n$ for subsets of $\mathbb{R}^n$ by the Isodiametric Inequality:

Theorem 1.B.5. *$\mathcal{H}^n = \mathcal{L}^n$ on $\mathbb{R}^n$ for $n \geq 1$.*

Proof. In order to prove

$$\mathcal{L}^n \leq \mathcal{H}^n, \tag{1.45}$$

it is sufficient to verify that if $E \subset \mathbb{R}^n$ is Borel measurable and $\varepsilon, \delta > 0$, then $\mathcal{L}^n(E) \leq \mathcal{H}^n_\delta(E) + \varepsilon$. We choose a countable covering $\mathcal{F}$ of E such that $\operatorname{diam} F < \delta$ for $F \in \mathcal{F}$ and $\sum_{F\in\mathcal{F}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n < \mathcal{H}^n_\delta(E) + \varepsilon$. Here $\mathcal{L}^n(F) \leq \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n$ for any $F \in \mathcal{F}$ according to the Isodiametric Inequality Theorem 1.10.6, and in turn, we conclude (1.45).

In the reverse direction, first we verify that

$$\mathcal{H}^n \leq \alpha_n \mathcal{L}^n \ \text{ for } \alpha_n = \omega_n n^{n/2} 2^{1-n}; \tag{1.46}$$

or equivalently, if $\delta > 0$ and $E \subset \mathbb{R}^n$ is compact, then $\mathcal{H}^n_\delta(E) \leq \alpha_n \mathcal{L}^n(E)$. Let us consider a tiling of $\mathbb{R}^n$ by translates of the cube $[0,a]^n$ for an $a > 0$ such that $\operatorname{diam}[0,a]^n = a\sqrt{n} < \delta$ and $\mathcal{L}^n(E + a\sqrt{n}\, B^n) \leq 2\mathcal{L}^n(E)$, and let $\mathcal{F}$ be the family of tiles intersecting E. It follows that $\#\mathcal{F} \leq 2\mathcal{L}^n(E)/a^n$, and hence $\mathcal{H}^n_\delta(E) \leq \#\mathcal{F} \cdot \omega_n \left(\frac{a\sqrt{n}}{2}\right)^n \leq \alpha_n \mathcal{L}^n(E)$.

Finally, we prove

$$\mathcal{H}^n \leq \mathcal{L}^n, \tag{1.47}$$

which is equivalent to verify that $\mathcal{H}^n_\delta(E) \leq \mathcal{L}^n(E) + \varepsilon$ holds for any $\varepsilon, \delta > 0$ and bounded open $E \subset \mathbb{R}^n$. The key statement is that if $\mathcal{B}$ is the family of balls contained in E and of diameter less than δ, then there exists a subfamily $\mathcal{F}' = \{B_1, B_2, \ldots\} \subset \mathcal{B}$ of pairwise disjoint balls such that

$$\mathcal{L}^n\left(E \backslash \cup \mathcal{F}'\right) = 0, \ \text{ and hence } \mathcal{L}^n(E) = \sum_{F\in F'} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n. \tag{1.48}$$

We construct B_m by induction on m where $B_1 \in \mathcal{B}$ is any ball. For $m \geq 2$, let d_m be the supremum of the diameters of the balls in E disjoint from $B_1, \ldots, B_{m-1}$, and let $B_m \in \mathcal{B}$ be any ball of diameter at least $\min\{\frac{1}{2}\,\delta, \frac{1}{2}\,d_m\}$ and disjoint from $B_1, \ldots, B_{m-1}$. To show that $\mathcal{F}' = \{B_m\}$ satisfies (1.48), let $B_m = z_m + r_m B^n$ for $r_m > 0$ where $\lim_{m\to\infty} \sum_{i=m}^\infty \mathcal{L}^n(B_i) = 0$ and $\lim_{m\to\infty} d_m = 0$ as E is bounded. Choose integer $m_0 \geq 2$ such that $d_m < \delta$ if $m \geq m_0$.

For $m \geq m_0$, if $x \in E \backslash \cup_{i=1}^{m-1} B_i$, then there exists some $k > m$ such that $B = x + d_k B^n \subset E \backslash \cup_{i=1}^{m-1} B_i$. If $j \in \{m, \ldots, k-1\}$ is the smallest index such that B intersects $\cup_{i=1}^j B_i$, then $\operatorname{diam} B \leq d_j$ and $r_j \geq \frac{d_j}{4}$, and hence $B \cap B_j \neq \emptyset$ yields that $x \in z_j + 5r_j B^n$. We conclude that $\mathcal{L}^n(E \backslash \cup_{i=1}^{m-1} B_i) \leq 5^n \sum_{i=m}^\infty \mathcal{L}^n(B_i)$, which in turn implies (1.48).

It follows from (1.46) that also $\mathcal{H}^n\left(E \backslash \cup \mathcal{F}'\right) = 0$, and hence there exists a countable covering $\widetilde{\mathcal{F}}$ of $E \backslash \cup \mathcal{F}'$ by subsets of $\mathbb{R}^n$ of diameter less than δ with the property that $\sum_{F\in\widetilde{\mathcal{F}}} \omega_n\left((\operatorname{diam} F)/2\right)^n < \varepsilon$. Therefore, the covering $\mathcal{F} = \widetilde{\mathcal{F}} \cup \mathcal{F}'$ of E satisfies

$$\mathcal{H}^n_\delta(E) \leq \sum_{F\in\mathcal{F}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n = \mathcal{L}^n(E) + \sum_{F\in\widetilde{\mathcal{F}}} \omega_n \left(\frac{\operatorname{diam} F}{2}\right)^n < \mathcal{L}^n(E) + \varepsilon,$$

proving (1.47). In turn, we conclude $\mathcal{H}^n = \mathcal{L}^n$ by (1.45). ∎

We note that (1.48) is a special case of Vitali's Covering Theorem (see Falconer [208]).

We observe that if $\delta > 0$, $t > s \geq 0$ and $\operatorname{diam} F < \delta$ for $F \subset \mathbb{R}^n$, then $(\operatorname{diam} F)^t \leq \delta^{t-s}(\operatorname{diam} F)^s$. We deduce that if $E \subset \mathbb{R}^n$ and $s > 0$, then

$$\begin{array}{rll} \text{if } \mathcal{H}^s(E) < \infty, & \text{then } \mathcal{H}^t(E) = 0 \text{ for} & t > s; \\ \text{if } \mathcal{H}^s(E) > 0, & \text{then } \mathcal{H}^t(E) = \infty \text{ for} & t \in [0, s). \end{array} \tag{1.49}$$

We conclude form Theorem 1.B.5 and (1.49) the following:

Proposition 1.B.6. *If $s > n$ then $\mathcal{H}^s \equiv 0$.*

It is ensured by (1.49) that the following definition makes sense:

Definition 1.B.7. For $E \subseteq \mathbb{R}^n$, the Hausdorff dimension $\dim_H(E)$ of E is defined by

$$\dim_H(E) := \inf\{s \in [0, \infty) \mid \mathcal{H}^s(E) = 0\}.$$

Remark. If $E \subset \mathbb{R}^n$, then

$$\mathcal{H}^s(E) = \begin{cases} 0 & \text{if } s > \dim_H(E), \\ \infty & \text{if } s < \dim_H(E). \end{cases}$$

Corollary 1.B.8. *For every $E \subset \mathbb{R}^n$, we have $\dim_H(E) \in [0, n]$.*

Proposition 1.B.9. *If $f : X \to \mathbb{R}^m$ is Lipschitz for $X \subset \mathbb{R}^n$ with Lipschitz constant $L \geq 0$; namely, $\|f(x) - f(y)\| \leq L\|x - y\|$ for $x, y \in X$, then $\mathcal{H}^s(f(E)) \leq L^s \mathcal{H}^s(E)$ for $E \subset X$ and $s \geq 0$.*

Proof. Let $\mathcal{F}_\delta$ be an almost optimal covering for $\mathcal{H}^s_\delta(E)$ by subsets of $\mathbb{R}^n$ of diameter less than δ. Then $\{f(F)\}_{F \in \mathcal{F}_\delta}$ is a covering of $f(E)$ and $\operatorname{diam}(f(F)) \leq L \cdot \operatorname{diam}(F)$ for every $F \in \mathcal{F}_\delta$, and hence

$$\mathcal{H}^s_{L\delta}(f(E)) \leq \omega_s \sum_{F \in \mathcal{F}_\delta} \left(\frac{\operatorname{diam}(f(E))}{2}\right)^s \leq L^s \omega_s \sum_{F \in \mathcal{F}_\delta} \left(\frac{\operatorname{diam}(F)}{2}\right)^s.$$

Letting $\delta > 0$ tend to zero proves the claim. ∎

Let $X \subset \mathbb{R}^n$ be convex and closed. We recall that according to Lemma 1.2.11, the closest point map $\Pi_X : \mathbb{R}^n \to X$ is a contraction; namely, $\|\Pi_X(y) - \Pi_X(z)\| \leq \|y - z\|$ for $y, z \in \mathbb{R}^n$.

Corollary 1.B.10. *If $X \subset \mathbb{R}^n$ is convex and closed and $E \subset \mathbb{R}^n$, then the closest point map Π_X satisfies $\mathcal{H}^s(\Pi_X(E)) \leq \mathcal{H}^s(E)$ for $s \geq 0$.*

Chapter 2

Surface area and the cone volume measure for convex bodies in $\mathbb{R}^n$

The surface of convex bodies is a more involved notion than the volume or diameter because it needs some basic properties Lipschitz surfaces from geometric measure theory, like that it is the first variation of the volume (cf. (2.7)). After establishing the main properties of the surface area, we verfiy the Isoperimetric and the Anisotropic Isoperimetric inequalities. We also introduce the surface area measure and cone volume measure on S^{n-1} associated to convex bodies that encode many important geometric properties.

2.1 Some integral formulas involving exterior unit normals

In order to establish some fundmental properties of the surface area of convex bodies in Chapter 2, we need the following basic properties of graphs of convex functions related to the exterior unit normals. The formula (2.1) and (iv) are well-known properties of convex functions (see for example Federer [212] or Rockafellar [498]), (i) is a direct consequence of the definition of gradient, and (ii) follows from (ii).

Remark 2.1.1. Let $u \in S^{n-1}$ and let $\varphi : \Omega \to \mathbb{R}$ convex where $\Omega \subset u^\perp$ is a relatively open convex set, and let $X = \{z + \varphi(z)u : z \in \Omega\}$ be the graph of φ. The points $x = z + \varphi(z)u \in X$ where $D\varphi(z)$ exists are denoted by X', and let $\nu(x) \in S^{n-1}$ be the unique exterior normal at $x \in X'$ with $\langle \nu(x), u\rangle < 0$. We note that $\mathcal{H}^{n-1}(X\backslash X') = 0$ by Rademacher's theorem or by Corollary 1.5.5.

(i) If $D\varphi(z)$ exists for $z \in \Omega$ and $x = z + \varphi(z)u \in X'$, then

$$\nu(x) = \frac{D\varphi(z)}{\sqrt{1 + \|D\varphi(z)\|^2}} + \frac{-1}{\sqrt{1 + \|D\varphi(z)\|^2}} \cdot u, \text{ and hence}$$
$$\langle \nu(x), u\rangle = \frac{-1}{\sqrt{1 + \|D\varphi(z)\|^2}}.$$

(ii) If $f : X \to \mathbb{R}$ is a bounded measurable function on X, then

$$\int_X f\, d\mathcal{H}^{n-1} = \int_\Omega f(z + \varphi(z)u)\sqrt{1 + \|D\varphi(z)\|^2}\, d\mathcal{H}^{n-1}(z). \tag{2.1}$$

(iii) If $g : \Omega \to \mathbb{R}$ is a bounded measurable function on Ω, then

$$\int_\Omega g\, d\mathcal{H}^{n-1} = \int_X g\,(\Pi_{u^\perp}x)\,\langle -\nu(x), u\rangle\, d\mathcal{H}^{n-1}(x). \tag{2.2}$$

(iv) If $\psi : \Omega \to \mathbb{R}$ convex and $D\varphi(z) = D\psi(z)$ for $\mathcal{H}^{n-1}$ a.e. $z \in \Omega$, then there exists $\gamma \in \mathbb{R}$ with $\psi(z) - \varphi(z) = \gamma$ for $z \in \Omega$.

We recall that according to Theorem 1.5.2 if $K \subset \mathbb{R}^n$ convex body, then the supporting hypeplane is unique at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$. Such an $x \in \partial K$ is called *regular*, and $\nu_K(x)$ denotes the unique exterior unit normal at x. The set of regular points $x \in \partial K$ is denoted by $\partial' K$ where $\mathcal{H}^{n-1}(\partial K \backslash \partial' K) = 0$. We note that the function $x \mapsto \nu_K(x)$ is continuous on $\partial' K$ (cf. Lemma 1.2.8), and hence it is measurable.

Lemma 2.1.2. *For a $u \in S^{n-1}$ and a $K \subset \mathbb{R}^n$ convex body, we have*

(i) $\displaystyle\int_{\partial' K} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = 0;$

(ii) $\displaystyle\int_{\partial' K} |\langle \nu_K, u\rangle| \, d\mathcal{H}^{n-1} = 2\mathcal{H}^{n-1}(K|u^\perp).$

Proof. $\Omega = (\mathrm{int}K)\,|u^\perp$. ψ_+ and ψ_- concave and convex on Ω such that $\mathrm{int}K = \{z + tu : z \in \Omega \;\&\; \psi_-(z) < t < \psi_+(z)\}$.

$$\begin{array}{rclllll} X_- &=& \{z + \psi_-(z)\, u : z \in \Omega\}, & \text{thus} & \langle \nu_K(x), u\rangle < 0 & \text{for} & x \in X_- \cap \partial' K \\ X_+ &=& \{z + \psi_+(z)\, u : z \in \Omega\}, & \text{thus} & \langle \nu_K(x), u\rangle > 0 & \text{for} & x \in X_+ \cap \partial' K \\ X_0 &=& \partial K \backslash (X_+ \cup X_-), & \text{thus} & \langle \nu_K(x), u\rangle = 0 & \text{for} & x \in X_0 \cap \partial' K. \end{array}$$

It follows from (2.2) that

$$\int_{X_- \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = -\int_{X_- \cap \partial K'} |\langle \nu_K, u\rangle| \, d\mathcal{H}^{n-1} = -\mathcal{H}^{n-1}(K|u^\perp);$$
$$\int_{X_+ \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(K|u^\perp);$$
$$\int_{X_0 \cap \partial K'} \langle \nu_K, u\rangle \, d\mathcal{H}^{n-1} = 0$$

proving both (i) and (ii). ∎

Corollary 2.1.3. *If $K \subset \mathbb{R}^n$ is a convex body, then* $\displaystyle\int_{\partial K} \nu_K \, d\mathcal{H}^{n-1} = o.$

It follows from Rademacher's theorem (see for example Federer [212] or Rockafellar [498]) that if $T = (T_1, \ldots, T_n) : \Omega \to \mathbb{R}^n$ is locally Lipschitz for an open $\Omega \subset \mathbb{R}^n$, then the $n \times n$ derivative $DT(x)$ exists at $\mathcal{H}^{n-1}$ a.e. $x \in \Omega$. At such an $x \in \Omega$, the divergence of T is $\mathrm{div}\, T(x) = \mathrm{tr}\, DT(x) = \sum_{i=1}^n \partial_i T_i(x)$, which does not depend on the choice of the orthonormal basis of $\mathbb{R}^n$.

Theorem 2.1.4 (Divergence Theorem, Federer [212], Theorem 4.5.6). *If a compact $X \subset \mathbb{R}^n$ has locally Lipschitz boundary and $T : X \to \mathbb{R}^n$ is Lipschitz, then*

$$\int_X \mathrm{div}\, T = \int_{\partial X} \langle T, \nu_X\rangle \, d\mathcal{H}^{n-1}.$$

In Section 2.A, we provide the elementary argument leading to the Divergence Theorem 2.1.4 in the case of convex bodies based on Remark 2.1.1 and the idea of the proof of Lemma 2.1.2.

Theorem 2.1.5. *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$|K| = \frac{1}{n}\int_{\partial K} \langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x) = \frac{1}{n}\int_{\partial K} h_K(\nu_K(x)) \, d\mathcal{H}^{n-1}(x). \tag{2.3}$$

Proof. Take $T(x) = x$ in the Divergence Theorem 2.1.4, and hence $\operatorname{div} T(x) = n$. ∎

Example 2.1.6 (Volume of an n-polytope P). If $F_1, \ldots, F_k$ are the facets of P with exterior unit vectors $u_1, \ldots, u_k$ (cf. Section 1.4), then considering the bounded "cones" $\operatorname{conv}\{o, F_i\}$ yields that

$$|P| = \frac{1}{n}\sum_{i=1}^{k} h_P(u_i)\,|F_i|. \tag{2.4}$$

This formula is consistent with Theorem 2.1.5 because if $x \in \operatorname{relint} F_i$, then $\nu_P(x) = u_i$ and $\langle \nu_P(x), x\rangle = h_P(u_i)$.

2.2 Parametrizing the boundary via the radial function

For a Lipschitz function $\varphi : S^{n-1} \to \mathbb{R}^d$, $d \geq 1$, we can consider its extension $\bar\varphi$ on $\mathbb{R}^n \backslash \{o\}$ by $\bar\varphi(tu) = \varphi(u)$ for $u \in S^{n-1}$ and $t > 0$ - that is also Lipschitz - and we can speak about the differential $\nabla\varphi(u) = D\bar\varphi(u)|_{u^\perp}$ at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$, which is the notion of differential with respect to a moving orthogonal frame used in differential geometry. For example, if $\varphi(u) = u$ for $u \in S^{n-1}$, then $\bar\varphi(x) = x/\|x\|$ for $x \in \mathbb{R}^n \backslash \{o\}$, and $\nabla\varphi$ at $u = (0, \ldots, 0, 1)$ is the $n \times (n-1)$ matrix whose full zero last row is below the $(n-1) \times (n-1)$ identity matrix.

We recall that $r_K(u) = \varrho_K(u)\, u \in \partial K \subset \mathbb{R}^n$ for $u \in S^{n-1}$ for the radial function $\varrho_K > 0$ of a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ (cf. Definition 1.9.5). Using the associated norm-type convex 1-homogeneous function $\|x\|_K = \min\{t \geq 0 : x \in tK\}$, we have $\varrho_K(u) = 1/\|u\|_K$ for $u \in S^{n-1}$; therefore, ϱ_K is Lipschitz, and hence differentiable at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$, and $\bar\varrho_K(x) = \|x\|/\|x\|_K$. Since ∂K consists of the points $x \in \mathbb{R}^n$ with $\|x\|_K = 1$, and $x \in \partial K$ satisfies $x = r_K(u)$ for $u = x/\|x\|$, we deduce the following.

Lemma 2.2.1. *For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, $x \in \partial K$ is a regular boundary point if and only if ϱ_K is differentiable at $x/\|x\|$. In addition, integer $k \geq 1$, ∂K is a C^k manifold - or in other words, ∂K is C^k - if and only if ϱ_K is C^k on S^{n-1}.*

To calculate the derivative of r_K at a $u \in S^{n-1}$ where ϱ_K is differentiable, let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $e_n = u$ and $\nabla\varrho_K(u) = \|\nabla\varrho_K(u)\|\,\tilde e_1$

where $\tilde{e}_i$ denotes e_i as a vector in $u^\perp$, $i = 1, \ldots, n-1$. It follows that

$$\nabla r_K(u) = u \cdot (\nabla \varrho_K(u))^t + \varrho_K(u)\, \nabla u = \|\nabla \varrho_K(u)\|\, e_n \cdot \tilde{e}_1^t + \varrho_K(u) \sum_{i=1}^{n-1} e_i \cdot \tilde{e}_i^t.$$

On the one hand, we deduce that $-\|\nabla \varrho_K(u)\| e_1 + \varrho_K(u)\, u$ is an exterior normal at $r_K(u) \in \partial K$, and hence we have the following.

Lemma 2.2.2. *For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, if ϱ_K is differentiable at a $u \in S^{n-1}$, then $x = r_K(u)$ is a regular boundary point of ∂K, and*

$$\langle \nu_K(x), u \rangle = \frac{\varrho_K(u)}{\sqrt{\|\nabla \varrho_K(u)\|^2 + \varrho_K(u)^2}}.$$

On the other hand, since $e_j^t \cdot e_i = 0$ for $j \neq i$ and $e_j^t \cdot e_j = 1$, it follows that $[(\nabla r_K(u))^t \nabla r_K(u)]$ has the eigenvalue $\|\nabla \varrho_K(u)\|^2 + \varrho_K(u)^2$ at e_1, and the other $n-2$ eigenvalues are $\varrho_K(u)^2$ if $n \geq 3$; therefore, the Jacobian of r_K at a $u \in S^{n-1}$ is

$$\sqrt{\det\,[(\nabla r_K(u))^t \nabla r_K(u)]} = \varrho_K(u)^{n-2} \sqrt{\|\nabla \varrho_K(u)\|^2 + \varrho_K(u)^2}.$$

In turn, we conclude the following.

Lemma 2.2.3. *For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ and $\mathcal{H}^{n-1}$ measurable function $\psi : \partial K \to [0, \infty)$, we have*

$$\int_{\partial K} \psi \, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi(r_K) \cdot \varrho_K^{n-2} \sqrt{\|\nabla \varrho_K\|^2 + \varrho_K^2}\, d\mathcal{H}^{n-1}$$

Finally, we discuss the connection between the support function and radial function. Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \operatorname{int} K$. As $h = h_K|_{S^{n-1}}$ is Lipschitz, it is differentiable at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$. In addition, if h; or equivalently, h_K is differentiable at a $u \in S^{n-1}$, then it is easy to see that $\nabla h(u) = D h_k(u)|_{S^{n-1}}$. According Lemma 1.6.6, if h_K is differentiable at a $u \in S^{n-1}$, then $D h_K(u) = x$ for the unique $x \in \partial K$ where u is an exterior normal, and we consider reverse radial Gauss image

$$\alpha_K^*(u) = x/\|x\|, \tag{2.5}$$

thus $x = r_K(\alpha_K^*(u))$. In turn, we deduce the following:

Lemma 2.2.4. *Let $h = h_K|_{S^{n-1}}$ for a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, and let h be differentiable at a $u \in S^{n-1}$. Then*

(i) $x = \nabla h(u) + h(u) \cdot u$ for the unique $x \in \partial K$ where u is an exterior normal;

(ii) $\varrho_K(\alpha_K^*(u)) = \sqrt{\|\nabla h(u)\|^2 + h(u)^2}$.

2.3 Surface area, Cauchy formula, continuity, monotonicity

This section discusses the surface area of any compact convex $K \subset \mathbb{R}^n$.

Definition 2.3.1 (Surface area of compact convex sets). If $K \subset \mathbb{R}^n$ is compact compact convex, then

$$S(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho\, B^n| - |K|}{\varrho}. \tag{2.6}$$

If $K \subset \mathbb{R}^n$ is a convex body, then the natural definition of surface area is $S(K) = \mathcal{H}^{n-1}(\partial K)$. We chose (2.6) as a definition of surface area of compact convex sets because it is continuous on compact convex sets (cf. Lemma 2.3.9), and if $\dim K = n$, then $S(K)$ as defined in (2.6) is known as Minkowski content, and actually

$$S(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho\, B^n| - |K|}{\varrho} = \mathcal{H}^{n-1}(\partial K), \tag{2.7}$$

see Schneider [522], Ambrosio, Colesanti, Villa [18] or Federer [212], Theorem 3.2.39.

Example 2.3.2 (Surface area of n-polytopes). Let P be a polytope with $\dim P = n$ and facets $F_1, \ldots, F_k$. Since $\beta(N_{F_i}) = \frac{1}{2}$, (1.7) and (1.8) yield

$$S(P) = \lim_{\varrho \to 0^+} \frac{|P + \varrho\, B^n| - |P|}{\varrho} = \sum_{i=1}^{k} \mathcal{H}^{n-1}(F_i).$$

Lemma 2.3.3. *Let $K \subset \mathbb{R}^n$ be compact convex.*

(i) If $\dim K = n$*, then* $S(K) = \mathcal{H}^{n-1}(\partial K)$.

(ii) If $\dim K = n - 1$*, then* $S(K) = 2\mathcal{H}^{n-1}(K)$.

(iii) If $\dim K \leq n - 2$*, then* $S(K) = 0$.

Proof. Let $d = \dim K$. Assume $d > 0$, as if $d = 0$, then $S(K) = 0$. We may assume $o \in \operatorname{relint} K$, and let $L = \operatorname{lin} K$, and let $r > 0$ such that $r(B^n \cap L) \subset K$.

If $d = n$, then (i) is just (2.7)

If $d \leq n - 1$, then for $\varrho > 0$, we have

$$\begin{aligned} K + \varrho B^n &\supset K + \varrho(B^n \cap L^\perp) \\ K + \varrho B^n &\subset K + \varrho(B^n \cap L) + \varrho(B^n \cap L^\perp) \subset \left(1 + \frac{\varrho}{r}\right) K + \varrho(B^n \cap L^\perp) \end{aligned}$$

where $B^n \cap L^\perp$ is $(n - d)$-dimensional. Thus

$$\mathcal{H}^d(K) \cdot \omega_{n-d}\varrho^{n-d} \leq |K + \varrho B^n| \leq \left(1 + \frac{\varrho}{r}\right)^d \mathcal{H}^d(K) \cdot \omega_{n-d}\varrho^{n-d}. \tag{2.8}$$

As $|K| = 0$ and $\omega_1 = 2$, we deduce from (2.8) that $S(K) = \lim_{\varrho \to 0^+} |K + \varrho\, B^n|/\varrho$. ∎

Now we prove the classical integral representation of the surface area. As we will soon see, integral formulas a very handy to prove monotonicity or continuity. We frequently write du for $d\mathcal{H}^{n-1}(u)$ when integrating on S^{n-1} in order to simplify the formulas.

Theorem 2.3.4 (Cauchy formula). *If $K \subset \mathbb{R}^n$ compact convex, then*

$$S(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, du. \tag{2.9}$$

Proof. If $\dim K = n$, then Lemma 2.1.2 applied first to K, then to B^n yields

$$\begin{aligned}
\int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, du &= \int_{S^{n-1}} \frac{1}{2} \int_{\partial K} |\langle \nu_K, u\rangle|\, d\mathcal{H}^{n-1}\, du \\
&= \int_{\partial K} \frac{1}{2} \int_{\partial B^n} |\langle \nu_K(x), u\rangle|\, du\, d\mathcal{H}^{n-1}(x) \\
&= \int_{\partial K} \mathcal{H}^{n-1}(B^n|\nu_K(x)^\perp)\, d\mathcal{H}^{n-1}(x) = \omega_{n-1} S(K).
\end{aligned}$$

If $\dim K = n-1$, then we may assume that $K \subset v^\perp$ for a $v \in S^{n-1}$, and hence (2.2) and Lemma 2.1.2 applied to B^n yield

$$\int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)\, d\mathcal{H}^{n-1}(u) = \int_{\partial B^n} |\langle v, u\rangle| \cdot \mathcal{H}^{n-1}(K)\, d\mathcal{H}^{n-1}(u) = 2\omega_{n-1}\mathcal{H}^{n-1}(K).$$

Finally, if $\dim K \le n-2$, then $\mathcal{H}^{n-1}(K|u^\perp) = 0$ for every $u \in S^{n-1}$. ∎

The Lebesgue measure is readily monotone on measurable sets. We can say more for convex bodies.

Lemma 2.3.5. *If $C \subset K \subset \mathbb{R}^n$ are compact convex sets with $\dim K = n$ and $C \neq K$, then $|C| < |K|$.*

Proof. Any $z \in \mathrm{int} K \setminus C$ can be strictly separated from C by a hyperplane. ∎

Definition 2.3.6. For $n \ge 1$, let $\mathcal{K}^n$ be the space of compact convex sets in $\mathbb{R}^n$ equipped with the Hausdorff metric δ_H.

Since orthogonal projection is a linear operation, we have the following.

Lemma 2.3.7. *If $X, Y \subset R^n$ and $L \subset \mathbb{R}^n$ is a linear subspace, then $(X+Y)|L = (X|L) + (Y|L)$.*

Remark. Choosing $Y = \varrho B^n$, we deduce that $\delta_H\Big((X|L), (Z|L)\Big) \le \delta_H(X, Z)$ for compact convex $X, Z \subset \mathbb{R}^n$.

The following lemma shows that the volume functional is not only continuous, but even locally Lipschitz on convex compact sets with respect to the Hausdorff distance.

Lemma 2.3.8. *If $K, C \subset R\,B^n$ are compact, convex for $R > 0$, then*

$$\big||C| - |K|\big| \leq 3^{n-1} n\omega_n \cdot R^{n-1} \cdot \delta_H(K, C). \tag{2.10}$$

Proof. Let $\varrho = \delta_H(K, C) \leq 2R$, and we may assume $|C| \geq |K|$. It follows from Lemma 2.3.3, the Cauchy formula (2.9) and $K + \varrho\,B^n \subset 3RB^n$ that yield

$$\begin{aligned}
|C| - |K| &\leq |K + \varrho\,B^n| - |K| = \int_0^{\varrho} S(K + r\,B^n)\,dr \\
&= \frac{1}{\omega_{n-1}} \int_0^{\varrho} \int_{S^{n-1}} \mathcal{H}^{n-1}\Big((K + rB^n)|u^{\perp}\Big)\,d\mathcal{H}^{n-1}(u)dr \\
&\leq 3^{n-1} n\omega_n \cdot R^{n-1} \cdot \varrho.
\end{aligned}$$

■

Now we ready to show that the surface area is continuous and monotone on compact convex sets.

Lemma 2.3.9. *The surface area $K \mapsto S(K)$ is continuous on $\mathcal{K}^n$, $n \geq 2$.*

Proof. Let $K, C \subset R\,B^n$ compact, convex and let $\varrho = \delta_H(K, C)$, and hence $K|u^{\perp}, C|u^{\perp} \subset R\,B^n$ for $u \in S^{n-1}$.

Lemma 2.3.7 yields that $\delta_H(K|u^{\perp}, C|u^{\perp}) \leq \varrho$ for $u \in S^{n-1}$, thus Cauchy formula (2.9) and Lemma 2.3.8 applied in $u^{\perp}$ for $u \in S^{n-1}$ imply that $|S(K) - S(C)| \leq c_n R^{n-2} \cdot \varrho$ for $c_n > 0$ depending on n. ■

Lemma 2.3.10. *If $K \subset C$ for compact convex sets in $\mathbb{R}^n$, then $S(K) \leq S(C)$. If, in addition,* $\dim C \geq n - 1$ *and $K \neq C$, then $S(K) < S(C)$.*

Proof. Since $\mathcal{H}^{n-1}(K|u^{\perp}) \leq \mathcal{H}^{n-1}(C|u^{\perp})$ for $u \in S^{n-1}$ in the Cauchy formula (2.9), we deduce that $S(K) \leq S(C)$.

If $\dim C \geq n - 1$ and $K \neq C$, then there exists a $z_0 \in \mathrm{relint} C \backslash K$ and $v_0 \in S^{n-1} \cap \mathrm{lin}\, C$ such that $\langle z_0, v_0 \rangle > h_K(v_0)$, and hence there exists a $\delta \in (0, \frac{\pi}{2})$ such that $\langle z_0, u \rangle > h_K(u)$ if $u \in S^{n-1}$ and $\angle(u, v_0) < \delta$. It follows that $\mathcal{H}^{n-1}(K|u^{\perp}) \leq \mathcal{H}^{n-1}(C|u^{\perp})$ if $u \in S^{n-1}$ and $\angle(u, v_0) < \delta$; therefore, $S(K) < S(C)$. ■

2.4 The Isoperimetric inequality and the Anisotropic Isoperimetric Inequality for convex bodies

First we verify the classical Isoperimetric Inequality Theorem 2.4.1 using Steiner's symmetrization honoring the first really deep argument in convexity, and then we provide an actually much simpler proof of the more general Anisometric Isoperimetric Inequality based on the Brunn-Minkowski inequality.

Theorem 2.4.1 (Isoperimetric Inequality for convex bodies). *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |rB^n|$ for $r > 0$, then $S(K) \geq S(rB^n)$ with equality if and only if K is a Euclidean ball.*

Remarks.

- Equivalently, $S(K) \geq n\omega_n^{\frac{1}{n}}|K|^{\frac{n-1}{n}}$, with equality if and only if K is a ball (see Theorem 8.6.2 and Corollary 8.6.3 for stability versions).
- The Isoperimetric Inequality is stated for sets with Lipschitz boundary in Theorem 4.1.5, and even more generally, for sets of finite perimeter in Theorem 5.2.1.

First we verify that Steiner symmetrization (cf. Definition 1.10.1) does not increase surface area.

Proposition 2.4.2. *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then $S(\Theta_{u^\perp}K) \leq S(K)$, with equality if and only if K is symmetric through a hyperplane parallel to $u^\perp$.*

Proof. We use Definition 1.10.1 (c) for Steiner symmetrization. Let $f, g : K|u^\perp \to \mathbb{R}$ be concave such that

$$K = \left\{z + t\,u : z \in K|u^\perp \text{ and } -g(z) \leq t \leq f(z)\right\},$$

and

$$\Theta_{u^\perp}K = \left\{z + t\,u : x \in K|u^\perp \text{ and } -\frac{f(z)+g(z)}{2} \leq t \leq \frac{f(z)+g(z)}{2}\right\}.$$

We write $\Omega = \operatorname{int} K|u^\perp$, $X = \{x \in \partial K : x|u^\perp \in \operatorname{relbd}\Omega\}$ and $\widetilde{X} = \{x \in \partial(\Theta_{u^\perp}K) : x|u^\perp \in \operatorname{relbd}\Omega\}$; therefore, $\mathcal{H}^{n-1}(\widetilde{X}) = \mathcal{H}^{n-1}(X)$ by Fubini's theorem. It follows from (2.1) that

$$S(K) = \int_\Omega \sqrt{1 + \|Df(z)\|^2} + \sqrt{1 + \|Dg(z)\|^2}\, d\mathcal{H}^{n-1}(z) + \mathcal{H}^{n-1}(X)$$
$$S(\Theta_{u^\perp}K) = \int_\Omega 2 \cdot \sqrt{1 + \left\|\frac{Df(z) + Dg(z)}{2}\right\|^2}\, d\mathcal{H}^{n-1}(z) + \mathcal{H}^{n-1}(\widetilde{X}).$$

Since $\|(1,a)\| + \|(1,b)\| \geq 2 \cdot \left\|\left(1, \frac{a+b}{2}\right)\right\|$ for $a, b \in \mathbb{R}^{n-1}$ by the triangle inequality with equality if and only if $a = b$, we deduce that $S(\Theta_{u^\perp}K) \leq S(K)$. In addition, if $S(\Theta_{u^\perp}K) = S(K)$, then $Df(z) = Dg(z)$ for $\mathcal{H}^{n-1}$ a.e. $z \in \Omega$; therefore, $\exists\, \gamma \in \mathbb{R}$ such that $f(z) = g(z) + \gamma$ by Remark 2.1.1 (iv), and hence K is symmetric through $u^\perp + \frac{\gamma}{2}u$. ∎

Proof of Theorem 2.4.1: We may assume that $\sigma_K = o$, and hence any hyperplane of symmetry for K contains o by the affine equivariance of the centroid (cf. Lemma 1.11.2). It follows that if K is symmetric through some hyperperplane parallel to $u^\perp$ for any $u \in S^{n-1}$, then $\varrho S^{n-1} \subset K$ for $\varrho > 0$ whenever $\varrho S^{n-1} \cap K \neq \emptyset$; therefore, K is a ball.

To prove Theorem 2.4.1, we assume that $\sigma_K = o$ and $K \neq rB^n$ where $r > 0$ and $|K| = |rB^n|$. The considerations above show that there exists $u^\perp$ such that K is not symmetric through any hyperperplane parallel to $u^\perp$, and hence $|K_0| = |K|$ and $S(K_0) < S(K)$ for $K_0 = \Theta_{u^\perp} K$ by Proposition 2.4.2. Now there exists a sequence of iterated Steined symmetrizations leading to rB^n by Theorem 1.10.7. Since the surface area is continuous, we deduce from Proposition 2.4.2 that $S(rB^n) \leq S(K_0) < S(K)$. ∎

Definition 2.4.3 (Anisotropic Surface area of compact convex sets). Given convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, if $K \subset \mathbb{R}^n$ is compact convex, then

$$P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho\, C| - |K|}{\varrho}.$$

Remark. As the volume is translation invariant, if $z \in \operatorname{int} C$, then

$$P_C(K) = P_{C-z}(K). \tag{2.11}$$

We will see in Remark 7.4.5 that $P_C(K) = nV(K, C; 1)$ is continuous and monotonic on the space of compact convex sets. If K is a convex body, then (see Schneider [522])

$$P_C(K) = \int_{\partial K} h_C(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{\partial K} \|\nu_K(x)\|_{C^*}\, d\mathcal{H}^{n-1}(x) \tag{2.12}$$

where C^* is the polar of C and $h_C(\nu_K(x)) = \|\nu_K(x)\|_{C^*}$ follows from Proposition 1.9.3.

Next we provide an actually simpler proof of the more general Anisotropic Isoperimetric Inequality based on the Brunn-Minkowski Inequality.

Theorem 2.4.4 (Anisotropic Isoperimetric Inequality). *If $C, K \subset \mathbb{R}^n$ is a convex bodies with $o \in \operatorname{int} C$, then*

$$P_C(K) \geq n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}},$$

with equality if and only if C and K are homothetic.

Proof. Recall that $f(t) = |K + tC|^{\frac{1}{n}}$ is concave for $t \in [0, 1]$ by the Brunn-Minkowski inequality (cf. Lemma 1.12.2), and linear if and only if K and C are homothetic. It follows that

$$\frac{1}{n}|K|^{\frac{1-n}{n}} P_C(K) = f'(0) \geq f(1) - f(0) = |K + C|^{\frac{1}{n}} - |K|^{\frac{1}{n}} \geq |C|^{\frac{1}{n}} \tag{2.13}$$

using the form $|K + C|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |C|^{\frac{1}{n}}$ of the Brunn-Minkowski inequality at the end (cf. Lemma 1.12.2), thus $P_C(K) \geq n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}$.

If $P_C(K) = n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}}$, then $f'(0) = f(1) - f(0)$, and hence f is linear on $[0, 1]$, which in turn yields that C and K are homothetic. ∎

2.5 Surface area measure

In this section, we show how to encode various properties of a compact convex set into its surface area measure on S^{n-1}. Besides the support function, the surface area measure is one of the most significant notions associated to a convex body because many other notions can be expressed with the help of it like mixed volumes (see Section 7.3) and L_p-surface area measures (see Section 9.3).

Remark 2.5.1. Let $K \subset \mathbb{R}^n$ be a convex body. We recall that the set $\partial' K$ of regular boundary points is Borel and satisfies $\mathcal{H}^{n-1}(\partial K \backslash \partial' K) = 0$ according to Theorem 1.5.2. In addition, the unique exterior unit normal $\nu_K(x)$ at an $x \in \partial K'$ is a continuous function on $\partial' K$ by Lemma 1.5.3. Therefore, $\nu_K^{-1}(\omega) \subset \partial' K$ is Borel as a subset of S^{n-1} for any Borel $\omega \subset S^{n-1}$.

Definition 2.5.2 (Surface area measure S_K on S^{n-1}). Let $K \subset \mathbb{R}^n$ be convex, compact.

- If K is convex body ($\dim K = n$), then

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\nu_K^{-1}(\omega)\right) \tag{2.14}$$

for a measurable $\omega \subset S^{n-1}$, which is well-defined according to Remark (2.5.1). In particular, if $g : S^{n-1} \to \mathbb{R}$ is a bounded measurable, then

$$\int_{S^{n-1}} g dS_K = \int_{\partial' K} g \circ \nu_K \, d\mathcal{H}^{n-1} = \int_{\partial K} g \circ \nu_K \, d\mathcal{H}^{n-1}. \tag{2.15}$$

- If $\dim K = n-1$ and $K \subset x + u^{\perp}$ for $u \in S^{n-1}$ and $x \in \mathbb{R}^n$, then $\operatorname{supp} S_K = \{u, -u\}$ and $S_K(\{u\}) = S_K(\{-u\}) = \mathcal{H}^{n-1}(K)$.
- If $\dim K \leq n-2$, then $S_K \equiv 0$.

Definition 2.5.3 (Pushforward of a measure). If X, Y are topological spaces, μ Borel measure on X and $\varphi : X \to Y$ Borel measurable, then $\varphi_* \mu$ is a Borel measure on Y satisfying that $\varphi_* \mu(\omega) = \mu(\varphi^{-1}\omega)$ for $\omega \subset Y$ Borel.

In particular, if $f : Y \to \mathbb{R}$ is Borel measurable, then $\int_Y f \, d\varphi_* \mu = \int f \circ \varphi \, d\mu$.

Remark. $S_K = \nu_{K,*}\left(\mathcal{H}^{n-1} \llcorner \partial' K\right)$ for a convex body $K \subset \mathbb{R}^n$.

Example 2.5.4. (see Section 8.2 for Examples of convex bodies with C^2_+ boundary)

- If K is an n-dimensional polytope with facets $F_1, \ldots, F_m$ and exterior unit normals $u_1, \ldots, u_m$, then

$$\operatorname{supp} S_K = \{u_1, \ldots, u_m\} \text{ and } S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i), i = 1, \ldots, m.$$

- If $K = rB^n$ for $r > 0$, then$S_K(\omega) = r^{n-1} \cdot \mathcal{H}^{n-1}(\omega)$.

The following properties are immidiate consequences of the definition of the surface area measure.

Lemma 2.5.5. *Let $K \subset \mathbb{R}^n$ be convex, compact.*

- *S_K is Borel measure on S^{n-1};*
- *$S_K(S^{n-1}) = S(K)$;*
- *$S_{\lambda K} = \lambda^{n-1} \cdot S_K$ for $\lambda > 0$ and $S_{K+x} = S_K$ for $x \in \mathbb{R}^n$.*

The significance of S_K for a convex body $K \subset \mathbb{R}^n$ is exhibited by the geometric properties of its support where $\operatorname{supp} S_K$ is the smallest closed $X \subset S^{n-1}$ such that $S_K(S^{n-1} \backslash X) = 0$. We need the following simple property of exterior normals (cf. Lemma 1.2.8): If $y_m \in \partial' K$ tends to $y \in \partial K$, then

$$u \in N_K(y) \text{ for any accumulation point } u \text{ of } \{\nu_K(y_m)\}. \tag{2.16}$$

Lemma 2.5.6. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) $\operatorname{supp} S_K = \operatorname{cl}\{\nu_K(x) : x \in \partial' K\}$*;*

(ii) $$\begin{aligned} K &= \{x \in \mathbb{R}^n : \langle x, \nu_K(y)\rangle \leq h_K(\nu_K(y)) \text{ for } y \in \partial' K\} \\ &= \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u) \text{ for } u \in \operatorname{supp} S_K\} \end{aligned};$$

(iii) $\operatorname{supp} S_K$ *is not contained in any closed hemisphere;*

(iv) For a closed $\Omega \subset S^{n-1}$ and $\varphi : \Omega \to \mathbb{R}$, if $K = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \varphi(u) \text{ for } x \in \Omega\}$, then $\operatorname{supp} S_K \subset \Omega$ *and Ω is not contained in any closed hemisphere.*

Proof. For $\Omega_0 = \operatorname{cl}\{\nu_K(x) : x \in \partial' K\}$, the definition of S_K (cf. (2.14)) yields that $S_K(S^{n-1} \backslash \Omega_0) = 0$, thus $\operatorname{supp} S_K \subset \Omega_0$. On the other hand, let us consider an open $U \subset S^{n-1}$ such that $\nu_K(x_0) \in U$ for an $x_0 \in \partial' K$. As ν_K is continuous on $\partial' K$ (cf. (2.16)), we deduce the existence of $\delta > 0$ such that $\nu_K(x) \in U$ if $x \in \partial' K \cap (x_0 + \delta B^n)$, and hence $S_K(U) > 0$, which in turn verifies (i).

Turning to (ii), let $K_0 = \{x \in \mathbb{R}^n : \langle x, \nu_K(y)\rangle \leq h_K(\nu_K(y)) \text{ for } y \in \partial' K\}$. Since readily $K \subset K_0$, (ii) follows if $\partial K \subset \partial K_0$. For $y \in \partial K$, there exist $y_m \in \partial' K$ tending to $y \in \partial K$ as $\mathcal{H}^{n-1}$ a.e. point of ∂K is in $\partial' K$, and we may assume that $\nu_K(y_m)$ tends to a $u \in S^{n-1}$. Now $\langle x, u\rangle \leq \langle y, u\rangle$ follows for any $x \in K_0$ from the estimates $\langle x, \nu_K(y_m)\rangle \leq \langle y_m, \nu_K(y_m)\rangle$ for every y_m; therefore, $y \in \partial K_0$.

Next (iii) is equivalent proving that for any $v \in S^{n-1}$, there exists $u \in \operatorname{supp} S_K$ with $\langle u, v\rangle > 0$. We consider an $x \in \operatorname{int} K$ and a $z = x + tv \notin K$ for large enough $t > 0$. It follows from (ii), that there exists $u \in \operatorname{supp} S_K$ such that $\langle z, u\rangle > \langle x, u\rangle$, and hence $\langle v, u\rangle > 0$.

For (iv), it is sufficient to prove that $\nu_K(y) \in \Omega$ for any $y \in \partial' K$ by (i) and (iii). The condition on φ yields that $\varphi(u) \geq h_K(u)$ for $u \in \Omega$, and for $k \geq 1$ and $z_k = y + \frac{1}{k}\nu_K(y) \notin K$, there exists $u_k \in \Omega$ such that $\langle z_k, u_k\rangle > \varphi(u_k)$. We deduce that $\langle z_k, u_k\rangle \geq h_K(u_k) \geq \langle y, u_k\rangle$ for $k \geq 1$. For a convergent subsequence $\{u_{k'}\} \subset \{u_k\}$, we have

$u = \lim_{k'\to\infty} u_{k'} \in \Omega$, and $\lim_{k'\to\infty} z_{k'} = y$ implies $\langle y, u\rangle = h_K(u)$, and hence u is a normal vector at y. Since $y \in \partial' K$, we conclude that $\nu_K(y) = u \in \Omega$. ∎

Next we consider some basic properties of the surface area measure with respect to integration:

Lemma 2.5.7. *If $K \subset \mathbb{R}^n$ is compact, convex, then*

$$\int_{S^{n-1}} u\, dS_K(u) = o; \tag{2.17}$$

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K\, dS_K. \tag{2.18}$$

Proof. If K is a convex body, then (2.3) and (2.15) yield the formulas. If $\dim K \le n-1$, then the formulas directly follow from the definition of S_K. ∎

In the proof of Proposition 2.5.9, we will need the following version of the coarea formula Theorem 10.4.8:

Theorem 2.5.8 (Coarea formula). *For $m \le k < q$ and Lipschitz embedded k-manifold $X \subset \mathbb{R}^q$, if $F : X \to \mathbb{R}^m$ is locally Lipschitz and $\varphi : X \to [0,\infty)$ measurable, then*

$$\int_X \varphi(x)\cdot J(F,x)\, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m}\int_{F^{-1}(y)} \varphi(x)\, d\mathcal{H}^{k-m}(x)\, d\mathcal{H}^m(y).$$

We recall that according to (2.12) about the anisotropic surface area, if $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in \operatorname{int} C$, then

$$\lim_{\varrho\to 0^+} \frac{|K+\varrho\, C| - |K|}{\varrho} = \int_{\partial K} h_C(\nu_K(x))\, d\mathcal{H}^{n-1}(x). \tag{2.19}$$

We now extend this formula to compact convex sets:

Proposition 2.5.9. *If $K, C \subset \mathbb{R}^n$ are compact and convex, then*

$$\lim_{\varrho\to 0^+} \frac{|K+\varrho C| - |K|}{\varrho} = \int_{S^{n-1}} h_C\, dS_K. \tag{2.20}$$

Proof. If $\dim K \le n-2$, then choose $R > 0$ such that $C \subset RB^n$, and hence Lemma 2.3.3 yields that

$$\lim_{\varrho\to 0^+} \frac{|K+\varrho C| - |K|}{\varrho} \le \lim_{\varrho\to 0^+} \frac{|K+\varrho RB^n| - |K|}{\varrho} = R^n \lim_{\varrho\to 0^+} \frac{|\frac{1}{R}K+\varrho RB^n| - |\frac{1}{R}K|}{\varrho} = 0.$$

Next, if $\dim K = n-1$, then we may assume that $o \in \operatorname{relint} K$ as the volume and S_K are translation invariant. Let $u \in S^{n-1}$ such that $u^\perp = \operatorname{lin} K$, and let $\tilde{x}, \tilde{y} \in C$ such that $\langle \tilde{x}, u\rangle = h_C(u)$ and $\langle \tilde{x}, -u\rangle = h_C(-u)$. According to the translation invariance of the volume and (2.17), we may also assume that $\frac{1}{2}(\tilde{x}+\tilde{y})=o$; or in other words, $\tilde{y}=-\tilde{x}$. For the segment $\theta=\mathrm{conv}\{\tilde{x},\tilde{y}\}\subset C$, we have

$$\int_{S^{n-1}} h_C\, dS_K = \mathcal{H}^{n-1}(K)(h_C(u)+h_C(-u)) = |K+\theta|. \tag{2.21}$$

On the one hand, $\theta\subset C$ yields that

$$\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho} \geq \lim_{\varrho\to 0^+}\frac{|K+\varrho\theta|}{\varrho} = |K+\theta|. \tag{2.22}$$

On the other hand, $o\in\mathrm{relint}\, K$ implies that $C\subset R\,K+\theta$ for some $R>0$, and hence

$$\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho} \leq \lim_{\varrho\to 0^+}\frac{|K+\varrho RK+\varrho\theta|}{\varrho} = \lim_{\varrho\to 0^+}(1+\varrho R)^{n-1}|K+\theta| = |K+\theta|. \tag{2.23}$$

Combining (2.21), (2.22) and (2.23) yields Proposition 2.5.9.

If $\dim K=n$ and also $\dim C=n$, then let $\widetilde{C}=C-z$ for a $z\in\mathrm{int}\, C$, and hence (2.19) and (2.15) yield that

$$\lim_{\varrho\to 0^+}\frac{|K+\varrho\widetilde{C}|-|K|}{\varrho} = \int_{S^{n-1}} h_{\widetilde{C}}\, dS_K.$$

As $h_{\widetilde{C}}(u)=h_C(u)-\langle z,u\rangle$ for $u\in S^{n-1}$, it follows from (2.17) that $\int_{S^{n-1}} h_C\, dS_K = \int_{S^{n-1}} h_{\widetilde{C}}\, dS_K$. Since the volume is translation invariant, we conlude (2.20).

Finally, we assume that $\dim K=n$ and $\dim C\leq n-1$ where we may also assume that $o\in\mathrm{relint}\, C$. Let $L=\mathrm{lin}\, C$, and we apply the previous case to $((x+L)\cap K)+\varrho_C$ for $\mathcal{H}^{n-d}$ a.e. $x\in\mathrm{int}\, K|L^\perp$ such that the $\mathcal{H}^{d-1}$ measures of $(x+L)\cap\partial K$ and $(x+L)\cap\partial' K$ coincide (in this case, $(x+L)\cap K$ and $x+C$ are convex bodies in $x+L$). Integrating over such $x\in\mathrm{int}\, K|L^\perp$ the analogue of (2.20) in $x+L$ (where the exterior normals to $(x+L)\cap\partial' K$ lying in L are used), and using the coarea formula Theorem 2.5.8 for $X=(L+\mathrm{int}\, K)\cap\partial K$ and F being the orthogonal projection onto $L^\perp$, we obtain (2.20). ■

We note that the Anisotropic Isoperimetric Inequality Theorem 2.4.4 is equivalent to the Minkowski inequality below as the expression $\lim_{\varrho\to 0^+}\frac{|K+\varrho C|-|K|}{\varrho}$ for convex bodies $K,C\subset\mathbb{R}^n$ is invariant under translations of C.

Theorem 2.5.10 (Minkowsi inequality). *If $K,C\subset\mathbb{R}^n$ convex bodies, then*

$$\int_{S^{n-1}} h_C\, dS_K \geq n\,|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{2.24}$$

with equality if and only if K and C are homothetic.

In turn, we deduce the characterization of the equality of surface area mauseres.

Theorem 2.5.11. *For convex bodies $K, C \subset \mathbb{R}^n$, $S_K = S_C$ if and only if K and C are translates.*

Proof. We deduce from (2.18), $S_K = S_C$ and the Minkowski inequality (2.24) that

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_C \geq |C|^{\frac{n-1}{n}} |K|^{\frac{1}{n}}; \tag{2.25}$$

therefore, $|K| \geq |C|$. Reversing the role of K and C in (2.25) implies $|C| \geq |K|$, and hence $|C| = |K|$. In turn, equality in (2.25) implies equality in the Minkowski inequality (2.24), which fact combined with $|C| = |K|$ yields that K and C are translates. ∎

In order to understand the significance and the use of the surface area measure, we list the fundamental properties that will be proved only later when we have the necessary tools.

Remark 2.5.12 (Additional properties of S_K discussed later in the book).

- S_K is weakly continuous (see Proposition 8.4.1, and also Proposition 2.6.12 if $\mathrm{int}K \neq \emptyset$). In particular, if $g : S^{n-1} \to \mathbb{R}$ is continuous and $K_m \to K$ for compact convex $K_m, K \subset \mathbb{R}^n$, then $\lim_{m\to\infty} \int_{S^{n-1}} g \, dS_{K_m} = \int_{S^{n-1}} g \, dS_K$.
- S_K can be considered as the first variation of the volume (see (2.20) and Aleksandrov's Lemma Theorem 7.5.2);
- $\mu = S_K$ on S^{n-1} for a finite Borel measure μ on S^{n-1} if and only if $\int_{S^{n-1}} u \, d\mu(u) = o$ and supp μ is not contained in any closed hemisphere (see Theorem 9.2.3 about the Minkowski problem where the necessity of the conditions have been proved in Lemma 2.5.6 and Lemma 2.5.7).
- If ∂K is C^2_+, then $dS_K = f_K d\mathcal{H}^{n-1}$ where $f_K(\nu_K(x)) = 1/\kappa(x)$ for $x \in \partial K$ and the Gauss curvature $\kappa(x) > 0$ (see Section 8.2).

2.6 Cone Volume measure

The cone volume volume measure, introduced by Firey [233] in 1974, has the significant property that while the suface area measure only interwines with orthogonal transformations, the cone volume measure intertwines with any linear transformation (cf. Proposition 2.6.15). In particular, it is an important tool in notions and problems intertwining with linear transformations (see Section 8.9), and has fundamental role in various Brunn-Minkowski type inequalities (see Section 8.7 and Section 8.8), and in various versions of the Minkowki problem; namely, in certain Monge-Ampère equations on the sphere with geometric significance (see Section 9.3 and Section 9.4). These sections discuss the fundamental uniqueness and characterization results and conjectures about the cone volume measure. What simple to show is that the cone volume measure is weakly continuous (cf. Proposition 2.6.11), which fact in turn directly yields the otherwise non-trivial weak continuity of surface area on convex bodies (cf. Proposition 2.6.12).

Definition 2.6.1 (Cone volume measure). If $K \subset \mathbb{R}^n$ is convex compact with $o \in K$, then V_K is the Borel measure on S^{n-1} satisfying $dV_K = \frac{1}{n} h_K \, dS_K$.

Remark. $V_K(S^{n-1}) = \frac{1}{n}\int_{S^{n-1}} h_K \, dS_K = |K|$ according to (2.18), and if $\dim K \leq n-1$, then $V_K(\omega) = 0$ for measurable $\omega \subset S^{n-1}$.

The name cone volume measure stems from the case of full dimensional polytopes.

Example 2.6.2 (Cone volume measure of an n-polytope). Let $P \subset \mathbb{R}^n$ be an n-polytope with $o \in \operatorname{int} P$, and let $F_1, \ldots, F_k$ be the facets of P, and $u_1, \ldots, u_k$ be the corresponding unit exterior normals. Then $\operatorname{supp} V_P = \{u_1, \ldots, u_k\}$, and $V_K(\{u_i\}) = \frac{1}{n} h_K(u_i)\mathcal{H}^{n-1}(F_i) = |\operatorname{conv}\{o, F_i\}|$; namely, the cone volume measure of an exterior normal of a facet is the volume of he corresponding "cone".

For many applications, one may assume that $o \in \operatorname{int} K$ when discussing the cone volume measure, which case is technically easier to handle. However, limits of convex bodies containing the origin in their interior might be a convex body that contains the origin on its boundary. Also, there exist convex body $K \subset \mathbb{R}^n$ such that $o \in \partial K$ and $dV_K = \varphi \, d\mathcal{H}^{n-1}$ for a positive $C^{0,\alpha}$ function φ on S^{n-1}, and such bodies are important for Monge-Ampère equations on S^{n-1} (see Chapter 9). Therefore we extend the definition of radial functions to allow the origin on the boundary.

Definition 2.6.3 (Radial function). For a convex body $K \subset \mathbb{R}^n$ with $o \in K$, if $u \in S^{n-1}$, then let $\varrho_K(u) = \max\{t \geq 0 : tu \in K\}$, and hence $\varrho_K(u) \geq 0$ and $\varrho_K(u)\, u \in \partial K$.

Remark. If $o \in \operatorname{int} K$, then ϱ_K is just the traditional radial function. If $o \in \partial K$, then we consider the open convex cone $\Sigma_K = \{(0, \infty)x : x \in \operatorname{int} K\}$. If $u \in \Sigma_K \cap S^{n-1}$, then $\varrho_K(u) = \max\{t \geq 0 : tu \in K\} > 0$, and ϱ_K is continuous on $\Sigma_K \cap S^{n-1}$. On the other hand, if $u \in S^{n-1} \backslash (\operatorname{cl} \Sigma_K)$, then $\varrho_K(u) = 0$, and hence ϱ_K is measurable on S^{n-1}. However, ϱ_K might be a "wild" function on $\partial\Sigma_K \cap S^{n-1}$, may not be Borel.

In order to understand the cone volume measure V_K on the sphere S^{n-1}, it is practical to consider an intimately connected measure on ∂K because it is equivariant under volume preserving linear transformations (cf. Proposition 2.6.15), and has the natural integral representations as in Lemma 2.6.6.

Definition 2.6.4 (Auxiliary Cone Volume measure). For a convex body $K \subset \mathbb{R}^n$ with $o \in K$, $\widetilde{V}_K$ is a Borel measure on ∂K such that if $\Xi \subset \partial K$ is Borel, then $\widetilde{V}_K(\Xi) = |\cup\{\operatorname{conv}\{o, x\} : x \in \Xi\}|$.

Remark. $\widetilde{V}_K(\partial K) = |K|$. The Auxialiary Cone Volume measure is connected by the formula $V_K = \nu_{K,*}\widetilde{V}_K$; namely, if $\omega \subset S^{n-1}$ is measurable, then

$$V_K(\omega) = \widetilde{V}_K\left(\{x \in \partial' K : \nu_K(x) \in \omega\}\right) \tag{2.26}$$
$$= \left|\bigcup \{\mathrm{conv}\{o,x\} : x \in \partial K \text{ and } N_K(x) \cap \omega \neq \emptyset\}\right|. \tag{2.27}$$

Many statements related to the cone volume measure uses the notion of radial projection onto the sphere.

Definition 2.6.5 (Radial projection). $\pi_{S^{n-1}}(x) = x/\|x\| \in S^{n-1}$ for $x \in \mathbb{R}^n\backslash\{o\}$.

Remark. Since the closest point map Π_{B^n} is Lipschitz (cf. Lemma 1.2.11), $\pi_{S^{n-1}}$ is locally Lipschitz on $\mathbb{R}^n\backslash\{o\}$.

Lemma 2.6.6. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in K$, and $\widetilde{\Xi} = \pi_{S^{n-1}}\Xi$ for a measurable $\Xi \subset \partial K$ Borel, then*

$$\widetilde{V}_K(\Xi) = \frac{1}{n}\int_{\widetilde{\Xi}} \varrho_K^n \, d\mathcal{H}^{n-1} = \frac{1}{n}\int_{\Xi} \langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x). \tag{2.28}$$

Proof. Integration using polar coordinates (cf. (1.26)) gives the first equality in (2.28).

To prove the second equality, let $\Sigma_K = \{(0,\infty)x : x \in \mathrm{int}\, K\}$, and hence $\partial\Sigma_K = \emptyset$ if $o \in \mathrm{int}K$. We consider the Borel measure $\mu(\Xi) = \frac{1}{n}\int_{\Xi}\langle \nu_K(x), x\rangle \, d\mathcal{H}^{n-1}(x)$ for measurable $\Xi \subset \partial K$ on ∂K, and hence the second equality in (2.28) is equivalent proving that $\mu(\Xi) = \widetilde{V}_K(\Xi)$ for any Borel set $\Xi \subset \partial K$. Since $\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\} \subset \partial\Sigma_K$ and $\widetilde{V}_K(\partial K \cap \partial\Sigma_K) = 0$ if $\partial\Sigma_K \neq \emptyset$, it is enough to consider the case when $\Xi \cap \Sigma_K = \emptyset$ and $C = \cup\{\mathrm{conv}\{o,x\} : x \in \Xi\}$ is a convex body in $\mathbb{R}^n$.

Now $\widetilde{V}_K(\Xi) = |C| = \frac{1}{n}\int_{\Xi}\langle \nu_X(x), x\rangle \, d\mathcal{H}^{n-1}(x)$ by (2.3) because $\langle \nu_C(x), x\rangle = 0$ for $x \in (\partial' C)\backslash\Xi$, which in turn yields the second equality in (2.28). ■

Lemma 2.6.6 suggests to introduce the so-called radial Gauss map.

Definition 2.6.7 (Radial Gauss map). Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $\Sigma_K = \{(0,\infty)x : x \in \mathrm{int}\, K\}$. We define

$$\alpha_K(u) = \nu_K(\varrho_K(u)\cdot u) \quad \text{for } u \in \pi_{S^{n-1}}(\partial' K \cap \Sigma_K),$$

and $\alpha_K(u) = u$ if $u \in S^{n-1}\backslash(\mathrm{cl}\Sigma_K)$.

Remark. $\alpha_K(u) \in S^{n-1}$ is well-defined for a.e. $u \in S^{n-1}$, and is a measurable and bounded function on S^{n-1}. To show this, we note that $\mathcal{H}^{n-1}S^{n-1}\backslash\Theta) = 0$ for

$$\Theta = \pi_{S^{n-1}}((\partial K\backslash\partial' K) \cup \left(S^{n-1}\backslash(\mathrm{cl}\Sigma_K)\right)$$

because $\pi_{S^{n-1}}$ is locally Lipschitz and $\mathcal{H}^{n-1}(\partial K\backslash\partial' K) = 0$, and α_K is continuous on $\Theta \cap \Sigma_K$ and on $\Theta\backslash\Sigma_K$.

The definition of the cone volume meausure, (2.26) and Lemma 2.6.6 yield the following.

Proposition 2.6.8. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $f : S^{n-1} \to \mathbb{R}$ be measurable such that f is bounded or non-negative.*

$$\int_{S^{n-1}} f \, dV_K = \frac{1}{n} \int_{S^{n-1}} f \cdot h_K \, dS_K \tag{2.29}$$

$$= \frac{1}{n} \int_{\partial K} f(\nu_K(x)) \langle \nu_K(x), x \rangle \, d\mathcal{H}^{n-1}(x) \tag{2.30}$$

$$= \frac{1}{n} \int_{S^{n-1}} f(\alpha_K(u)) \cdot \varrho_K(u)^n \, d\mathcal{H}^{n-1}(u). \tag{2.31}$$

Proposition 2.6.8 allows us to express an integral with respect to the surface area measure in terms of the radial function.

Corollary 2.6.9. *If $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, and $g : S^{n-1} \to \mathbb{R}$ is measurable where g is bounded or non-negative, then*

$$\int_{S^{n-1}} g \, dS_K = \int_{S^{n-1}} g(\alpha_K(u)) \cdot \frac{\varrho_K(u)^n}{h_K(\alpha_K(u))} \, d\mathcal{H}^{n-1}(u). \tag{2.32}$$

Proof. Apply (2.31) to $f = \frac{g}{h_K}$. ∎

While (2.29) and (2.30) directly follow from the definition of the cone volume measure, the formulas (2.31) and (2.32) first appeared probably in Huang, Lutwak, Yang, Zhang [331].

It is easy to see that ν_K is continuous on $\partial' K$ for a convex body K (cf. Lemma 1.2.8). Lemma 2.6.10 below, needed in the proof of the weak convergence of the cone volume measure (cf. Proposition 2.6.11), extends this property:

Lemma 2.6.10. *For convex bodies $K_m, K \subset \mathbb{R}^n$, if $x \in \partial' K$ and $x_m \in \partial K_m$ satisfy $\lim_{m\to\infty} x_m = x$, and $u_m \in S^{n-1}$ exterior normal to K_m at x_m, then $\lim_{m\to\infty} u_m = \nu_K(x)$.*

Proof. As S^{n-1} is compact and $\nu_K(x)$ is the unique exterior unit normal to K at x, it is enough to prove that if $\lim_{m\to\infty} u_m = u$, then u is an exterior unit normal to K at x.

For $z \in \operatorname{int} K$, we have $z \in K_m$ for large m, and hence $\langle u_m, z - x_m \rangle \leq 0$ for large m. It follows that $\langle u, z - x \rangle \leq 0$; therefore, u is an exterior unit normal to K at x. ∎

We recall that for finite Borel measures μ_m, μ on S^{n-1}, μ_m tends weakly to μ if and only if $\lim_{m\to\infty} \int_{S^{n-1}} f \, d\mu_m = \int_{S^{n-1}} f \, d\mu$ for any continuous function $f : S^{n-1} \to \mathbb{R}$.

Proposition 2.6.11. *If the convex compact sets K_m tend to a convex compact set K in $\mathbb{R}^n$ with $o \in K_m$, $o \in K$, then V_{K_m} tends weakly to V_K.*

Proof. Let $f : S^{n-1} \to \mathbb{R}$ continuous. If $\dim K \leq n-1$, then $\lim_{m\to\infty} |K_m| = 0$ by the continuity of volume (cf. Lemma 1.7.4). Since $V_{K_m}(S^{n-1}) = |K_m|$ and f is bounded, we deduce that $\lim_{m\to\infty} \int_{S^{n-1}} f\, dV_{K_m} = 0 = \int_{S^{n-1}} f\, dV_K$.

Therefore let K be a convex body, and hence K_m is convex body, as well, for large m. For $\Sigma_K = \{(0,\infty)x : x \in \operatorname{int} K\}$ and $\Theta = \pi_{S^{n-1}}((\partial K \backslash \partial' K) \cup (S^{n-1} \backslash (\operatorname{cl}\Sigma_K))$, we have $\mathcal{H}^{n-1}(S^{n-1})(S^{n-1} \backslash \Theta) = 0$ (cf. the Remark after Definition 2.6.7). Since $\lim_{m\to\infty} \varrho_{K_m}(u) = \varrho_K(u)$ and $\lim_{m\to\infty} \alpha_{K_m}(u) = \alpha_K(u)$ for $u \in \Theta \cap \Sigma_K$ by Lemma 2.6.10, and $\lim_{m\to\infty} \varrho_{K_m}(u) = 0 = \varrho_K(u)$ if $u \in \Theta \backslash \Sigma_K$, we conclude Proposition 2.6.11 by (2.31) and the continuity of f. ∎

Proposition 2.6.12. *If convex bodies K_m tend to a convex body K in $\mathbb{R}^n$, then S_{K_m} tends weakly to S_K.*

Remark. Proposition 8.4.1 verifies this for any convex compact sets.

Proof. We may assume that $o \in \operatorname{int} K$, and hence also $o \in \operatorname{int} K_m$.

Let $f : S^{n-1} \to \mathbb{R}$ continuous. It follows from Lemma 2.6.10 that if $\varrho_K(u) \cdot u \in \partial' K$ for $u \in S^{n-1}$, then $f(\alpha_{K_m}(u)) \cdot \frac{\varrho_{K_m}(u)^n}{h_{K_m}(\alpha_{K_m}(u))}$ tends to $f(\alpha_K(u)) \cdot \frac{\varrho_K(u)^n}{h_K(\alpha_K(u))}$, and hence (2.32) yields that $\lim_{m\to\infty} \int_{S^{n-1}} f\, dS_{K_m} = \int_{S^{n-1}} f\, dS_K$. ∎

For a general linear map $\Phi \in \mathrm{GL}(n)$, typically $\Phi(u) \notin S^{n-1}$. Still, we associate the map $\widetilde{\Phi} : S^{n-1} \to S^{n-1}$, $\widetilde{\Phi}(u) = \frac{\Phi(u)}{\|\Phi(u)\|}$ to Φ, and as a notational abuse, we write $\Phi_*\mu = \widetilde{\Phi}_*$.

Definition 2.6.13 ("Pushforward" of a measure by a linear map)**.** For $\Psi \in \mathrm{GL}(n)$, let $\widetilde{\Psi} : S^{n-1} \to S^{n-1}$ defined by $\widetilde{\Psi}(u) = \frac{\Psi(u)}{\|\Psi(u)\|}$. If μ is finite Borel measure on S^{n-1}, then we write $\Psi_*\mu$ to denote finite Borel measure $\widetilde{\Psi}_*\mu$ on S^{n-1}; namely,

$$\Psi_*\mu(\omega) = \mu\left(\left\{\frac{\Psi^{-1}(u)}{\|\Psi^{-1}(u)\|} : u \in \omega\right\}\right) \quad \text{for any Borel } \omega \subset S^{n-1} \tag{2.33}$$

$$\int_{S^{n-1}} f\, d\Psi_*\mu = \int_{S^{n-1}} f\left(\frac{\Psi(u)}{\|\Psi(u)\|}\right) d\mu \quad \text{for Borel } f : S^{n-1} \to \mathbb{R}. \tag{2.34}$$

Lemma 2.6.14 follows as $\langle \Phi^{-t}x, \Phi y\rangle = \langle x, y\rangle$ for $x, y \in \mathbb{R}^n$ and $\Phi \in \mathrm{GL}(n)$.

Lemma 2.6.14. *If $K \subset \mathbb{R}^n$ is a convex body and $\Phi \in \mathrm{GL}(n)$, then $\Phi^{-t}\nu_K(x)$ is an exterior normal at $\Phi x \in \partial(\Phi K)$ for $x \in \partial' K$, and $h_{\Phi K}(u) = h_K(\Phi^t u)$ for $u \in \mathbb{R}^n$.*

Proposition 2.6.15 (Intertwining with Linear Transformations)**.** *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in K$, and let $\Phi \in \mathrm{GL}(n)$.*

(i) $\widetilde{V}_{\Phi K}(\Phi\Xi) = |\det \Phi| \cdot \widetilde{V}_K(\Xi)$ for measurable $\Xi \subset \partial K$, and hence $\widetilde{V}_{\Phi K} = \Phi_* \widetilde{V}_K$ if in addition $|\det \Phi| = 1$;

(ii) $V_{\Phi K} = |\det \Phi| \cdot (\Phi^{-t})_* V_K$*, and hence if* $f \geq 0$ *is measurable on* S^{n-1}*, then*

$$\int_{S^{n-1}} f\, dV_{\Phi K} = |\det \Phi| \cdot \int_{S^{n-1}} f\left(\frac{\Phi^{-t}u}{\|\Phi^{-t}u\|}\right) dV_K(u).$$

Proof. (i) holds by the very definition of $\widetilde{V}_K$ and by change of variables.

For (ii), if $\omega \subset \partial K$ is Borel, then (i), (2.26), Lemma 2.6.14 and (2.33) yield for $\Xi = \{x \in \partial'(\Phi K) : \nu_{\Phi K}(x) \in \omega\}$ that

$$\begin{aligned} V_{\Phi K}(\omega) &= \widetilde{V}_{\Phi K}(\Xi) = |\det \Phi| \cdot \widetilde{V}_K(\Phi^{-1}\Xi) = |\det \Phi| \cdot V_K(\{\nu_K(\Phi^{-1}x) : x \in \Xi\}) \\ &= |\det \Phi| \cdot V_K\left(\left\{\frac{\Phi^t u}{\|\Phi^t u\|} : u \in \omega\right\}\right) = |\det \Phi| \cdot (\Phi^{-t})_* V_K. \end{aligned}$$

■

Remark 2.6.16 (Characterization and uniqueness of the cone volume measure). Concerning the characterization of cone volume measure, the problem proposed by Firey [233] in 1974, only partial results exist (see Section 9.3 for a detailed account). For example, any non-trivial absolutely continuous measure on S^{n-1} is a cone volume measure according to Chen, Li, Zhu [157], who actually verify that any finite Borel measure μ on S^{n-1} satisfying the "subspace concentration conditions" (i) and (ii) below is a cone volume measure (cf. Theorem 9.C.1). As a necessary condition (see (9.19) for a simplified statement), Böröczky, Hegedűs [102] characterized the restriction to a pair of antipodal points; namely, if $o \in K$, $u \in S^{n-1}$, $\alpha = \frac{V_K(u)}{V_K(S^{n-1})}$ and $\beta = \frac{V_K(-u)}{V_K(S^{n-1})}$, then

$$\alpha + \beta + \min_{\varrho > 0} \sum_{i=1}^{n-1} (\alpha \varrho^{-2i} + \beta \varrho^{2i}) \leq 1. \tag{2.35}$$

Following partial results by Chou, Wang [162], He, Leng, Li [304], Henk, Schürman, Wills [308], Stancu [537, 538], Xiong [575], the paper Böröczky, Lutwak, Yang, Zhang [111] characterized even cone volume measures by the following so-called subspace concentration condition (i) and (ii) (cf. Theorem 9.B.5): A finite even non-trivial Borel measure μ on S^{n-1} is a cone volume measure if and only if

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any proper linear subspace $L \subset \mathbb{R}^n$;

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with $\operatorname{supp} \mu \subset L \cup L'$.

We note that V_K for an o-symmetric convex body $K \subset \mathbb{R}^n$ satisfies (ii) if and only if $K = C + C'$ where $C \subset L^\perp$ and $C' \subset L'^\perp$ are o-symmetric compact convex sets (cf. Theorem 9.B.5). Böröczky, Henk [104] prove that the subspace concentration conditions (i) and (ii) hold for the cone volume measure of any centered convex body (centroid is the origin), but in general, cone volume measure of a centered convex body satisfies some additional conditions (however, characterization is not known, not even in the plane).

Actually, Firey [233] was also interested in the uniqueness of the cone volume measure. According to Chen, Li, Zhu [157], there exists an absolutely continuous measure on S^{n-1} that is the cone volume measure of two different convex bodies. However, in line with the "Worn Stone" problem initiated by Firey [233], if the cone volume measure of a convex body K is the Lebesgue measure on S^{n-1}, then K is a centered ball (see see Brendle, Choi, Daskalopoulos [126], Ivaki, Milman [350] and Saroglou [510] for different arguments). The question of uniqueness of the cone volume measure of o-symmetric convex bodies is a major open problem called Logarithmic Minkowski Conjecture, asking, for example, whether $V_K = V_C$ for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^1 boundaries implies that $K = C$ (see Section 9.4).

2.7 Comments to Section 2

The Cauchy formula is due to Cauchy [144, 145] in $\mathbb{R}^2$ and $\mathbb{R}^3$ in the middle of the 19th century. It was put in the modern setting in the 1920's, for example, by Kubota [388].

If a bounded $X \subset \mathbb{R}^n$ non-empty interior and Lipschitz boundary, then the Divergence Theorem 2.1.4 yields the analogue of Theorem 2.1.5, Lemma 2.1.2 (i) (taking $T(x) = u$), and hence Corollary 2.1.3, as well; namely;

- $\displaystyle\int_{\partial X} \langle \nu_X, u\rangle \, d\mathcal{H}^{n-1} = 0$ for $u \in S^{n-1}$;
- $\displaystyle\int_{\partial X} \nu_X \, d\mathcal{H}^{n-1} = o$;
- $\displaystyle|X| = \frac{1}{n}\int_{\partial X} \langle \nu_X(x), x\rangle \, d\mathcal{H}^{n-1}(x).$

The extremal property of balls with respect to the isoperimetric problem was known to the ancient Greeks; for example, Zenodorus (circa 200 BC - 140 BC) suggested an argument using polygons in the plane, and even claimed that spheres are optimal in three dimensions (cf. Blasjö [75]). In the Euclidean spaces, the Isoperimetric Inequality for convex bodies was proved by the work of Steiner, Schwarz, Weierstrass and Minkowski in the 19th century (see Gruber [276]). Minkowski used the Brunn-Minkowski inequality in order to prove the Isoperimetric Inequality, while Steiner famously provided a symmetrization method showing that given the volume, only balls can be the minimizers of the surface area. However, Steiner did not prove the existence of a minimizer, which was verified by Weierstrass and Schwarz, still in the 19th century. Theorem 8.6.2 and Corollary 8.6.3 provide stability versions of the Isoperimetric inequality in terms of volume difference, essentially due to Fusco, Maggi, Pratelli [251], and in terms of the Hausdorff distance (see Groemer [272] for a survey on the latter type of estimates).

Extending Minkowski's ideas [464, 465], the basic properties of the surface area measure have been established by Aleksandrov [2–4,7], and Schneider [522] provides a thorough discussion (see also Hug, Weil [343] for a more direct account). Surface area measures were characterized, together with its uniqueness was provided by Minkowski [463, 464] if the measure μ is discrete (and hence the convex body is a polytope) or absolutely continuous. Minkowski's characterisation was extended to any general measure μ by Aleksandrov [3,4,7] (see Remark 2.5.12 and Theorem 9.2.3). Stability of the surface area measure in terms of the closeness of the convex bodies up to translation is discussed by Hug, Schneider [340, 342].

Livshyts [418] generalized the notion of surface area measure to the weighted case. Given a measure $d\mu = \varphi\, d\mathcal{H}^n$ for a continuous density function φ on $\mathbb{R}^n$, the *weighted surface area measure* $S_{\mu,K}$ of a convex body $K \subset \mathbb{R}^n$ is defined in a way such that

$$S_{\mu,K}(\omega) = \int_{\nu_K^{-1}(\omega)} \varphi\, \mathcal{H}^{n-1}$$

for a measurable $\omega \subset S^{n-1}$ (note that this notion is unrelated to the L_p surface area measure $S_{K,p}$, $p \in \mathbb{R}$, discussed in Section 9.3). For any convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, Livshyts [418] obtains the variational formula that

$$\lim_{\varrho \to 0^+} \frac{\mu(K + \varrho C) - \mu(K)}{\varrho} = \int_{S^{n-1}} h_C\, dS_{\mu,K}$$

similarly to the proof of Proposition 2.5.9. Various results about the weighted surface area measure are proved by Kryvonos, Langharst [387] and Fradelizi, Langharst, Madiman, Zvavitch [246].

Cone volume measure was introduced by Firey [233], and has been a widely used tool since the paper Gromov, Milman [273], see for example Barthe, Guédon, Mendelson, Naor [54], Ludwig [426], Naor [469], Paouris, Werner [482]. The still open Logarithmic Minkowski Problem about characterization of Cone Volume measures was posed by Firey [233], and it has been solved in the even case by Böröczky, Lutwak, Yang, Zhang [111] (see Chapter 9 for more details about the Cone Volume Measure).

2.A Supplement: Divergence Theorem for convex bodies

In this section, we present the elementary argument leading to the Divergence Theorem 2.1.4 in the case of convex bodies.

Theorem 2.A.1 (Divergence Theorem for Convex bodies). *If $K \subset \mathbb{R}^n$ is aconvex body, and $T : K \to \mathbb{R}^n$ is Lipschitz, then*

$$\int_K \operatorname{div} T = \int_{\partial K} \langle T, \nu_K \rangle \, d\mathcal{H}^{n-1}.$$

Proof. Let $u_1, \dots, u_n$ orthonormal basis of $\mathbb{R}^n$, and let $T_i(x) = \langle T(x), u_i \rangle$ for $x \in K$ and $\nu_i(x) = \langle \nu_K(x), u_i \rangle$ for $x \in \partial' K$, $i = 1, \dots, n$. It is sufficient to prove that

$$\int_K \partial_i T = \int_{\partial K} T_i \cdot \nu_i \, d\mathcal{H}^{n-1} \quad \text{for } i = 1, \dots, n. \tag{2.36}$$

Let $\Omega = (\mathrm{int} K)\,|u_i^\perp$, and let ψ_+ and ψ_- be concave and convex on Ω such that $\mathrm{int} K = \{z + tu_i : z \in \Omega \;\&\; \psi_-(z) < t < \psi_+(z)\}$. It follows that

$$\begin{array}{llll} \text{for } X_- = & \{z + \psi_-(z)\, u_i \,:\, z \in \Omega\}, & \nu_i(x) < 0 & \text{for} \quad x \in X_- \cap \partial' K \\ \text{for } X_+ = & \{z + \psi_+(z)\, u_i \,:\, z \in \Omega\}, & \nu_i(x) > 0 & \text{for} \quad x \in X_+ \cap \partial' K \\ \text{for } X_0 = & \partial K \backslash (X_+ \cup X_-), & \nu_i(x) = 0 & \text{for} \quad x \in X_0 \cap \partial' K. \end{array}$$

Setting $T_+ = T_i \circ \psi_+$ and $T_- = T_i \circ \psi_-$ on Ω, we deduce by Fubini's theorem, Newton-Leibniz and (2.2) that

$$\begin{aligned} \int_K \partial_i T &= \int_\Omega \int_{\psi_-(z)}^{\psi_+(z)} \partial_i T(z + tu_i)\, dt dz = \int_\Omega T_+(z) - T_-(z)\, dz = \\ &= \int_{X^+} T_i \langle \nu_K, u_i \rangle \, d\mathcal{H}^{n-1} - \int_{X^-} T_i \langle \nu_K, -u_i \rangle \, d\mathcal{H}^{n-1} = \int_{\partial K} T_i \cdot \nu_i \, d\mathcal{H}^{n-1}. \end{aligned}$$

■

2.B The Projection Body and the Isoperimetric Inequality

Let $K \subset \mathbb{R}^n$ be a convex body. It follows from Lemma 2.1.2 and the definition of the surface area measure that there exists a so-called *projection body* ΠK whose support function at $u \in \mathbb{R}^n$ is area $\mathcal{H}^{n-1}(K|u^\perp)$ of the projection onto $u^\perp$; namely

$$h_{\Pi K}(u) = \mathcal{H}^{n-1}(K|u^\perp) = \frac{1}{2} \int_{\partial' K} |\langle \nu_K, u \rangle| \, d\mathcal{H}^{n-1} = \frac{1}{2} \int_{S^{n-1}} \langle v, u \rangle| \, dS_K(v).$$

In particular, the projection body ΠK is an o-symmetric zonoid, which is a zonotope if K is a polytope (see Example 1.6.3). The existence of the projection body was probably known to Minkowski himself, and it is discussed in the 1934 classic Bonnesen, Fenchel [81]. In this section, we survey just a few properties of the projection body, see Gardner [254] and Scheider [522] for more information in general, Kryvonos, Langharst [387] and Langharst, Roysdon, Zvavitch [390] for recent extensions of the notion.

A characteristic property of the projection body proved by Petty [484] is that if $\Phi \in \mathrm{SL}(n)$, then

$$\Pi(\Phi K) = \Phi^{-t}\Pi K.$$

Partially due to this linear invariance, the projection body has a central role in affine invariant inequalities. For example, according to the Zhang projection body inequality in Zhang [577] (the lower bound) and the Petty projection inequality in Petty [485] (the upper bound), writing $\Pi^*K = (\Pi K)^*$ to denote the polar of the projection body, we have

$$\binom{2n}{n} n^{-n} \le |K|^{n-1} \, |\Pi^* K| \le \left(\frac{\omega_n}{\omega_{n-1}}\right)^n \tag{2.37}$$

where equality holds in the lower bound if and only if K is a simplex, and equality holds in the upper bound if and only if K is an ellipsoid.

What may sound surprising, the Petty projection inequality (the upper bound in (2.37)) for the affine invariant quantity $|K|^{n-1} \, |\Pi^* K|$ yields the Isoperimetric Inequality $S(K) \ge n\omega_n^{\frac{1}{n}} |K|^{\frac{n-1}{n}}$ (cf. Theorem 2.4.1), as the Petty projection inequality is the $p = -n$ case of the inequality (cf. Proposition 1.9.3 and Lemma 1.11.6)

$$\left(\frac{|K|}{\omega_n}\right)^{\frac{n-1}{n}} \le \frac{1}{\omega_{n-1}} \left(\frac{1}{n\omega_n} \int_{S^{n-1}} \mathcal{H}^{n-1}(K|u^\perp)^p \, du\right)^{\frac{1}{p}}, \tag{2.38}$$

while the $p = 1$ case - that follows from the $p = -n$ case by the Hölder inequality, - is the Isoperimetric Inequality by the Cauchy formula (2.9).

Finally, we close the section with the 50 years old fundamental conjecture by Petty [485] from 1972 stating that

$$|\Pi K| \ge \frac{\omega_{n-1}^n}{\omega_n^{n-2}} \cdot |K|^{n-1}$$

where equality holds if K is an ellipsoid.

Chapter 3

The Brunn-Minkowski inequality and the Prékopa-Leindler inequality for measurable sets and functions

The main goal of this chapter is to prove the Brunn-Minkowski inequality for measurable sets, and its functional analogue, the Prékopa-Leindler inequality. We also provide some typical applications of these inequalities.

3.1 The Brunn-Minkowski inequality for measurable sets

Since this chapter can be read essentially independently from the previous ones, we recall some definitions from Chapter 1 about convexity.

Definition 3.1.1 (Convex sets).

- $X \subset \mathbb{R}^n$ convex if $(1-\lambda)x + \lambda y \in X$ for $x, y \in X$ and $\lambda \in [0,1]$;
- $K \subset \mathbb{R}^n$ is a convex body if K convex, compact and $\operatorname{int} K \neq \emptyset$;
- Convex bodies $K, C \subset \mathbb{R}^n$ are homothetic if $C = \gamma K + z$ for some $z \in \mathbb{R}^n$ and $\gamma > 0$.

Definition 3.1.2 (Minkowski combination). If $X, Y \subset \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$, then

$$\alpha X + \beta Y = \{\alpha x + \beta y : x \in X \text{ and } y \in Y\}.$$

Remark. $\alpha X + \beta Y$ may not be measurable even if X and Y are measurable (but measurable, if X and Y Borel, cf. Section 3.9). However, it is compact if X, Y are compact, it is convex if X, Y are convex, and it is open if either X or Y is open and $\alpha, \beta \neq 0$. We lso note that if $X \subset \mathbb{R}^n$ is convex and $\lambda \in (0,1)$, then $(1-\lambda)X + \lambda X = X$.

We write $|X|_*$ to denote the inner Lebesgue measure of an $X \subset \mathbb{R}^n$; namely, it is the maximum of the Lebesgue measure of any measurable subset of X (see Appendix Chapter 10). We have already provided two proofs of the core Brunn-Minkowski inequality below in the case of convex bodies in Chapter 1, are going to provide several other proofs of the general case - due to Lusternik [431] in 1935 - in this chapter.

Theorem 3.1.3 ("Classical" Brunn-Minkowski inequality). *If $X, Y \subset \mathbb{R}^n$ measurable and $\alpha, \beta \geq 0$, then*

$$|\alpha X + \beta Y|_*^{\frac{1}{n}} \geq \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}. \tag{3.1}$$

Equality holds assuming $\alpha, \beta, |X|, |Y| > 0$ if and only if there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$.

Remark. The "essential sum" $Z = \{z \in \mathbb{R}^n : |(z - \alpha X) \cap \beta Y\}| > 0\}$ is measurable assuming $\alpha, \beta, |X|, |Y| > 0$, and $|Z|^{\frac{1}{n}} \geq \alpha|X|^{\frac{1}{n}} + \beta|Y|^{\frac{1}{n}}$.

While the Brunn-Minkowski inequality is one of the most widely used geometric inequalities, Theorem 1.11.7 shows that it is far from optimal if for example $|K| = |C|$ and $|K \cap C|$ is small for centered convex bodies $K, C \subset \mathbb{R}^n$ (see also Section 3.6).

Lemma 3.1.4 (Equivalent forms of Brunn-Minkowski inequality). *The following statements are equivalent assuming that they hold for any measurable $X, Y \subset \mathbb{R}^n$ with $|X|, |Y| > 0$.*

(i) $|\alpha X + \beta Y|_*^{\frac{1}{n}} \geq \alpha|X|^{\frac{1}{n}} + \beta|Y|^{\frac{1}{n}}$ *for* $\alpha, \beta > 0$*;*

(ii) $|X + Y|_*^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$*;*

(iii) $|(1-\lambda) X + \lambda Y|_*^{\frac{1}{n}} \geq (1-\lambda)|X|^{\frac{1}{n}} + \lambda|Y|^{\frac{1}{n}}$ *for* $\lambda \in (0, 1)$*;*

(iv) $|(1-\lambda) X + \lambda Y|_* \geq |X|^{1-\lambda}|Y|^{\lambda}$ *for* $\lambda \in (0, 1)$.

Equality in (i), (ii) or (iii) if and only if there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$. Equality in (iv) if and only if there exist $z \in \mathbb{R}^n$ and convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$, $|C \backslash Y| = 0$ and $K = C + z$.

Proof. (i), (ii) and (iii) are equivalent as $|\alpha X| = \alpha^n |X|$ for $\alpha > 0$, and (iii) yields (iv) by the AM-GM inequality.

To show that (iv) implies (i), set $\alpha_0 = |X|^{\frac{1}{n}}$, $\beta_0 = |Y|^{\frac{1}{n}}$, $X_0 = X/\alpha_0$, $Y_0 = Y/\beta_0$, and hence $|X_0| = |Y_0| = 1$. For $\lambda = \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0}$, (iv) yields $|(1-\lambda) X_0 + \lambda Y_0|_* \geq 1$, and hence

$$|\alpha X + \beta Y|_*^{\frac{1}{n}} = (\alpha\alpha_0 + \beta\beta_0) \left| \frac{\alpha\alpha_0}{\alpha\alpha_0 + \beta\beta_0} X + \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0} Y \right|_*^{\frac{1}{n}} \geq \alpha\alpha_0 + \beta\beta_0. \qquad \blacksquare$$

We provide one proof of the Brunn-Minkowski inequality (3.1) *via* the Prékopa-Leindler inequality Theorem 3.4.2 that also characterizes the equality case, and various other proofs without equality characterization; namely, the probably most elegant one due to Hadwiger and Ohman in Section 3.2, the one *via* Steiner symmetrization in Section 3.8, and the one *via* induction together with the Prékopa-Leindler inequality in Section 3.A.

Figalli, van Hintum, Tiba [223] have proved an essentially optimal stability version of the Brunn-Minkowski Inequality:

Theorem 3.1.5 (Stability of the Brunn-Minkowski Inequality, Figalli, van Hintum, Tiba). *For $n \geq 2$ and $t \in (0, 1/2]$, there exist $c_n, d_{n,t} > 0$ depending on n and n, t such that if the measurable sets $X, Y \subset \mathbb{R}^n$ satisfy that*

$$|(1-t)X + tY| \leq (1+\delta)|X| \text{ and } |X| = |Y| > 0 \tag{3.2}$$

for $\delta \in (0, d_{n,t})$, then there exists a convex body K such that $X, Y - y \subset K$ for some $y \in \mathbb{R}^n$, and

$$|K\backslash X| = |K\backslash(Y - y)| \le c_n t^{-1/2}\delta^{1/2}|X|. \tag{3.3}$$

Remarks.

- The exponents of t and δ are optimal in (3.3). For example, take $X = [0,1]^n$ and $Y = [0, 1+a] \times \left[0, \frac{1}{1+a}\right] \times [0,1]^{n-2}$ for small $a > 0$, and hence $|(1-t)X + tY| \le (1+\delta)|X|$ for $\delta = ta^2$, while $|X\backslash(Y - y)| \ge a/2$ for any $y \in \mathbb{R}^n$.
- The condition that $\delta < d_{n,t}$ in Theorem 3.1.5 for some $d_{n,t} > 0$ depending on n and t is necessary. For example, take $X = [0,1]^n \cap \{p\}$ and $Y = [0,1]^n$ where $\|p\| > 2n$. Then $|X| = |Y| = 1$ and $|(1-t)X + tY| = (1+t^n)|X|$ but $|\mathrm{conv}\, X|$ can be arbitrary large, and hence $d_{n,t} \le t^n$. Actually, $d_{n,t} = t^n$ according to van Hintum, Keevash [312].
- Figalli, van Hintum, Tiba [223] verified an even stronger estimate if we compare the X and Y satisfying (3.2) to their respective convex hulls:

$$|\mathrm{conv} X\backslash X| \le c_{n,t}\delta\, |X| \text{ and } |\mathrm{conv} Y\backslash Y| \le c_{n,t}\delta\, |Y| \tag{3.4}$$

  where $c_{n,t} > 0$ depends on n, t.

As a direct application of the the Brunn-Minkowski inequality (3.1), we prove the Isodiametric inequality Theorem 3.1.8.

Definition 3.1.6 (Diameter). If $X \subset \mathbb{R}^n$ is bounded, then $\mathrm{diam}\, X = \sup_{x,y\in X} \|x - y\|$.

Lemma 3.1.7 (Difference body). *Let $X \subset \mathbb{R}^n$ be bounded and measurable with $|X| > 0$. Then*

- $\frac{1}{2}(X - X) = \frac{1}{2}X + \frac{1}{2}(-X)$ *is origin symmetric;*
- $\mathrm{diam}\frac{1}{2}(X - X) = \mathrm{diam}\, X$*;*
- $|\frac{1}{2}(X - X)|_* \ge |X|$, *with equality if and only if there exists centrally symmetric convex body $K \subset X$ with $|K\backslash X| = 0$.*

Proof. (i) and (ii) follows from the definition, and (iii) from the Brunn-Minkowski inequality (3.1). ■

Theorem 3.1.8 (Isodiametric inequality). *If $X \subset \mathbb{R}^n$ is bounded measurable with $|X| = |rB^n|$ for $r > 0$, then* $\mathrm{diam}\, X \ge 2r$*; or in other words,* $\mathrm{diam}\, X \ge 2\omega_n^{-1/n}|X|^{1/n}$.

Equality holds if and only if there exists $z \in \mathbb{R}^n$ with $X \subset z + rB^n$.

Proof. If $\mathrm{diam}\, X = D$, then $\frac{1}{2}(X - X) \subset \frac{D}{2}B^n$, and $|\frac{1}{2}(X - X)|_* \ge |X| = |rB^n|$, verifying $D \ge 2r$. If $\mathrm{diam}\, X = 2r$, then we may assume that $X \subset K$ for an o-symmetric convex body K with $|K| = |rB^n|$ and $\mathrm{diam}\, K = 2r$ by the equality case of the Brunn-Minkowski inequality (3.1), and hence $K = rB^n$. ■

3.2 The Hadwiger-Ohman proof of the Brunn-Minkowski inequality

In this section, we provide the probably simplest and most elegant proof of the Brunn-Minkowski inequality (3.1) due to Hadwiger and Ohman. We sketch the argument for the characterization of the equality case in Section 3.B because that is more involved.

Theorem 3.2.1. *If $X, Y \subset \mathbb{R}^n$ measurable, then $|X+Y|_*^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$*

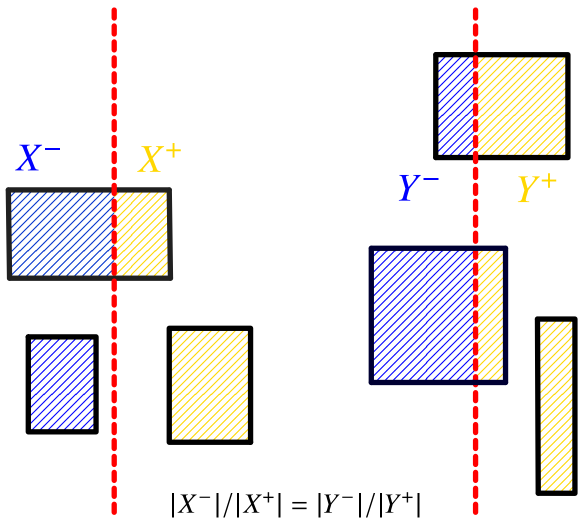
Proof. First we reduce the problem to the case when X and Y are unions of boxes.

We may assume that X and Y are compact by the regularity of the Lebesgue measure (see the Appendix Chapter 10, and hence $X+Y$ compact

Next let $G = \operatorname{int} B^n$. For $\varepsilon > 0$, there exists $\varrho > 0$ such that $|X+Y+\varrho G| < |X+Y| + \varepsilon$, and hence $\left|(X + \frac{\varrho}{2} G) + (Y + \frac{\varrho}{2} G)\right| < |X+Y| + \varepsilon$, thus we may assume that X and Y are bounded open. In turn we may assume that X and Y are unions of finitely many pairwise non-overlapping boxes with edges parallel to vectors in a fixed orthonormal basis $v_1, \ldots, v_n$.

If this is the case, then we prove the Brunn-Minkowski theorem by induction on the total number of boxes used for X and Y. First let the total number of boxes is 2. If X is a box box with edge-lengths $e_1, \ldots, e_n > 0$, and Y is box with edge-lengths $f_1, \ldots, f_n > 0$, then $X+Y$ is a box with edge-lengths $e_1 + f_1, \ldots, e_n + f_n$; therefore, the AM-GM inequality yields

$$\frac{|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}}{|X+Y|^{\frac{1}{n}}} = \left(\prod_{i=1}^{n} \frac{e_i}{e_i + f_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{f_i}{e_i + f_i}\right)^{\frac{1}{n}} \leq \frac{1}{n} \sum_{i=1}^{n} \frac{e_i}{e_i + f_i} + \frac{1}{n} \sum_{i=1}^{n} \frac{f_i}{e_i + f_i} = 1.$$



Finally, let us assume that the total number of boxes is $k \geq 3$. We may assume that X consists of $m \geq 2$ pairwise non-overlapping boxes $B^{(j)} = \oplus_{i=1}^{n} [a_i^{(j)}, b_i^{(j)}] v_i$, and hence there exists $\ell \in \{1, \ldots, n\}$ such that $(a_\ell^{(m-1)}, b_\ell^{(m-1)}) \cap (a_\ell^{(m)}, b_\ell^{(m)}) = \emptyset$. We may assume that $b_\ell^{(m-1)} \le a_\ell^{(m)}$.

Let $X^- = \{x \in X : \langle x, v_\ell \rangle < a_\ell^{(m)}\}$ and $X^+ = \{x \in X : \langle x, v_\ell \rangle > a_\ell^{(m)}\}$, and hence both $\mathrm{cl}\, X^-$ and $\mathrm{cl}\, X^+$ are unions of at most $m-1$ boxes as $B^{(m)} \cap X^- = \emptyset$ and $B^{(m-1)} \cap X^+ = \emptyset$. Choose $\gamma \in \mathbb{R}$ such that for $Y^- = \{y \in Y : \langle y, v_\ell \rangle < \gamma\}$ and $Y^+ = \{y \in Y : \langle y, v_\ell \rangle > \gamma\}$, we have $|X^+|/|X| = q = |Y^+|/|Y| \in (0,1)$, thus $|X+Y| \ge |X^+ + Y^+| + |X^- + Y^-|$ because if $z \in X^+ + Y^+$, then $\langle z, v_\ell \rangle > a_\ell^{(m)} + \gamma$ and if $w \in X^- + Y^-$, then $\langle w, v_\ell \rangle < a_\ell^{(m)} + \gamma$. Now we apply the induction hypothesis to $X^+ + Y^+$ and $X^- + Y^-$ as total number of boxes for $\mathrm{cl}\, X^+$ and $\mathrm{cl}\, Y^+$ is at most $k-1$, and the same holds for $\mathrm{cl}\, X^-$ and $\mathrm{cl}\, Y^-$. We deduce that

$$\begin{aligned}\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n &= q\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n + (1-q)\left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n \\ &= \left(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}\right)^n + \left(|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}}\right)^n \\ &\le |X^+ + Y^+| + |X^- + Y^-| \le |X+Y|.\end{aligned}$$

■

3.3 An application of the Brunn-Minkowski theorem: the Green-Tao theorem on sumsets

For finite $A \subset \mathbb{R}^n$, we write $\#A$ to denote its cardinality.

Proposition 3.3.1. *If $A, B \subset \mathbb{R}^n$ are finite, then $\#(A + B + \{0,1\}^n) \ge 2^n \min\{\#A, \#B\}$.*

Proof. First we consider the case when $A, B \subset \mathbb{Z}^n$. Since $\{0,1\}^n + [0,1]^n = [0,2]^n = [0,1]^n + [0,1]^n$ and $A + B + \{0,1\}^n \subset \mathbb{Z}^n$, Brunn-Minkowski inequality (3.1) yields

$$\begin{aligned}\#(A + B + \{0,1\}^n) &= |A + B + \{0,1\}^n + [0,1]^n| = |A + [0,1]^n + B + [0,1]^n| \\ &\ge \left(|A + [0,1]^n|^{\frac{1}{n}} + (|B + [0,1]^n|^{\frac{1}{n}}\right)^n \\ &\ge 2^n \min\{|A + [0,1]^n|, |B + [0,1]^n|\} = 2^n \min\{\#A, \#B\}\end{aligned}$$

Now let $A, B \subset \mathbb{R}^n$ be finite sets, let cosets of A with respect to $\mathbb{Z}^n$ be $A_1, \ldots, A_m$ where $A_j \subset a_j + \mathbb{Z}^n$ for $a_j \in A$ and $j = 1, \ldots, m$, and let the cosets of B with respect to $\mathbb{Z}^n$ be $B_1, \ldots, B_k$ where $B_i \subset b_i + \mathbb{Z}^n$ for $b_i \in B$ and $i = 1, \ldots, k$. We may assume that $\#A_1 \geq \#B_i$, $i = 1, \ldots, k$; therefore, the previous case on subsets of $\mathbb{Z}^n$ yields

$$\begin{aligned}\#\left(A + B + \{0,1\}^n\right) \geq \#\left(A_1 + B + \{0,1\}^n\right) &= \sum_{i=1}^{k} \#\left(A_1 + B_i + \{0,1\}^n\right) \\ &\geq \sum_{i=1}^{k} 2^n \min\{\#A_1, \#B_i\} \geq \sum_{i=1}^{k} 2^n \cdot \#B_i = 2^n \cdot \#B \\ &\geq 2^n \min\{\#A, \#B\}.\end{aligned}$$

■

Remarks.

- Proposition 3.3.1 is optimal, take $A = B = [0,k]^n \cap \mathbb{Z}^n$ for $k \in \mathbb{N}$
- We recall (see for example Tao, Vu TaV06) that if A, B are a finite subsets of a torsion free Abelian group, then $\#(A+B) \geq \#A + \#B - 1$, where assuming $\#A, \#B \geq 2$, equality holds if and only if A and B are arithmetic progressions of the same difference.
  Proposition 3.3.1 is a key tool in the improvement of Green, Tao [269] on the Freiman-Bilu theorem stating that if $\#(A + A) \leq K \cdot \#A$ for a finite subset A of a torsion free Abelian group and $K > 2$, then A can be covered by a few low dimensional progressions of size at most $\#A$ (where the meaning of "a few" and "low dimensional" depends on K).

3.4 The Prékopa-Leindler inequality

In this section, we introduce the Prékopa-Leindler inequality - that is a functional version of the Brunn-Minkowski inequality - in several equivalent forms. The Prékopa-Leindler inequality is proved together with the chracterization of the equality *via* optimal transport in Section 3.4, and using a more elementary argument - that does not characterize the equality case but more clearly shows the equivalence of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality - in Section 3.A. Let us first introduce the fundamental notion of log-concave functions, which are essentially the extremizers in the Prékopa-Leindler inequality.

Definition 3.4.1 (Log-concave functions). A function $f : \mathbb{R}^n \to [0, \infty)$ is *log-concave* if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ holds for $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$.

Remarks.

- f is log-concave if and only if $f = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$.
- For a log-concave function f, the level sets $\{f > t\}$ are convex for $t \in \mathbb{R}$, and if f is log-concave on an open convex $\Omega \subset \mathbb{R}^n$, then f is continuous on Ω.

- Typical examples are $e^{-\pi\|x\|^2}$ (Gaussian), or $f(x) = e^{-\|x\|_K}$, or $f(x) = e^{-\|x\|_K^2}$ for $\|x\|_K = \min\{t \geq 0 : x \in tK\}$ for a convex body K with $o \in \operatorname{int} K$.
- $f = \mathbf{1}_X$ log-concave for $X \subset \mathbb{R}^n$ if and only if X is convex.

When not signalled, integration is always with respect to the Lebesgue measure on $\mathbb{R}^n$.

Theorem 3.4.2 (Prékopa-Leindler inequality, equality by Dubuc). *Given $\lambda \in (0,1)$, measurable $f, g, h : \mathbb{R}^n \to [0,\infty)$, $n \geq 1$, with $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, we have*

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.5}$$

Equality holds assuming $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$ if and only if for $a = \int_{\mathbb{R}} f / \int_{\mathbb{R}} g$, there exist a log-concave function ψ and $w \in \mathbb{R}^n$ with

- *$h(x) = \psi(x)$ for a.e.x and $h \geq \psi$;*
- *$f(x) = a^\lambda \psi(x + \lambda w)$ for a.e.x and $f(x) \leq a^\lambda \psi(x + \lambda w)$ for each $x \in \mathbb{R}^n$;*
- *$g(x) = a^{-(1-\lambda)} \psi(x - (1-\lambda)w)$ for a.e.x and $g(x) \leq a^{-(1-\lambda)} \psi(x - (1-\lambda)w)$ for each $x \in \mathbb{R}^n$.*

Remark. See Section 3.A for an elementary proof, not yielding the equality case, of the Prékopa-Leindler inequality (3.5).

Using $\int_{*,\mathbb{R}^n}$ to denote the inner integral, the Prékopa-Leindler inequality can be written in the following form:

Theorem 3.4.3 (Prékopa-Leindler inequality for only f, g). *For $\lambda \in (0,1)$ and non-negative $f, g \in L_1(\mathbb{R}^n)$, $n \geq 1$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dz \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.6}$$

Equality holds assuming $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$ if and only if there exist $w \in \mathbb{R}^n$ and log-concave φ such that for $a = \int_{\mathbb{R}^n} f / \int_{\mathbb{R}^n} g$, we have $g(x) = \varphi(x)$ and $f(x) = a \cdot \varphi(x + w)$ for a.e. x, and $g(x) \leq \varphi(x)$ and $f(x) \leq a \cdot \varphi(x + w)$ for each $x \in \mathbb{R}^n$.

Remark. The Prékopa-Leindler inequality (3.6) yields the Brunn-Minkowski inequality Theorem 3.1.3 taking $f = \mathbf{1}_X$ and $g = \mathbf{1}_Y$ as $\sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda = \mathbf{1}_{(1-\lambda)X+\lambda Y}(z)$ in this case.

While $z \mapsto \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda$ may not be measurable even if f and g are measurable (see Sierpiński [533]), it is measurable if f, g are Borel, and is log-concave, and hence measurable, if f, g log-concave. We observe that Theorem 3.4.3 is slightly stronger than Theorem 3.4.2 because Theorem 3.4.2 yields directly (3.6) only for the outer integral.

Corollary 3.4.4 (Prékopa-Leindler inequality for multiple functions). *For non-negative $f_1, \ldots, f_k \in L_1(\mathbb{R}^n)$ and $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=\sum_{i=1}^k \lambda_i x_i} \prod_{i=1}^k f_i(x_i)^{\lambda_i}\, dz \geq \prod_{i=1}^k \left(\int_{\mathbb{R}^n} f_i \right)^{\lambda_i}. \tag{3.7}$$

Equality holds assuming each $\int f_i > 0$ if and only if there exist log-concave function φ, $a_1, \ldots, a_k > 0$ and $w_1, \ldots, w_k \in \mathbb{R}^n$ such that $\sum_{i=1}^k \lambda_i w_i = o$ and $f_i(x) = a_i \varphi(x + w_i)$ for a.e. x and $f_i(x) \leq a_i \varphi(x + w_i)$ for each $x \in \mathbb{R}^n$.

3.5 Proof of Prékopa-Leinder via optimal transport

The following property of the Brenier map in optimal transport can be found for example in Villani [558], Theorem 4.14:

Theorem 3.5.1 (Brenier, McCann, Cafferelli). *If f, g are positive, bounded, C^1 probability densities on $\mathbb{R}^n$, then there exists a C^2 convex function φ on $\mathbb{R}^n$ such that $T = D\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is a diffeomorphism satisfying*

$$f(x) = g(T(x)) \cdot \det DT(x) = g(D\varphi(x)) \cdot \det D^2\varphi(x), \tag{3.8}$$

or in other words, $\int_A g = \int_{T^{-1}A} f$ for measurable $A \subset \mathbb{R}^n$.

Remarks.

- $DT = D^2\varphi$ is symmetric positive definite (because φ is convex).
- $\det D^2\varphi = \dfrac{f}{g \circ T}$ is a Monge-Ampère equation for φ.
- If $n = 1$, then T can be simply chosen to be the monotone increasing function satisfying $\int_{-\infty}^x f = \int_{-\infty}^{T(x)} g$, and hence readily $\frac{f(x)}{g(T(x))} = T'(x)$.

Lemma 3.5.2. *If $\lambda \in (0, 1)$, and A, B are symmetric positive definite $n \times n$ matrices, then*

$$\det((1 - \lambda)A + \lambda B) \geq (\det A)^{1-\lambda} (\det B)^{\lambda}, \tag{3.9}$$

with equality if and only if $A = B$.

Proof. There exists $M \in \mathrm{GL}(n)$ such that $I_n = M^t A M$ and $M^t B M$ is positive definite diagonal matrix; therefore, we may assume that $A = I_n$ and $B = \mathrm{diag}[b_1, \ldots, b_n]$. We deduce that $(1 - \lambda)A + \lambda B = \mathrm{diag}[1 - \lambda + \lambda b_1, \ldots, 1 - \lambda + \lambda b_n]$, and in turn deduce (3.9) from the AM-GM inequality applied to the eigenvalues. ∎

Lemma 3.5.3. *If $a > 0$ and the C^1 map $T : \mathbb{R}^n \to \mathbb{R}^n$ satisfies that $DT(x)$ is symmetric and all of its eigenvalues are at least a, then T is a diffeomorphism.*

Proof. For $x \neq y$, the function $f(s) = \langle y - x, T((1-s)x + sy)\rangle$ satisfies that $f'(s) \geq a\|y-x\|^2$, and hence $f(1) > f(0)$, which in turn yields that $T(y) \neq T(x)$. Therefore T is a diffeomorphism onto an open set $U \subset \mathbb{R}^n$.

Let $w_0 = T(o) \in U$. As U is open, $U = \mathbb{R}^n$ follows from the following claim: If $v \in S^{n-1}$ and $\tilde{s} > 0$, and $w_0 + sv \in U$ for $s \in [0, \tilde{s})$, then $w_0 + \tilde{s}v \in U$. We observe that the gradient of the inverse $T^{-1} : U \to \mathbb{R}^n$ of T is also always a symmetric matrix, and all of its eigenvalues are at most a^{-1}. It follows that $\|x_k\| \leq a^{-1}\tilde{s}$ for $x_k = T^{-1}(w_0 + (1 - \frac{1}{k})\tilde{s}v)$ and $k \geq 2$; therefore, a subsequence $\{x_{k'}\}$ tends to a x satisfying $Tx = w_0 + \tilde{s}v$. ■

Remark. For measurable $f, g : \mathbb{R}^n \to [0,\infty)$ with $0 < \int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g < \infty$ in the Prékopa-Leindler inequality, f or g can be freely shifted, as if $h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda$ and $w \in \mathbb{R}^n$, then $h(z + \lambda w) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y+w)^\lambda$ satisfies

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y+w)^\lambda \, dz = \int_{*,\mathbb{R}^n} h(z)\, dz. \tag{3.10}$$

Proposition 3.5.4 (Prékopa-Leindler inequality (3.6) for positive C^1 functions). *For positive C^1 functions $f, g \in L_1(\mathbb{R}^n)$ and $\lambda \in (0,1)$,*

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dz \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.11}$$

Equality holds if and only if f, g are log-concave and $f(x) = a \cdot g(x+w)$ for $a > 0$ and $w \in \mathbb{R}^n$.

Proof. We may assume that $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g = 1$ and f, g are bounded as if f or g is unbounded, then $\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dx = \infty$.

Let $T = D\varphi$ be the Brenier map satisfying (3.8). Since $D\left((1-\lambda)I_n + \lambda T\right)$ is symmetric, and all of its eigenvalues are at least $1-\lambda$, Lemma 3.5.3 yields that $(1-\lambda)I_n + \lambda T$ is a diffeomorphism onto $\mathbb{R}^n$.

For $z \in \mathbb{R}^n$, we define $h(z) = f(x)^{1-\lambda} g(T(x))^\lambda$ if $z = (1-\lambda)x + \lambda T(x)$ for $x \in \mathbb{R}^n$. We use Lemma 3.5.2 and substitution $z = (1-\lambda)x + \lambda T(x)$ with $dz = \det((1-\lambda)I_n + \lambda\, DT)(x)\, dx$ to obtain

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} f(x)\, dx = \int_{\mathbb{R}^n} f(x)^{1-\lambda} g(T(x))^\lambda \det DT(x)^\lambda \, dx \\ &\leq \int_{\mathbb{R}^n} h\Big((1-\lambda)x + \lambda T(x)\Big) \det[(1-\lambda)I_n + \lambda DT(x)]\, dx \\ &= \int_{\mathbb{R}^n} h(z)\, dz \leq \int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^\lambda \, dz. \end{aligned}$$

If equality holds in (3.11), then $DT(x) = I_n$ for $x \in \mathbb{R}^n$ by Lemma 3.5.2, and hence there exists $w \in \mathbb{R}^n$ such that $T(x) = x + w$. It follows that $g(x+w) = f(x)$ for $x \in \mathbb{R}^n$.

We may assume $w = o$ by (3.10)), thus $h(z) = f(z)$. Since equality in (3.11) yields $f(z) = h(z) = \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$, we conclude that f is log-concave. ∎

We recall that for $f, \tilde{f} \in L_1(\mathbb{R}^n)$, their convolution is $f * \tilde{f}(z) = \int_{\mathbb{R}^n} f(x)\tilde{f}(z-x)\,dx$ where $\int_{\mathbb{R}^n} f * \tilde{f} = (\int_{\mathbb{R}^n} f)(\int_{\mathbb{R}^n} \tilde{f})$ (cf. the Appendix Chapter 10).

Proof of Prékopa-Leindler inequality (3.6) for measurable f and g. We may assume $\int f, \int g > 0$. Approximating f and g by step functions from below (a step function is of the form $\sum_{i=1}^{k} t_i \mathbf{1}_{A_i}$ for $t_1, \dots, t_k > 0$ and pairwise disjoint measurable $A_1, \dots, A_k$), we may assume that $f = \sum_{i=1}^{k} s_i \mathbf{1}_{B_i}$ for $s_1, \dots, s_k > 0$ and pairwise disjoint compact $B_1, \dots, B_k$ and $g = \sum_{i=1}^{m} r_i \mathbf{1}_{C_i}$ for $r_1, \dots, r_m > 0$ and pairwise disjoint compact $C_1, \dots, C_k$, and hence $z \mapsto \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ is measurable.

Next we approximate these f and g by decreasing sequence of positive C^∞ functions $\{f_m\}$ and $\{g_m\}$ from above. To construct such approximating sequences, enough to find a decreasing positive C^∞ sequence $\{\psi_m\}$ for $\mathbf{1}_X$ where X is compact. Let open U_m be such that $\cap_m U_m = X$ and $\mathrm{cl} U_m \subset U_{m-1}$, thus there exists $r_m > 0$ such that $U_m + r_m B^n \subset U_{m-1}$ and $U_{m+1} + r_m B^n \subset U_m$. Now we define $\phi_m = \mathbf{1}_{U_m} * h_m$ where $h_m : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is a C^1 function such that $\mathrm{supp} h_m \subset r_m B^n$ and $\int_{\mathbb{R}^n} h_m = 1$, and hence $\mathbf{1}_{U_{m-1}} \leq \phi_m \leq \mathbf{1}_{U_{m+1}}$. Therefore, we can choose $\psi_m = \phi_{2m}(x) + \frac{1}{m} e^{-\|x\|^2}$.

Since f and g have compact support, $\sup_{z=(1-\lambda)x+\lambda y} f_m(x)^{1-\lambda} g_m(y)^{\lambda}$ tends to $\sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}$ for any $z \in \mathbb{R}^n$. Therefore Proposition 3.5.4 applied to f_m, g_m completes the proof of Prékopa-Leindler inequality (3.6). ∎

Extremizers: Non-negative $f, g \in L_1(\mathbb{R}^n)$ extremizers for the Prékopa-Leindler inequality (3.6) for $\lambda \in (0,1)$ if $\int f, \int g > 0$ and

$$\int_{*,\mathbb{R}^n} \sup_{z=(1-\lambda)x+\lambda y} f(x)^{1-\lambda} g(y)^{\lambda}\,dz = \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$

For $F : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $\int_{*,\mathbb{R}^n} F = \int_{\mathbb{R}^n} \widetilde{F}$ for measurable $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ if $\widetilde{F} \leq F$, and the set $\{F > \widetilde{F}\}$ has no measurable subset with positive measure. In this case, we say that $\widetilde{F}$ is a witness for F.

Lemma 3.5.5. *Given $\lambda \in (0,1)$, if $f_1, f_2 \in L_1(\mathbb{R}^n)$ are extremizers for the Prékopa-Leindler inequality (3.6), and the same holds for $g_1, g_2 \in L_1(\mathbb{R}^n)$ then the convolutions $f_1 * g_1, f_2 * g_2$ are also extremizers for (3.6).*

Proof. To simplify the formulas, we set $\lambda = c_1$ and $1 - \lambda = c_2$, and may assume that $\int_{\mathbb{R}^n} f_1 = \int_{\mathbb{R}^n} f_2 = \int_{\mathbb{R}^n} g_1 = \int_{\mathbb{R}^n} g_2 = 1$. We define (assuming that $x_i, y_i, z_i \in \mathbb{R}^n$)

$$F(x) = \sup_{x=\sum_{i=1}^{2} c_i x_i} \prod_{i=1}^{2} f_i(x_i)^{c_i} \text{ and } G(y) = \sup_{y=\sum_{i=1}^{2} c_i y_i} \prod_{i=1}^{2} g_i(y_i)^{c_i}.$$

Possibly F and G are not measurable but as f_1, f_2 and g_1, g_2 are extremizers, there exist measurable $0 \le \widetilde{F} \le F$ and $0 \le \widetilde{G} \le G$ such that $\int_{\mathbb{R}^n} \widetilde{F}(x)\,dx = \int_{\mathbb{R}^n} \widetilde{G}(x)\,dx = 1$, and neither $\{F > \widetilde{F}\}$ nor $\{G > \widetilde{G}\}$ contains a subset of $\mathbb{R}^n$ with positive measure. We write any point of $\mathbb{R}^{2n}$ in the form (x, y) for $x, y \in \mathbb{R}^n \Longrightarrow (z, y) \mapsto \widetilde{F}(z)\widetilde{G}(y)$ is a witness for $(z, y) \mapsto F(z)G(y)$, and hence $(x, y) \mapsto \widetilde{F}(x - y)\widetilde{G}(y)$ is a witness for $(x, y) \mapsto F(x - y)G(y)$.

Setting $x_i = z_i + y_i$ in (3.12) , we have

$$\begin{aligned}
1 &= \int_{\mathbb{R}^n} \widetilde{F} * \widetilde{G} = \int_{\mathbb{R}^{2n}} \widetilde{F}(x-y)\widetilde{G}(y)\,d(x,y) \\
&= \int_{*,\mathbb{R}^{2n}} \sup_{\substack{x-y=\sum_{i=1}^2 c_i z_i \\ z_i \in \mathbb{R}^n}} \sup_{\substack{y=\sum_{i=1}^2 c_i y_i \\ y_i \in \mathbb{R}^n}} \prod_{i=1}^2 f_i(z_i)^{c_i} \prod_{i=1}^2 g_i(y_i)^{c_i}\,d(x,y) \qquad (3.12) \\
&= \int_{*,\mathbb{R}^{2n}} \sup_{\substack{x=\sum_{i=1}^2 c_i x_i \\ x_i \in \mathbb{R}^n}} \sup_{\substack{y=\sum_{i=1}^2 c_i y_i \\ y_i \in \mathbb{R}^n}} \prod_{i=1}^2 f_i(x_i - y_i)^{c_i} \prod_{i=1}^2 g_i(y_i)^{c_i}\,d(x,y) \\
&\ge \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^k c_i x_i} \int_{*,\mathbb{R}^n} \sup_{y=\sum_{i=1}^k c_i y_i,\, y_i \in \mathbb{R}^n} \prod_{i=1}^2 \left(f_i(x_i - y_i) g_i(y_i)\right)^{c_i}\,dydx,
\end{aligned}$$

and hence using Prékopa-Leindler inequality (3.6) yields

$$\begin{aligned}
1 &\ge \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^2 c_i x_i} \prod_{i=1}^k \left(\int_{E_i} f_i(x_i - y_i) g_i(y_i)\,dy_i \right)^{c_i} dx \\
&= \int_{*,\mathbb{R}^n} \sup_{x=\sum_{i=1}^2 c_i x_i} \prod_{i=1}^2 \left(f_i * g_i(x_i)\right)^{c_i} dx \ge \prod_{i=1}^2 \left(\int_{\mathbb{R}^n} f_i * g_i \right)^{c_i} = 1;
\end{aligned}$$

therefore, $f_i * g_i$, $i = 1, 2$, are also extremizers. ■

Characterizing equality in Prékopa-Leindler (3.6) for measurable f, g. For $h \in L_1(\mathbb{R}^n)$, let $\hat{h}$ be the Fourier transform $\hat{h}(z) = \int_{\mathbb{R}^n} h(x) e^{-2\pi i \langle x, z\rangle}\,dx$.

For $\psi(x) = e^{-\pi\|x\|^2}$, we have $\int_{\mathbb{R}^n} \psi = 1$ and the second partial derivatives $\partial_{\alpha\beta}\psi$ bounded for $\alpha, \beta = 1, \ldots, n$. For $r > 0$, let $\psi_r(x) = r^{-n}\psi(x/r)$, and hence ψ_r is log-concave (and hence equality holds for $f = g = \psi$ in the Prékopa-Leindler inequality (3.6)), $\int_{\mathbb{R}^n} \psi_r = 1$, and $\widehat{\psi_r}(x) = \psi(rx)$, and for any $h \in L_1(\mathbb{R}^n)$, $h * \psi_r$ is C^1 for $r > 0$ (cf. Lemma 10.2.2), and

$$\lim_{r \to 0^+} h * \psi_r(z) = h(z) \text{ if } z \in \mathbb{R}^n \text{ is a density point of } h. \qquad (3.13)$$

Now let f, g be measurable extremizers for the Prékopa-Leindler inequality (3.6) with $\int f, \int g > 0$. We may assume that $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} g$. Lemma 3.5.5 yields that for

$r > 0$, the positive C^1 functions $f * \psi_r, g * \psi_r \in L_1(\mathbb{R}^n)$ are extremizers, and $\int_{\mathbb{R}^n} f * \psi_r = \int_{\mathbb{R}^n} g * \psi_r$. We deduce from Proposition 3.5.4 that there exists $w \in \mathbb{R}^n$ such that $f * \psi_r(x) = g * \psi_r(x + w)$ log-concave.

We may assume that $w = 0$ (otherwise we can exchange g by $x \mapsto g(x + w)$ by (3.10)). It follows $f * \psi_r = g * \psi_r$, and hence $\hat{f} \cdot \widehat{\psi_r} = \widehat{f * \psi_r} = \widehat{g * \psi_r} = \hat{g} \cdot \widehat{\psi_r}$. We conlude that $\hat{f} = \hat{g}$; therefore, $f(x) = g(x)$ for a.e.$x \in \mathbb{R}^n$.

Finally have to show that f is a.e. log-concave. We deduce from Proposition 3.5.4 that $f * \psi_r$ log-concave, thus for $t \in (0, 1)$ and $x, y \in \mathbb{R}^n$, we have $f * \psi_r((1 - t)x + ty) \geq (f * \psi_r)(x)^{1-t}(g * \psi_r)(y)^t$, and hence (3.13) for $t \in (0, 1)$ yields

$$f((1 - t)x + ty) \geq f(x)^{1-t} g(y)^t \quad \text{if } (1 - t)x + ty, x, y \in Z \tag{3.14}$$

where Z is the set of density points of f with $|\mathbb{R}^n \backslash Z| = 0$.

For $x \in \mathbb{R}^n$, let $\varphi(x) = \limsup_{r\to 0^+} \frac{\int_{x+rB^n} f}{|rB^n|}$ where φ measurable and $\varphi(z) = f(z)$ for $z \in Z$, thus (3.14) implies that φ is log-concave. ■

3.6 Coordinatewise product of unconditional convex bodies

In this section, we show how the Prékopa-Leindler inequality (see Section 3.4) for log-concave functions yields a strengthening of the Brunn-Minkowski inequality (3.1.3) for unconditional convex bodies.

Definition 3.6.1. A convex body $K \subset \mathbb{R}^n$ is unconditional if $(\pm x_1, \ldots, \pm x_n) \in K$ for $(x_1, \ldots, x_n) \in K$.

The Minkowski linear combination $\frac{1}{2} X + \frac{1}{2} Y$ can be considered as an arithmetic mean of $X, Y \subset \mathbb{R}^n$. We now introduce an operation that can be considered as the geometric mean of two unconditional convex bodies.

Definition 3.6.2 (Coordinatewise product). If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0, 1)$, then

$$K^{1-\lambda} \cdot C^\lambda = \left\{ \left(\pm|x_1|^{1-\lambda}|y_1|^\lambda, \ldots, \pm|x_n|^{1-\lambda}|y_n|^\lambda \right) : (x_1, \ldots, x_n) \in K \;\&\; (y_1, \ldots, y_n) \in C \right\}$$

Remark. Readily, $K^{1-\lambda} \cdot C^\lambda \subset (1 - \lambda)K + \lambda C$. To show that $M = K^{1-\lambda} \cdot C^\lambda$ is an unconditional convex body, first we observe that if $0 \leq s_i \leq t_i$ for $i = 1, \ldots, n$, and $(t_1, \ldots, t_n) \in M$, then $(s_1, \ldots, s_n) \in M$. From this property the convexity of M easily follows.

Example 3.6.3. $K^{1-\lambda} \cdot C^\lambda$ might be much smaller than $(1 - \lambda)K + \lambda C$. For example, if $K = [a, a] \times [\frac{1}{a}, \frac{1}{a}]$ and $C = [\frac{1}{a}, \frac{1}{a}] \times [a, a]$ for $a > 1$, then

$$\begin{aligned} K^{\frac{1}{2}} \cdot C^{\frac{1}{2}} &= [-1, 1]^2 \\ \tfrac{1}{2} K + \tfrac{1}{2} C &\supset [-\tfrac{a}{2}, \tfrac{a}{2}]^2. \end{aligned}$$

In particular, $|\frac{1}{2} K + \frac{1}{2} C|$ can be arbirary large if a is large, while $|K^{\frac{1}{2}} \cdot C^{\frac{1}{2}}| = 4$.

Example 3.6.3 shows that the following inequality can be a substantial improvement on the Brunn-Minkowski inequality (3.1.3).

Theorem 3.6.4 (Uhrin-Bollobas-Leader, equality by Saroglou). *If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$\left|K^{1-\lambda} \cdot C^{\lambda}\right| \geq |K|^{1-\lambda}|C|^{\lambda}. \tag{3.15}$$

Equality holds if and ony if $K = \Phi C$ for a positive definit diagonal matrix Φ.

Proof. To prove (3.15), we apply the Prékopa-Leindler inequality (3.5) to the functions

$$\begin{aligned} f(x_1,\ldots,x_n) &= \mathbf{1}_K(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n} \\ g(x_1,\ldots,x_n) &= \mathbf{1}_C(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n} \\ h(x_1,\ldots,x_n) &= \mathbf{1}_{K^{1-\lambda}\cdot C^{\lambda}}(e^{x_1},\ldots,e^{x_n})e^{x_1+\ldots+x_n} \end{aligned}$$

If equality holds in (3.15), then we may assume that $|K| = |C|$, and hence the equality conditions in the Prékopa-Leindler inequality (3.5) imply that $f(x_1,\ldots,x_n) = g(x_1+b_1,\ldots,x_n+b_n)$ for $(b_1,\ldots,b_n) \in \mathbb{R}^n$; therefore, $\Phi = \mathrm{diag}(e^{b_1},\ldots,e^{b_n})$. ■

3.7 Hausdorff metric on the space of compact subsets

We have already seen in Section 1.7 that the Hausdorff distance between compact convex sets provides a natural topology on the space of compact convex subsets of $\mathbb{R}^n$. In this section, we show that the natural habitat of the Hausdorff distance is the space of compact subsets, as Hausdorff introduced it more than a century ago.

Definition 3.7.1. Let (Ξ, d) be metric space where each closed ball is compact.

Diameter If $X \subset \Xi$ compact, then $\mathrm{diam}\, X = \max\{d(x,y) : x, y \in X\}$.

Parallel domain For compact $X \subset \Xi$ and $\varrho \geq 0$, let

$$X^{(\varrho)} = \{z \in \Xi : \exists x \in X \text{ with } d(x,y) \leq \varrho\},$$

which is compact, as well, as each closed ball is compact.

Hausdorff distance For compact $X, Y \subset \Xi$, their Hausdorff distance is

$$\delta_H(X,Y) = \min\left\{\varrho \geq 0 : X \subset Y^{(\varrho)} \text{ and } Y \subset X^{(\varrho)}\right\}.$$

It is equivalent to fiding the minimal $\varrho \geq 0$ such that for any $x \in X$, there exists $y \in Y$ with $d(x,y) \leq \varrho$, and for any $y \in Y$, there exists $x \in X$ with $d(x,y) \leq \varrho$.

Remark. δ_H is readily a metric on the space of compact subsets of Ξ. For compact $X_m, Y \subset \Xi$, we write $\lim_{m\to\infty} X_m = Y$ (X_m tends to Y), if $\lim_{m\to\infty} \delta_H(X_m, Y) = 0$.

The following property directly follows from the definition:

Lemma 3.7.2. *Let (Ξ, d) be a metric space where each closed ball is compact. For compact $X_m, Y \subset \Xi$, $\lim\limits_{m\to\infty} X_m = Y$ if and only if*

- *$\{X_m\}$ bounded (contained in a fixed ball);*
- *for any $y \in Y$, there exists $x_m \in X_m$ such that $\lim_{m\to\infty} x_m = y$;*
- *for any sequence $x_m \in X_m$, any accumulation point y of $\{x_m\}$ lies in Y.*

Theorem 3.7.3 (Hausdorff). *If (Ξ, d) metric space where each closed ball is compact, and $X_m \subset \Xi$ is a bounded sequence of compact subsets, then there exists a subsequence $\{X_{m'}\} \subset \{X_m\}$ and a compact $Z \subset \Xi$ with $\lim_{m'\to\infty} X_{m'} = Z$.*

Proof. We may assume that (Ξ, d) is compact, and consider the compact meric ball $B(z, r)\{x \in \Xi : d(x, y) \le r\}$ for $z \in \Xi$ and $r > 0$.

For any integer $k \ge 1$, we consider an $1/k$-net $Y_k \subset \Xi$, and hence Y_k is finite. We take subsequences of $\{X_m\}$ for $k = 1, 2, \ldots$ by induction on k to evetually construct a $\{X_{m'}\} \subset \{X_m\}$ with the following property: For any $k \ge 1$, there exists a threshold N_k and $Y'_k \subset Y_k$ such that if $m' > N_k$, then $B(y, \frac{1}{k}) \cap X_{m'} \neq \emptyset$ for $y \in Y'_k$ and $B(z, \frac{1}{k}) \cap X_{m'} = \emptyset$ for $z \in Y_k \backslash Y'_k$, and hence $X_{m'} \subset \cup_{y\in Y'_k} B(y, \frac{1}{k}) = Z_k$. It follows that $Z = \cap_{k=1}^{\infty} Z_k$ is a nn-empty compact set, and $\lim_{m'\to\infty} X_{m'} = Z$. ■

Lemma 3.7.4. *Let (Ξ, d) be metric space where each closed ball is compact, and let μ be a Borel measure on Ξ such that for compact $X \subset \Xi$, $\mu(X) < \infty$ and $\mu(X) = \inf\{\mu(U) : U \supset X \text{ open}\}$. Let $\lim_{m\to\infty} X_m = Y$ for compact $X_m, Y \subset \Xi$.*

(i) $\operatorname{diam} Y = \lim_{m\to\infty} \operatorname{diam} X_m$;

(ii) $\mu(Y) \ge \limsup_{m\to\infty} \mu(X_m)$;

(iii) $\mu(Y^{(\varrho)}) \le \liminf_{m\to\infty} \mu(X_m^{(\varrho+\varepsilon)})$ *for* $\varrho, \varepsilon > 0$.

Proof. (i) follows from the definition of δ_H and sequential compactness.

For (ii), choose any open $U \subset \Xi$ with $Y \subset U$, and choose $\varrho > 0$ with $Y^{(\varrho)} \subset U$. Since $X_m \subset Y^{(\varrho)} \subset U$ for large m, we deduce that $\mu(X_m) \le \mu(U)$.

For (iii), the set $U = \left\{z \in \Xi : \exists y \in Y \text{ with } d(z, y) < \varrho + \frac{\varepsilon}{2}\right\}$ is open and $Y^{(\varrho)} \subset U \subset X_m^{(\varrho+\varepsilon)}$ for large m. ■

3.8 Steiner symmetrization of compact sets and the Brunn-Minkowski inequality

We have seen in Section 1.10 how to use Steiner symmetrization to prove the Brunn-Minkowski inequality for convex bodies. In this section, we extend the method to compact subsets of $\mathbb{R}^n$. First we prove the Brunn-Minkowski inequality in $\mathbb{R}$ for compact sets.

Lemma 3.8.1 (One-dimensional Brunn-Minkowski). *If $A, B \subset \mathbb{R}$ compact, then*

$$\mathcal{H}^1(A+B) \geq \mathcal{H}^1(A) + \mathcal{H}^1(B). \tag{3.16}$$

Proof. Assume $\max A = \min B = 0$, and hence $A, B \subset A + B$ and $A \cap B = \{0\}$. ∎

Definition 3.8.2 (Steiner symmetrization of compact sets). If $X \subset \mathbb{R}^n$ is compact, and $u \in S^{n-1}$, then

$$\Theta_{u^\perp} X = \bigcup \left\{ x + [-q, q]u : x \in X|u^\perp \text{ and } \mathcal{H}^1\Big(X \cap (x + \mathbb{R}u)\Big) = 2q \right\}.$$

Lemma 3.8.3. *Let $X, Y \subset \mathbb{R}^n$ be compact, and let $u \in S^{n-1}$.*

(i) $\Theta_{u^\perp} X$ is symmetric through $u^\perp$, and $\Theta_{u^\perp} X = rB^n$ if $X = rB^n$ for $r > 0$;

(ii) $\Theta_{u^\perp} X$ is compact, and $\Theta_{u^\perp} X$ is convex if X is convex;

(iii) $\operatorname{diam} \Theta_{u^\perp} X \leq \operatorname{diam} X$*;*

(iv) $|\Theta_{u^\perp} X| = |X|$;

(v) $|\alpha \Theta_{u^\perp} X + \beta \Theta_{u^\perp} Y)| \leq |\alpha X + \beta Y|$ for $\alpha, \beta > 0$.

Proof. (i) follows by definition, and (iv) by Fubini's theorem. For (ii), $\Theta_{u^\perp} X$ is compact by Lemma 3.7.4 (ii), and if X is convex, then Lemma 1.10.3 yields that $\Theta_{u^\perp} X$ convex.

For (iii), as $\Theta_{u^\perp} X$ is symmetric through $u^\perp$, there exist $a_1, a_2 \in X|u^\perp$ and $t_1, t_2 \geq 0$ such that $a_i \pm t_i u \in \Theta_{u^\perp} X$ and $\operatorname{diam} \Theta_{u^\perp} X = \|(a_1 + t_1 u - (a_2 - t_2 u)\| = \|(a_1 - a_2) + (t_1 + t_2)u\|$, thus there exist $r_i \geq s_i$ with $r_i \geq s_i + 2t_i$ and $a_i + r_i u, a_i + s_i u \in X$ for $i = 1, 2$. Assuming $r_1 - s_2 \geq r_2 - s_1$ (otherwise switch a_1 and a_2), we have $r_1 - s_2 \geq t_1 + t_2$, and hence $\operatorname{diam} X \geq \|(a_1 + r_1 u - (a_2 + s_2 u)\| = \sqrt{\|a_1 - a_2\|^2 + (r_1 - s_2)^2} \geq \operatorname{diam} \Theta_{u^\perp} X$.

For (v), we may assume that $\alpha = \beta = 1$, and it is enough to prove that $\Theta_{u^\perp} X_1 + \Theta_{u^\perp} X_2 \subset \Theta_{u^\perp}(X_1 + X_2)$ for compact $X_1, X_2 \subset \mathbb{R}^n$ by (iv) where $\Theta_{u^\perp} X_1 + \Theta_{u^\perp} X_2$ is symmetric through $u^\perp$.

Let $a_i + t_i u \in \Theta_{u^\perp} X_i$ for $a_i \in X_i|u^\perp$ and $t_i \geq 0$, and hence $\mathcal{H}^1\Big(X_i \cap (a_i + \mathbb{R}u)\Big) \geq 2t_i$, which in turn yields that $\mathcal{H}^1\Big((X_1 + X_2) \cap (a_1 + a_2 + \mathbb{R}u)\Big) \geq 2(t_1 + t_2)$ by the one-dimensional Brunn-Minkowski inequality (3.16); therefore, $a_1 + a_2 + (t_1 + t_2)u \in \Theta_{u^\perp}(X_1 + X_2)$. ∎

Lemma 3.8.4. *If $u_1, \ldots, u_n$ orthonormal basis of $\mathbb{R}^n$ and $X \subset \mathbb{R}^n$ compact, then $\Theta_{u_n^\perp} \circ \ldots \circ \Theta_{u_1^\perp} X$ is symmetric through $u_1^\perp, \ldots, u_n^\perp$ and is o-symmetric.*

Proof. If $Y \subset \mathbb{R}^n$ compact, $1 \leq i < j \leq n$ and Y is symmetric through $u_i^\perp$, then $\Theta_{u_j^\perp} Y$ is also symmetric through $u_i^\perp$. ∎

Theorem 3.8.5 (Iterated Steiner symmetrizations). *If $X \subset \mathbb{R}^n$ is compact with $|X| = |rB^n|$ for $r > 0$, and $\varepsilon \in (0, \frac{1}{2})$, then there exist finitely many Steiner symmetrizations* *starting with X producing a X' with $|X' \cap rB^n| \geq (1-\varepsilon)|rB^n|$. In particular, there exists a sequence $\{X_m\}$ of compact sets tending to a compact set $\widetilde{X}$ with $rB^n \subset \widetilde{X}$ where $X_0 = X$ and $X_{m+1} = \Theta_{u_m^\perp} X_m$, $u_m \in S^{n-1}$.*

Proof. According to Lemma 3.8.4, we may assume that X is o-symmetric, and let $X \subset RB^n$ for $R > 0$. Writing $\mathcal{F}_X$ to denote the family of convex bodies resulting from finitely many iterated Steiner symmetrizations starting from X, we have $|C| = |rB^n|$ for $C \in \mathcal{F}_X$ by Lemma 3.8.3, and Theorem 3.8.5 is equivalent to proving that

$$\Xi = \sup\{|C \cap rB^n| : C \in \mathcal{F}_X\} = |rB^n|. \tag{3.17}$$

The argument is indirect, so we suppose that $\Xi < |rB^n|$, and seek a contradiction. For $k \geq 2$, let $C_k \in \mathcal{F}_X$ such that $|C_k| > \Xi - \frac{1}{k}$, and hence

$$\lim_{k\to\infty} |C_k \cap rB^n| = \Xi. \tag{3.18}$$

As $C_k \subset RB^n$, we may assume that $C_k \to C_0$ for an o-symmetric compact set C_0 by Theorem 3.7.3. In addition, we may assume that the compact sets $C_k' = C_k \backslash \mathrm{int}(rB^n)$ tend to some compact set, and the compact sets $C_k \cap rB^n$ tend to some compact set. We claim that

$$|C_0| = |rB^n|. \tag{3.19}$$

$|C_k| = |X|$ for $k \geq 2$ and Lemma 3.7.4 (ii) yield that $|C_0| \geq |rB^n|$, and hence all we need to verify is that $|C_0| \leq |rB^n| + \eta$ for any $\eta > 0$. Choose $\alpha > 0$ such that $X + \alpha B^n < |X| + \eta$, and hence $C_k + \alpha B^n < |X| + \eta$ for $k \geq 2$ by Lemma 3.8.3 (v), which estimate implies $|C_0| \leq |rB^n| + \eta$, and in turn (3.19) by Lemma 3.7.4 (iii).

Next we claim that

$$|C_0 \cap rB^n| = \Xi. \tag{3.20}$$

Since $\lim_{k\to\infty} C_k' \subset C_0 \backslash \mathrm{int}(rB^n)$ and $\lim_{k\to\infty} |C_k'| = |rB^n| - \Xi$ by (3.18), we deduce from Lemma 3.7.4 (ii) that $|C_0 \backslash \mathrm{int}(rB^n)| \geq |rB^n| - \Xi$. Moreover Lemma 3.7.4 (ii), (3.18) and $\lim_{k\to\infty} C_k \cap rB^n \subset C_0 \cap rB^n$ imply $|C_0 \cap rB^n| \geq \Xi$, and hence we conclude (3.20) by $|C_0 \backslash \mathrm{int}(rB^n)| \geq |rB^n| - \Xi$.

As $|C_0 \cap rB^n| = \Xi < |rB^n|$, there exist $z_0 \in \mathrm{int}(rB^n) \backslash C_0$ and $\varrho > 0$ such that $z_0 + \varrho B^n \subset \mathrm{int}(rB^n) \backslash C_0$ and $|(r + 3\varrho)B^n| - |rB^n| < \frac{1}{2}(|rB^n| - \Xi)$. It follows from $|C_k| = |rB^n|$ and $|C_k \cap rB^n| \leq \Xi$ that $|\widetilde{C}_k| \geq \frac{1}{2}(|rB^n| - \Xi)$ for

$$\widetilde{C}_k = C_k \backslash (r + 3\varrho)B^n$$

and $k \geq 2$. Considering the averge of the integral of $|(z + \varrho B^n) \cap \widetilde{C}_k|$ for $z \in (R + \varrho)B^n \backslash (r + 2\varrho)B^n$, we deduce that existence of $z_k \in (R + \varrho)B^n \backslash (r + 2\varrho)B^n$ such that

$$|(z_k + \varrho B^n) \cap \widetilde{C}_k| \geq \gamma |\varrho B^n| \ \text{ for } \gamma = \frac{\frac{1}{2}(|rB^n| - \Xi)}{|(R + \varrho)B^n \backslash (r + 2\varrho)B^n|}, \tag{3.21}$$

while $(z_k + \varrho B^n) \cap rB^n = \emptyset$ as $z_k \notin (r + 2\varrho)B^n$.

There exists k large enough such that $\frac{1}{k} < \gamma\,|\varrho B^n|$ and $z_0 + \varrho B^n \subset (rB^n)\backslash C_k$; therefore, if $u_k = \frac{z_0 - z_k}{\|z_0 - z_k\|}$ and $\|x - x_k\| < \varrho$ for $x \in u_k^\perp$ and $x_k = z_k|u_k^\perp$, then $\ell_x = x + \mathbb{R}u_k$ satifies

$$\begin{aligned}\mathcal{H}^1\left(\ell_x \cap rB^n \cap \Theta_{u_k^\perp} C_k\right) &= \min\left\{\mathcal{H}^1\left(\ell_x \cap rB^n\right), \mathcal{H}^1\left(\ell_x \cap C_k\right)\right\}\\ &\geq \mathcal{H}^1\left(\ell_x \cap rB^n \cap C_k\right) + \mathcal{H}^1\left(\ell_x \cap (z_k + \varrho B^n) \cap C_k\right),\end{aligned}$$

and hence (3.21) implies

$$\left|rB^n \cap \Theta_{u_k^\perp} C_k\right| \geq |C_k| + \gamma\,|\varrho B^n| > \Xi,$$

which is a contradiction verifying (3.17). ∎

Theorem 3.8.6 (Brunn-Minkowski via Steiner symmetrization).
If $X, Y \subset \mathbb{R}^n$ are measurable and $\alpha, \beta \geq 0$, then

$$|\alpha X + \beta Y|_*^{\frac{1}{n}} \geq \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}.$$

Proof. As the Lebesgue is regular, we may assume that X, Y compact with $|X| = |rB^n|$ and $|Y| = |RB^n|$ for $r, R > 0$, and as the Lebesgue is n homogeneous, we may assume that $\alpha = \beta = 1$. It is sufficient to prove that for any $\varepsilon > 0$, we have

$$|X + Y| + \varepsilon \geq \left(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}\right)^n. \tag{3.22}$$

Choose $\varrho > 0$ such that $|X + Y + \varrho B^n| < |X + Y| + \varepsilon$.

We consider the sequences X_m, Y_m of compact sets provided by Theorem 3.8.5 such that $X = X_0$ and $Y = Y_0$, X_{m+1} and Y_{m+1} are Steiner symmetrials of X_m and Y_m, respectively, and X_m and Y_m tend to compact sets $\widetilde{X}$ and $\widetilde{Y}$, respectively, satisfying $rB^n \subset \widetilde{X}$ and $RB^n \subset \widetilde{Y}$. It follows from Lemma 3.7.2 that $X_m + Y_m$ tends to $\widetilde{X} + \widetilde{Y}$. Applying Lemma 3.7.4 (iii), and then Lemma 3.8.3 (v) yields that

$$\begin{aligned}|(r+R)B^n| &= \left(|rB^n|^{\frac{1}{n}} + |RB^n|^{\frac{1}{n}}\right)^n \leq \left(|\widetilde{X}|^{\frac{1}{n}} + |\widetilde{Y}|^{\frac{1}{n}}\right)^n\\ &\leq \liminf_{m\to\infty} |X_m + Y_m + \varrho B^n| \leq |X + Y + \varrho B^n| < |X + Y| + \varepsilon,\end{aligned}$$

proving (3.22). ∎

Actually, there is a stronger version of Theorem 3.8.5 where one uses a fixed set of hyperplanes independent of the convex body.

Theorem 3.8.7 (Bianchi, Gardner, Gronchi [71,72]). *Let $v_1, \ldots, v_n \in S^{n-1}$ be independent such that $\langle v_i, v_j\rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for irrational $\alpha \in (0,1)$, and let $u_{kn+i} = v_i$ for $k \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$.*

If $X \subset \mathbb{R}^n$ is a compact set with $|X| = |rB^n|$ for $r > 0$, then X_m tends to rB^n where $X_0 = X$ and $X_{m+1} = \Theta_{u_m^\perp} X_m$.

3.9 Comments to Chapter 3

The Brunn-Minkowski inequality for convex bodies was proved by Brunn [131] in dimensions $n = 2, 3$, and by Minkowski in any dimensions (see Section 3.B for their argument). It was Minkowski's work [465] where the importance of the inequality was recognized, and it has found its place within a whole, now called Brunn-Minkowski, theory.

There is a trivial way to construct examples of measurable subsets $X, Y \subset \mathbb{R}^2$ such that $X + Y$ is not measurable; namely, for an orthornormal basis $e_1, e_2 \in \mathbb{R}^2$, let $X \subset \mathbb{R}e_1$ and $Y \subset \mathbb{R}e_2$ be non-measurable subsets with respect to the one-dimensional Lebesgue measure. To construct X and Y spanning $\mathbb{R}^2$, one can use the measurable subsets $A, B \subset \mathbb{R}$ constructed by Sierpiński [533] such that $A + B$ is non-measurable, and take $X = A \times [0, 1]$ and $Y = B \times [0, 1]$.

However, Minkowski linear combination of Borel sets is analytic (in the sense of Descriptive Set Theory, see Kechris [363]), and hence measurable. Here a set is analytic if it is the continuous image of a Borel subset of a complete separable metric space, and if $X, Y \in \mathbb{R}^n$ Borel and $\alpha, \beta > 0$, then $\alpha X + \beta Y$ is the image of $X \times Y$ by the map $(x, y) \mapsto \alpha x + \beta y$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^n$.

One of the early arguments proving the Brunn-Minkowski inequality for convex bodies is due to Hilbert, and is based on a spectral gap estimate for a differential operator (see Bonnesen, Fenchel [81] for the original argument of Hilbert). This approach was further developed by Aleksandrov [3, 7] leading to the generalization Aleksandrov-Fenchel inequality of the Brunn-Minkowski inequality (see Section 8.5.2 based on additional ideas by van Handel, Shenfeld [300]), and also to the L_p-Minkowski inequality Theorem 8.8.5 improving the Brunn-Minkowski inequality for origin symmetric convex bodies by Chen, Huang, Li, Liu [154], based on the local result by Kolesnikov, Milman [381]. Another fundamental approach proving the Brunn-Minkowski inequality for convex bodies is initiated by Gromov's influential appendix to Milman, Schechtman [461] (using ideas by Knothe [376]) providing a proof of the Isoperimetric Inequality Theorem 2.4.1 using optimal transport, and the argument can be readily extended to the Brunn-Minkowski inequality and the Prékopa-Leindler inequality. This approach lead even to the stability versions Theorem 8.6.4 and Corollary 8.6.5 of the Brunn-Minkowski inequality by Figalli, Maggi, Pratelli [224, 225]. We note that the original argument of Brunn and Minkowski (see Section 3.B) can be also considered as a version of the mass transportation approach.

The Brunn-Minkowski inequality for measurable subsets of $\mathbb{R}^n$ is proved by Lusternik [431] in 1935, and the equality case is clarified by Henstock, Macbeath [309] much later. The beautiful proof in Section 3.2 is due to Hadwiger, Ohmann [297]. This idea also leads to a simple proof of the Brunn-Minkowski inequality with exponent $\frac{1}{n}$ in a simply connected nilpotent Lie group of topological dimension n (see Pozuelo [491] and (3.24)), and is also extensively used in the study of log-concave measures by Borell [86].

The application of the Brunn-Minkowski inequality in Section 3.3 is due to Green, Tao [269] (see also Tao, Vu [548]).

The Prékopa-Leindler inequality was proved by Prékopa [492,493], Leindler [399], Borell [86], and the case of equality was originally characterized by Dubuc [195], and later Balogh, Kristály [44] provided an argument using displayment convexity in optimal transport. The idea to use optimal transport to prove the Brunn-Minkowski inequality is due to Knoethe [376], and was popularized later by Gromov (see Milman, Schechtman [461]). The argument characterizing equality in the Prékopa-Leindler inequality in Section 3.4 is new. Stability versions of the Prékopa-Leindler inequality were verified by Böröczky, De [94] for log-concave functions, and by Böröczky, Figalli, Ramos [98] in general.

A generalization of the Prékopa-Leindler inequality is the Borell-Brascamp-Lieb inequality proved by Borell [86] and Brascamp-Lieb [124]. For $p \in \mathbb{R}$, $a, b > 0$ and $\lambda \in (0,1)$, let $\mathcal{M}_\lambda^p(a,b) = ((1-\lambda)a^p + \lambda\, b^p)^{\frac{1}{p}}$ and let $\mathcal{M}_\lambda^{-\infty}(a,b) = \min\{a,b\}$. Now the Borell-Brascamp-Lieb inequality says that if $p \geq -\frac{1}{n}$, $\lambda \in (0,1)$ and measurable $f, g, h : \mathbb{R}^n \to [0,\infty)$ with positive integrals satisfy that $h((1-\lambda)x + \lambda\, y) \geq \mathcal{M}_\lambda^p(f(x), g(y))$ whenever $f(x), g(y) > 0$, then

$$\int_{\mathbb{R}^n} h \geq \mathcal{M}_\lambda^{\frac{p}{1+pn}}\left(\int_{\mathbb{R}^n} f, \int_{\mathbb{R}^n} g\right). \tag{3.23}$$

If $p = 0$, then we get back the Prékopa-Leindler inequality. Equality in the Borell-Brascamp-Lieb inequality is charactherized by Balogh, Kristály [44] using displayment convexity in optimal transport.

The Prékopa-Leindler inequality, and even the Borell-Brascamp-Lieb inequality was generalized to Riemannian manifolds by Cordero-Erausquin, Schmuckenschläger, McCann [177] (see Cordero-Erausquin [173] for the spherical and hyperbolic case) where equality is charactherized by Balogh, Kristály [44]. In these papers, $h((1-\lambda)x + \lambda\, y)$ in the condition of the Prékopa-Leindler inequality (or in the more general Borell-Brascamp-Lieb inequality) is replaced by $h(z_\lambda(x,y))$ for $\lambda \in (0,1)$ where $z_\lambda(x,y)$ is the point of the geodesic segment between x and y dividing the length of the geodesic segment in ratio $\lambda : (1-\lambda)$.

Tao [547] also verified the following analogue of the Prékopa-Leindler inequality for a simply connected nilpotent group G of topological dimension n similarly to the argument in Section 3.A. Let μ be a Haar measure on G (μ is unimodular in this case; namely, both left and right invariant) and let $\lambda \in (0,1)$. If non-negative measurable functions f, g, h on G satisfy that $h(xy) \geq f(x)^{1-\lambda} g(y)^\lambda$ for x, y, then

$$\int_G h\, d\mu \geq \frac{1}{(1-\lambda)^{n(1-\lambda)}\lambda^{n\lambda}} \left(\int_G f\, d\mu\right)^{1-\lambda} \left(\int_G g\, d\mu\right)^{\lambda}.$$

Actually, this unusual Prékopa-Leindler type inequality still yields the Brunn-Minkowski inequality in G according to Tao [547]; namely, if $X \cdot Y \subset Z$ for measurable $X, Y, Z \subset G$, then

$$\mu(Z)^{\frac{1}{n}} \geq \mu(X)^{\frac{1}{n}} + \mu(Y)^{\frac{1}{n}}. \tag{3.24}$$

In the proof of the Prékopa-Leindler inequality in Section 3.5, one can also use the analogue of Knothe map initated by Knothe [376] instead of the Brenier map (see i.e. Maggi [439]). For positive C^1 probability measures f, g on $\mathbb{R}^n$, the Knothe map $T(x_1, \ldots, x_n) = (T_1(x_1), T_2(x_1, x_2), \ldots, T_n(x_1, \ldots, x_n))$ is defined as follows:

$$\int_{\{(z_1,\ldots,z_n):z_1 \leq T_1(x_1)\}} f = \int_{\{(y_1,\ldots,y_n):y_1 \leq x_1\}} g;$$

$$\int_{\{(z_1,\ldots,z_n):z_1 \leq T_1(x_1),\ z_2 \leq T_2(x_1,x_2)\}} f = \int_{\{(y_1,\ldots,y_n):y_1 \leq x_1,\ y_2 \leq x_2\}} g;$$

and so on. In particular, we have again (3.8). Since DT is lower triangular, one can use the analogue of Lemma 3.5.2 for lower triangle matrices.

Theorem 3.6.4 for the coordinatewise product is proved independently by Uhrin [556], Bollobás, Leader [80] and Cordero-Erausquin, Fradelizi, Maurey [174], and the equality case was clarified by Saroglou [508].

For in depth studies on Steiner symmetrization and Schwarz symmetrization of compact sets, see Bianchi, Gardner, Gronchi [71, 72].

3.A Supplement: Common elementary proof of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality by induction

We present the probably most elementary proof of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality (using now outer measure, not the inner measure). While this argument exhibits the equivalence of the Brunn-Minkowski inequality and the Prékopa-Leindler inequality, it does not seem to lead to a characterization of the equality case.

Theorem 3.A.1. *Let $\lambda \in (0, 1)$ and $\alpha, \beta \geq 0$.*

Brunn-Minkowski inequality *For measurable $X, Y, Z \subset \mathbb{R}^n$, if $\alpha X + \beta Y \subset Z$, then*

$$|Z|^{\frac{1}{n}} \geq \alpha|X|^{\frac{1}{n}} + \beta|Y|^{\frac{1}{n}}; \tag{3.25}$$

Prékopa-Leindler inequality *for measurable $f, g, h : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, if $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$, then*

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^{\lambda}. \tag{3.26}$$

Proof. **Step 1** Brunn-Minkowski inequality when $n = 1$.

We may assume that X and Y are compact by the regularity of the Lebesgue measure, and $\max X = \min Y = 0$. Then $\alpha X, \beta Y \subset Z$, and $\alpha X \cap \beta Y = \{0\}$, yielding (3.25).

Step 2 Prékopa-Leindler inequality when $n = 1$.

We may assume that $f, g \in L_1(\mathbb{R})$, are bounded and $\sup f = \sup g = 1$. As the condition on f, g, h yields that $(1-\lambda)\{f > t\} + \lambda\{g > t\} \subset \{h > t\}$ for $t \in (0,1)$; therefore, the layer cake formula, the Brunn-Minkowski inequality when $n = 1$ and AM-GM inequality imply

$$\int_R h \geq \int_0^1 \mathcal{H}^1(\{h > t\})\, dt \geq \int_0^1 (1-\lambda)\mathcal{H}^1(\{f > t\}) + \lambda\mathcal{H}^1(\{g > t\})\, dt$$
$$= (1-\lambda)\int_{\mathbb{R}} f + \lambda \int_{\mathbb{R}} g \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$

Step 3 Prékopa-Leindler inequality for $n \geq 1$ by induction.

Assume that the inequality holds in $\mathbb{R}^{n-1}$ for $n \geq 2$. For a $u \in S^{n-1}$ and $t \in \mathbb{R}$, let

$$F(t) = \int_{u^\perp + tu} f\, d\mathcal{H}^{n-1}, \quad G(t) = \int_{u^\perp + tu} g\, d\mathcal{H}^{n-1}, \quad H(t) = \int_{u^\perp + tu} h\, d\mathcal{H}^{n-1}.$$

Since $h((1-\lambda)x + \lambda y + tu) \geq f(x + ru)^{1-\lambda} g(y + su)^{\lambda}$ for $x, y \in u^\perp$ and $r, s, t \in \mathbb{R}$ with $(1-\lambda)r + \lambda s = t$, the Prékopa-Leindler inequality in $u^\perp$ implies that $H((1-\lambda)r + \lambda s) \geq F(r)^{1-\lambda} G(s)^{\lambda}$ for $r, s \in \mathbb{R}$. In turn the Fubini theorem and the one-dimensional Prékopa-Leindler inequality for F, G, H yield

$$\int_{\mathbb{R}^n} h = \int_{\mathbb{R}} H \geq \left(\int_{\mathbb{R}} F\right)^{1-\lambda}\left(\int_{\mathbb{R}} G\right)^{\lambda} = \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda}\left(\int_{\mathbb{R}^n} g\right)^{\lambda}.$$

Step 4 Brunn-Minkowski inequality for $n \geq 1$

If $\lambda \in (0,1)$ and measurable $X, Y, Z \subset \mathbb{R}^n$ satisfy $(1-\lambda)X + \lambda Y \subset Z$, then the Prékopa-Leindler inequality for $f = \mathbf{1}_X$, $g = \mathbf{1}_Y$ and $h = \mathbf{1}_Z$ implies that $|Z| \geq |X|^{1-\lambda}|Y|^{\lambda}$. The general Brunn-Minkowski inequality follows by the n-homogeneity of the Lebesgue measure (see Lemma 3.1.4). ■

3.B Supplement: Equality in the Brunn-Minkowski inequality for measurable sets *via* Henstock-Macbeath and Hadwiger-Ohmann

This section sketches the argument to classify equality in the Brunn-Minkowski inequality (3.1.3) based on the papers Henstock, Macbeath [309] and Hadwiger, Ohmann [297]. One of the equivalent forms of the Brunn-Minkowski inequality (cf. Lemma 3.1.4) is that if $X, Y \subset \mathbb{R}^n$ are measurable, then

$$|X + Y|_*^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}.$$

Theorem 3.B.1 (Equality in Brunn-Minkowski inequality). *If $X, Y \subset \mathbb{R}^n$ measurable with $|X|, |Y| \in (0, \infty)$ and $|X+Y|_*^{\frac{1}{n}} = |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$, then there exist homothetic convex bodies $K \supset X$ and $C \supset Y$ with $|K \backslash X| = 0$ and $|C \backslash Y| = 0$.*

In order to reduce the equality case of Brunn-Minkowski inequality to compact sets, we use the notion of inner density points following Henstock, Macbeath [309]. For bounded $Z \subset \mathbb{R}^n$, we say that $z \in \mathbb{R}^n$ is an inner density point if

$$\lim_{\varrho \to 0^+} \frac{|Z \cap (z + \varrho\, B^n)|_*}{|\varrho\, B^n|} = 1.$$

Let Z_* be the set of inner density points associated to Z, and hence $|Z_*| = |Z_* \cap Z| = |Z|_*$ follows from using Lebesgue's density theorem to a measurable set of maximal measure in Z. We also observe that if Z is compact, then $Z_* \subset Z$. The following simple observation is due to Henstock, Macbeath [309]:

Lemma 3.B.2 (Henstock, Macbeath). *If $X, Y \subset \mathbb{R}^n$ are bounded measurable, then $X_* + \mathrm{cl}\, Y \subset (X+Y)_*$.*

To handle the equality case of Brunn-Minkowski for compact sets, we also need the following estimates due to Hadwiger, Ohmann [297] H^+, H^- are complementrary closed halfspaces if $H^+ \cap H^-$ is the common boundary.

Lemma 3.B.3. *If $|X|, |Y| > 0$ for compact $X, Y \subset \mathbb{R}^n$, and $z \in \mathbb{R}^n$ and complementary closed halfspace H^+, H^- satisfy that $0 < |X^+|/|X| = |Y^+|/|Y| < 1$ for $X^+ = X \cap H^+$, $X^- = X \cap H^-$, $Y^+ = Y \cap (z + H^+)$ and $Y^- = Y \cap (z + H^-)$, then*

(i) $\frac{|X+Y|^{\frac{1}{n}}}{|X|^{\frac{1}{n}}+|Y|^{\frac{1}{n}}} \geq \min\left\{\frac{|X^++Y^+|^{\frac{1}{n}}}{|X^+|^{\frac{1}{n}}+|Y^+|^{\frac{1}{n}}}, \frac{|X^-+Y^-|^{\frac{1}{n}}}{|X^-|^{\frac{1}{n}}+|Y^-|^{\frac{1}{n}}}\right\}$

(ii) Equality holds in (i) if and only if $|X+Y| = |X^+ + Y^+| + |X^- + Y^-|$ and $\frac{|X^++Y^+|^{\frac{1}{n}}}{|X^+|^{\frac{1}{n}}+|Y^+|^{\frac{1}{n}}} = \frac{|X^-+Y^-|^{\frac{1}{n}}}{|X^-|^{\frac{1}{n}}+|Y^-|^{\frac{1}{n}}}$

Proof. We may assume that $o \in \partial H^+$ and $z = o$, and hence $X^+ + Y^+ \subset H^+$ and $X^- + Y^- \subset \mathrm{cl}(\mathbb{R}^n \backslash H^+)$. It follows that $|X+Y| \geq |X^+ + Y^+| + |X^- + Y^-|$ and

$$\begin{aligned}
\frac{|X+Y|}{(|X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}})^n} &= \frac{|X+Y|}{(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}})^n + (|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}})^n} \\
&\geq \frac{|X^+ + Y^+| + |X^- + Y^-|}{(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}})^n + (|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}})^n} \\
&\geq \min\left\{\frac{|X^+ + Y^+|}{\left(|X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}\right)^n}, \frac{|X^- + Y^-|}{\left(|X^-|^{\frac{1}{n}} + |Y^-|^{\frac{1}{n}}\right)^n}\right\}.
\end{aligned}$$

The characterization of the case of equality in (i) readily follows. ∎

Sketch of the proof of Theorem 3.B.1. X and Y are bounded because otherwise $|X + Y|_* = \infty$. We may replace X and Y by their closures by Lemma 3.B.2, and hence we may assume that X and Y are compact.

We claim that

$$X \subset \mathrm{cl}\,\mathrm{conv}X_* \text{ and } Y \subset \mathrm{cl}\,\mathrm{conv}Y_*. \tag{3.27}$$

The argument is indirect, we suppose that there exist for example an $x_0 \in X\backslash\widetilde{X}$ for $\widetilde{X} = \mathrm{cl}\,\mathrm{conv}X_*$ and $\widetilde{Y} = \mathrm{cl}\,\mathrm{conv}Y_*$. For the closest point $\tilde{x} \in \widetilde{X}$ to x_0 (cf. Lemma 1.2.2), let $\tilde{y} \in \partial\widetilde{Y}$ be a boundary point with exterior normal $x_0 - \tilde{x}$, and let $y \in Y_*$ with $\|y - \tilde{y}\| < \frac{1}{2}\|x_0 - \tilde{x}\|$. Then $x_0 + y \in X + Y$ is a density point not contained in $\widetilde{X} + \widetilde{Y}$, which fact leads to a contradiction as $|\widetilde{X} + \widetilde{Y}|^{\frac{1}{n}} \geq |X|^{\frac{1}{n}} + |Y|^{\frac{1}{n}}$ by the Brunn-Minkowski inequality, verifying (3.27).

Next we observe that

$$x_1, x_2 \in X_* \text{ and } \mathrm{conv}\{x_1, x_2\} \subset X \text{ yield } \mathrm{conv}\{x_1, x_2\} \subset X_* \tag{3.28}$$

by the Brunn-Minkowski inequality, and the analogous statement holds for Y.

Now we prove that

$$X = \mathrm{cl}\,\mathrm{conv}X_* \text{ and } Y = \mathrm{cl}\,\mathrm{conv}Y_*. \tag{3.29}$$

The argument is indirect, we suppose that for example $X \neq \mathrm{cl}\,\mathrm{conv}X_*$. It follows from (3.27) and (3.28) that there exist $x_1, x_2 \in X_*$ and $\lambda \in (0,1)$ with $w = (1-\lambda)x_1 + \lambda x_2 \notin X$, and hence writing $e_1, \ldots, e_n$ to denote an orthonomal basis of $\mathbb{R}^n$ with $e_n = \frac{x_1 - x_2}{\|x_1 - x_2\|}$, there exists $a > 0$ such that $(w + [-a,a]^n) \cap X = \emptyset$ and $w + [-a,a]^n \subset \mathrm{int}\,\mathrm{conv}X_*$. By translating X, we may assume that $w = o$.

Setting $X_0 = X$ and $Y_0 = Y$, we construct $X_1, \ldots, X_{2n-2}$ and $Y_1, \ldots, Y_{2n-2}$ by induction on $i = 1, \ldots, 2n-2$ such that $X_i = X_{i-1} \subset H_i^+$ for $H_i^+ = \{x : \langle x, e_i\rangle \leq a\}$ if $i = 1, \ldots, n-1$ and $H_i^+ = \{x : \langle x, -e_i\rangle \leq a\}$ if $i = n, \ldots, 2n-2$, and compact $Y_1, \ldots, Y_{2n-2}$ are constructed using Lemma 3.B.3 in a way such that $0 < |X_i|/|X| = |Y_i|/|Y| < 1$ and $|X_i + Y_i|^{\frac{1}{n}} = |X_i|^{\frac{1}{n}} + |Y_i|^{\frac{1}{n}}$ for $i = 1, \ldots, 2n-2$.

Set $\overline{X} = X_{2n-2}$, $\overline{Y} =, X_{2n-2}$, and hence $X^+ = H^+ \cap \overline{X}$ and $X^- = H^- \cap \overline{X}$ for $H^+ = \{x : \langle x, e_n\rangle \geq 0\}$ and $H^- = \{x : \langle x, e_n\rangle \leq 0\}$ satisfy that $\alpha^+ = \min\{\langle x, e_n\rangle : x \in X^+\} \geq a$, $\alpha^- = \max\{\langle x, e_n\rangle : x \in X^-\} \leq -a$, $|X^+| > 0$ and $|X^-| > 0$ as $x_1 \in X^+$ and $x_2 \in X^-$. Let us translate Y in a way such that $0 < |X^+|/|X| = |Y^+|/|Y| < 1$ for $Y^+ = Y \cap H^+$ and $Y^- = Y \cap H^-$, and let $\beta^+ = \min\{\langle y, e_n\rangle : y \in Y^+\} \geq 0$ and $\beta^- = \max\{\langle y, e_n\rangle : y \in Y^-\} \leq 0$. We deduce from Lemma 3.B.3 that $|\overline{X} + \overline{Y}| = |X^+ + Y^+| + |X^- + Y^-|$ and $|X^+ + Y^+|^{\frac{1}{n}} = |X^+|^{\frac{1}{n}} + |Y^+|^{\frac{1}{n}}$.

Since $Y^+ \subset \mathrm{cl}\,\mathrm{conv}Y_*^+$ by (3.27), there exists $y \in Y_*^+$ with $\beta_+ < \langle y, e_n\rangle < \beta_+ + a$. Choose $x \in X^-$ with $\langle x, e_n\rangle = \alpha_-$, and hence

$$\max\{\langle z, e_n\rangle : z \in X^- + Y^-\} \leq \alpha^- < \langle x + y, e_n\rangle < \beta_+ + a \leq \min\{\langle z, e_n\rangle : z \in X^+ + Y^+\}.$$

Therefore, $x + y$ is a density point of $\overline{X} + \overline{Y}$ contained neither in $X^- + Y^-$ nor in $X^+ + Y^+$, contradicting $|\overline{X} + \overline{Y}| = |X^+ + Y^+| + |X^- + Y^-|$, and verifying (3.29).

Finally, (3.29) implies that X and Y are convex bodies. We deduce from the equality case of the Brunn-Minkowski inequality for convex bodies (see Theorem 1.12.3) that X and Y are homothetic. ■

Chapter 4

The Isoperimetric inequality in the case of Lipschitz boundary

4.1 The Isoperimetric inequality in the Euclidean space

The isoperimetric inequality; namely, that the surface area of a body of given volume is minimized by balls, is probably the most funcdamental geometric inequality. However, while what the notion of volume should be has been essentially clear for millenia - that is the Lebesgue measure in today's terms, - the suitable notions of surface area have been understood only the last decades. In the main part of the book, surface area in $\mathbb{R}^n$ is the $(n-1)$-dimensional Hausdorff measure of Lipschitz hypersurfaces. We note that the "right notion" seems to be the more technical Finite Perimeter, which is reviewed in Section 5.1.

First we recall from Appendix Chapter 10 the notions we need to speak about surface area in this chapter.

Definition 4.1.1 (Hausdorff measure $\mathcal{H}^\alpha$). For $\alpha > 0$, $\varepsilon > 0$ and $X \subset \mathbb{R}^n$, let

$$\mathcal{H}^\alpha_{(\varepsilon)}(X) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} Z_i)^\alpha : X \subset \cup_{i=1}^{\infty} Z_i \text{ and } \forall \operatorname{diam} Z_i < \varepsilon\right\}.$$

The Hausdorff outer measure is $\mathcal{H}^{*,\alpha}(X) = \lim_{\varepsilon\to 0^+}\frac{\omega_\alpha}{2^\alpha}\cdot\mathcal{H}^\alpha_{(\varepsilon)}(X)$, and $\mathcal{H}^\alpha$ is the corresponding Borel measure (here $\omega_n = |B^n|$).

Remark. Concerning normalization, $\mathcal{H}^n(X) = |X|$ for Borel $X \subset \mathbb{R}^n$, which follows by the Isodiameteric Inequality Theorem 3.1.8.

The key objects in our study are the Lipschitz functions and their images.

Lemma 4.1.2 (Lipschitz function and Rademacher's theorem).
Let $L > 0$, $f : X \to Z$ be surjective for $X \subset \mathbb{R}^n$ and $Z \subset \mathbb{R}^m$ such that $\|f(x) - f(y)\| \le L\cdot\|x - y\|$ for $x, y \in X$.

- *$\mathcal{H}^\alpha(Z) \le L^\alpha\cdot\mathcal{H}^\alpha(X)$ for any $\alpha > 0$. In particular, $\mathcal{H}^\alpha(Z) = 0$ if $\mathcal{H}^\alpha(X) = 0$.*
- *If X is open, then f is $\mathcal{H}^n$ a.e differentiable; namely, there exists $m\times n$ matrix $Df(z)$ at a.e. $z \in X$ with*

$$f(x) = f(z) + Df(z)(x - z) + o(\|x - z\|);$$

or in other words, $\lim_{x\to z}\frac{f(x)-f(z)-Df(z)(x-z)}{\|x-z\|} = 0.$

Basic properties of compact sets with rectifiable boundary are discussed for example in Federer [212] and Ambrosio, Colesanti, Villa [18]:

Definition 4.1.3 (Rectifiable boundary). For a compact $X \subset \mathbb{R}^n$ that is the closure of its non-empty interior, we say that X has rectifiable boundary if and ∂X is the union of finitely many sets that are Lipschitz images of compact subsets of $\mathbb{R}^{n-1}$. In this case, $0 < \mathcal{H}^{n-1}(\partial X) < \infty$, and for $\mathcal{H}^{n-1}$ a.e. $x \in \partial X$, there exists a unique exterior unit normal $\nu_X(x) \in S^{n-1}$ to X at x, and hence $x - t\nu_X(x) \in \operatorname{int} X$ for small $t > 0$. In particular, a small neighbourhood of x on ∂X is a graph of a Lipschitz function $F : U \to \mathbb{R}^n$ on an open neighbourhood $U \subset \mathbb{R}^{n-1}$ of $o \in \mathbb{R}^{n-1}$ with $F(o) = x$, the differential $DF(o)$ exists and is of rank $n-1$, and $\nu_X(x)$ is ortogonal to the image of $DF(o)$.

Remarks. For example, any convex body has rectifiable boundary (cf. Lemma 1.5.6). Compact sets with rectiafiable boundary are sets of finite perimeter (cf. Example 5.1.4).

In this book, we mostly use the notion of surface area; namely, the Minkowski content, that is derived from volume of the *parallel domain* $X^{(\varrho)} = X + \varrho B^n$ of a compact X with rectifiable boundary. The following stament is proved for example in Federer [212], Theorem 3.2.39, and in Ambrosio, Colesanti, Villa [18].

Theorem 4.1.4 (Minkowski content). *For a compact $X \subset \mathbb{R}^n$ that is the closure of its non-empty interior and has rectifiable boundary, its Minkowski content is*

$$S(X) = P(X) = \lim_{\varrho \to 0^+} \frac{|X^{(\varrho)}| - |X|}{\varrho} = \mathcal{H}^{n-1}(\partial X).$$

Remarks.

- Theorem 4.1.4 yields for example that $S(B^n) = n|B^n| = n\omega_n$.
- $S(X) = P(X)$ has various names, it is called; for example, perimeter, Minkowski content or surface area of X (see Chapter 5 for the generalization of the notion to sets of finite perimeter).
- $S(\gamma X) = \gamma^{n-1} S(X)$ for $\gamma > 0$ and $S(\Phi X + z) = S(X)$ for $\Phi \in O(n)$ and $z \in \mathbb{R}^n$.

Theorem 4.1.5 (Isoperimetric Inequality in $\mathbb{R}^n$). *For a compact $X \subset \mathbb{R}^n$ that is the closure of its non-empty interior and has rectifiable boundary, and $|X| = |rB^n|$ for $r > 0$, then $S(X) \geq S(rB^n)$; or equivalently,*

$$S(X) \geq n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}}. \tag{4.1}$$

Remark. According to Theorem 5.2.1, equality holds in (4.1) if and only if X is a Euclidean ball, and the Isoperimetric Inequality (4.1) holds even if X has finite perimeter (using the appropriate definition of surface area). The Isoperimetric Inequality for convex bodies is discussed in Theorem 2.4.1.

Proof. The Brunn-Minkowski inequality (3.1) yields

$$\lim_{\varrho\to 0^+}\frac{|X+\varrho B^n|-|X|}{\varrho}\geq \lim_{\varrho\to 0^+}\frac{\left(|X|^{\frac{1}{n}}+\varrho|B^n|^{\frac{1}{n}}\right)^n-|X|}{\varrho}$$
$$=\lim_{\varrho\to 0^+}\frac{|(r+\varrho)B^n|-|rB^n|}{\varrho}=S(rB^n).$$

■

After various attempts, satisfactory stability version of the Isoperimetric Inequality Theorem 4.1.5 was provided by Fusco, Maggi, Pratelli [251] (actually, even in the more general framework of sets of finite perimeter, cf. Theorem 5.2.2), whose estimate was improved by Figalli, Maggi, Pratelli [225]. To state the result, for compact sets $X,Y\subset\mathbb{R}^n$, let $\alpha=|X|^{\frac{-1}{n}}$ and $\beta=|Y|^{\frac{-1}{n}}$, and let

$$A(X,Y)=\min\left\{|\alpha X\Delta(z+\beta Y)|\,:\,z\in\mathbb{R}^n\right\}.$$

Theorem 4.1.6 (Figalli, Fusco, Maggi, Pratelli). *For $\theta_n=2^{-16}n^{-17}$, if a compact $X\subset\mathbb{R}^n$ has rectifiable boundary, then*

$$S(X)\geq n\omega_n^{\frac{1}{n}}|X|^{\frac{n-1}{n}}\left[1+\theta_n\cdot A(B^n,X)^2\right]. \tag{4.2}$$

Remark. Here the exponent 2 of $A(K,E)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

4.2 Sobolev inequality

We write $C^1_c(\mathbb{R}^n)$ to denote the space of C^1 functions on $\mathbb{R}^n$ with compact support. The following fundamental inequality on the gradient of a C^1 function is esentially equivalent with the Isoperimetric Inequality:

Theorem 4.2.1 (Sobolev inequality). *If $f\in C^1_c(\mathbb{R}^n)$ and $n'=\frac{n}{n-1}$, then*

$$\int_{\mathbb{R}^n}\|Df\|\geq n\omega_n^{\frac{1}{n}}\|f\|_{n'}. \tag{4.3}$$

Remark 4.2.2.

(i) Our proof of the Sobolev inequality shows that it holds for any Lipschitz $f\in L_{n'}(\mathbb{R}^n)$ such that the level sets $\{|f|\geq t\}$ are bounded and satisfy the isoperimetric inequality (4.1) for a.e.$t>0$; for example, if the typical level sets are bounded and convex.

(ii) Factor $n\omega_n^{\frac{1}{n}}$ is optimal, as if $\varrho > 0$, and

$$f_\varrho(x) = \begin{cases} \mathbf{1}_{B^n}(x) & \text{if } \|x\| \le 1 \text{ or } \|x\| \ge 1+\varrho; \\ 1 - \frac{1}{\varrho}(\|x\|-1) & \text{if } 1 \le \|x\| \le 1+\varrho, \end{cases}$$

then $\|Df_\varrho(x)\| = \frac{1}{\varrho}$ for $x \in (1+\varrho)B^n \backslash B^n$, and $\|Df_\varrho(x)\| = 0$ otherwise; therefore,

$$\lim_{\varrho\to 0^+} \int_{\mathbb{R}^n} \|Df_\varrho\| = \mathcal{H}^{n-1}(\partial B^n) = n\omega_n^{\frac{1}{n}} |B^n|^{1/n'} = n\omega_n^{\frac{1}{n}} \lim_{\varrho\to 0^+} \left(\int_{\mathbb{R}^n} |f_\varrho|^{n'}\right)^{1/n'}.$$

(iii) Sobolev inequality holds for the larger class of BV functions (functions of bounded variations) (cf. Theorem 5.3.1) where equality holds if and only if $f = a\mathbf{1}_{x+rB^n}$ for $a, r > 0$ and $x \in \mathbb{R}^n$.

(iv) Sobolev inequality yields the Isoperimetric Inequality for any comvex body $X \subset \mathbb{R}^n$. For $\varrho > 0$, let $f_\varrho(x) = \mathbf{1}_X(x)$ if $x \in X$ or $x \notin X^{(\varrho)}$, and let $f_\varrho(x) = 1 - \frac{d(x,X)}{\varrho}$ if $x \in X^{(\varrho)}\backslash X$ where $d(x, X)$ is the distance of x from X. Then $\|Df_\varrho(x)\| = \frac{1}{\varrho}$ if $x \in X^{(\varrho)}\backslash X$, and $\|Df_\varrho(x)\| = 0$ otherwise; therefore,

$$\mathcal{H}^{n-1}(\partial X) = \lim_{\varrho\to 0^+} \int_{\mathbb{R}^n} \|Df_\varrho\| \ge n\omega_n^{\frac{1}{n}} \lim_{\varrho\to 0^+} \left(\int_{\mathbb{R}^n} |f_\varrho|^{n'}\right)^{1/n'} = n\omega_n^{\frac{1}{n}} |X|^{\frac{n-1}{n}}.$$

In order to prove the Sobolev inequality (4.3), we need some additional tools from analysis:

Lemma 4.2.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz.*

Sard's Theorem: $\mathcal{H}^1\left(\left\{t \in \mathbb{R} : \exists x \in f^{-1}(t) \text{ such that } Df(x) = 0 \text{ or } \nexists Df(x)\right\}\right) = 0.$

Coarea formula: $\displaystyle\int_{\mathbb{R}^n} \|Df\| = \int_{\mathbb{R}} \mathcal{H}^{n-1}\left(f^{-1}(t)\right) dt = \int_{\mathbb{R}} S(\{f > t\}))\, dt.$

Lemma 4.2.4. *If $F : [0,\infty) \to [0,\infty)$ decreasing function with $F \not\equiv 0$, and $\alpha > 1$, then*

$$\alpha \int_0^\infty t^{\alpha-1} F(t)^\alpha dt \le \left(\int_0^\infty F(t)\, dt\right)^\alpha. \tag{4.4}$$

Equality holds if and only if $F(t) = c\mathbf{1}_{[0,T]}(t)$ for some $c, T > 0$.

Proof. Since F is decreasing, $tF(t) \le \int_0^t F(s)\, ds$. Hence,

$$\alpha\big(tF(t)\big)^{\alpha-1} F(t) \le \alpha\left(\int_0^t F(s)\, ds\right)^{\alpha-1} F(t) = \frac{d}{dt}\left(\int_0^t F(s)\, ds\right)^\alpha.$$

Integrating the inequality above over $(0,\infty)$, yields the inequality (4.4). ■

Proof of Theorem 4.2.1. We can assume that f is not identically zero. Since $f \in C_0^1(\mathbb{R}^n)$, $|f|^{-1}(t)$ is empty or is the C^1, and hence rectifiable, boundary of $\{|f| > t\}$ for a.e. $t > 0$, we deduce from the Isoperimetric Inequality Theorem 4.1.5 that

$$\mathcal{H}^{n-1}\left(|f|^{-1}(t)\right) \geq n\omega_n^{\frac{1}{n}} |\{|f| > t\}|^{1/n'} \tag{4.5}$$

for a.e. $t > 0$.

Combining the coarea formula Lemma 4.2.3 with (4.5) and Lemma 4.2.4 (with $F(t) = |\{f > t\}|^{1/n'}$ and $\alpha = n'$), we have

$$\begin{aligned}\int_{\mathbb{R}^n} \|Df\| &= \int_{\mathbb{R}} \mathcal{H}^{n-1}\left(f^{-1}(t)\right)\, dt = \int_0^\infty \mathcal{H}^{n-1}\left(|f|^{-1}(t)\right)\, dt \\ &\geq n\omega_n^{1/n}\left(n' \int_0^\infty t^{n'-1} |\{|f| > t\}|\, dt\right)^{1/n'} = n\omega_n^{1/n}\left(\int_0^\infty |\{|f|^{n'} > s\}|\, ds\right)^{1/n'} \\ &= n\omega_n^{1/n}\|f\|_{n'},\end{aligned}$$

where the last equalities follow from $\{|f| > t\} = \{|f|^{n'} > t^{n'}\}$, the substitution $s = t^{n'}$ and the layer cake formula. ∎

4.3 The Anisotropic Isoperimetric inequality

In order to define the Anisotropic Perimeter of a compact set with rectifiable boundary, we need some notions in convexity. For a convex body $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$, and for $x \in \mathbb{R}^n$, Section 1.6 introduces the convex and one homogeneous support function $h_K(x) = \max\{\langle y, x\rangle : y \in K\}$ on $\mathbb{R}^n$, and Section 1.9 discusses the associated norm $\|x\|_K = \min\{t \geq 0 : x \in tK\}$, the polar convex body $K^* = \{z \in \mathbb{R}^n : \langle z, y\rangle \leq 1\ \forall\, y \in K\}$, and the relation $h_K(x) = \|x\|_{K^*}$. According to Definition 4.1.3, if a compact $X \subset \mathbb{R}^n$ has rectifiable boundary, then $0 < \mathcal{H}^{n-1}(x) < \infty$, and for $\mathcal{H}^{n-1}$ a.e. $x \in \partial X$, there exists a unique exterior normal $\nu_X(x) \in S^{n-1}$. Extending the ideas of Federer about the Minkowski content, the following statement is due to Chambolle, Lisini, Lussardi [147] and Lussardi, Villa [430]:

Theorem 4.3.1 (Anisotropic Minkowski content). *For a compact $X \subset \mathbb{R}^n$ that is the closure of its non-empty interior and has rectifiable boundary, if $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(X) = \lim_{\varrho \to 0^+} \frac{|X + \varrho K| - |X|}{\varrho} = \int_{\partial X} h_K(\nu_X)\, d\mathcal{H}^{n-1} = \int_{\partial X} \|\nu_X\|_{K^*}\, d\mathcal{H}^{n-1}.$$

Remarks. $P_K(X)$ is called anisotropic perimeter or anisotropic Minkowski content.

- Theorem 4.3.1 yields that $P_K(K) = n|K|$.

- $P_K(\gamma X) = \gamma^{n-1} P_K(X)$ for $\gamma > 0$.
- $P_K(X)$ does not depend on translating K as the Lebesgue measure is translation invariant, and hence $\int_{\partial X} \langle u, \nu_X \rangle \, d\mathcal{H}^{n-1} = 0$ for $u \in \mathbb{R}^n$.

The Anisotropic Isoperimetric Inequality is a consequence of the Brunn-Minkowski inequality (3.1) stating that if $X, Y \subset \mathbb{R}^n$ compact and $\alpha, \beta, |X|, |Y| > 0$, then

$$|\alpha X + \beta Y|^{\frac{1}{n}} \geq \alpha |X|^{\frac{1}{n}} + \beta |Y|^{\frac{1}{n}}. \tag{4.6}$$

Theorem 4.3.2 (Anisotropic Isoperimetric Inequality). *For a compact $X \subset \mathbb{R}^n$ that is the closure of its non-empty interior and has rectifiable boundary, if $K \subset \mathbb{R}^n$ is convex body with $o \in \operatorname{int} K$, and $|X| = |rK|$ for $r > 0$, then $P_K(X) \geq P_K(rK)$. Equivalently,*

$$P_K(X) \geq n|K|^{\frac{1}{n}} |X|^{\frac{n-1}{n}}. \tag{4.7}$$

Remark. Assuming $X = \operatorname{cl} \operatorname{int} X$, equality holds if and only if $X = z + rK$, as it follows from Theorem 5.2.1 even dealing with the case of sets of finite perimeter. The case when X is a convex body has been already considered in Theorem 2.4.4.

Proof. The Brunn-Minkowski inequality (4.6) yields that

$$\frac{|X + \varrho K| - |X|}{\varrho} \geq \frac{\left(|X|^{\frac{1}{n}} + \varrho |K|^{\frac{1}{n}}\right)^n - |X|}{\varrho} = \frac{\left(|rK|^{\frac{1}{n}} + |\varrho K|^{\frac{1}{n}}\right)^n - |rK|}{\varrho}$$
$$= \frac{|rK + \varrho K| - |rK|}{\varrho} = P_K(rK).$$

■

After various attempts, satisfactory stability version of the Anisotropic Isoperimetric Inequality Theorem 4.3.2 was provided by Figalli, Maggi, Pratelli [225] (actually, even in the more general framework of sets of finite perimeter, cf. Theorem 5.2.2). We recall that for compact sets $X, Y \subset \mathbb{R}^n$, if $\alpha = |X|^{\frac{-1}{n}}$ and $\beta = |Y|^{\frac{-1}{n}}$, then

$$A(X, Y) = \min \{ |\alpha X \Delta (z + \beta Y)| : z \in \mathbb{R}^n \}.$$

Theorem 4.3.3 (Figalli, Maggi, Pratelli). *For $\theta_n = 2^{-16} n^{-17}$, if a compact $X \subset \mathbb{R}^n$ is the closure of its non-empty interior and has rectifiable boundary, and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(X) \geq n|K|^{\frac{1}{n}} |X|^{\frac{n-1}{n}} \left[1 + \theta_n \cdot A(K, X)^2\right]. \tag{4.8}$$

Remark. Here the exponent 2 of $A(K, X)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

In some sense, the following Anisotropic Sobolev inequality inequality is the functional version of the Anisotropic Inisotropic inequality:

Theorem 4.3.4 (Anisotropic Sobolev inequality). *If $f \in C_0^1(\mathbb{R}^n)$, $n' = \frac{n}{n-1}$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$\int_{\mathbb{R}^n} \|Df\|_{K^*} \geq n|K|^{\frac{1}{n}} \|f\|_{n'}.$$

Remark 4.3.5.

- Proof of the Anisotropic Sobolev inequality runs as the argument for the Sobolev inequality Theorem 4.2.1, only the Coarea formula has to exchanged into the Anisotropic Coarea formula:

$$\int_{\mathbb{R}^n} \|Df\|_{K^*} = \int_{\mathbb{R}} P_K\left(f^{-1}(t)\right)\, dt,$$

  and the classical Isoperimetric inequality is replaced by the Anisotropic Inisotropic inequality (4.7). This proof yields the Anisotropic Sobolev inequality for any Lipschitz $f \in L_{\frac{n}{n-1}}(\mathbb{R}^n)$ such that the level sets $\{|f| \geq t\}$ have rectifiable boundary for a.e. $t > 0$.
- Factor $n|K|^{\frac{1}{n}}$ is optimal, and the Anisotropic Sobolev inequality yields the Anistropic Isoperimetric Inequality.
- According Theorem 5.3.1 (see also Figalli, Maggi, Pratelli [227]), the Anistropic Sobolev inequality has a natural extension to functions of bounded variation where equality holds if and only if $f = a\mathbf{1}_{x+rK}$ for $a, r > 0$ and $x \in \mathbb{R}^n$, and [227] even verifies a stability version.

4.4 Wulff shape and minimizing surface energy

We recall that a function $\varrho : S^{n-1} \to \mathbb{R}_{>0}$ is lower semicontinous if $u_k \to u$ for $u, u_k \in S^{n-1}$ yields that $\varrho(u) \leq \liminf_{k\to\infty} \varrho(u_k)$. In this case, $\inf \varrho > 0$.

Definition 4.4.1 (Wulff shape). Given bounded lower semicontinous $\varrho : S^{n-1} \to \mathbb{R}_{>0}$, the corresponding Wulff shape is

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \varrho(u)\ \forall u \in S^{n-1}\}.$$

Remark. W_ϱ is a convex body with $o \in \operatorname{int} W_\varrho$, and $h_{W_\varrho}(u) \leq \varrho(u)$ for $u \in S^{n-1}$.

Wulff shapes occur for example in the solution of the Minkowski Problem (see Section 9.2) *via* the variational method, and in the Logarithmic Brunn-Minkowski Conjecture 8.7.1 and in the L_p Brunn-Minkowski Conjecture 8.8.1 for $p \in (0, 1)$, which are conjectured stregthenings of the Brunn-Minkowski inequality for o-symmetric convex bodies. However, originally, Wulff shape was considered as a model for possible shapes of a crystal *via* minimzing certain surface tenson given the volume.

Example 4.4.2 (Modelling crystal growth). Given à lattice $\Lambda = \sum_{i=1}^n \mathbb{Z}\, v_i$ for independent $v_1, \dots, v_n \in \mathbb{R}^n$, and $T, c > 0$ such that $\|v\| > \frac{c}{T}$ for $v \in \Lambda \backslash o$, if $u \in S^{n-1}$, then the “surface tension" is

$$\varrho(u) = \begin{cases} T & \text{if } \mathbb{R}\, u \cap \Lambda = \{o\}; \\ T - \frac{c}{\alpha} & \text{if } \alpha u \in \Lambda \text{ for } \alpha > 0 \text{ and } (0,\alpha)\, u \cap \Lambda = \emptyset. \end{cases}$$

Then Diophantine approximation yields that the corresponding Wulff shape W_ϱ is a polytope; namely, there exist “short" $w_1, \dots, w_k \in \Lambda \backslash o$ such that

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le \varrho(u_i),\ i = 1, \dots, k\} \quad \text{for } u_i = \frac{w_i}{\|w_i\|}.$$

Actually, W_ϱ is a possible shape for a crystal whose underlying crystallographic structure is the lattice $\Lambda^* = \{z \in \mathbb{R}^n : \langle z, y \rangle \in \mathbb{Z} \text{ for } y \in \Lambda\}$.

We note that for the Wulff shape W_ϱ defined as in Definition 4.4.1, it can happen that $h_{W_\varrho}(u) < \varrho(u)$ for the typical $u \in S^{n-1}$. However, these two functions coincide at the exiterior normals at the regular boundary points of the Wulff shape, and hence the support function can be used instead of ϱ in calculating the surface tension of W_ϱ.

Lemma 4.4.3 (Aleksandrov). *If* $\varrho : S^{n-1} \to (0, \infty)$ *is bounded and lower semicontinous, then*

$$\int_{\partial W_\varrho} \varrho(\nu_{W_\varrho})\, d\mathcal{H}^{n-1} = \int_{\partial W_\varrho} h_{W_\varrho}(\nu_{W_\varrho})\, d\mathcal{H}^{n-1}.$$

Proof. Set $W = W_\varrho$. It is sufficient to prove that

$$\varrho(\nu_W(x)) = h_W(\nu_W(x)) \quad \text{for } x \in \partial' W. \tag{4.9}$$

If $x \in \partial' W$, then there exists $x_k \notin W$ with $x_k \to x$, and hence there exists $v_k \in S^{n-1}$ satisfying $\langle v_k, x_k \rangle > \varrho(v_k)$ by $x_k \notin W$. We may assume that $v_k \to v \in S^{n-1}$, thus the lower semicontinuity of ϱ implies that

$$\varrho(v) \le \liminf_{k\to\infty} \varrho(v_k) \le \liminf_{k\to\infty} \langle v_k, x_k \rangle = \langle v, x \rangle$$

where $x \in W$ yields $\langle v, x \rangle \le h_W(v) \le \varrho(v)$. We deduce that v is exterior normal at the $x \in \partial' W$ with $h_W(v) = \varrho(v)$, and in turn (4.9) follows. ∎

For measurable and bounded $\varrho : S^{n-1} \to (0, \infty)$ and compact $X \subset \mathbb{R}^n$ with rectifiable boundary and $|X| > 0$, the associated surface energy of X is

$$\mathcal{E}_\varrho(\partial X) = \int_{\partial X} \varrho(\nu_X)\, d\mathcal{H}^{n-1}$$

where $\nu_X(x)$ is the unique exterior unit normal at $\mathcal{H}^{n-1}$ a.e.$x \in \partial X$ (see Theorem 5.2.5 for the version for sets of finite perimeter).

Theorem 4.4.4 (Minimizing surface energy). *Let $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semicontinous. If a compact $X \subset \mathbb{R}^n$ is the closure of its non-empty interior and has rectifiable boundary, and $|X| = \left|rW_\varrho\right|$ for $r > 0$, then*

$$\mathcal{E}_\varrho(\partial X) \geq \mathcal{E}_\varrho\left(\partial(rW_\varrho)\right) = n\left|W_\varrho\right|^{\frac{1}{n}} \cdot |X|^{\frac{n-1}{n}}. \tag{4.10}$$

Equality holds in (4.10) if and only if $X = z + rW_\varrho$ for $z \in \mathbb{R}^n$.

Proof. For $W = W_\varrho$, $\varrho \geq h_W$, the Anisotropic Isoperimetric Inequality and Lemma 4.4.3 yield

$$\begin{aligned}
\mathcal{E}_\varrho(\partial X) &= \int_{\partial X} \varrho \circ \nu_X \, d\mathcal{H}^{n-1} \geq \int_{\partial X} h_W \circ \nu_X \, d\mathcal{H}^{n-1} \\
&\geq \int_{\partial(rW)} h_W \circ \nu_{rW} \, d\mathcal{H}^{n-1} = \int_{\partial(rW)} \varrho \circ \nu_{rW} \, d\mathcal{H}^{n-1} = \mathcal{E}_\varrho\left(\partial(rW)\right)
\end{aligned}$$

where $\int_{\partial(rW)} h_W \circ \nu_{rW} \, d\mathcal{H}^{n-1} = n\left|W_\varrho\right|^{\frac{1}{n}} \cdot |X|^{\frac{n-1}{n}}$.

Equality in (4.10) yields equality in the Anisotropic Isoperimetric Inequality; therefore, X is a translate of rW. In this case, we do have equality in (4.10) by Lemma 4.4.3. ∎

The stability version Theorem 4.3.3 of the Anisotropic Isoperimetric Inequality due to by Figalli, Maggi, Pratelli [225] and the argument above yields the following stability version of Wulff's theorem:

Theorem 4.4.5 (Figalli, Maggi, Pratelli). *Let $\theta_n = 2^{-16}n^{-17}$, and $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If a compact $X \subset \mathbb{R}^n$ is the closure of its non-empty interior and has rectifiable boundary, then*

$$\mathcal{E}_\varrho(\partial X) \geq n|W_\varrho|^{\frac{1}{n}}|X|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(W_\varrho, X)^2\right]. \tag{4.11}$$

Remark. Figalli, Zhang [229] proved an even stronger stability estimate in the crystalline case when W_ϱ is a polytope.

4.5 Isodiametric and Isoperimetric inequalities in the Hyperbolic and the Spherical spaces

Our next main goal is to prove the Isoperimetric inequality and the Isodiametric inequality in the Spherical and the Hyperbolic space. Equality in the Isodimetric inequality will be characterized in Section 4.B, but we do not consider the equality case in the Isoperimetric inequality in this monograph.. For the fundamental properties of the spherical space S^n and the hyperbolic space H^n, see Section 4.A.

Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. The k-dimensional Hausdorff measure with respect to the instrinsic metric of $\mathcal{M}^n$ is denoted by $\mathcal{H}^k_{\mathcal{M}^n}(\cdot)$ is ; for example, $\mathcal{H}^n_{\mathcal{M}^n}(\cdot) = |\cdot|$ is the volume on $\mathcal{M}^n$. We note that for the standard embedding $S^n \subset \mathbb{R}^{n+1}$, we have $\mathcal{H}^k_{S^n}(X) = \mathcal{H}^k_{\mathbb{R}^{n+1}}(X)$ for Borel set $X \subset S^n$ and $0 < k \leq n$. We write $d(x, y)$ to denote the geodesic distance of $x, y \in \mathcal{M}^n$, and write $B(z, r) = \{x \in \mathcal{M}^n : d(x, z) \leq r\}$ to denote the metric ball of radius r centered at x where we assume $r < \pi$ if $\mathcal{M}^n = S^n$. For compact $X \subset \mathcal{M}^n$ and $\varrho > 0$, the corresponding parallel domain is

$$X^{(\varrho)} = \{y \in \mathcal{M}^n : \exists x \in X \text{ with } d_{\mathcal{M}^n}(x, y) \leq \varrho\} = \cup \{B_{\mathcal{M}^n}(x, \varrho) : x \in X\}.$$

If $X \subset \mathcal{M}^n$ has non-empty interior and its boundary is rectifiable (finite union of images of Lipschitz functions from compact sets in $\mathcal{M}^{n-1}$), then its surface area is

$$S(X) = \lim_{\varrho \to 0^+} \frac{\left|X^{(\varrho)}\right| - |X|}{\varrho} = \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X). \tag{4.12}$$

To state Theorem 4.5.1, we fix a $z_0 \in \mathcal{M}^n$.

Theorem 4.5.1. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. If $X \subset \mathcal{M}^n$ is bounded and measurable, and $|X| = |B(z_0, r)|$ for $r > 0$ ($r < \pi$ if $\mathcal{M}^n = S^n$), then*

(a) Isodiametric Inequality: diam $X \geq 2r$ *(where $r \leq \frac{\pi}{2}$ if $\mathcal{M}^n = S^n$) where equality holds if and only if $X \subset B(y, r)$ for some $y \in \mathcal{M}^n$;*

(b) Isoperimetric Inequality (Parallel domains): $\left|X^{(\varrho)}\right| \geq |B(z, r + \varrho)|$ *for $\varrho > 0$, and equality holds (assuming $r + \varrho < \pi$ if $\mathcal{M}^n = S^n$) if and only if $X \subset B(y, r)$ for some $y \in \mathcal{M}^n$;*

(c) Isoperimetric Inequality (classical): *assuming that X is compact and ∂X is rectifiable, we have $S(X) \geq S(B(z_0, r))$ where assuming in addition that $X = \operatorname{cl}\operatorname{int} X$, equality holds if and only if $X = B(y, r)$ for some $y \in \mathcal{M}^n$.*

Remarks.

- In the two dimensional case, the Isoperimetric Inequality has the following nice form based on formulas of the perimeter and area of circular discs in Section 4.A: If $\mathcal{M}^2$ is either $H^2, \mathbb{R}^2$ or S^2 with curvature $\kappa = -1, 0, 1$, respectively, and $X \subset \mathcal{M}^2$ is compact with rectifiable boundary, then

$$S(X)^2 \geq 4\pi|X| - \kappa\,|X|^2.$$

- In this monograph, the equality case is only considered for the Isodiametric Inequality in H^n, S^n and $\mathbb{R}^n$ but assuming $r < \frac{\pi}{4}$ if $\mathcal{M}^n = S^n$.

The main method to prove Theorem 4.5.1 is the so-called two-point symmetrization (sometimes called polarization). For a closed closed halfspace H^+ in $\mathcal{M}^n$, in this section we write $H = \partial H^+$ to denote the bounding hyperplane, and H^- to denote the complementary closed halfspace where $H = H^+ \cap H^-$.

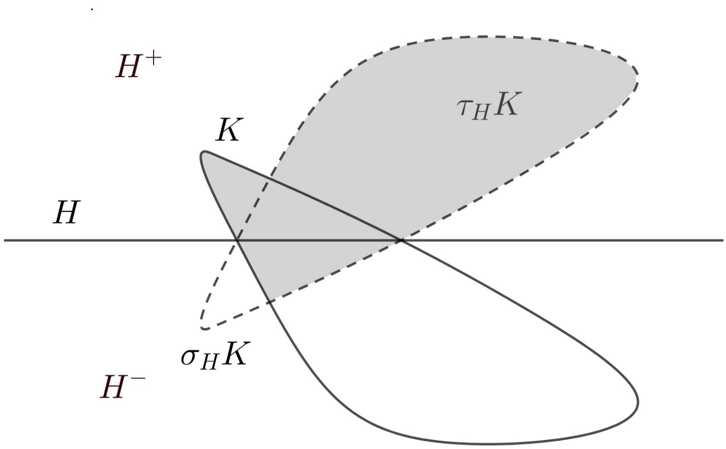


Figure 4.1

Definition 4.5.2 (Two-point symmetrization). Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $X \subset \mathcal{M}^n$ be compact. The two-point symmetrial of X is

$$\tau_{H^+} X = \big((H^+ \cap X) \cup (H^+ \cap \xi_H X)\big) \bigcup ((H^- \cap X) \cap (H^- \cap \xi_H X))$$

Remark. We frequently write only $\tau_H X$ to denote $\tau_{H^+} X$ because $\tau_{H^+} X$ and $\tau_{H^-} X$ are symmetric through H.

While $\tau_H X$ may not look more symmetric than X, the two-symmetrization actually keeps balls. We note that if $K \subset \mathcal{M}^n$ is a convex body such that $\tau_H K$ is convex for any hyperplane H, then K is a ball (cf. Aubrun, Fradelizi [31] and Böröczky, Sagmeister [115]), which property we prove in later sections if either all boundary points are regular (cf. Proposition 4.B.2), or $n = 2$ ((cf. Lemma **??**).

Lemma 4.5.3. *If $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, and H^+ is a closed closed halfspace, then*

(i) $\tau_{H+} B(z, r)$ is a ball, and if $z \in H^+$, then $\tau_{H+} B(z, r) = B(z, r)$;

(ii) $\tau_{H+} Y \subset \tau_{H+} Z$ for compact $Y \subset Z$.

For a hyperplane $H \subset \mathcal{M}^n$, the reflected image of an $x \in \mathcal{M}^n$ is denoted by $\xi_H x$ (see Section 4.A). The following properties follow from the very definition of the two-point symmetrization, where we also use use Lemma 4.A.5 (ii) in the case of (v):

Lemma 4.5.4. *If $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, H^+ is a closed closed halfspace and $X \subset \mathcal{M}^n$ is compact where* $\operatorname{diam}_{\mathcal{M}^n}(X) < \pi$ *provided $\mathcal{M}^n = S^n$, then*

(i) $\tau_H X = \tau_H(\xi_H X)$ *compact;*

(ii) $(\tau_H X) \cap H^+ = \left(X \cup \xi_H X\right) \cap H^+$*;*

(iii) $(\tau_H X) \cap H^- = \left(X \cap \xi_H X\right) \cap H^-$*;*

(iv) $|\tau_H X| = |X|$*;*

(v) $\operatorname{diam}_{\mathcal{M}^n}(\tau_H X) \leq \operatorname{diam}_{\mathcal{M}^n}(X)$.

The property why two-point symmetrization is so useful for us is that it does not increase the volume of the paralleldomain.

Lemma 4.5.5 (Benyamini). *If $\varrho > 0$, $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, H^+ is a closed closed halfspace and $X \subset \mathcal{M}^n$ is compact, then $\left|(\tau_H X)^{(\varrho)}\right| \leq \left|X^{(\varrho)}\right|$.*

Proof. It follows from Lemma 4.5.4 (iv) applied to $X^{(\varrho)}$ that it is sufficient

$$(\tau_{H^+} X)^{(\varrho)} \subset \tau_{H^+}\left(X^{(\varrho)}\right). \tag{4.13}$$

Therefore, let $z \in (\tau_{H^+} X)^{(\varrho)}$, thus there exists some $y \in \tau_{H^+} X$ such that $d(y,z) \leq \varrho$. Since the role of X and $\xi_H X$ are symmetric in the definition of two-point symmetrization, we may assume that $y \in X$, and hence

$$y \in X \cap \tau_{H^+} X;\ d(y,z) \leq \varrho,\ \text{thus } z \in X^{(\varrho)}. \tag{4.14}$$

If $z \in H^+$, then Lemma 4.5.4 (ii) applied to $X^{(\varrho)}$ yields $z \in \tau_{H^+}(X^{(\varrho)})$.

Finally, we assume that $z \notin H^+$, and claim that

$$z \in \xi_H\left(X^{(\varrho)}\right) = (\xi_H X)^{(\varrho)}. \tag{4.15}$$

Here (4.15) readily holds if $y \in \xi_H X$, while if $y \notin \xi_H X$, then $y \in H^+$ by Lemma 4.5.4 (ii), thus $d(z, \xi_H y) \leq d(z,y) \leq \varrho$ by Lemma 4.A.5 (i), verifying (4.15).

We deduce from (4.14), (4.15), and Lemma 4.5.4 (iii) applied to $X^{(\varrho)}$ that $z \in H^- \cap X^{(\varrho)} \cap \xi_H(X^{(\varrho)}) \subset \tau_{H^+}(X^{(\varrho)})$, completing the proof of Lemma 4.5.5. ■

We also need the property that starting from any $X \subset \mathcal{M}^n$, there is a sequence of two-point symmetrizations that leads to a ball of equal volume. More precisely, we prove a slightly weaker, but equally useful property.

Lemma 4.5.6 (Benyamini). *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, H^+ be a closed closed halfspace and $X \subset \mathcal{M}^n$ be compact with $|X| = |B(z_0, r)|$ for $r > 0$. If $\mathcal{F}$ is the smallest family of compact sets containing X and closed under two-point symmetrization, taking limit and isometries of $\mathcal{M}^n$, then*

(i) $\operatorname{diam} Y \leq \operatorname{diam} X$ *for $Y \in \mathcal{F}$;*

(ii) $\left|Y^{(\varrho)}\right| \le \left|X^{(\varrho)}\right|$ *for* $\varrho > 0$ *and* $Y \in \mathcal{F}$*;*

(iii) $|Y| = |X|$ *for* $Y \in \mathcal{F}$*;*

(iv) there exists $Z_0 \in \mathcal{F}$ *with* $B(z_0, r) \subset Z_0$.

Proof. We may asume that $z_0 \in X$, and it is enough to prove (i)-(iv) for $\mathcal{F}_0 = \{Z \in \mathcal{F} : B(z_0, r) \cap Z \neq \emptyset\}$.

Now (i) and (ii) follow from Lemmas 3.7.4, 4.5.4 and 4.5.5, and for (iii), use also (ii).
For (iv), Lemma 3.7.4 yields that there $Z_0 \in \mathcal{F}_0$ with $|B(z_0, r) \cap Z_0|$ is maximal.

Indirectly, we suppose that $B(z_0, r) \not\subset Z_0$, and hence $|B(z_0, r) \cap Z_0| < |B(z_0, r)|$. Since $|Z_0 \backslash B(z_0,r)| = |B(z_0,r) \backslash Z_0| > 0$ by (iii), there exist density points $x \in Z_0 \backslash B(z_0, r)$ and $y \in (\mathrm{int} B(z_0, r)) \backslash Z_0$. We define the closed halfspace H^+ by the properties $\xi_H x = y$ and $y \in H^+$, and hence $|B(z_0, r) \cap \tau_{H^+} Z_0| > |B(z_0, r) \cap Z_0|$ by Lemma 4.5.3, which is a contradiction verifying Lemma 4.5.6. ∎

Proof of Theorem 4.5.1 without the equality cases. As for the Z_0 in Lemma 4.5.6 (iv), we have diam $Z_0 \ge 2r$ and $|Z_0^{(\varrho)}| \ge |B(z_0, r + \varrho)|$ for $\varrho > 0$, we deduce (i) and (ii) in Theorem 4.5.1 (the Isodiametric inequality and the Isoperimetrimetric inequality for parallel domains) from Lemma 4.5.6 (i) and (ii). In turn, we conclude Theorem 4.5.1 (iii) (the classical Isoperimetrimetric inequality) by $|Z_0| = |X| = |B(z_0, r)|$ and (4.12). ∎

4.6 The Gaussian Isoperimetric inequality

In this section we show how the Spherical Isoperimetric inequality Theorem 4.5.1 (ii) yields the Gaussian Isoperimetric inequality. In particular, we consider the version

$$\gamma_n(X) = \int_X e^{-\pi \|x\|^2}\, dx \text{ for measurable } X \subset \mathbb{R}^n.$$

of the Gaussian probability measure because the formulas are simpler than in the case of normal Gaussian distribution. Actually, the the Gaussian Isoperimetric inequality Theorem 4.6.2 holds whatever Gaussian density functon we use.

Definition 4.6.1. For closed set $X \subset \mathbb{R}^n$, we say that X has locally Lipschitz boundary if int $X \neq \emptyset$, and for any $x \in \partial X$, there exists $\varepsilon > 0$ such that $(x + \varepsilon B^n) \cap X$ is the graph of a Lipschitz function.

Remark. In this case, if $p : \mathbb{R}^n \to [0, \infty)$ is locally Lipschitz, then (cf. Federer [212])

$$\lim_{\varrho \to 0^+} \frac{\int_{X^{(\varrho)}} p - \int_X p}{\varrho} = \int_{\partial X} p \, d\mathcal{H}^{n-1}. \tag{4.16}$$

In particular, if $p(x) = e^{-\pi\|x\|^2}$, then

$$\lim_{\varrho\to 0^+} \frac{\gamma_n(X^{(\varrho)}) - \gamma_n(X)}{\varrho} = \int_{\partial X} e^{-\pi\|x\|^2}\, d\mathcal{H}^{n-1}(x). \tag{4.17}$$

Theorem 4.6.2 (Gaussian Isoperimetric inequality). *If $\varrho > 0$, $X \subset \mathbb{R}^n$ is measurable and H_+ is a closed half space with $\gamma_n(H_+) = \gamma_n(X) \in (0,1)$, then*

$$\gamma_n\left(X^{(\varrho)}\right) \geq \gamma_n\left(H_+^{(\varrho)}\right).$$

Remarks.

(i) $\psi^{-1}\left(\gamma_n\left(X^{(\varrho)}\right)\right) \geq \psi^{-1}\left(\gamma_n(X)\right) + \varrho$ is an equivalent form of the Gaussian Isoperimetric inequality where $\psi(s) = \int_{-\infty}^{s} e^{-\pi t^2}\, dt$ is the cumulative distribution function of the one-dimensional Gaussian γ_1.

(ii) Another form of the Gaussian Isoperimetric inequality - that follows from (4.17), - is that if $X \subset \mathbb{R}^n$ has locally Lipschitz boundary, and $\gamma_n(X) = \gamma_n(H_+)$ for a closed halffspace H^+, then

$$\int_{\partial X} e^{-\pi\|x\|^2}\, dx \geq \int_{\partial H_+} e^{-\pi\|x\|^2}\, dx. \tag{4.18}$$

The idea of the proof of the Gaussian inequality is to embed R^n into $\mathbb{R}^{k+1}$ as a linear subspace where $k \to \infty$, and consider the embeddded submanifold $r_k S^k \subset \mathbb{R}^{k+1}$, and the orthogonal projection $\pi_k : r_k S^k \to \mathbb{R}^n$ satisfying $\|\pi_k(x) - \pi_K(y)\| \leq d_{r_k S^k}(x,y))$ for

$$r_k = \sqrt{\frac{k}{2\pi}}.$$

Noting that the density of uniform probability measure on $r_k S^k$ is $\frac{1}{\mathcal{H}^k(r_k S^k)}$, the core claim is that

$$\gamma_n(Z) = \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\pi_k^{-1}(Z)\right)\ \forall\, Z \subset \mathbb{R}^n \text{ with } |Z| > 0. \tag{4.19}$$

Here $\pi_k^{-1}(H_+)$ is a spherical cap - that is a spherical ball, - for large k, which is the extremal body for the Spherical Isoperimetric Inequality.

Now the claim (4.19) yields Theorem 4.6.2 via the following argument:

$$\begin{aligned}
\gamma_n\left(X^{(\varrho)}\right) &= \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\pi_k^{-1}\left(X^{(\varrho)}\right)\right) && (4.20)\\
&\geq \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\left(\pi_k^{-1} X\right)^{(\varrho)}\right) && (4.21)\\
&\geq \lim_{k\to\infty} \frac{1}{\mathcal{H}^k(r_k S^k)} \cdot \mathcal{H}^k\left(\left(\pi_k^{-1} H_+\right)^{(\varrho)}\right) && (4.22)\\
&= \gamma_n\left(H_+^{(\varrho)}\right). && (4.23)
\end{aligned}$$

Before explaining how to get from (4.20) to (4.23) based on (4.19), we remark that if $z \in S^k$ and $x = \pi_k(z)$ with $\|x\| < r_k$, and $v \in T_z$, then

$$D\pi_k v = v|\mathbb{R}^n. \tag{4.24}$$

Firstly, (4.19) directly yields (4.20). In turn (4.20) implies (4.21) as π_k contraction, and hence $\left(\pi_k^{-1}X\right)^{(\varrho)} \subset \pi_k^{-1}\left(X^{(\varrho)}\right)$. Next (4.22) follows from (4.21) and the Spherical Isoperimetric Inequality Theorem 4.5.1 (ii), as if $H_+ = \{x \in \mathbb{R}^n : \langle x, w\rangle \geq s\}$ for $w \in S^{n-1}$ and $s \in \mathbb{R}$, and for large k, $H_{+,k} = \{x \in \mathbb{R}^n : \langle x, w\rangle \geq s_k\}$ is the halfspace for $s_k \in \mathbb{R}$ such that $\mathcal{H}^k\left(\pi_k^{-1}X\right) = \mathcal{H}^k\left(\pi_k^{-1}H_{+,k}\right)$, then (4.19) yields $\lim_{k\to\infty} s_k = s$.

In order to show that (4.22) implies (4.23), writing $\left(\pi_k^{-1}H_+\right)^{(\varrho)}$ to denote the parallel domain of $\pi_k^{-1}H_+$ on S^k of spherical radius ϱ, we prove that for given $\varepsilon \in (0, \varrho)$, if k is large, then

$$\pi_k^{-1}\left(H_+^{(\varrho-\varepsilon)}\right) \subset \left(\pi_k^{-1}H_+\right)^{(\varrho)} \subset \pi_k^{-1}\left(H_+^{(\varrho)}\right). \tag{4.25}$$

To verify (4.25), let $z \in \partial\pi_k^{-1}H_+$, $x = \pi_k z$ and $v \in T_z$ be the exterior unit normal to $\pi_k^{-1}H_+$, and hence $z_0 = z\cos\frac{\varrho}{r_k} + vr_k\sin\frac{\varrho}{r_k} \in \left(\pi_k^{-1}H_+\right)^{(\varrho)}$ and if k is large, then

$$\langle\pi_k z_0, w\rangle = \langle z_0, w\rangle = s\cdot\cos\frac{\varrho}{r_k} + \sqrt{1-\frac{s^2}{r_k^2}}\cdot r_k\sin\frac{\varrho}{r_k} > s + \varrho - \varepsilon,$$

completing the the proof of (4.25). In turn, combining our core claim (4.19) with (4.25) implies (4.23).

Therefore all we are left to do is to prove the core claim (4.19). First we recall the coarea formula in the following form (cf. Corollary 10.4.9):

Remark 4.6.3 (Coarea formula). For $n \leq k < q$ and C^2 embedded Riemannian k-manifold $X^k \subset \mathbb{R}^q$, if $F : X^k \to \mathbb{R}^n$ is locally Lipschitz and $\psi : \mathbb{R}^n \to \mathbb{R}$ is measurable, then

$$\int_{X^k}\psi(F(z))\cdot J(F,z)\,d\mathcal{H}^k(z) = \int_{\mathbb{R}^n}\psi(y)\cdot\mathcal{H}^{k-n}\left(F^{-1}(y)\right)\,d\mathcal{H}^n(y). \tag{4.26}$$

The following statement collects some facts about the projection π_k:

Claim 4.6.4. *Let $x \in \mathrm{int}(r_k B^n)$ with $x \neq o$, and let and $x^\perp = \{y \in \mathbb{R}^k : \langle x, y\rangle = 0\}$.*

(a) *$\pi_k^{-1}(x) = x^\perp \cap S^k$ is a $(k-n)$-sphere of radius $\sqrt{r_k^2 - \|x\|^2}$;*

(b) *for $z \in \pi_k^{-1}(x)$, and $v \in T_z$, we have $\|v|\mathbb{R}^n\| = \sqrt{1-\frac{\|x\|^2}{r_k^2}}\cdot\|v\|$ if v is normal to $x^\perp \cap T_z$, and $\|D\pi_k v\| = \|v|\mathbb{R}^n\| = \|v\|$ if $v \in T_z \cap x^\perp \cap \mathbb{R}^n$, and $\|D\pi_k v\| = \|v|\mathbb{R}^n\| = 0$ if $v \in T_z \cap (\mathbb{R}^n)^\perp$ (cf. (4.24)). In particular. $J(\pi_k, z) = \left(1-\frac{\|x\|^2}{r_k^2}\right)^{\frac{1}{2}}$ in the Coarea formula (4.26) with $F = \pi_k$.*

As the final preparation for the proof of the core claim (4.19), we note that for $m \geq 3$, (10.1) stating $\sqrt{\frac{m}{2\pi}} < \frac{\omega_{m-1}}{\omega_m} < \sqrt{\frac{m+1}{2\pi}}$ yields

$$\frac{\mathcal{H}^{m-1}(S^{m-1})}{\mathcal{H}^m(S^m)} = \frac{(m-1)\omega_{m-1}}{m\omega_m} = \left(1 + O\left(\frac{1}{m}\right)\right)\sqrt{\frac{m}{2\pi}}. \tag{4.27}$$

Proof of the core claim (4.19)*, and in turn of Theorem* 4.6.2. For $Z \subset \mathbb{R}^n$ with $|Z| > 0$, $k \geq 3n$, $r_k = \sqrt{\frac{k}{2\pi}}$, we deduce from Claim 4.6.4 that first using $\psi(x) = 1/J(\pi_k, z)$ for $x \in Z \cap \operatorname{int}(r_k B^n)$ and $z \in \pi_k^{-1}Z$ and $\psi(x) = 0$ if $x \notin Z \cap \operatorname{int}(r_k B^n)$ in the Coarea formula (4.26) with $F = \pi_k$, and later applying (4.27), we have

$$\begin{aligned}
\frac{\mathcal{H}^k\left(\pi_k^{-1}(Z)\right)}{\mathcal{H}^k(r_k S^k)} &= \frac{\mathcal{H}^{k-n}(S^{k-n})}{\mathcal{H}^k(r_k S^k)} \cdot \int_{Z \cap r_k B^n} \left(1 - \frac{\|x\|^2}{r_k^2}\right)^{\frac{-1}{2}} \left(r_k^2 - \|x\|^2\right)^{\frac{k-n}{2}} dx \\
&= \frac{\mathcal{H}^{k-n}(S^{k-n})}{r_k^n \mathcal{H}^k(S^k)} \cdot \int_{Z \cap r_k B^n} \left(1 - \frac{\|x\|^2}{r_k^2}\right)^{\frac{k-n-1}{2}} dx \\
&= \left(1 + O\left(\frac{n}{k}\right)\right) \int_{Z \cap r_k B^n} \left(1 - \frac{2\pi\|x\|^2}{k}\right)^{\frac{k-n-1}{2}} dx.
\end{aligned} \tag{4.28}$$

Now we use the simple estimates

$$1 + t \leq e^t \quad \text{and if } |t| \leq \tfrac{1}{2}, \text{ then } 1 + t = e^{t + O(t^2)} \text{ and } e^t = 1 + O(t). \tag{4.29}$$

We deduce for the $Z \subset \mathbb{R}^n$ with $|Z| > 0$ from (4.28) and (4.29) that

$$\lim_{k \to \infty} \frac{\mathcal{H}^k\left(\pi_k^{-1}(Z \backslash k^{\frac{1}{8}} B^n)\right)}{\mathcal{H}^k(r_k S^k)} = 0; \tag{4.30}$$

$$\lim_{k \to \infty} \gamma_n\left(Z \backslash k^{\frac{1}{8}} B^n\right) = 0, \tag{4.31}$$

and hence combining (4.28), (4.30) and (4.31) leads to

$$\begin{aligned}
\lim_{k \to \infty} \frac{\mathcal{H}^k\left(\pi_k^{-1}(Z)\right)}{\mathcal{H}^k(r_k S^k)} &= \lim_{k \to \infty} \left(1 + O\left(\frac{n}{k}\right)\right) \int_{Z \cap k^{\frac{1}{8}} B^n} e^{-\pi\|x\|^2 + O\left(\frac{n}{k^{3/4}}\right) + O\left(\frac{1}{k^{1/2}}\right)} dx \\
&= \gamma_n(Z).
\end{aligned}$$

Therefore, we conclude the core claim (4.19), and in turn of Theorem 4.6.2. ■

Definition 4.6.5 (Median). If μ is a probability measure on $\mathbb{R}^n$ and $f \in L_1(\mu)$, then a median $m \in \mathbb{R}$ satisfies $\mu(\{f > m\}) \leq \frac{1}{2}$ and $\mu(\{f < m\}) \leq \frac{1}{2}$.

4.7 The Kannan-Lovász-Simonovits conjecture, Cheeger constant and a Poincaré inequality

After solving the isoperimetric problem with respect to a Gaussian density in $\mathbb{R}^n$, we consider the version with respec to any log-concave density where $p : \mathbb{R}^n \to [0, \infty)$ is log-concave if $p = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. In particular, such p is locally Lipschitz, and hence a.e. differentiable (see Section 10.9 in the Appendix for a survey on log-concave functions). Naturally, we do not expect exact solution in general only approximate solution in this case. Since many results about the corresponding Cheeger constant holds for any locally Lipschitz density function p, we also discuss that more general framework whenever it is appropriate.

First we define what we mean by weighted surface area of a set with locally Lipschitz boundary, and what type of isoperimetric ratio, bounded by the so-called Cheeger constant, we are considering, then review the necessary tools to show how the isoperimetric type problem relates to integral inequalities in terms of the gradient, like Proposition 4.7.6 and the Poicaré type inequality Theorem 4.7.8. This is followed by the main part of the section discussing bounds for the Cheeger constant; in particular, the Kannan-Lovász-Simonovits conjecture about the Cheeger constant.

Definition 4.7.1. If $d\mu = p\, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for locally Lipschitz p, and $X \subset \mathbb{R}^n$ has locally Lipschitz boundary (cf. Definition 4.6.1 and (4.16)), then

$$\mu_+(\partial X) = \lim_{\varrho \to 0^+} \frac{\mu(X^{(\varrho)}) - \mu(X)}{\varrho} = \int_{\partial X} p\, d\mathcal{H}^{n-1}. \tag{4.32}$$

Definition 4.7.2 (Cheeger constant). If $d\mu = p\, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for locally Lipschitz p, then $C_{\mathrm{Che}}(\mu) > 0$ is defined to be minimal such that for every closed $X \subset \mathbb{R}^n$ with locally Lipschitz boundary, we have

$$C_{\mathrm{Che}}(\mu) \cdot \mu_+(\partial X) \geq \min\{\mu(X), 1 - \mu(X)\}. \tag{4.33}$$

Remark 4.7.3. (i) If p is log-concave, then it is enough to consider the case when $\mu(X) = \frac{1}{2}$ in (4.33) according to Milman [458].

(ii) *(Gaussian measure)* If $p(x) = e^{-\pi\|x\|^2}$, then $C_{\mathrm{Che}}(\mu) = \frac{1}{2}$ by (i) and the Gaussian Isoperimetric inequality (4.18).

(iii) *(Uniform measure on the unit ball)* If $d\mu = \omega_n^{-1}\mathbf{1}_{B^n}\, d\mathcal{H}^n$, then

$$C_{\mathrm{Che}}(\mu) = \frac{\omega_n}{2\omega_{n-1}} < \sqrt{\frac{\pi}{2n}}. \tag{4.34}$$

To prove (4.34), one may assume that $|X| = \frac{1}{2}|B^n|$ for the $X \subset B^n$ with locally Lipschitz boundary by (i), and the task is to minimize $\mathcal{H}^{n-1}((\partial X) \cap \operatorname{int} B^n)$. This isoperimetric type problem was solved by Almgren [14] and Bokowski, Sperner

[79], and the optimal X is a half ball (intersection by a half space), yielding (4.34) (see (10.1) for the estimate on $\frac{\omega_n}{\omega_{n-1}}$).

We note that in some papers, what called Cheeger constant is the reciprocal $1/C_{\text{Che}}(\mu)$. We choose our normalization because it fits better the integral inequalities Proposition 4.7.6 and Theorem 4.7.8. Now we collect the tools we need to prove Proposition 4.7.6 that is the integral version of the definition of the Cheeger constant. We need the following version of the Coarea formula (see Theorem 10.4.8):

Lemma 4.7.4 (Coarea formula). *If $f : \mathbb{R}^n \to \mathbb{R}$ locally Lipschitz and $p \in L_1(\mathbb{R}^n)$ is non-negative, then*

$$\int_{\mathbb{R}^n} \|Df\| \cdot p = \int_{\mathbb{R}} \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \, dt. \tag{4.35}$$

Remark. As $f : \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz, Sard's Theorem yields that

$$\mathcal{H}^1 \left(\left\{ t \in \mathbb{R} : \exists x \in f^{-1}(t) \text{ s.t.no } Df(x) \text{ or } Df(x) = 0 \right\} \right) = 0.$$

Definition 4.7.5 (Median). If μ is a probability measure on $\mathbb{R}^n$ and $f \in L_1(\mu)$, then a median $m \in \mathbb{R}$ satisfies $\mu(\{f > m\}) \leq \frac{1}{2}$ and $\mu(\{f < m\}) \leq \frac{1}{2}$.

Proposition 4.7.6. *If $d\mu = p \, d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for locally Lipschitz p, and $f \in L_1(\mu)$ is localy Lipschitz with median m, then*

$$\int_{\mathbb{R}^n} |f - m| \, d\mu \leq C_{\text{Che}}(\mu) \cdot \int_{\mathbb{R}^n} \|Df\| \, d\mu. \tag{4.36}$$

Proof. It follows from the Layer Cake formula and the Coarea formula (4.35) that

$$\begin{aligned}
\int_{\mathbb{R}^n} |f - m| \, d\mu &= \int_{m(f)}^{\infty} \mu(\{f > t\}) \, dt + \int_{-\infty}^{m(} \mu(\{f < t\}) \, dt \\
&= \int_{\mathbb{R}} \min\{\mu(\{f > t\}), 1 - \mu(\{f > t\})\} \, dt \\
&\leq C_{\text{Che}}(\mu) \int_{\mathbb{R}} \int_{f^{-1}(t)} p \, d\mathcal{H}^{n-1} \, dt = C_{\text{Che}}(\mu) \int_{\mathbb{R}^n} \|Df\| \, d\mu.
\end{aligned}$$

■

The Poincaré type inequality Theorem 4.7.8 uses the the notion of variance that we recall for the reader's convenience.

Definition 4.7.7. If $d\mu = p \, d\mathcal{H}^n$ probability measure on $\mathbb{R}^n$ for locally Lipschitz p, and $f \in L_2(\mu)$ is locally Lipschitz, then

$$\text{Var}_\mu(f) = \int_{\mathbb{R}^n} f^2 \, d\mu - \left(\int_{\mathbb{R}^n} f \, d\mu \right)^2.$$

Theorem 4.7.8 (Poincaré inequality). *If $d\mu = p\, d\mathcal{H}^n$ probability measure on $\mathbb{R}^n$ for locally Lipschitz p, and $f \in L_2(\mu)$ is locally Lipschitz, then*

$$\mathrm{Var}_\mu(f) \le 4C_{\mathrm{Che}}(\mu)^2 \cdot \int_{\mathbb{R}^n} \|Df\|^2\, d\mu. \tag{4.37}$$

Remark. It is equivalent to saying that if $\int_{\mathbb{R}^n} f\, d\mu = 0$, then

$$\int_{\mathbb{R}^n} f^2\, d\mu \le 4C_{\mathrm{Che}}(\mu)^2 \cdot \int_{\mathbb{R}^n} \|Df\|^2\, d\mu.$$

Proof. We observe that if m is a median for f, then 0 is a the median for $\tilde{f} = (f - m)^2 \mathrm{sgn}\,(f - m)$ with respect to μ where $\mathrm{sgn}\, t$ is $+1, 0, -1$ provided $t > 0$, $t = 0$ or $t < 0$, respectively. Therefore, (4.36), $D(f - m) = Df$ and the Hölder inequality yield

$$\begin{aligned}
\int_{\mathbb{R}^n} |f - m|^2\, d\mu = \int_{\mathbb{R}^n} |\tilde{f}| &\le C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} \|D\tilde{f}\|\, d\mu = C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} \|D(f - m)^2\|\, d\mu \\
&= 2C_{\mathrm{Che}}(\mu) \int_{\mathbb{R}^n} |f - m| \cdot \|Df\|\, d\mu \\
&\le 2C_{\mathrm{Che}}(\mu) \sqrt{\int_{\mathbb{R}^n} |f - m|^2\, d\mu} \cdot \sqrt{\int_{\mathbb{R}^n} \|Df\|^2\, d\mu}.
\end{aligned}$$

We conclude (4.37) because $\mathrm{Var}_\mu(f) \le \int_{\mathbb{R}^n} |f - m|^2\, d\mu$ as $\int_{\mathbb{R}^n} |f - t|^2\, d\mu$ is minimized by the mean $t = \int_{\mathbb{R}^n} f\, d\mu$. ■

Remark 4.7.9 (Poincaré inequality as a spectral gap estimate). Consider the heat equation $\partial_t u = Lu$ on $[0, \infty) \times \mathbb{R}^n$ where $p = e^{-V}$ for $V : \mathbb{R}^n \to (0, \infty]$, and for suitable $f \in L_2(\mu)$,

$$Lf = \Delta f - \langle Df, DV\rangle.$$

Then $-L$ is a non-negative self adjoint operator in $L_2(\mu)$, satisfying

$$\int_{\mathbb{R}^d} (-Lf) \cdot f\, d\mu = \int_{\mathbb{R}^d} \|Df\|^2\, d\mu.$$

Readily, the constant functions form the kernel Ker$L$. If $\int_{\mathbb{R}^n} f\, d\mu = 0$, then f is orthogonal to Ker$L$, and hence the smallest positive eigenvalue λ_μ of $-L$ satisfies that λ_μ is the largest with the property.

$$\lambda_\mu \mathrm{Var}_\mu(f) \le \int_{\mathbb{R}^n} \|Df\|^2\, d\mu \quad \text{for locally Lipschitz } f \in L_2(\mu). \tag{4.38}$$

Now (4.37) yields that

$$\lambda_\mu \ge C_{\mathrm{Che}}(\mu)^{-2}/4, \tag{4.39}$$

which is known as Cheeger's inequality (see for example Ledoux [393]). If p is log-concave, then De Ponti, Mondino [189] verified a matching bound on λ_μ, which in turn yields

$$\tfrac{1}{4} \le \lambda_\mu \cdot C_{\mathrm{Che}}(\mu)^2 \le \pi. \tag{4.40}$$

Now we turn to bounds on the Cheeger cosntant. We observe that if $d\mu = p\,d\mathcal{H}^n$ where $p = \mathbf{1}_{R_a}$ for $a > 1$ and rectangular box $R_a = [0, \frac{1}{a}]^{n-1} \times [0, a^{n-1}]$, then $C_{\mathrm{Che}}(\mu) \ge a^{n-1}/2$. In particular, we can't expect an upper bound on $C_{\mathrm{Che}}(\mu)$ depending only on the dimension n in general even if p is log-concave.

Definition 4.7.10 (Covariance matrix, Isotropic measure).

(i) If $d\mu = p\,d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ with $\int_{\mathbb{R}^n} x\,d\mu(x) = o$, then the *covariance matrix* is $\mathrm{Cov}(\mu) = [m_{ij}]$ where $m_{ij} = \int_{\mathbb{R}^n} x_i x_j\,d\mu(x_1, \dots, x_n)$.

(ii) μ is *isotropic* if in addition $\mathrm{Cov}(\mu) = I_n$; or equivalently, if $\int_{\mathbb{R}^n} \langle v, x\rangle^2\,d\mu(x) = \|v\|^2$ for $v \in \mathbb{R}^n$.

Remarks.

(i) If $d\mu = p\,d\mathcal{H}^n$ is probability measure on $\mathbb{R}^n$, then there exists $\Phi \in \mathrm{GL}(n)$ such that the probability measure $\tilde{\mu}(X) = |\det \Phi|\mu(\Phi(X + b))$ is isotropic for $b = \int_{\mathbb{R}^n} x\,d\mu(x)$.

(ii) $\mu_+(\partial X)$ in (4.32) and $\|Df\|$ in (4.36) are defined in terms of the given Euclidean structure, and the importance of the measure being isotropic is that the measure "matches" the Euclidean structure in that case.

(iii) $d\mu(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}}\,dx$ is the isotropic Gaussian measure, and $C_{\mathrm{Che}}(\mu) = \sqrt{\frac{\pi}{2}}$ by the Gaussian Isoperimetric inequality (4.18).

Kannan, Lovász, Simonovits [361] stated the following fundamental conjecture in 1995:

Conjecture 4.7.11 (KLS conjecture). *If $d\mu = p\,d\mathcal{H}^n$ is an isotropic probability measure on $\mathbb{R}^n$ for a log-concave p, then $C_{\mathrm{Che}}(\mu) \le c$ for an absolute constant c.*

After intense research for more than two decades (see the Comments Section 4.8), Klartag [373] proved the currently best upper bound on $C_{\mathrm{Che}}(\mu)$ in the KLS Conjecture 4.7.11.

Theorem 4.7.12 (Klartag). *If $d\mu = p\,d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p, then $C_{\mathrm{Che}}(\mu) \le c\sqrt{\log n}$ for an absolute constant c.*

Remark. It follows that if $d\mu = p\,d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p, then $C_{\mathrm{Che}}(\mu) \le c\sqrt{\log n}\sqrt{\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}}$, c absolute constant, where $\|\mathrm{Cov}(\mu)\|_{\mathrm{op}}$ is the operator norm (largest eigenvalue).

In Definition 4.7.2 of the Cheeger constant, we have no information on what X to take. However, for algorithms, one would take halfspaces as X. The main motivation of Kannan, Lovász, Simonovits [361] to state the KLS conjecture 4.7.11 was the hope is that halfspaces are reasonably close to be optimal (see Remark 4.7.14).

The following results about how halfspaces divide a log-concave measure with zero mean were proved by Lovász, Vempala [423] and Fradelizi [239].

Theorem 4.7.13 (Lovász, Vempala, Fradelizi). *If $d\mu = p\,d\mathcal{H}^n$ is a probability measure on $\mathbb{R}^n$ for a log-concave p with mean $\int_{\mathbb{R}^n} x\,p(x)\,dx = o$, $n \geq 1$, and $H^+ = \{x \in \mathbb{R}^n : \langle x, u\rangle \geq 0\}$ for $u \in S^{n-1}$, then*

$$\int_{H^+} p \geq \frac{1}{e}; \tag{4.41}$$

$$\mu_+(tu + \partial H^+) \leq e \cdot \mu_+(\partial H^+) \;\; \text{for } t \in \mathbb{R}; \tag{4.42}$$

$$\frac{1}{150} < \mu_+(\partial H^+)^2 \cdot \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\,dx < \frac{2}{3}. \tag{4.43}$$

Remark. (4.41) and (4.42) are optimal (for example, consider $p(x) = e^{-x-1}$ if $x \geq -1$, and $p(x) = 0$ if $x < -1$ for $n = 1$). Instead of (4.43), Fradelizi [239] proves the optimal estimates

$$\frac{1}{12} \leq \mu_+(\partial H^+)^2 \cdot \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\,dx \leq \frac{1}{2}, \tag{4.44}$$

and also verifies that $\max_{t\in\mathbb{R}} \mu_+(tu + \partial H^+) \int_{\mathbb{R}^n} \langle u, x\rangle^2 p(x)\,dx \leq 1$.

Proof. We observe that $f(t) = \int_{tu+u^\perp} p\,d\mathcal{H}^{n-1}$ is long-concave for $t \in \mathbb{R}$ by the Prékopa-Leindler inequality (3.5), $\int_{\mathbb{R}} f = 1$ and $\int_{\mathbb{R}} t\,f(t)\,dt = 0$. Therefore our problem problem is essentially one-dimensional, it is equivalent to prove that if $t \in \mathbb{R}$, then

$$\int_0^\infty f \geq \frac{1}{e} \text{ and } \int_{-\infty}^0 f \geq \frac{1}{e} \text{ and } f(t) \leq e \cdot f(0); \tag{4.45}$$

$$\frac{1}{150} < f(0)^2 \cdot \int_{\mathbb{R}} t^2 f(t)\,dt < \frac{2}{3}. \tag{4.46}$$

We may assume that $f(0) = 1$ after replacing f by $x \mapsto \lambda f(\lambda x)$ for $\lambda = 1/f(0)$. Let $\int_0^\infty f = a = \int_0^\infty e^{-t/a}\,dt$ where $a \in (0, 1)$.

Since f is log-concave and $\int_{-\infty}^0 f = \int_{-\infty}^0 e^{-t/a}\,dt$, there exists $\alpha > 0$ such that $f(t) \geq e^{-t/a}$ if $t \in [0, \alpha]$ and $f(t) \leq e^{-t/a}$ if $t > \alpha$. In particular, we have

$$\begin{aligned}\int_0^\infty te^{-t/a}\,dt - \int_0^\infty tf &= -\int_0^\alpha t(f(t) - e^{-t/a})dt + \int_\alpha^\infty t(e^{-t/a} - f(t))dt \\ &\geq -\alpha \int_0^\alpha f(t) - e^{-t/a}\,dt + \alpha \int_\alpha^\infty e^{-t/a} - f(t)\,dt = 0.\end{aligned}$$

We define $\beta < 0$ with the property $e^{-\beta/a} = 1/a$, and hence $\int_\beta^\infty e^{-t/a}\,dt = 1$, and $f(t) \leq e^{-t/a}$ if $t \in [\beta, 0]$ as f is log-concave. Let $g(t) = e^{-t/a}$ if $t \geq \beta$, and $g(t) = 0$ if $t < \beta$, thus $\int_{-\infty}^0 g = 1 - a = \int_{-\infty}^0 f$. We deduce that $\int_{-\infty}^0 t\,g(t)\,dt \geq \int_{-\infty}^0 t\,f(t)\,dt$, and hence $\int_{\mathbb{R}} t\,g(t)\,dt \geq \int_{\mathbb{R}} t\,f(t)\,dt = 0$, which in turn yields that $\beta + a = \int_{\mathbb{R}} t\,g(t)\,dt \geq 0$. It follows that $e \geq e^{-\beta/a} = 1/a$; or equivalently, $\int_0^\infty f = a \geq \frac{1}{e}$, and similar argument implies $\int_{-\infty}^0 f \geq \frac{1}{e}$. In turn, we conlude the first estimates $\min\{a, 1-a\} \geq \frac{1}{e}$ of (4.45).

To prove the third estimate in (4.45), we may assume that $f(t) > 1$ and $t < 0$. Since f is log-concave, $f(t) = e^{-\gamma t}$ for $\gamma = \frac{\log f(t)}{|t|}$ and $f(0) = e^{-\gamma \cdot 0}$, we deduce that $f(s) \geq e^{-\gamma s}$ if $s \in [t, 0]$ and $f(s) \leq e^{-\gamma s}$ if $s \geq 0$, and hence

$$f(t) = \frac{e^{-\gamma t}}{e^{-\gamma\cdot 0}} = \frac{\int_t^\infty e^{-\gamma s}\,ds}{\int_0^\infty e^{-\gamma s}\,ds} = \frac{\int_t^0 e^{-\gamma s}\,ds}{\int_0^\infty e^{-\gamma s}\,ds} + 1 \leq \frac{\int_t^0 f}{\int_0^\infty f} + 1 \leq \frac{1-a}{a} + 1 \leq e. \quad (4.47)$$

For the upper bound in (4.43), the argument above yields

$$\int_0^\infty t^2 f(t)\,dt \leq \int_0^\infty t^2 e^{-t/a}\,dt = 2a^3,$$

and similarly $\int_{-\infty}^0 t^2 f(t)\,dt \leq 2(1-a)^3$. Since $\min\{a, 1-a\} \geq 1/e$, we deduce that $\int_{\mathbb{R}} t^2 f(t)\,dt \leq 2a^3 + 2(1-a)^3 \leq 2(\frac{1}{e^3} + (1-\frac{1}{e})^3) < \frac{2}{3}$.

For the lower bound in (4.43), we use that $f = e^{-V}$ for a convex function V with $V(0) = 0$. Let $b \in \partial V(0)$ from the subdifferential of V at 0 (cf. Definition 1.5.4), and hence $V(t) \geq bt$ where we may assume that $b \geq 0$. It follows that $f(t) \leq e^{-bt}$, thus $\int_0^\infty e^{-bt}\,dt \geq \int_0^\infty f = a \geq e^{-1}$ implies that $b \leq \frac{1}{a} \leq e$. We deduce from $e^{-bt} \leq 1$ for $t \geq 0$ that there exists $q \in [\frac{1}{e}, \infty]$ with $\int_0^q e^{-bt} = a$, and hence $f(t) \leq e^{-bt}$ yields

$$\int_0^\infty t^2 f(t)\,dt \geq \int_0^q t^2 e^{-bt}\,dt \geq \int_0^{1/e} t^2 e^{-et}\,dt = \frac{2 - 5\cdot e^{-1}}{e^3} > \frac{1}{150},$$

completing the proof of (4.43). ∎

Remark 4.7.14 (Halfspaces are efficient for Cheeger constant). Let $d\mu = p\,d\mathcal{H}^n$ be an isotropic probability measure for a log-concave function p on $\mathbb{R}^n$. On the one hand, writing $\mathcal{F}_\mu$ to denote the family of $X \subset \mathbb{R}^n$ with locally Lipschitz boundary also satisfying $\mu_+(\partial X) > 0$ and $\mu(X) \in (0, 1)$, we have

$$C_{\mathrm{Che}}(\mu) = \sup_{X \in \mathcal{F}_\mu} \frac{\min\{\mu(X), 1 - \mu(X)\}}{\mu_+(\partial X)} \leq c\sqrt{\log n}$$

for an abolute constant $c > 0$ according to Theorem 4.7.12 by Klartag [373].

On the other hand, if H^+ is any halfspace with $o \in \partial H^+$, then (4.41) and (4.43) yield that

$$C_{\mathrm{Che}}(\mu) \geq \frac{\min\{\mu(H^+), 1 - \mu(H^+)\}}{\mu_+(\partial H^+)} \geq \frac{\sqrt{3/2}}{e}.$$

Therefore, one can use hyperplanes to subdivide a measure almost as efficiently as general hypersurfaces. This is very important for algorithms, and this was the original motivation of Kannan, Lovász and Siminovits to study the problem (see Alonso-Gutiérrez, Bastero [15] and Lee, Vempala [394]).

Finally, we discuss the smallest positive eigenvalue λ_μ in the Poincaré inequality (4.38) for an isotropic probability measure $d\mu = p\, d\mathcal{H}^n$ for a log-concave function p on $\mathbb{R}^n$. Plugging the test function $f(x) = \|x\|^2$ into (4.38) shows that

$$\lambda_\mu \mathrm{Var}_\mu(\|x\|^2) \leq \int_{\mathbb{R}^n} \|2x\|^2 \, d\mu(x) = 4\sum_{i=1}^n \int_{\mathbb{R}^n} \langle e_i, x\rangle^2 \, d\mu(x) = 4n \tag{4.48}$$

using an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. Using his method "stochastic localization", Eldan [199] prove that this specific test function gives a very good estimate on λ_μ.

Theorem 4.7.15 (Eldan). *Writing $\mathcal{M}^n$ to denote the family of isotropic log-concave probability measures on $\mathbb{R}^n$,*

$$\sup_{\mu\in\mathcal{M}^n} \frac{\mathrm{Var}_\mu(\|x\|^2)}{4n} \leq \sup_{\mu\in\mathcal{M}^n} \frac{1}{\lambda_\mu} \leq c\cdot(\log n)^2 \sup_{\mu\in\mathcal{M}^n} \frac{\mathrm{Var}_\mu(\|x\|^2)}{n}$$

for an absolute constant $c > 0$.

Remark. According to the Variance conjecture due to Anttila, Ball, Perissinak [25] and Bobkov, Koldobsky [78], $\mathrm{Var}_\mu(\|x\|^2) \leq c_0 n$ for $\mu \in \mathcal{M}^n$ where $c_0 > 0$ is an absolute constant. We deduce from (4.48), Cheeger's inequality (4.39) and Theorem 4.7.12 by Klartag [373] that

$$\mathrm{Var}_\mu(\|x\|^2) \leq \frac{4n}{\lambda_\mu} \leq 16n\cdot C_{\mathrm{Che}}(\mu)^2 \leq c_1 n\log n$$

for an absolute constant $c_1 > 0$.

4.8 Comments to Chapter 4

The extremal property of balls with respect to the isoperimetric problem was known to the ancient Greeks; for example, Zenodorus (circa 200 BC - 140 BC) suggested an argument first proving that regular polygons are optimal in the plane, and even claimed that spheres are optimal in three dimensions (cf. Blasjö [75]). In higher dimensional spaces, the Isoperimetric Inequality for convex bodies was proved by the work of Steiner, Schwarz, Weierstrass and Minkowski in the 19th century (see Gruber [276]).

For properties of rectifiable sets in $\mathbb{R}^n$; more precisely, $(n-1)$-rectifiable sets, see Federer [212] and Ambrosio, Colesanti, Villa [18] in the classical setting, and

Chambolle, Lisini, Lussardi [147] and Lussardi, Villa [430] concerning the Anisotropic Perimeter. Actually, there are papers where $Z \subset \mathbb{R}^n$ being $(n-1)$-rectifiable means that Z is the union of countable many Lipschitz images of $(n-1)$-dimensional compact sets up to a set of $\mathcal{H}^{n-1}$-measure zero, but Theorem 4.1.4 does not hold in this generality (see [18]).

See Bianchi, Gardner, Gronchi [71, 72] for properties of Steiner-type symmetrisations of compact sets.

The right framework for the isoperimetric inequality is sets of finite perimeter (see Chapter 5, or for example Ambrosio, Fusco, Pallara [19], Maggi [439]) that includes sets with rectifiable boundary. Talenti [546] provided a proof of the isperimetric inequality for sets of finite perimeter based on Steiner Symmetrization, also characterizing the equality case. Fusco, Maggi, Pratelli [251] even managed to prove a stability version of the isoperimetric inequality.

The optimal factor in Sobolev's inequality Theorem 4.2.1 is verified by Federer, Fleming [211] using symmetrization, and the stability version of the Sobolev inequality of optimal order for functions of bounded variation is due to Figalli, Maggi, Pratelli [227]. Actually, there exists an L_p version of the Sobolev inequality for $1 < p < n$, as well, where the optimal factor has been determined by Talenti [545], and the stability version of optimal order is due to Bianchi, Egnell [70] if $p = 2$, and to Figalli, Zhang [228] if $1 < p < n$.

As the Anisotropic Isoperimetric for convex bodies is a direct consequence of the Brunn-Minkowski theorem, it was already known to Minkowski (see [464, 465]) even if not using this term.

The notion of Wulff shape originates from the paper Wulff [568] related to Crystallography, and see Maggi [439] for a dicussion of Wulff's theorem for sets of finite perimeter, the papers Taylor [549], Miracle-Sole [466] and Figalli, Maggi [226] for the role of Wulff shape within crystallography, and Figalli, Zhang [229] for a strong stability version of the Wulff inequality. Wulff type isoperimetric inequalities within a convex cone are discussed by Cabré, Ros-Oton,· Serra [134]. Many examples of Wulff shapes and the relation to the underlying periodic and quasi-periodic structure are discussed in Böröczky, Schnell, Wills [118]. In particular, Wulff shapes are also successful models of certain quasi-crystals.

Many properties of the Wulff-shape have been established by Aleksandrov in the 1930s, like Lemma 4.4.3 and Aleksandrov's Lemma Theorem 7.5.2 (see Aleksandrov [7]).

For in depth studies on Steiner symmetrization and Schwarz symmetrization, see Bianchi, Gardner, Gronchi [71, 72]. They provide a broder class of n hyperplanes for the iterated Steiner Symmetrization than Theorem 1.A.3; namely, $v_1, \ldots, v_n \in S^{n-1}$ are independent in a way such that $\langle v_i, v_j \rangle \neq 0$ for $i \neq j$, and $\angle(v_1, v_2) = \alpha\pi$ for irrational $\alpha \in (0, 1)$.

For the fundamental properties of hyperbolic and spherical spaces, see Berger [62] or Vinberg [559]. The isoperimetric inequality in the spherical and hyperbolic spaces is due to E. Schmidt [514]. We provide the elegant argument by Benyamini [60] because it works simultaneously in all spaces of constant curvature, and yields also the Isodiametric Inequality. Stability versions of the Isoperimetric Inequality in terms of the volume difference were proved by Bögelein, Duzaar, Fusco [82] in the spherical case, and by Bögelein, Duzaar, Scheven [83] in the hyperbolic case.

The Isodiametric Inequality in $\mathbb{R}^n$ is due to P. Urysohn [557], and proved by Schmidt [515, 516] in the spherical space S^n and the hyperbolic space H^n (see also Böröczky, Sagmeister [115] for discussion of equality, and [116] for a stability version).

Two-point symmetrization appeared first in Wolontis [566] in the framework of conformal functions. It is applied to prove the isoperimetric inequality in the spherical space by Benyamini [60], whose argument is adapted to the hyperbolic space by Böröczky, Sagmeister [116]. Two-point symmetrization leads to the spherical analogue of the Blaschke-Santaló inequality (6.25) by Gao, Hug, Schneider [252]. Proposition 4.B.2 for any convex body K that may have non-regular boundary points is verified by Aubrun, Fradelizi [31] in the spherical and by Böröczky, Sagmeister [115] in the hyperbolic case.

A hyperbolic version of the Brunn-Minkowski inequality has been proved by Assouline, Klartag [30] in H^2. A horocycle Σ in H^2 has constant curvature one, and is an orthogonal trajectory of a pencil of pairwise parallel lines, which is the boundary of an ellipse touching the boundary of the hyperbolic plane in the Bertrami-Cayley-Klein model; therefore, the orentiation of H^2 induces an orientation of Σ. For any $x \neq y \in H^2$, there exists exactly one horocycle cycle arc connecting x and y where x comes first before y according to the orientation of the horocycle. For measurable $X, Y \subset H^2$ and $\lambda \in (0,1)$, [30] defines their "hyperbolic Minkowski combination" analogously to the Euclidean case as

$$\begin{aligned}(1-\lambda)\, X : \lambda Y \;\; = \;\; & \{\sigma(\lambda) \,:\, \sigma : [0,1] \to H^2 \text{ oriented constant-speed} \\ & \text{horocycle arc with } \sigma(0) \in X \text{ and } \sigma(1) \in Y\},\end{aligned}$$

and proves

$$\sqrt{|(1-\lambda)\, X : \lambda Y|_*} \geq (1-\lambda)\sqrt{|X|} + \lambda\sqrt{|Y|}.$$

For a smooth even convex function $g : \mathbb{R} \to (0, \infty)$, let $f(x) = e^{g(\|x\|)}$ for $x \in \mathbb{R}^n$. According to the Log-Convex Density Theorem by Chambers [146], if $X \subset \mathbb{R}^n$ is of finite perimeter and $\int_X f = \int_{rB^n} f$ for $r > 0$, then

$$\int_{\partial X} f \, d\mathcal{H}^{n-1} \geq \int_{rS^{n-1}} f \, d\mathcal{H}^{n-1}.$$

Silini [535] verified the hyperbolic version of the Log-Convex Density Theorem in H^n.

We note that the isoperimetric inequality in the complex hyperbolic space $H^n_{\mathbb{C}}$ is still open (see J.R. Parker [479] for a beautiful introduction into complex hyperbolic geometry). It has been verified for the so-called Hopf-symmetric sets by Silini [535] using the hyperbolic Log-Convex Density Theorem.

Another famous open isoperimetric problem is the Isoperimetric Inequality in the Heisenberg group $\mathbb{H}^d$. It is the "simplest" simply connected nilpotent Lie-group whose product structure is the easiest to describe on $\mathbb{C}^d \times \mathbb{R}$, and hence its topological dimension is $n = 2d + 1$ (see Franceschi, Leonardi, Monti [247] for the known results about the Isoperimetric Problem in the Heisenberg group $\mathbb{H}^d$). We note that the Brunn-Minkowski (3.24) with exponent $\frac{1}{n}$ does not help in this case because (3.24) does not hold with exponent $\frac{1}{Q}$ for $Q = 2d + 2 = n + 1$, and the isoperimetric problem asks for the minimal $C > 0$ such that $\mu(X)^{\frac{Q-1}{Q}} \leq C \cdot P(X)$ where μ is a Haar measure and $P(X)$ is the corresponding finite perimeter.

The Gaussian Isoperimetric Inequality was proved independently by Borell [85] and Sudakov, Tsirelson [544] using the Spherical Isoperimetric Inequality as in Section 4.6. The significance of the Gaussian Isoperimetric Inequality is shown also by the act how different methods are used to prove it; for example, Bobkov [77], Bakry, Ledoux [43], Barthe, B. Maurey [55]. Cianchi, Fusco, Maggi, Pratelli [164] verified a stability version of the Gaussian Isoperimetric inequality of optimal order using Ehrhard's symmetrization method to prove his inequality (4.49) (see also Barchiesi, Brancolini, Julin [48] for a stability version of the Gaussian Isoperimetric inequality).

Both the Spherical and the Gaussian Isoperimetric inequalities can be considered as results about measure concentration: If X has at least half of the measure of the space, then the complement of $X^{(\varrho)}$ has "small" measure.

An important Brunn-Minkowski-type generalization (even if not on a trivial way, see Livshyts [421]) of the Gaussian Isoperimetric Inequality is the Ehrhard inequality, proved by Ehrhard [197,198] for convex bodies using a symmetrization method, and by Borell [88] (see also van Handel [298]) for measurable sets, and the cases of equality is clarified by Shenfeld, van Handel [532]. The inequality says that any measurable sets $X, Y \subset \mathbb{R}^n$ satisfy

$$\psi^{-1}\left(\gamma_n\left((1-\lambda)X + \lambda Y\right)\right) \geq (1-\lambda)\psi^{-1}\left(\gamma_n(X)\right) + \lambda\,\psi^{-1}\left(\gamma_n(Y)\right) \tag{4.49}$$

for $\lambda \in (0, 1)$ and $\psi(s) = \int_{-\infty}^{s} e^{-\pi t^2}\, dt$.

A famous open problem, posed by Barthe [51], is the Gaussian Isoperimetric Inequality for origin symmetric sets. Here the conjectured optimal set depends on the fixed "volume" (Gaussian measure) $V \in (0, 1)$ of the origin symmetric set. The only known case is due to the recent paper Barchiesi, Julin [49] when the fixed volume is close to one. More precisely, there exists an absolute constant $c \in (0, 1)$ (actually very close to one) such that if $V \in (c, 1)$, then among origin symmetric closed sets X with locally Lipschitz boundary and satisfying $\gamma_n(X) = V$, the origin symmetric slabs bounded by two parallel hyperplanes minimize the Gaussian perimeter. For the current state of art of the problem, see Barchiesi, Julin [49] and Livshyts [421].

Possible strengthenings of the Ehrhard inequality for symmetric sets are discussed by Livshyts [421], stating explicit conjectures.

A Brunn-Minkowski-type inequality for the Gaussian measure that only holds for origin symmetric convex sets $X, Y \subset \mathbb{R}^n$ is due to Eskenazis, Moschidis [203]; namely,

$$\gamma_n\left((1-\lambda)X + \lambda Y\right)^{\frac{1}{n}} \geq (1-\lambda)\gamma_n(X)^{\frac{1}{n}} + \lambda\,\gamma_n(Y)^{\frac{1}{n}} \tag{4.50}$$

holds for $\lambda \in (0, 1)$. Both of the conditions convex and origin symmetric are important; for example, if X is a fixed origin symmetrix convex body and Y is a translate by a vector whose length tends to infinity, then $\gamma_n\left((1-\lambda)X + \lambda Y\right)$ can be arbitrarily small.

The KLS conjecture about the Cheeger constant $C_{\mathrm{Che}}(\mu)$ was stated in the groundbreaking paper Kannan, Lovász, Simonovits [361] in 1995. The paper [361] proved $C_{\mathrm{Che}}(\mu) \leq c\sqrt{n}$, c absolute constant, for an isotropic log-concave measue μ on $\mathbb{R}^n$, and even after 20 years of intense research (see Alonso-Gutiérrez, Bastero [15], Artstein-Avidan, Giannopoulos, Milman [28,29] and Klartag [373]), the best upper bound still stayed at $cn^{\frac{1}{4}}$. The breakthrough came by Yuansi Chen in 2020 verifying that the bound $cn^{\frac{1}{4}}$ can be replaced by $n^{o(1)}$, and the currently best upper bound is the $c\sqrt{\log n}$ of Theorem 4.7.12 due to Klartag [373].

A related problem is discussed by Alter, Caselles [16]. For the uniform probability measure on a convex body $K \subset \mathbb{R}^n$, [16] considers the infimum of $\mathcal{H}^{n-1}(\partial X)/|X|$ for rectifiable subsets of $X \subset K$, more precisely, they consider sets of finit perimeter. Alter, Caselles [16] prove that there is essentially a unique optimal set $X \subset K$, and it is a convex body.

4.A Supplement: The Spherical Space and the Hyperpolic Space

In this section, we summarize the basic notions related to the spherical space and the hyperbolic space. We recall that $\mathrm{pos}\{x, y\} = \{\alpha x + \beta y : \alpha, \beta \geq 0\}$ for $x, y \in \mathbb{R}^{n+1}$.

First we introduce the spherical space $S^n \subset \mathbb{R}^{n+1}$. Fix an $e \in \mathbb{R}^{n+1}$ with $\|e\| = 1$.

Remark 4.A.1 (Notions related to the spherical space $S^n \subset \mathbb{R}^{n+1}$).

Spherical Space: $S^n = \{x \in \mathbb{R}^{n+1} : \langle x, x\rangle = 1\}$.

Distance: $d_{S^n}(x, y) = \arccos\langle x, y\rangle$ for $x, y \in S^n$.

Isometries: $O(n+1)$ - acts transitively (any $x \in S^n$ can be mapped to any other $y \in S^n$ by an isometry), stabilizer of a point is isomorphic to $O(n)$.

Tangent space at $z \in S^n$*:* $T_z = \{y \in \mathbb{R}^{n+1} : \langle y, z\rangle = 0\}$, T_z is equipped withthe scalar product $\langle\cdot,\cdot\rangle$.

Geodesic segment: (also called geodesic arc) $[x, y]_{S^n} = \mathrm{pos}\{x, y\} \cap S^n$ for $x, y \in S^n$ provided $y \neq \pm x$.
If $x, y \in S^n$ with $y \neq \pm x$, then there exists unique "directional vector" $v \in T_x$ with $\langle v, v\rangle = 1$ and $y = x\cos r + v \sin r$ where $r = d_{S^n}(x, y) \in (0, \pi)$, and we have $[x, y]_{S^n} = \{x \cos t + v \sin t : t \in [0, r]\}$.

Angle: If $x, y, z \in S^n$ with $y \neq \pm x$, $z \neq \pm x$, and $y = x\cos r + v \sin r$, $z = x \cos s + w \sin s$ for $r, s \in (0, \pi)$ and $v, w \in T_x$ with $\langle v, v\rangle = \langle w, w\rangle = 1$, then the spherical angle $\angle(y, x, z) = \arccos \langle v, w\rangle = \alpha$, which is also the angle of $[x, y]_{S^n}$ and $[x, z]_{S^n}$, and $\cos d_{S^n}(x, y) = \cos r \cos s + \sin r \sin s \cos \alpha$ according to the *Spherical Law of Cosines for sides.*

"Hyperplane": (or Great subsphere) $H = \{x \in S^n : \langle x, u\rangle = 0\}$ for a $u \in S^n$.

"Closed Halfspace": (or closed Hemisphere) $H^+ = \{x \in S^n : \langle x, u\rangle \geq 0\}$ for a $u \in S^n$, and the corresponding open hemisphere is $\{x \in S^n : \langle x, u\rangle > 0\}$.

Reflection through the hyperplane H: If $H = \{x \in S^n : \langle x, u\rangle = 0\}$ for $u \in S^n$, then the reflected image of an $x \in S^n$ through H is $\xi_H x = x - 2\langle x, u\rangle \cdot u$.

- $\xi_H x = x$ for $x \in H^n$ if and only if $x \in H$;
- ξ_H is an isometry such that $\xi_H(\xi_H x) = x$ for $x \in S^n$;
- for any $y, z \in S^n$, there exists a hyperplane H such that $\xi_H y = z$ ("perpendicular bisector" of $[y, yz]_{S^n}$ provided $y \neq \pm z$).

Spherically convex sets: $X \subset S^n$ is convex, if X contained in an open hemisphere, and $[x, y]_{S^n} \subset X$ for any $x, y \in X$ where the latter property is equivalent to saying that $X = C \cap S^n$ for a Euclidean convex cone $C \subset \mathbb{R}^{n+1}$.

Bi-Lipschitz map to Euclidean geometry:
$S^n_+ = \{x \in S^n : \langle x, e\rangle > 0\}$ fixed open hemisphere. The radial projection $x \mapsto \frac{1}{\langle x,e\rangle} x$ is a bi-Lipschitz map $S^n_+ \to e^\perp + e$ (actually, C^∞ diffeomorphism) mapping spherical segments and convex sets in S^n_+ onto Euclidean segments and convex sets in $e^\perp + e$.

Now we turn to basic properties of the hyperbolic space H^n. One of its characteristic properties how paralell lines occur. We observe that on S^2, any two lines (great circles) intersect in two (antipodal) points, and in $\mathbb{R}^2$, if $\ell \subset \mathbb{R}^2$ is line an $x \in \mathbb{R}^2$ does not lie on ℓ, then there exists exactly one line through x not intersecting ℓ, which is called parallel. Now if ℓ is a line in the hyperbolic plane H^2, and $x \in H^2$ does not lie on ℓ, then there exist exactly two lines ℓ_1 and ℓ_2 through x that does not intersect ℓ, but gets asymptotically arbitrary close to ℓ (see the Beltrami-Cayley-Klein model below). In addition, any line through x between ℓ_1 and ℓ_2 (in the suitable sense) not simply avoids ℓ, but diverges away at both ends.

The hyperbolic space has various useful models, and we discuss two. We start with the Hyperboloid model, which exhibits the metrixcproperties of H^n analogously to S^n. After that we briefly discuss Beltrami-Cayley-Klein model, whose advantage is that hyperbolic lines are Euclidean segments in the model, and it is very transparent how ideal points of the Hyperbolic Space and parallelism work.

To define the Hyperboloid model of H^n in $\mathbb{R}^{n+1}$, fix an $e \in \mathbb{R}^{n+1}$ with $\|e\| = 1$. We consider the following symmetric bilinear form $\mathcal{B}(\cdot,\cdot)$ on $\mathbb{R}^{n+1}$ with eigenvalues $1, -1, \ldots, -1$: If $x = \tilde{x} + te \in \mathbb{R}^{n+1}$ and $y = \tilde{y} + se \in \mathbb{R}^{n+1}$ for $\tilde{x}, \tilde{y} \in e^{\perp}$ and $t, s \in \mathbb{R}$, then

$$\mathcal{B}(x, y) = ts - \langle \tilde{x}, \tilde{y} \rangle.$$

Remark 4.A.2 (Hyperboloid Model of the hyperbolic space H^n in $\mathbb{R}^{n+1}$).

Hyperbolic Space: $H^n \subset \mathbb{R}^{n+1}$.
$H^n = \{x = \tilde{x} + te : \mathcal{B}(x, x) = 1 \text{ and } \tilde{x} \in e^{\perp} \text{ and } t > 0\}$.

Distance: $d_{H^n}(x, y) = \operatorname{arccosh} \mathcal{B}(x, y)$ for $x, y \in H^n$.

Isometries: $\{\Phi \in \mathrm{GL}(n+1) : \mathcal{B}(\Phi x, \Phi y) = \mathcal{B}(x, y) \text{ and } \langle \Phi e, e \rangle > 0\}$.
The isometry group is transitive, and the stabilizer of a point is isomorphic to $O(n)$.

Tangent space at $z \in H^n$: $T_z = \{y \in \mathbb{R}^{n+1} : \mathcal{B}(y, z) = 0\}$. T_z is equipped with the scalar product $-\mathcal{B}(\cdot,\cdot)$.

Geodesic segment: $[x, y]_{H^n} = \operatorname{pos}\{x, y\} \cap H^n$ for $x, y \in H^n$ with $y \neq x$.There exists a unique "directional vector" $v \in T_x$ with $\mathcal{B}(v, v) = -1$ and $y = x \cosh r + v \sinh r$ for $r = d_{H^n}(x, y) > 0$, and $[x, y]_{H^n} = \{x \cosh t + v \sinh t : t \in [0, r]\}$.

Hyperbolic line: $\ell = \Pi \cap H^n$ where $\Pi \subset \mathbb{R}^{n+1}$ two-dimensional linear subspace with $\Pi \cap H^n \neq \emptyset$.
If $x \in \ell$ and $v \in T_x \cap \Pi$ with $\mathcal{B}(v, v) = -1$, then $\ell = \{x \cosh r + v \sinh r : r \in \mathbb{R}\}$. In particular, $w_1 = x + v \in \Pi$ and $w_2 = x - v \in \Pi$ satisfy $\mathcal{B}(w_i, w_i) = 0$, and $\mathbb{R}w_1$ and $\mathbb{R}w_2$ represent the two ideal points of ℓ where the "ideal point" linear subspace $\mathbb{R}w_i$ is part of the "asymptotic cone" $\{w \in \mathbb{R}^{n+1} : \mathcal{B}(w, w) = 0\}$ of the hyperboloid in $\mathbb{R}^{n+1}$.

Parallel lines: Lines ℓ_1 and ℓ_2 are parallel if there exists $w \in \mathbb{R}^{n+1}$ with $\mathcal{B}(w, w) = 0$ such that $\mathbb{R}w$ is an ideal point of both lines.

Angle: If $x, y, z \in H^n$ with $y \neq x, z \neq x$, then $y = x \cosh r + v \sinh r$, $z = x \cosh s + w \sinh s$ for $r, s > 0$ and $v, w \in T_x$ with $\mathcal{B}(v, v) = \mathcal{B}(w, w) = -1$, and the hyperbolic angle is $\angle(y, x, z) = \operatorname{arccosh}(-\mathcal{B}(v, w)) = \alpha$, which is also the angle of $[x, y]_{H^n}$ and $[x, z]_{H^n}$, and satisfies $\cosh d_{H^n}(x, y) = \cosh r \cosh s - \sinh r \sinh s \cos \alpha$ according to the *Hyperbolic Law of Cosines for sides*

Horosphere: Let $w \in \mathbb{R}^{n+1}$ with $\mathcal{B}(w, w) = 0$ and $\langle w, e \rangle > 0$. For $t > 0$, $\Sigma = \{x \in H^n : \mathcal{B}(w, x) = t\}$ is a horosphere centered at the ideal point $\mathbb{R}w$. If $x \in \Sigma$, then $v = x - \frac{1}{t} w \in T_x$ with $\mathcal{B}(v, v) = -1$, and hence $\mathbb{R}w$ ideal point of the line $\ell = \{x \cosh r + v \sinh r : r \in \mathbb{R}\}$ orthogonal to Σ; or in other words, Σ is the orthogonal trajectory to the pencil of parallel lines at $\mathbb{R}w$.

Hyperplane: $H = \{x \in H^n : \mathcal{B}(x, u) = 0\}$ where $\mathcal{B}(u, u) = -1$, $u \in \mathbb{R}^{n+1}$.

Closed Halfspace: $H^+ = \{x \in H^n : -\mathcal{B}(x, u) \geq 0\}$ where $\mathcal{B}(u, u) = -1$, $u \in \mathbb{R}^{n+1}$.

Reflection through the hyperplane H: $\xi_H x = x + 2\mathcal{B}(x, u) \cdot u$ for $x \in H^n$ provided $H = \{x \in H^n : \mathcal{B}(x, u) = 0\}$ where $\mathcal{B}(u, u) = -1$, $u \in \mathbb{R}^{n+1}$.

- ξ_H is an isometry such that $\xi_H(\xi_H x) = x$ for $x \in H^n$;
- $\xi_H x = x$ for $x \in H^n$ if and only if $x \in H$;
- if $y \neq z$, then there exists a unique hyperplane H with $\xi_H y = z$ ("perpendicular bisector" of $[y, z]_{H^n}$).

Hyperbolic convex sets: $X \subset H^n$ is convex if $[x, y]_{H^n} \subset X$ for any $x, y \in X$; or equivalenty, if $X = C \cap H^n$ for a Euclidean convex cone $C \subset \mathbb{R}^{n+1}$.

Bi-Lipschitz map between Hyperbolic and Euclidean geometry: The radial projection $x \mapsto \frac{1}{\langle x, e\rangle} x$ is a bi-Lipschitz map $H^n \to e^\perp + e$ (actually, C^∞ diffeomorphism) mapping hyperbolic segments and convex sets in H^n onto Euclidean segments and convex sets in $e^\perp + e$.

For the Beltrami-Cayley-Klein model of H^n, we only discuss a few notions to indicate how ideal points and paralellism work in the Hyperbolic World.

Remark 4.A.3 (Beltrami-Cayley-Klein model of H^n).

Beltrami-Cayley-Klein model: $\operatorname{int} B^n$ is set of points of the model.

Hyperbolic segment: For $x, y \in \operatorname{int} B^n$, the Euclidean segment $\operatorname{conv}\{x, y\}$ coincides with the hyperbolic sedment $[x, y]_{H^n}$.

Ideal points: ∂B^n is the set of ideal points.

Hyperbolic line: $\operatorname{conv}\{w_1, w_2\} \backslash \{w_1, w_2\}$ for the ideal points $w_1, w_2 \in \partial B^n$.

Parallel Hyperbolic lines: Hyperbolic lines $\ell_1, \ell_2 \subset \operatorname{int} B^n$ are parallel if ℓ_1 and ℓ_2 have common ideal point $w \in \partial B^n$.

Horosphere: $\partial E \backslash \{w\}$ for certain Euclidean ellipsoid $E \subset B^n$ touching B^n from inside at a $w \in \partial B^n$.

Isometry with the Hyperboloid model: $\tilde{\pi} : H^n \to \operatorname{int} B^n$, $\tilde{\pi}(x) = \frac{1}{\langle x, e\rangle} x - e$.

Finally, we discuss properties of the three geometries that work similarly in all three of them. Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n. The following properties readily follow from the definition of convexity in $\mathcal{M}^n$.

Lemma 4.A.4. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n.*

(i) Intersection of convex sets is convex.

(ii) Closed half spaces are convex if $\mathcal{M}^n = \mathbb{R}^n, H^n$.

(iii) Intersection of a convex set and closed half space is convex if $\mathcal{M}^n = S^n$.

The metric ball centered at $x \in \mathcal{M}^n$ and of radius $r > 0$ (where $r < \pi$ if $\mathcal{M}^n = S^n$) is $B_{\mathcal{M}^n}(x, r) = \{y \in \mathcal{M}^n : d_{\mathcal{M}^n}(x, y) \le r\}$. For compact $X \subset \mathcal{M}^n$ and $\varrho > 0$, the corresponding parallel domain is

$$X^{(\varrho)} = \{y \in \mathcal{M}^n : \exists x \in X \text{ with } d_{\mathcal{M}^n}(x, y) \le \varrho\} = \cup \{B_{\mathcal{M}^n}(x, \varrho) : x \in X\}.$$

Lemma 4.A.5. *Let* $\mathcal{M}^n$ *be either* $\mathbb{R}^n$, H^n *or* S^n, *let* H^+ *be a closed half-space in* $\mathcal{M}^n$ *bounded by the hyperplane* H, *and let* H^- *be the complementary closed half-space with* $H^+ \cap H^- = H$.

(i) (Triangle inequality) $d_{\mathcal{M}^n}(x, y) + d_{\mathcal{M}^n}(y, z) \ge d_{\mathcal{M}^n}(x, z)$ *for* $x, y, z \in \mathcal{M}^n$, *with equality if and only if* $y \in [x, z]_{\mathcal{M}^n}$.

(ii) Given $x \in H^+$, $y \in H^+$ *if and only if* $d_{\mathcal{M}^n}(y, x) \le d_{\mathcal{M}^n}(y, \xi_H x)$.

(iii) If $H \cap \mathrm{int} B_{\mathcal{M}^n}(z, r) \ne \emptyset$ *and* $z \in H^+$, *then* $\xi_H(H^- \cap B_{\mathcal{M}^n}(z, r)) \subset H^+ \cap B_{\mathcal{M}^n}(z, r)$ *where we assume that* $r < \pi$ *if* $\mathcal{M}^n = S^n$.

(iv) $B_{\mathcal{M}^n}(x, r)$ *convex for* $r > 0$ *where we assume that* $r < \frac{\pi}{2}$ *if* $\mathcal{M}^n = S^n$.

(v) If X *compact and* $\varrho > 0$, *then* $\mathrm{diam}\, X^{(\varrho)} = 2\varrho + \mathrm{diam}\, X$ *where we assume that* $2\varrho + \mathrm{diam}\, X < \pi$ *if* $\mathcal{M}^n = S^n$.

(vi) A convex body K *in* $\mathcal{M}^n$ *is the intersection of half spaces, and there exists supporting hyperplane* H *at any* $x \in \partial K$ *(*$x \in H$ *and* $K \subset H^+$*).*

Proof. In the case of $\mathbb{R}^n$, these properties are well-known or proved in Chapter 1; therefore, we only present the argument when $\mathcal{M}^n$ is either H^n or S^n.

(i) follows from the Law of Cosines for sides, and in turn (i) yields (ii), and (ii) implies (iii). For (iv), we may assume that $x = e$, and for $B(e, r)_{\mathcal{M}^n}$, and we use the radial projection $\tilde{\pi}_{\mathcal{M}^n}$ and the convexity of Euclidean balls.

For (v), $\mathrm{diam}\, X^{(\varrho)} \le 2\varrho + \mathrm{diam}\, X$ follows from (i). If $\mathrm{diam}\, X = d(x, y)$ for $x, y \in X$, choose $x', y' \in X^{(\varrho)}$ such that $x, y \in [x', y']$ and $d(x, x') = d(y, y') = \varrho$, and hence $d(x', y') = 2\varrho + \mathrm{diam}\, X$.

(vi) follows by the Euclidean case (cf.Lemma 1.2.3), and by the use of radial projection $\tilde{\pi}_{\mathcal{M}^n}$. ■

Remark 4.A.6 ("Volume" in $\mathcal{M}^n$). The "canonical" measure on $\mathcal{M}^n$ is just the n-dimensional Hausdorff measure $\mathcal{H}^n_{\mathcal{M}^n}$ determined by the metric of $\mathcal{M}^n$, which coincides with the Euclidean $\mathcal{H}^n$ in the case of S^n (but this does hold in the hyperbolic case). For measurable $X \subset \mathcal{M}^n$, its "canonical" measure is denoted by $|X|$. This measure is also the suitably normalized Haar measure on $\mathrm{Iso}(\mathcal{M}^n)/O(n)$ for the isometry group $\mathrm{Iso}(\mathcal{M}^n)$ of $\mathcal{M}^n$. The "canonical" measure is regular (cf. Appendix Chapter 10);

namely, if $X \subset \mathcal{M}^n$ is meaurable with $|X| < \infty$, then

$$|X| = \inf\{|U| : X \subset U \ \&\ U \text{ open}\} = \sup\{|C| : C \subset X \ \&\ C \text{ compact}\}.$$

Remark 4.A.7 (Surface area and regular boundary points in $\mathcal{M}^n$). A compact $X \subset \mathcal{M}^n$ has rectifiable boundary if $\mathrm{int}X \neq \emptyset$ and ∂X is the union of finitely many sets that are Lipschitz images of compact subsets of $\mathbb{R}^{n-1}$. In this case, $0 < \mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) < \infty$, and

$$\mathcal{H}^{n-1}_{\mathcal{M}^n}(\partial X) = \lim_{\varrho \to 0^+} \frac{|X^{(\varrho)}| - |X|}{\varrho}. \tag{4.51}$$

$\mathcal{H}^{n-1}_{\mathcal{M}^n}$ a.e. $y \in \partial K$ is a regular boundary point where supporting hyperplane even to $K \cap B(y, \varepsilon)$ for small $\varepsilon > 0$ is unique.

We note that for $x \in \mathcal{M}^n$ and $r > 0$, we have

$$S\left(B_{\mathcal{M}^n}(x,r)\right) = \begin{cases} n\omega_n r^{n-1} & \text{if } \mathcal{M}^n = \mathbb{R}^n; \\ n\omega_n (\sin r)^{n-1} & \text{if } \mathcal{M}^n = S^n; \\ n\omega_n (\sinh r)^{n-1} & \text{if } \mathcal{M}^n = H^n, \end{cases}$$

which formulas follow by considering $B_{\mathcal{M}^n}(e,r)$ if $\mathcal{M}^n$ is either S^n or H^n, and hence the volume of the ball is $|B_{\mathbb{R}^n}(x,r)| = \omega_n r^n$, $|B_{S^n}(x,r)| = n\omega_n \int_0^r (\sin r)^{n-1}$ and $|B_{H^n}(x,r)| = n\omega_n \int_0^r (\sinh r)^{n-1}$. We observe that if $\mathcal{M}^n$ is either $\mathbb{R}^n$, H^n or S^n, then locally the geometry is close to Euclidean; in particular,

$$\lim_{r\to 0^+} \frac{|B_{\mathcal{M}^n}(x,r)|}{\omega_n r^n} = 1 \text{ and } \lim_{r\to 0^+} \frac{S(B_{\mathcal{M}^n}(x,r))}{n\omega_n r^{n-1}} = 1.$$

Even if we promised to discuss the similarities among the three fundamental spaces of constant curvature in the last part of this section, let us close the section with some fun facts showing how the curvature $\kappa = +1, 0, -1$ of S^n, $\mathbb{R}^n$ and H^n, respectively, influences the geometry. For example, S^n is compact, $\lim_{r\to\infty} \frac{S(B_{\mathbb{R}^n}(x,r))}{|B_{\mathbb{R}^n}(x,r)|} = 0$, while the hyperbolic space "expands", namely, $\lim_{r\to\infty} \frac{S(B_{H^n}(x,r))}{|B_{H^n}(x,r)|} = 1$. In addition, if T is a triangle in $\mathcal{M}^2$; namely, convex hull of three non-collinear points x, y, z, then ∂T consists of the three segments determined by x, y, z, and the three angles α, β, γ at x, y, z satisfy

$$\alpha + \beta + \gamma = \pi + \kappa \cdot |T|.$$

4.B Supplement: Equality in the Isodiametric Inequality in H^n and S^n

In this section, we characterize equality in the Isodiametric Inequality Theorem 4.5.1 in H^n and S^n. The argument actually works in $\mathbb{R}^n$, as well. Therefore let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $S^n_+ = \{x \in S^n : \langle x, e\rangle > 0\}$ for the fixed $e \in \mathbb{R}^{n+1}$ in the spherical case. First relate the diameter of a compact set to its convex hull.

Lemma 4.B.1. *Let $\mathcal{M}^n$ be either $\mathbb{R}^n$, H^n or S^n, and let $X \subset \mathcal{M}^n$ be compact where $X \subset S^n_+$ if $\mathcal{M}^n = S^n$.*

(i) There exists a minimal compact convex set $\mathrm{conv}X$ *containing X that is the intersection of all compact convex sets containing X;*

(ii) $\mathrm{diam}\,\mathrm{conv}\, X = \mathrm{diam}\, X$ *provided* $\mathrm{diam}\, X \le \frac{\pi}{2}$ *if $\mathcal{M}^n = S^n$;*

(iii) $|X| > 0$ *and* $\mathrm{conv}\, X \neq X \Longrightarrow |\mathrm{conv}\, X| > |X|$.

Remark. (ii) does not hold if $\mathcal{M}^n = S^n$ and $\mathrm{diam}\, X > \frac{\pi}{2}$.

Proof. For (i), the only non-trivial part is wether some compact convex set contains X if $\mathcal{M}^n = S^n$. However, $X \subset S^n_+$ yields that $X \subset B(e, r)$ for some $r \in (0, \frac{\pi}{2})$.

For (ii), let $D = \mathrm{diam}\, X$. Lemmas 4.A.4 and 4.A.5 yield that it is sufficient to prove the claim that $d(p, q) \le D$ for $p, q \in \widetilde{X}$ where $\widetilde{X} = \bigcap\{B(y, D) : X \subset B(y, D)\}$.

To verify the claim, we observe that $p \in B(x, D)$ for any $x \in X$, and hence $d(p,x) \le D$ for any $x \in X$, which in turn yields that $X \subset B(p, D)$. It follows that $q \in B(p, D)$, thus $d(p, q) \le D$, proving the claim, and in turn (ii).

For (iii), if $|X| > 0$ and $\mathrm{conv}\, X \neq X$, then $(\mathrm{int}\,\mathrm{conv}\, X) \setminus X \neq \emptyset$, implying that $|\mathrm{conv}\, X| > |X|$. ■

For the two-point symmetrization (cf. Definition 4.5.2), we prove at least in the case of a "smooth" convex body K that if $\tau_H K$ is convex for any hyperplane H, then K is a ball.

Proposition 4.B.2. *If $K \subset \mathcal{M}^n$ convex body such that every boundary point is regular and $\tau_H K$ is convex for any hyperplane H, then K is a ball.*

Remark. Proposition 4.B.2 holds even if K is any convex body that may have non-regular boundary points, as well (see Aubrun, Fradelizi [31] and Böröczky, Sagmeister [115]).

Proof. For any $p \in \partial K$, we write H_p to denote the unique supporting hyperplane to K at p. Let $x, y \in K$ satisfy that $d(x, y) = \mathrm{diam}\, K$, and hence H_x and H_y are ortogonal to $[x, y]$, and $w \in \mathrm{int}\, K$ for the midpoint w of $[x, y]$.

Let $r > 0$ be maximal such that $B(w, r) \subset K$. Then $r \le d(w, x) = d(w, y)$, and there exists $z \in \partial K \cap \partial B(w, r)$ such that $z \neq x, y$. As $B(w, r) \subset K$, the unique supporting hyperplane H_z at z is ortogonal to $[w, z]$.

Let H be the perpendicular bisector hyperplane of the segment $[x, z]$, and let H^+ be the corresponding closed halfspace with $z \in H^+$. It follows that $\xi_H x = z$, and Lemma 4.A.4 (vi), the fact that $\tau_{H^+} K$ is convex, and considering the supporting hyperplane of $\tau_{H^+} K$ at $z = \xi_H x$ yield that $\xi_H H_x = H_z$. Since $[w, x]$ and $[w, z]$ are orthogonal to H_x and H_z, the lines of the segments $[w, x]$ and $[w, z]$ are mapped onto each other by ξ_H, and hence $\xi_H w = w$. We deduce that $d(x, w) = d(z, w) = r$, thus $B(w, r) \subset K$, which in turn yields that $B(w,r) = K$ because the diameters of $B(w,r)$ and K are both $d(x,y)$. ∎

Theorem 4.B.3 (Isodiametric Inequality with equality). *Let $\mathcal{M}^n$ be either H^n or S^n, and let $z_0 \in \mathcal{M}^n$. If $X \subset \mathcal{M}^n$ is bounded and measurable with $|X| \geq |B(z_0,r)|$ and* $\operatorname{diam} X \leq 2r$ *for $r > 0$ where $r < \frac{\pi}{4}$ if $\mathcal{M}^n = S^n$, then $X \subset B(y,r)$ for some $y \in \mathcal{M}^n$.*

Proof. Since $\operatorname{cl} X$ has the same diameter as X, we may assume that X is compact.

Fix $\varrho > 0$ where in the spherical case also assume that $r + \varrho < \frac{\pi}{4}$, and hence $2\varrho + \operatorname{diam} X < \frac{\pi}{2}$.

Isoperimetric Inequality Theorem 4.5.1 and Lemma 4.A.5 (v) yield

$$\left|X^{(\varrho)}\right| \geq |B(z_0, r+\varrho)| \text{ and } \operatorname{diam} X^{(\varrho)} \leq 2r + 2\varrho.$$

It follows from the Isodiametric Inequality Theorem 4.5.1 and Lemma 4.B.1 that $X^{(\varrho)}$ is a convex body, thus Proposition 4.B.2 implies that $X^{(\varrho)} = B(y, r+\varrho)$ for some $y \in \mathcal{M}^n$. In turn, we conclude that $X \subset B(y,r)$. ∎

Chapter 5

The Isoperimetric Inequality for sets of Finite Perimeter and the Sobolev inequality for BV functions

The natural set up for the Isoperimetric Inequality in $\mathbb{R}^n$ is in the framework of sets of finite perimeter (that includes bounded sets of Lipschitz boundary) partially because the equality case can be characterized in a natural way for them. In addition, functions of bounded variations (BV functions) - that are the functional analogues of sets of finite perimeter, - form the natural family for the Sobolev inequality, and in this family, unlike in the case of C_c^1 functions, the Sobolev inequality does have extremizers; namely, multiples of characteristic functions of balls.

For related properties of the Haussdor measure, see Section 1.B. Given the technicalities in the subject, this chapter is mostly survey. For in depth study of sets of Finite Perimeter and BV functions, see, for example, Ambrosio, Fusco, Pallara [19] and Maggi [439].

Let us describe an idea how to prove the Isoperimetric Inequality for the surface area $P(E)$ of a convex body $E \subset \mathbb{R}^n$ with C_+^2 boundary (see Section 8.6.1 for the detailed argument). We may assume that $|E| = |B^n|$, and hence Caffarelli's [137] theorem yields the existence of a C^1 diffeomorphism (Brenier map) $T : E \longrightarrow B^n$ such that $D(T)(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in E$. It follows from the AM-GM inequality for the eigenvalues of DT that $\operatorname{div} T(x) \geq n(\det DT(x))^{\frac{1}{n}} = n$, with equality if and only if $DT(x) = I_n$. We deduce from the Divergence Theorem and $\|T\| \leq 1$ that

$$n|B^n|^{\frac{1}{n}}|E|^{\frac{n-1}{n}} = \int_E n\,dx \leq \int_E \operatorname{div} T(x)dx = \int_{\partial E} \langle T(x), \nu_E(x)\rangle\, d\mathcal{H}^{n-1}(x) \leq P(E).$$

If equality holds, then $\operatorname{div} T(x) = n$ for $x \in \operatorname{int} E$; therefore, $DT \equiv I_n$. We conclude that $T(x) = x + z$ for a $z \in \mathbb{R}^n$, and hence $B^n = E + z$.

Now this simple argument does not extend even to the case of convex bodies because typically the Brenier map does not extend to the boundary. However, T is a function of bounded variation (BV function) allowing to use a generalized version of the Divergence Theorem (see Section 5.2.1 for the proof of the Isoperimetric Inequality in its natural settings). Since the notions of BV functions and sets of finite perimeter are essentially equivalent, we develop the two theories in parallel.

5.1 Sets of Finite Perimeter and BV functions

In order to motivate the definition of sets of finite perimeter, we present an example of a set of finite Lebesgue measure in $\mathbb{R}^n$ whose boundary has infinity Lebesgue measure.

Example 5.1.1. For a dense sequence $\{(x_k\}_{k\in\mathbb{N}}$ of points in $\mathbb{R}^n$, we consider $E = \bigcup_{k\in\mathbb{N}} B(x_k, 2^{-k})$. Then $|E| \le \sum_{k\in\mathbb{N}} |B(x_k, 2^{-k})| < \infty$. However, E is dense in $\mathbb{R}^n$, so $|\partial E| = |\overline{E} \setminus E| = \infty$.

We write $C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ to denote the space of C^1 vector fields $T : \mathbb{R}^n \to \mathbb{R}^n$ with compact support.

Idea to define perimeter: If E is a convex body with C^2 boundary with $o \in \operatorname{int} E$, then the Divergence Theorem 2.1.4 (known also as Gauss-Green theorem) yields that

$$\int_E \operatorname{div}(T)\, dx = \int_{\partial E} \langle T, \nu_{\partial E}\rangle d\mathcal{H}^{n-1} \le \mathcal{H}^{n-1}(\partial E)$$

for any vector field $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$ such that $\|T\| \le 1$. On the other hand, if T is the vertor field $T(tz) = \psi(t)\,\nu_E(z)$ for $z \in \partial E$ and $t \ge 0$ where $\psi : \mathbb{R} \to [0,1]$ is C^1 with compact support and $\psi(1) = 1$ and $\psi(0) = 0$, then $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, $\|T\| \le 1$ and

$$\int_E \operatorname{div}(T)\, dx = \int_{\partial E} \langle T, \nu_{\partial E}\rangle d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial E);$$

therefore,

$$\mathcal{H}^{n-1}(\partial E) = \sup\left\{\int_E \operatorname{div}(T) d\mathcal{H}^n \mid T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \|T\| \le 1\right\}.$$

Definition 5.1.2. A measurable set $E \subseteq \mathbb{R}^n$ is a set of *finite perimeter* if

$$P(E) := \sup\left\{\int_E \operatorname{div}(T) dx \mid T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \|T\| \le 1\right\} < \infty.$$

A measurable set $E \subseteq \mathbb{R}^n$ is *locally of finite perimeter* if, for every bounded open set A, it holds

$$P(E;A) := \sup\left\{\int_E \operatorname{div}(X) dx \mid T \in C_c^1(A;\mathbb{R}^n),\ \|T\| \le 1\right\} < \infty.$$

Proposition 5.1.3. *A set $E \subset \mathbb{R}^n$ has locally finite perimeter if and only if there exists a $\mathbb{R}^n$-valued Radon measure $\mu_E = \nu_E\, d|\mu_E|$ on $\mathbb{R}^n$ such that $\|\nu_E\| = 1$ $|\mu_E|$ a.e. in $\mathbb{R}^n$ and*

$$\int_E \operatorname{div}(T) = \int_{\mathbb{R}^n} \langle T, d\mu_E\rangle = \int_E \langle T, \nu_E\rangle d|\mu_E|$$

for any $T \in C_c^1(A;\mathbb{R}^n)$.

Proof. Assume that E has locally finite perimeter, and consider the linear functional

$$L : C_c^1(\mathbb{R}^n;\mathbb{R}^n) \longrightarrow \mathbb{R}, \qquad T \longmapsto \int_E \operatorname{div}(T).$$

Given an open set A with compact closure, if $T \in C_c^1(A;\mathbb{R}^n)$ then

$$|L(T)| = \left|\int_E \operatorname{div}(T)\right| \leq \|T\|_\infty \left|\int_E \operatorname{div}\left(\frac{T}{\|T\|}\right)\right| \leq \|T\|_\infty P(E;A).$$

As $|L(T)| \leq \|T\|_\infty P(E;A)$ for any $T \in C_c^1(A;\mathbb{R}^n)$, it follows that for every open A with compact closure,

$$L : (C_c^1(A;\mathbb{R}^n), \|\cdot\|_\infty) \longrightarrow \mathbb{R}$$

is a bounded densely defined linear functional which thus extends uniquely to a bounded functional on $C_c(A;\mathbb{R}^n)$ with respect to $\|\cdot\|_\infty$. Now the Riesz Representation Theorem 10.1.5 gives the existence of an $\mathbb{R}^n$ valued Radon measure $\mu_E = \nu_E d|\mu_E|$ such that $L(T) = \int_{\mathbb{R}^n} \langle T, \nu_E\rangle d|\mu_E|$ for every $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$, $\nu_E : \mathbb{R}^n \to \mathbb{R}^n$ is $|\mu_E|$ measurable, and $\|\nu_E\| = 1$ $|\mu_E|$ a.e. in $\mathbb{R}^n$.

For the other direction, if $\mu_E = \nu_E d|\mu_E|$ exists, then given $T \in C_c^1(A;\mathbb{R}^n)$ with $\|T\| \leq 1$ we have

$$\left|\int_E \operatorname{div}(T)\right| = \left|\int_{\mathbb{R}^n} \langle T, \nu_E\rangle d|\mu_E|\right| = \left|\int_A \langle T, \nu_E\rangle d|\mu_E|\right| \leq |\mu_E|(A)$$

so $P(E;A) \leq |\mu_E|(A) < \infty$. ■

Example 5.1.4. If $\operatorname{int} E \neq \emptyset$ and ∂E is locally Lipschitz for $E \subset \mathbb{R}^n$, then E is a set of locally finite perimeter, and even is a set of finite perimeter provided E is bounded. In addition, $P(E) = \mathcal{H}^{n-1}(\partial E)$, $|\mu_E| = \mathcal{H}^{n-1}\llcorner_{\partial E}$ and ν_E is the unit exterior normal to ∂E at $\mathcal{H}^{n-1}$ a.e. $x \in \partial E$ according to the Divergence Theorem 2.1.4. If $A \subset \mathbb{R}^n$ is open bounded, then $P(E;A) = \mathcal{H}^{n-1}(A \cap \partial E)$.

Actually, compact $E \subset \mathbb{R}^n$ with rectifiable boundary and $\operatorname{int} E \neq \emptyset$ is also a set of finite perimeter because the Divergence Theorem 2.1.4 holds also in this setting according to Federer [212].

The following facts are contained in Ambrosio, Fusco, Pallara [19] and Maggi [439]:

Remark 5.1.5. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter.

- $P(\lambda E) = \lambda^{n-1} P(E)$ for every $\lambda > 0$.
- $P(x + \Phi E) = P(E)$ for every $\Phi \in O(n)$ and $x \in \mathbb{R}^n$.
- $\mathbb{R}^n \backslash E$ is also a set of finite perimeter, $P(\mathbb{R}^n \backslash E) = P(E)$ and $\mu_{\mathbb{R}^n\backslash E} = -\mu_E$.
- $\operatorname{supp} \mu_E = \{x \in \mathbb{R}^n : 0 < |E \cap (x + rB^n)| < |rB^n|\} \subset \partial E$.

- $P(E;A) = |\mu_E|(A)$ and $P(E) = |\mu_E|(\mathbb{R}^n)$ for any bounded open $A \subset \mathbb{R}^n$.
- For any measurable $F \subset \mathbb{R}^n$ with $|E\Delta F| = 0$ (here Δ stands for the symmetric difference), F is a set of finite perimeter with $\mu_E = \mu_F$, and hence $P(E) = P(F)$.
- There exists a Borel set $F \subset \mathbb{R}^n$ with $|E\Delta F| = 0$ such that $\operatorname{supp} \mu_E = \partial F$.

5.1.1 Rectifiability of sets of locally finite perimeter

It follows from the Lebesgue-Besicovitch theorem on differentiation of measures that for $|\mu_E|$-a.e. x, it holds

$$\lim_{r\to 0^+} \frac{\mu_E(x + rB^n)}{|\mu_E|(rB^n)} = \nu_E(x) \quad \text{and} \quad |\nu_E(x)| = 1. \tag{5.1}$$

Definition 5.1.6. The set of points x such that (5.1) holds is call the *reduced boundary of E* and denoted by $\partial^* E$. Also, at points of the reduced boundary, ν_E is the *measure theoretic outer unit normal to E*.

Theorem 5.1.7 (De Giorgi Rectifiability Theorem, I). *If $E \subset \mathbb{R}^n$ is a set of finite perimeter, then $|\mu_E| = \mathcal{H}^{n-1} \llcorner_{\partial^* E}$ and $\mu_E = \nu_E \, d\mathcal{H}^{n-1} \llcorner_{\partial^* E}$; namely,*

$$\int_E \operatorname{div} T = \int_{\partial^* E} \langle T, \nu_E \rangle d\mathcal{H}^{n-1}$$

for every $T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$. Moreover, $\partial^* E = N \cup \left(\bigcup K_j\right)$ where $\mathcal{H}^{n-1}(N) = 0$, the sets K_j are compact, and $K_j \subseteq M_j$ where M_j is a C^1 manifold of dimension $n-1$ and for every $x \in K_j$, $\nu_E(x)$ is normal to $T_x M_j$.

The importance of the reduced boundary is clarified by the following result (cf. [19, Theorem 3.59]). Here we use the L^1_{loc} convergence of sets, defined by setting $E_h \to E$ if, for every compact set C, we have $|C \cap (E_h \Delta E)| \to 0$.

Theorem 5.1.8 (De Giorgi Rectifiability Theorem, II). *If E is a set of finite perimeter and $x \in \partial^* E$, then*

$$\frac{(E - x)}{r} \to \{y \in \mathbb{R}^n : \langle \nu_E(x), y - x \rangle < 0\} \tag{5.2}$$

as $r \to 0^+$. Moreover, the following representation formulas hold true:

$$\mu_E = \nu_E \, d\mathcal{H}^{n-1} \llcorner_{\partial^* E}\,, \qquad |\mu_E|(\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial^* E)\,. \tag{5.3}$$

Starting from (5.3) and the distributional Divergence Theorem, one finds that, if E is a set of finite perimeter, then

$$\int_E \operatorname{div} T(x) dx = \int_{\partial^* E} \langle T, \nu_E \rangle \, d\mathcal{H}^{n-1} \tag{5.4}$$

for every vector field $T \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$. We shall need a refinement of this result, as discussed below. The first step now is to introduce the space of functions with bounded variation.

5.1.2 Functions of bounded variation (BV functions)

Sets of finite perimeter can be thought as those sets whose indicator functions has a distributional derivative which is a measure. This concept can be generalizes to arbitrary functions, giving rise to the notion of BV functions.

Definition 5.1.9. A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is *BV* if

$$\sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{div}(X)dx \mid X \in C_c^1(\mathbb{R}^n;\mathbb{R}^n),\ \|X\| \leq 1\right\} < \infty.$$

A measurable function $f : \mathbb{R}^n \to$ is *locally BV* if, for every bounded open set A, it holds

$$\sup\left\{\int_{\mathbb{R}^n} f\,\mathrm{div}(X)dx \mid X \in C_c^1(A;\mathbb{R}^n),\ \|X\| \leq 1\right\} < \infty.$$

As for sets of finite perimeter, BV functions enjoy a series of structure theorem. First of all, being a (locally) BV function is equivalently to asking that the distributional derivative of f is a (locally) finite Radon measure.

Proposition 5.1.10. *A measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is* locally BV *if and only if there exist n Radon measures $\{\mu_{i,f}\}_{1\leq i\leq n}$, such that*

$$\int_{\mathbb{R}^n} f\,\partial_i\varphi\,dx = -\int \varphi\,d\mu_{i,f} \qquad \forall\,\varphi \in C_c^1(\mathbb{R}^n).$$

These measures are usually denoted by $\{D_i f\}_{1\leq i\leq n}$, and one writes $Df = (D_1 f,\ldots,D_n f)$, so that the following identity holds:

$$\int_{\mathbb{R}^n} f\,\mathrm{div}(T)\,dx = -\int_{\mathbb{R}^n} \langle T, d(Df)\rangle = -\sum_{i=1}^{n}\int_{\mathbb{R}^n} T^i\,d(D_i f)$$

for all $T = (T^1,\ldots,T^n) \in C_c^1(\mathbb{R}^n;\mathbb{R}^n)$.

Remark. If E is a set of locally finite perimeter, then $\mu_E = -D\mathbf{1}_E$.

Since Df is a (vector-valued) measure, we can define its total variation as

$$|Df|(E) := \sup\left\{\sum_{i\in\mathbb{N}}\left|\int_{E_i} d(Df)\right| \ :\ E_i \cap E_j = \emptyset,\ \bigcup_{i\in\mathbb{N}} E_i \subset E\right\}$$

for all Borel sets $E \subset \mathbb{R}^n$. BV functions and sets of finite perimeter are intrinsically related by the coarea formula [19, Theorem 3.40]: if $A \subset \mathbb{R}^n$ is open and $f \in BV(\mathbb{R}^n)$, then

$$\int_A |Df| = \int_{\mathbb{R}} P(\{f > t\}; A)dt. \tag{5.5}$$

This is just a particular case of the coarea formula:

Theorem 5.1.11 (Coarea Formula). *If $f \in BV(\mathbb{R}^n)$ and $\psi : \mathbb{R}^n \to [0,\infty]$ is a Borel function, then*

$$\int_{\mathbb{R}^n} \psi \, d|Df| = \int_{\mathbb{R}} \int_{\partial^*\{f>t\}} \psi \, d\mathcal{H}^{n-1} \, dt. \tag{5.6}$$

Definition 5.1.12. The Sobolev space $W^{1,1}(\mathbb{R}^n)$ is the set of $f \in BV(\mathbb{R}^n)$ such that df is absolutely continuous with respect to the Lebesgue measure.

Example 5.1.13 (BV functions on an interval). For a function $f : [a,b] \to \mathbb{R}$, f is a BV function if and only if its total variation in the classical sense is finite; namely,

$$|Df|([a,b]) = \sup\left\{\sum_{i=1}^{k} |f(x_i) - f(x_{i-1})| : \, a = x_0 < x_1 < \ldots < x_k = b\right\} < \infty.$$

Note that if f is Lipschitz, then $|Df|([a,b]) = \int_a^b |f'|$ (actually, this also works for any $f \in W^{1,1}(\mathbb{R})$).

5.1.3 Anisotropic perimeter

All previous concepts can be generalized to the setting of anisotropic perimeter. More precisely, given convex body $K \subset \mathbb{R}^n$ with $o \in \mathrm{int}K$, h_K its support function, and $\|\cdot\|_K$ its norm, we define

$$P_K(E) := \sup\left\{\int_E \mathrm{div}(T)dx \mid T \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), \ \|T\|_K \le 1\right\},$$

and, for every open set A,

$$P_K(E;A) := \sup\left\{\int_E \mathrm{div}(T)dx \mid T \in C^1_c(A;\mathbb{R}^n), \ \|T\|_K \le 1\right\}.$$

Note that, since $c_1\|\cdot\| \le \|\cdot\|_K \le c_2\|\cdot\|$ for $c_2 \ge c_1 > 0$, the notion of sets of finite perimeter is independent of the choice of the norm.

Given a (vector-valued) measure μ, we can define its total variation with respect to the dual norm $\|\cdot\|_K$ as

$$\|-Df\|_{K^*}(E) := \sup\left\{\sum_{i\in\mathbb{N}}\left\|\int_{E_i} d(Df)\right\|_{K^*} : \, E_i \cap E_j = \emptyset, \, \bigcup_{i\in\mathbb{N}} E_i \subset E\right\}.$$

Then, (5.5) and (5.6) generalize as follows:

Theorem 5.1.14 (Anistropic Coarea Formula). *If $f \in BV(\mathbb{R}^n)$, $\psi : \mathbb{R}^n \to [0,\infty]$ is a Borel function, $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$ and $A \subset \mathbb{R}^n$ open, then*

$$\int_A d\|-Df\|_{K^*} = \int_{\mathbb{R}} P_K(\{f > t\}; A)dt, \tag{5.7}$$

$$\int_{\mathbb{R}^n} \psi\, d\|-Df\|_{K^*} = \int_{\mathbb{R}} \int_{\partial^*\{f>t\}} \psi\, h_K(\nu_{\{f>t\}})\, d\mathcal{H}^{n-1}\, dt. \tag{5.8}$$

We deduce from (5.7) and (5.8) applied to $\psi = \mathbf{1}_A$ for $A = \mathbb{R}^n$ and $f = \mathbf{1}_E$ for a set of finite perimeter $E \subset \mathbb{R}^n$ the following simple representation of P_K:

Corollary 5.1.15. *If $E \subset \mathbb{R}^n$ is a set of finite perimeter and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(E) = \int_{\partial^* E} \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} = \int_{\partial^* E} h_K(\nu_E)\, d\mathcal{H}^{n-1}. \tag{5.9}$$

5.1.4 A divergence theorem for BV vector fields on sets of finite perimeter

Our goal here it to generalize (5.4) to the case of vector fields $X \in BV(\mathbb{R}^n; \mathbb{R}^n)$.

If E is a Borel set and $\ell \in [0,1]$, we denote by $E^{(\ell)}$ the set of points x of $\mathbb{R}^n$ having density ℓ with respect to E, i.e., $x \in E^{(\ell)}$ if

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} = \ell\,.$$

We use the notation $\partial_{1/2}E$ for $E^{(1/2)}$. A theorem by Federer [19, Theorem 3.61] relates the reduced boundary $\partial^* E$ to the set of points of density $1/2$, ensuring that, if E is a set of finite perimeter then these sets are $\mathcal{H}^{n-1}$-equivalent. More precisely,

$$\mathcal{H}^{n-1}(\mathbb{R}^n \setminus (E^{(1)} \cup E^{(0)} \cup \partial^* E)) = 0\,, \tag{5.10}$$

$$\mathcal{H}^{n-1}(\partial^* E \Delta \partial_{1/2} E) = 0\,. \tag{5.11}$$

Let now E and F be sets of finite perimeter. By [19, Proposition 3.38, Example 3.68, Example 3.97], $E \cap F$ is a set of finite perimeter and, if we let

$$J_{E,F} = \{x \in \partial^* E \cap \partial^* F : \nu_E(x) = \nu_F(x)\}\ , \tag{5.12}$$

then, up to $\mathcal{H}^{n-1}$-null sets,

$$\partial^*(E \cap F) = J_{E,F} \cup [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(1)}]\,, \tag{5.13}$$

Moreover, at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap F$ we find

$$\nu_{E\cap F}(x) = \begin{cases} \nu_E(x)\,, & \text{if } x \in \partial^* E \cap F^{(1)}\,,\\ \nu_F(x)\,, & \text{if } x \in \partial^* F \cap E^{(1)}\,,\\ \nu_E(x) = \nu_F(x)\,, & \text{if } x \in J_{E,F}\,. \end{cases} \tag{5.14}$$

In the particular case that $F \subseteq E$, (5.13) and (5.14) reduce to

$$\partial^* F = [\partial^* F \cap \partial^* E] \cup [\partial^* F \cap E^{(1)}]\,, \tag{5.15}$$

$$\nu_F(x) = \nu_E(x)\,, \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* F \cap \partial^* E, \tag{5.16}$$

where (5.15) is valid up to $\mathcal{H}^{n-1}$-null sets. We shall also use the following lemma concerning the union of two sets of finite perimeter:

Lemma 5.1.16. *Let E and F be sets of finite perimeter with $|E \cap F| = 0$. Then*

$$\nu_{E\cup F}\, d\mathcal{H}^{n-1} \llcorner \partial^*E \cup F) = \nu_E\, d\mathcal{H}^{n-1} \llcorner (\partial^* E \setminus \partial^* F) + \nu_F\, d\mathcal{H}^{n-1} \llcorner (\partial^* F \setminus \partial^* E)\,, \tag{5.17}$$

and $\nu_E(x) = -\nu_F(x)$ at $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E \cap \partial^* F$.

Proof. As $|E \cap F| = 0$, we have $1_{E\cup F} = 1_E + 1_F$. Therefore, by (5.3),

$$\begin{aligned} \nu_{E\cup F}\, d\mathcal{H}^{n-1} \llcorner \partial^* E \cup F) &= D1_{E\cup F} = D1_E + D1_F \\ &= \nu_E\, d\mathcal{H}^{n-1} \llcorner \partial^* E + \nu_F\, d\mathcal{H}^{n-1} \llcorner \partial^* F\,. \end{aligned} \tag{5.18}$$

Since $\partial_{1/2} E \cap \partial_{1/2} F \subseteq (E \cup F)^{(1)}$, we have $\mathcal{H}^{n-1}(\partial^* E \cup F) \cap \partial^* E \cap \partial^* F) = 0$ by (5.11). In particular, (5.17) follows from (5.18). Moreover,

$$0 = \int_C (\nu_E + \nu_F)\, d\mathcal{H}^{n-1}\,, \quad \text{for every Borel set } C \subseteq \partial^* E \cap \partial^* F\,,$$

i.e. $\nu_E = -\nu_F$ at $\mathcal{H}^{n-1}$-a.e. point in $\partial^* E \cap \partial^* F$. ∎

Let us endow the space of $n \times n$ tensors $\mathbb{R}^{n\times n}$ with the metric $|L| = \sqrt{\mathrm{trace}(L^t L)}$. In particular, if $T \in BV_{loc}(\mathbb{R}^n; \mathbb{R}^n)$ and DT is its $\mathbb{R}^{n\times n}$-valued distributional derivative (which is a measure), then we denote by $|DT|(C)$ the total variation of DT on the Borel set C defined with respect to this metric.

Since DT is a measure, we can decompose it into its absolutely continuous and singular part with respect to the Lebesgue measure. More precisely, we denote by ∇T the density of DT with respect to Lebesgue measure, and by $D_s T$ the corresponding singular part, so that $DT = \nabla T\, dx + D_s T$. If $\mathrm{Div}\, T$ is the distributional divergence of T and if DT takes values in the set of $n \times n$-tensors that are symmetric and positive definite, then $\mathrm{Div}\, T$ is a non-negative Radon measure on $\mathbb{R}^n$, which is bounded above and below by the total variation of T: for every Borel set C in $\mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \int_C d(\mathrm{Div}\, T) \le \int_C d|DT| \le \int_C d(\mathrm{Div}\, T), \tag{5.19}$$

as a consequence of the inequality $n^{-1/2} \sum_{i=1}^n \ell_i \le (\sum_{i=1}^n \ell_i^2)^{1/2} \le \sum_{i=1}^n \ell_i$ whenever $\ell_i \ge 0$. Moreover, if we set $\mathrm{div}\, T(x) = \mathrm{trace}(\nabla T(x))$, then

$$\mathrm{Div}\, T = \mathrm{div}\, T\, dx + (\mathrm{Div}\, T)_s\,, \quad (\mathrm{Div}\, T)_s = \mathrm{trace}(D_s T) \ge |D_s T|\,, \tag{5.20}$$

Note that, as a consequence of (5.20), $\operatorname{Div} T - \operatorname{div} T\, dx$ is a non-negative Radon measure.

Whenever $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ there exists a vector $\operatorname{tr}_E(T)(x) \in \mathbb{R}^n$ such that

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)\cap\{y:(y-x)\cdot\nu_E(x)<0\}} |T(y) - \operatorname{tr}_E(T)(x)|dy = 0\,, \tag{5.21}$$

called the *inner trace of T on E*, see [19, Theorem 3.77]. Note that, as a byproduct of (5.2) we have in fact

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B_r(x)\cap E} |T(y) - \operatorname{tr}_E(T)(x)|dy = 0\,. \tag{5.22}$$

Moreover, as a consequence of [19, Example 3.97] (applied to the pair of functions T and 1_E) the Divergence Theorem holds true in the following form:

Theorem 5.1.17 (Divergence Theorem for BV vector fields). *If $T \in BV(\mathbb{R}^n;\mathbb{R}^n)$ and E is a set of finite perimeter, then*

$$\int_{E^{(1)}} d(\operatorname{Div} T) = \int_{\partial^* E} \langle \operatorname{tr}_E(T), \nu_E\rangle\, d\mathcal{H}^{n-1}. \tag{5.23}$$

5.2 Characterization of isoperimetric sets

5.2.1 The Anisotropic Isoperimetric inequality

In section we wish to characterize isoperimetric sets for the anisotropic isoperimetric inequality. This argument is contained in Figalli, Maggi, Pratelli [225] in a more general setting.

Theorem 5.2.1 (Anisotropic Isoperimetric Inequality with Equality). *Let E be a set of finite perimeter with $0 < |E| < \infty$. Then $P_K(E) \geq n|K|^{1/n}|E|^{(n-1)/n}$, with equality if and only if $|E\Delta(x_0 + rK)| = 0$ for some $x_0 \in \mathbb{R}^n$ and $r > 0$.*

After various attempts, satisfactory stability version of the Anisotropic Isoperimetric Inequality Theorem 5.2.1 was provided by Figalli, Maggi, Pratelli [225]. We recall that for measurable sets $X, Y \subset \mathbb{R}^n$ with $|X|, |Y| > 0$, if $\alpha = |X|^{\frac{-1}{n}}$ and $\beta = |Y|^{\frac{-1}{n}}$, then

$$A(X,Y) = \min\{|\alpha X \Delta(z + \beta Y)| \,:\, z \in \mathbb{R}^n\}\,.$$

Theorem 5.2.2 (Figalli, Maggi, Pratelli). *For $\theta_n = 2^{-16}n^{-17}$, if $E \subset \mathbb{R}^n$ has finite perimeter and $|E| > 0$, and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then*

$$P_K(E) \geq n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(K,E)^2\right]. \tag{5.24}$$

Remark. Here the exponent 2 of $A(K, E)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

To prove Theorem 5.2.1, we shall need some preliminary results.

A set of finite perimeter E is said *indecomposable* if for every $F \subseteq E$ having finite perimeter and such that

$$\mathcal{H}^{n-1}(\partial^* E) = \mathcal{H}^{n-1}(\partial^* F) + \mathcal{H}^{n-1}(\partial^*(E \setminus F)), \tag{5.25}$$

we have that $\min\{|F|, |E \setminus F|\} = 0$. We shall need the following lemma.

Lemma 5.2.3. *Let E be an indecomposable set and let $f \in BV(\mathbb{R}^n)$. If $\int_{E^{(1)}} d|Df| = 0$, then there exists $c \in \mathbb{R}$ such that $f(x) = c$ for a.e. $x \in E$.*

Proof. Let $F_t = E \cap \{f > t\}$. As E is indecomposable, it suffices to show that (5.25) holds with $F = F_t$ for a.e. $t \in \mathbb{R}$. In fact it is enough to prove that

$$\mathcal{H}^{n-1}(\partial^* E) \geq \mathcal{H}^{n-1}(\partial^* F_t) + \mathcal{H}^{n-1}(\partial^*(E \setminus F_t)), \tag{5.26}$$

for a.e. $t \in \mathbb{R}$, as the converse inequality follows from the subadditivity of the distributional perimeter [19, Proposition 3.38 (d)]. To this end we start by noticing that

$$\{f \leq t\}^{(1)} = \{f > t\}^{(0)}, \quad \partial_{1/2}\{f > t\} = \partial_{1/2}\{f \leq t\}, \tag{5.27}$$

$$\mathcal{H}^{n-1}(\partial^*\{f > t\} \Delta \partial^*\{f \leq t\}) = 0, \quad \mathcal{H}^{n-1}(J_{E,\{f>t\}} \cap J_{E,\{f \leq t\}}) = 0, \tag{5.28}$$

where (5.27) is trivially checked, and where (5.28) follows from (5.27), Lemma 5.1.16 and (5.11). We now come to the proof of (5.26). By (5.13), as $E \setminus F_t = E \cap \{f \leq t\}$, we have that, up to $\mathcal{H}^{n-1}$-null sets,

$$\begin{aligned} \partial^* F_t &= J_{E,\{f>t\}} \cup [E^{(1)} \cap \partial^*\{f > t\}] \cup [\partial^* E \cap \{f > t\}^{(1)}], & (5.29)\\ \partial^*(E \setminus F_t) &= J_{E,\{f \leq t\}} \cup [E^{(1)} \cap \partial^*\{f \leq t\}] \cup [\partial^* E \cap \{f \leq t\}^{(1)}], & (5.30) \end{aligned}$$

(the notation $J_{E,F}$ was introduced in (5.12)). Hence, by the Coarea Formula (5.6) we get

$$0 = \int_{E^{(1)}} d|Df| = \int_{\mathbb{R}} \mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f > t\}) dt,$$

i.e. $\mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f > t\}) = 0$ for a.e. $t \in \mathbb{R}$. By (5.28), we also have $\mathcal{H}^{n-1}(E^{(1)} \cap \partial^*\{f \leq t\}) = 0$ for a.e. $t \in \mathbb{R}$. Therefore, thanks to (5.27) we eventually deduce from (5.29) and (5.30), that

$$\begin{aligned} \mathcal{H}^{n-1}(\partial^* F_t) + \mathcal{H}^{n-1}(\partial^*(E \setminus F_t)) &= \mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f \leq t\}}) & (5.31)\\ &\quad + \mathcal{H}^{n-1}(\partial^* E \cap [\{f > t\}^{(1)} \cup \{f > t\}^{(0)}]). \end{aligned}$$

Since (5.28) implies $\mathcal{H}^{n-1}(J_{E,\{f>t\}}) + \mathcal{H}^{n-1}(J_{E,\{f \leq t\}}) \leq \mathcal{H}^{n-1}(\partial^* E \cap \partial^*\{f > t\})$, (5.26) follows from (5.10) and (5.31). ∎

Before proving Theorem 5.2.1, we first give a rigorous proof of the anisotropic isoperimetric inequality itself for sets of finite perimeter using the Brenier map.

Theorem 5.2.4. *Whenever $|E| < \infty$, we have*

$$P_K(E) \geq n|K|^{1/n}|E|^{1/n'} .$$

Proof. By scaling, we can assume that $|E| = |K|$. The Brenier-McCann Theorem [127, 443] ensures the existence of a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that, if we set $T = \nabla\varphi$, then $T(x)$ belongs to K for a.e. $x \in \mathbb{R}^n$ and $T_\#(1_E(x)dx) = 1_K(y)dy$, i.e.

$$\int_K h(y)dy = \int_E h(T(x))dx\,, \tag{5.32}$$

for every Borel function $h : \mathbb{R}^n \to [0,\infty]$. As T is the gradient of convex function, its distributional derivative DT takes values in the set of symmetric and non-negative definite $n \times n$-tensors. Therefore $T \in BV(\mathbb{R}^n; K)$ (see e.g. [1, Proposition 5.1]) and (5.19) and (5.20) hold. Moreover, as a consequence of (5.32), $\det \nabla T(x) = 1$ for a.e. $x \in E$, see [444]. Since $\nabla T(x)$ is a positive semi-definite symmetric tensor for a.e. $x \in \mathbb{R}^n$, we can define measurable functions $\ell_k : \mathbb{R}^n \to [0,\infty)$ and $e_k : \mathbb{R}^{n n-1}$, $k = 1, \dots, n$, such that

$$0 < \ell_k \leq \ell_{k+1}\,, \quad e_i \cdot e_j = \delta_{i,j}\,, \quad \nabla T = \sum_{k=1}^n \ell_k e_k \otimes e_k\,.$$

Then the arithmetic-geometric mean inequality implies that, for a.e. $x \in E$,

$$n = n(\det \nabla T(x))^{1/n} = n\left(\prod_{k=1}^n \ell_k(x)\right)^{1/n} \leq \sum_{k=1}^n \ell_k(x) = \mathrm{div} T(x)\,. \tag{5.33}$$

By (5.33), (5.20), and (5.23)

$$\begin{aligned} n|K|^{1/n}|E|^{1/n'} &= n|E| = \int_E n(\det \nabla T(x))^{1/n}dx \\ &\leq \int_E \mathrm{div} T(x)dx = \int_{E^{(1)}} \mathrm{div}\, T(x)dx \qquad (5.34) \\ &\leq \int_{E^{(1)}} d(\mathrm{Div}\, T) = \int_{\partial^* E} \langle \mathrm{tr}_E(T), \nu_E\rangle\, d\mathcal{H}^{n-1}\,, \qquad (5.35) \end{aligned}$$

By (5.21), since T takes values in K, we find $\|\mathrm{tr}_E(T)(x)\|_K \leq 1$ for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$. Hence,

$$\begin{aligned} n|K|^{1/n}|E|^{1/n'} &\leq \int_{\partial^* E} \|\mathrm{tr}_E(T)\|_K \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} \\ &\leq \int_{\partial^* E} \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} = P_K(E), \end{aligned} \tag{5.36}$$

which proves the anisotropic isoperimetric inequality. ∎

Proof of Theorem 5.2.1. Assume that $P(E) = n|K|^{1/n}|E|^{(n-1)/n}$. We first claim that E is indecomposable.

Indeed, let F be a set of finite perimeter contained in E and such that (5.25) holds true. The usual considerations based on (5.10), (5.11) and (5.13) serves to show that

$$\mathcal{H}^{n-1}(\partial^* F) = \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* E \cap \partial^* F)\,,$$
$$\mathcal{H}^{n-1}(\partial^* (E \setminus F)) = \mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) + \mathcal{H}^{n-1}(\partial^* E \setminus \partial^* F)\,.$$

Thus, by (5.25), we deduce $\mathcal{H}^{n-1}(\partial^* F \cap E^{(1)}) = 0$. In particular

$$\begin{aligned}
P_K(F) + P_K(E \setminus F) &= \int_{\partial^* F \cap \partial^* E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* F \cap E^{(1)}} \|\nu_F\|_* d\mathcal{H}^{n-1} \\
&\quad + \int_{\partial^* E \setminus \partial^* F} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* F \cap E^{(1)}} \|-\nu_F\|_* d\mathcal{H}^{n-1} \\
&= \int_{\partial^* F \cap \partial^* E} \|\nu_E\|_* d\mathcal{H}^{n-1} + \int_{\partial^* E \setminus \partial^* F} \|\nu_E\|_* d\mathcal{H}^{n-1} = P_K(E).
\end{aligned}$$

Hence, by the anisotropic isoperimetric inequality

$$\begin{aligned}
P_K(E) = P_K(F) + P_K(E \setminus F) &\geq n|K|^{1/n}(|F|^{(n-1)/n} + |E \setminus F|^{(n-1)/n}) \\
&\geq n|K|^{1/n}(|F| + |E \setminus F|)^{(n-1)/n} = P_K(E)\,,
\end{aligned}$$

and so, by strict concavity, $\min\{|F|, |E \setminus F|\} = 0$. Thus, E is indecomposable.

We now assume without loss of generality that $|E| = |K|$, and repeat the proof of Theorem 5.2.4. As $P(E) = n|K|^{1/n}|E|^{1/n'}$, we deduce in particular that the Brenier map T between E and K satisfies

$$0 = \int_E \left\{ \frac{\operatorname{div} T(x)}{n} - (\det \nabla T(x))^{1/n} \right\} dx + \int_{E^{(1)}} \frac{d(\mathrm{Div} T)_s}{n}\,.$$

In particular, since the last term vanishes, we deduce that $T \in W^{1,1}(\mathbb{R}^n; K)$. As $\det \nabla T(x) = 1$ a.e. on E, the equality $\frac{\operatorname{div} T(x)}{n} = (\det \nabla T(x))^{1/n}$ implies that $\nabla T(x) = \mathrm{Id}$ at a.e. $x \in E$, or equivalently $\nabla S = 0$ a.e. in E, with $S = T - x \in W^{1,1}(\mathbb{R}^n; \mathbb{R}^n)$. Thus, applying Lemma 5.2.3 to each component of the vector field S we deduce the existence of a vector $x_0 \in \mathbb{R}^n$ such that $T(x) = x - x_0$ for a.e. $x \in E^{(1)}$. As $T(x) \in K$ for a.e. $x \in E$, we deduce that $E^{(1)}$ is a subset of $x_0 + K$, and since $|K| = |E| = |E^{(1)}|$ we conclude the proof. ∎

5.2.2 Wulff theorem on minimizing surface energy

For a lower semicontimuous and bounded $\varrho : S^{n-1} \to (0, \infty)$ and $E \subset \mathbb{R}^n$ of finite perimeter, the associated surface energy of E is

$$\mathcal{E}_\varrho(\partial E) = \int_{\partial^* E} \varrho(\nu_E)\, d\mathcal{H}^{n-1}.$$

Extending the results of Section 4.4, we characterize sets minimizing the surface energy $\mathcal{E}_\varrho(\partial E)$ among sets of finite perimeter of given volume. The corresponding Wulff shape is

$$W_\varrho = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \varrho(u)\ \forall u \in S^{n-1}\}. \tag{5.37}$$

Since $\inf \varrho > 0$ as ϱ is lower semi-continuous, W_ϱ is a convex body with $o \in \operatorname{int} W_\varrho$. Examples for possible ϱ and W_ϱ; moreover, the connection of the problem to crystallography is discussed in Section 4.4 and in the Comments to Chapter 4.

Theorem 5.2.5 (Wulff's inequality on minimizing surface energy). *Let $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If $E \subset \mathbb{R}^n$ is a set of finite perimeter with $0 < |E| < \infty$ and $|E| = \left|rW_\varrho\right|$ for $r > 0$, then*

$$\mathcal{E}_\varrho(\partial E) \geq \mathcal{E}_\varrho\left(\partial(rW_\varrho)\right) = n\left|W_\varrho\right|^{\frac{1}{n}} \cdot |E|^{\frac{n-1}{n}}. \tag{5.38}$$

Equality holds in (5.38) if and only if $|E\Delta(z + rW_\varrho)| = 0$ for some $z \in \mathbb{R}^n$.

In order to prove Theorem 5.2.5, we recall two facts. First, we deduce from the Anisotropic Isoperimetric inequality Theorem 5.2.1 and (5.9) that if K is a convex body with $o \in \operatorname{int} K$ and $E \subset \mathbb{R}^n$ is a set of finite perimeter with $0 < |E| < \infty$, then

$$\int_{\partial^* E} h_K(\nu_E)\, d\mathcal{H}^{n-1} \geq n|K|^{1/n}|E|^{(n-1)/n}, \tag{5.39}$$

with equality if and only if $|E\Delta(z + rK)| = 0$ for some $z \in \mathbb{R}^n$ and $r > 0$. Secondly, Lemma 4.4.3 says that if $\varrho : S^{n-1} \to \mathbb{R}_{>0}$ is bounded and lower semi-continuous, then

$$\int_{\partial W_\varrho} \varrho(\nu_{W_\varrho})\, d\mathcal{H}^{n-1} = \int_{\partial W_\varrho} h_{W_\varrho}(\nu_{W_\varrho})\, d\mathcal{H}^{n-1}. \tag{5.40}$$

Proof of Theorem 5.2.5. For $W = W_\varrho$, we have $\varrho \geq h_W$ by the definition of W_ϱ. Therefore, the Anisotropic Isoperimetric Inequality (5.39) and (5.40) yield

$$\begin{aligned}
\mathcal{E}_\varrho(\partial E) &= \int_{\partial^* E} \varrho \circ \nu_E\, d\mathcal{H}^{n-1} \geq \int_{\partial^* E} h_W \circ \nu_E\, d\mathcal{H}^{n-1} \\
&\geq \int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \int_{\partial(rW)} \varrho \circ \nu_{rW}\, d\mathcal{H}^{n-1} = \mathcal{E}_\varrho\left(\partial(rW)\right)
\end{aligned}$$

where $\int_{\partial(rW)} h_W \circ \nu_{rW}\, d\mathcal{H}^{n-1} = n\left|W_\varrho\right|^{\frac{1}{n}} \cdot |E|^{\frac{n-1}{n}}$.

Equality in (5.38) yields equality in the Anisotropic Isoperimetric Inequality; therefore, $|E\Delta(z + rW_\varrho)| = 0$ for some $z \in \mathbb{R}^n$. ■

The stability version Theorem 5.2.2 of the Anisotropic Isoperimetric Inequality due to by Figalli, Maggi, Pratelli [225] and the argument above yields the following stability version of Wulff's theorem:

Theorem 5.2.6 (Figalli, Maggi, Pratelli). *Let $\theta_n = 2^{-16}n^{-17}$, and $\varrho : S^{n-1} \to (0,\infty)$ be bounded and lower semi-continuous. If $E \subset \mathbb{R}^n$ has finite perimeter and $|E| > 0$, then*

$$\mathcal{E}_\varrho(\partial E) \geq n|W_\varrho|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(W_\varrho, E)^2\right]. \tag{5.41}$$

Remark. Figalli, Zhang [229] proved an even stronger stability estimate in the crystalline case when W_ϱ is a polytope.

5.3 The anisotropic Sobolev inequality for BV functions

The isoperimetric inequality has a correspondent for BV functions: define the anisotropic total variation of a BV function f as

$$TV_K(f) := \int_{\mathbb{R}^n} \|-Df\|_{K^*}.$$

As a consequence of the Anisotropic Isoperimetric inequality Theorem 5.2.1 and of the Coarea formula (5.7), one can prove the following anisotropic Sobolev inequality:

Theorem 5.3.1 (Anisotropic Sobolev inequality for BV functions). *If $f \in L^1_{loc}(\mathbb{R}^n)$ vanishes at infinity; namely,*

$$|\{x \in \mathbb{R}^n : |f(x)| > t\}| < \infty, \quad \forall t > 0, \tag{5.42}$$

and $K \subset \mathbb{R}^n$ is a convex body with $o \in \operatorname{int} K$, then

$$TV_K(f) \geq n|K|^{1/n}\|f\|_{n'}, \qquad n' = \frac{n}{n-1}, \tag{5.43}$$

with equality if and only if $f = a\mathbf{1}_{x+rK}$ for $a, r > 0$ and $x \in \mathbb{R}^n$.

We show here a proof of this inequality, with a characterization of the equality case. One of the advantages of allowing functions with bounded variation is that some functions do satify the equality condition unlike in the case of of classical Sobolev inequality Theorem 4.2.1 and Theorem 4.3.4. We first need the following elementary result:

Lemma 5.3.2. *If $F : [0,\infty) \to [0,\infty)$ decreasing function with $F \not\equiv 0$, and $\alpha > 1$, then*

$$\alpha \int_0^\infty t^{\alpha-1}F(t)^\alpha dt \leq \left(\int_0^\infty F(t)\, dt\right)^\alpha. \tag{5.44}$$

Equality holds if and only if $F(t) = c\mathbf{1}_{[0,T]}(t)$ for some $c, T > 0$.

Proof. Since F is decreasing, $tF(t) \le \int_0^t F(s)\,ds$. Hence,

$$\alpha\big(tF(t)\big)^{\alpha-1}F(t) \le \alpha\bigg(\int_0^t F(s)\,ds\bigg)^{\alpha-1}F(t) = \frac{d}{dt}\bigg(\int_0^t F(s)\,ds\bigg)^{\alpha}.$$

Integrating the inequality above over $(0,\infty)$, yields the inequality (5.44).

Now if equality holds in (5.44), then for a.e. $t \in (0,\infty)$ we have

$$\text{either} \quad tF(t) = \int_0^t F(s)\,ds \quad \text{or} \quad F(t) = 0.$$

Since F is decreasing,we deduce that $F(t) = c\mathbf{1}_{[0,T]}(t)$ for some $c, T \ge 0$. ∎

We can now prove the Anisotropic Sobolev inequality (5.43), including the characterization of the equality case.

Proof of Theorem 5.3.1. We can assume that f is not identically zero.

Combining the coarea formula (5.7) (with $A = \mathbb{R}^n$) with Theorem 5.2.1 (applied to each set $\{|f| > t\}$) and Lemma 5.3.2 (with $F(t) = |\{|f| > t\}|^{1/n'}$ and $\alpha = n'$), we have

$$\begin{aligned} TV_K(f) &= \int_{-\infty}^{\infty} P_K(\{f > t\})\,dt = \int_0^{\infty} P_K(\{|f| > t\})\,dt \ge \int_0^{\infty} n|K|^{1/n}|\{|f| > t\}|^{1/n'}\,dt \\ &\ge n|K|^{1/n}\bigg(n'\int_0^{\infty} t^{n'-1}|\{|f| > t\}|\,dt\bigg)^{1/n'} = n|K|^{1/n}\|f\|_{n'}, \end{aligned}$$

where the last equality follow from $\{|f| > t\} = \{|f|^{n'} > t^{n'}\}$, the substitution $s = t^{n'}$ and the layer cake formula. This proves (5.43).

Now, if equality holds then Theorem 5.2.1 implies that, for a.e. t, up to sets of measure zero the sets $\{f > t\}$ are of the form $x_t + r_tK$ for some $x_t \in \mathbb{R}^n$ and $r_t > 0$. Secondly, Lemma 5.3.2 implies that the function $F(t) = |\{f > t\}|^{1/n'}$ is equal to $c1_{[0,T]}$ for some constants $c, T > 0$, which implies the existence of a radius $r > 0$ such that $r_t = r$ for $t \in [0,T]$ and $r_t = 0$ for $t > T$. Hence, up to sets of measure zero,

$$\{f > t\} = x_t + rK \quad \text{for } t \in [0,T], \qquad \{f > t\} = \emptyset \quad \text{for } t > T.$$

However, since the sets $\{f > t\}$ are contained one inside the other, the only possibility is that x_t is constant, thus $f = T\mathbf{1}_{x+rK}$. ∎

5.4 Comments to Chapter 5

The material of this chapter is discussed in the monographs Federer [212], Ambrosio, Fusco, Pallara [19] and Maggi [439], and in the paper Figalli, Maggi, Pratelli [225].

Continuous functions of bounded variations were seemingly first considered by Camille Jordan in his studies about the convergence Fourier series in 1881. General functions of bounded variation in several variable we treated by Lamberto Cesari around 1936, and the intimate connections between BV functions and sets of finite perimeter were established by Renato Caccioppoli and Ennio de Giorgi in the 1950s. Sets of finite perimeter are sometimes called Caccioppoli sets. De Giorgi [187] proved the classical isperimetric inequality for sets of finite perimeter *via* Steiner Symmetrization, also characterizing the equality case in 1958 (see also Talenti [546]). The Anisotropic Isoperimetric inequality for sets of finite perimeter was proved by Taylor [549] in 1978, and later Brothers, Morgan [130] and Fonseca, Müller [238] presented simplified arguments.

The notion of Wulff shape originates from the paper Wulff [568] related to Crystallography, and see Maggi [439], Brothers, Morgan [130] and Fonseca, Müller [238] for a dicussion of Wulff's theorem for sets of finite perimeter, and the papers Taylor [549], Miracle-Sole [466] and Figalli, Maggi [226] for the role of Wulff shape within crystallography. Even a strong stability version of the Wulff inequality is verified by Figalli, Zhang [229]. Many examples of Wulff shapes and the relation to the underlying periodic and quasi-periodic structure are discussed in Böröczky, Schnell, Wills [118]. In particular, Wulff shapes are also successful models of certain quasi-crystals.

The optimal factor in Sobolev's inequality Theorem 4.2.1 is verified by Federer, Fleming [211] using symmetrization, and the stability version of the Sobolev inequality of optimal order for functions of bounded variation is due to Figalli, Maggi, Pratelli [227]. Actually, there exists an L_p version of the Sobolev inequality for $1 < p < n$, as well, where the optimal factor has been determined by Talenti [545], and the stability version of optimal order is due to Bianchi, Egnell [70] if $p = 2$, and to Figalli, Zhang [228] if $1 < p < n$.

Chapter 6

Associated ellipsoids, Blaschke-Santaló inequality and the Reverse Isoperimetric Inequality

This chapter focuses on affine invariant properties of convex bodies in $\mathbb{R}^n$ centered around the study of some affine equivariant associated ellipsoids; namely, the maximal volume inscribed so-called John Ellipsoid, and the minimal volume circumscribed so called Löwner Ellipsoid are introduced in the first section, the Ellipsoid of Inertia in Section 6.4, and the so-called M-Ellipsoid in the closing Section 6.9. One of the main significance of these ellipsoids that they approximate rather well the corresponding convex body. We discuss various related inequalities, like the Blaschke-Santaló inequality, and the Reverse forms of the Isoperimetric and the Blaschke-Santaló inequalities where the associated ellipsoids have significant roles in the arguments.

6.1 John and Löwner ellipsoid

We recall that an ellipsoid E is of the form $E = \Phi\, B^n + v$ for $\Phi \in \mathrm{GL}(n)$ and $v \in \mathbb{R}^n$ where v is the center of E. Actually, there exists a positive definite symmetric matrix A such that $E = AB^n + v$ (see Section 6.A). In this section, we introduce the inscribed John and the circumscribed Löwner ellipsoid that approximate best in terms of volume difference. We only provide the arguments in the case of the John ellipsoid because they are analogous but slightly more technical in the case of the Löwner ellipsoid. The following statement is proved in Proposition 6.1.5 and Proposition 6.1.7:

Theorem 6.1.1 (John). *If $K \subset \mathbb{R}^n$ is a convex body, then there exists a unique "John" ellipsoid E of maximal volume contained in K, and writing x_0 to denote the center of E, we have*

$$E \subset K \subset x_0 + n(E - x_0).$$

If K is origin symmetric ($K = -K$), then $x_0 = o$, and $E \subset K \subset \sqrt{n}\, E$.

Remark. ΦE is the John ellipsoid of ΦK for any $\Phi \in \mathrm{GL}(n)$.

For $u \in S^{n-1}$, $u \otimes u = u \cdot u^t$ is a rank one positive semidefinite $n \times n$ symmetric matrix with $\mathrm{tr}(u \otimes u) = 1$. We postpone the proof of John's condition Theorem 6.1.2 characterizing the maximal volume inscribed ellipsoid by the contact points to Section 6.A because the argument is somewhat lengthy even if natural.

Theorem 6.1.2 (John). *If B^n is the John ellipsoid inside a convex body K in $\mathbb{R}^n$, then $B^n \subset K$ and there exist $u_1, \ldots, u_k \in S^{n-1} \cap \partial K$ and $c_1, \ldots, c_k > 0$, $k \leq \frac{n(n+3)}{2}$, such* *that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.1}$$

$$\sum_{i=1}^{k} c_i u_i = o \tag{6.2}$$

where I_n *denotes the* $n \times n$ *identity matrix.*

If K *is origin symmetric* ($K = -K$)*, then* (6.1) *is sufficient.*

Remarks.

- (6.1) yields that $\langle x, y\rangle = \sum_{i=1}^{k} c_i \langle x, u_i\rangle\langle y, u_i\rangle$ for $x, y \in \mathbb{R}^n$, and hence the discrete measure μ on S^{n-1} concentrated on $\{u_1, \dots, u_k\}$ with $\mu(u_i) = c_i$ is called isotropic.
- $\sum_{i=1}^{k} c_i = n$ follows by comparing traces in (6.1).
- $\langle x, u_i\rangle \leq 1$ for $x \in K$ and $i = 1, \dots, k$ as K and B^n share the same supporting hyperplanes at $u_1, \dots, u_k$.

Next we consider the minimum volume circumscribed so-called Löwner ellipsoid.

Theorem 6.1.3 (John, Löwner). *If* $K \subset \mathbb{R}^n$ *convex body, then there exists a unique "Löwner" ellipsoid* E *of minimal volume containing* K*, and writing* x_0 *to denote the center of* E*, we have*

$$x_0 + \frac{1}{n}(E - x_0) \subset K \subset E.$$

If K *is origin symmetric* ($K = -K$)*, then* $x_0 = o$ *and* $\frac{1}{\sqrt{n}} \cdot E \subset K \subset E$.

Remark. ΦE is the Löwner ellipsoid of ΦK for any $\Phi \in \mathrm{GL}(n, \mathbb{R})$.

Theorem 6.1.4 (John). *If* B^n *is the Löwner ellipsoid containing a convex body* K *in* $\mathbb{R}^n$*, then* $K \subset B^n$ *and there exist* $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ *and* $c_1, \dots, c_k > 0$, $k \leq \frac{n(n+3)}{2}$*, such that*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.3}$$

$$\sum_{i=1}^{k} c_i u_i = o. \tag{6.4}$$

If K *is origin symmetric* ($K = -K$)*, then* (6.3) *is sufficient.*

In the remaining of this section, we discuss Theorem 6.1.1. We prove directly the uniqueness of the John ellipsoid (see Proposition 6.1.5), and verify two consequences of it; namely, that the conjugates of the orthogonal group are maximal compact subgroups of $\mathrm{GL}(n)$, and Brunn's characterization of ellipsoids. After that we show how the conditions (6.1) and (6.2) yield that John's ellipsoid is a good approximation of the convex body (see Proposition 6.1.7).

Proposition 6.1.5. *For any convex body $K \subset \mathbb{R}^n$, there exists a unique ellipsoid of maximal volume contained in K.*

Proof. The arguments is indirect, we suppose that there exist two ellipsoids of maximal volume $A\,B^n + x \neq B\,B^n + y$ contained in K for positive definite matrices A, B and $x, y \in \mathbb{R}^n$, and hence $\det A = \det B$. We may assume that $y = -x$ and $B = I_n$.

Now $(\frac{1}{2}A + \frac{1}{2}I_n)B^n \subset K$ as K convex and $y = -x$ where $\det(\frac{1}{2}A + \frac{1}{2}I_n) > 1$ unless $A = I_n$ according to (3.5.2), and hence $A = I_n$ and $x \neq o$. We deduce that $E \subset K$ for the ellipsoid $E = \Phi B^n$ for the $\Phi \in \mathrm{GL}(n)$ where $\Phi x = (1 + \frac{1}{2}\|x\|)x$ and $\Phi w = w$ for $w \in x^\perp$. Since $|E| = (1 + \frac{1}{2}\|x\|)|B^n|$, $E \subset K$ contradicts the maximality of the volume of the ellipsoid $A\,B^n + x$. ■

Theorem 6.1.6. *For any compact subgroup $G \subset \mathrm{GL}(n)$, there exists a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$ that is invariant under G. In addition, the maximal compact subgroups of* $\mathrm{GL}(n, \mathbb{R})$ *are the conjugates of $O(n)$.*

Proof. Let $v_1, \ldots, v_n$ be any basis of $\mathbb{R}^n$, and let

$$K = \mathrm{conv}\{\pm\Phi v_i : \Phi \in G \text{ and } i = 1, \ldots, n\}.$$

Then K is an o-symmetric convex body (as G is compact) invariant under G, and hence the John ellipsoid E of G is also invariant under G. Choose the scalar product $\langle\cdot,\cdot\rangle$ in a way such that E is the unit ball of the corresponding Euclidean norm, thus $G \subset \Phi\, O(n)\, \Phi^{-1}$ where $E = \Phi\, B^n$, $\Phi \in \mathrm{GL}(n)$. ■

Based on the conditions (6.1) and (6.2) for the John ellipsoid, we prove how well the John ellipsoid approximates the convex body using the simple argument due to Keith Ball:

Proposition 6.1.7 (John). *For convex body $K \subset \mathbb{R}^n$, if x_0 is the center of the unique John ellipsoid E of maximal volume contained in K, then*

$$E \subset K \subset x_0 + n(E - x_0).$$

If, in addition, K is o-symmetric, then $x_0 = o$, and $E \subset K \subset \sqrt{n}\,E$.

Proof. We assume that B^n is the John ellipsoid, and use the notation of Theorem 6.1.2.

If $K = -K$ and $x \in K$, then the Remarks after Theorem 6.1.2 yield that

$$\|x\|^2 = \sum_{i=1}^{k} c_i \langle u_i, x\rangle^2 \leq \sum_{i=1}^{k} c_i = n.$$

For general convex body K, we need to prove that $\|x\| \le n$ for $x \in K$. Let $\|x\| = r$, and hence $-r \le \langle x, u_i\rangle \le 1$ for $i = 1, \ldots, k$ by the Remarks after Theorem 6.1.2. It also follows using these Remarks and (6.2) that

$$\begin{aligned}
0 &\le \sum_{i=1}^{k} c_i(1 - \langle x, u_i\rangle)(r + \langle x, u_i\rangle) \\
&= r\sum_{i=1}^{k} c_i + (1-r)\sum_{i=1}^{k} c_i\langle x, u_i\rangle - \sum_{i=1}^{k} c_i\langle x, u_i\rangle^2 \\
&= rn - \|x\|^2 = \|x\|(n - \|x\|),
\end{aligned}$$

and hence $\|x\| \le n$. ■

6.2 Some characterizations of ellipsoids

In this section, we discuss some characterizations of elipsoids based on some properties of sections (where the first two are due to Brunn [131] from 1889!) that are used in the later parts of the book. These properties are useful for example when discussing the equality cases of inequalities when Steiner symmetrisation is applied during the argument.

Theorem 6.2.1 (Brunn). *A convex body $K \subset \mathbb{R}^n$, $n \ge 2$, is an ellipsoid if and only if for any $u \in S^{n-1}$, the midpoints of the secants of K parallel to u are contained in a hyperplane.*

Proof. We observe that the property decribed in Lemma 6.2.1 is invariant under affine tranformations, and hence any ellipsod satisfies this property. On the other hand, if K satisfies this property, then we may assume that B^n is the maximum volume John ellipsoid in K.

For a $u \in S^{n-1}$, let H be the hyperplane containing the midpoints of the secants of K parallel to u. If Φ is the affine transformation that fixes each point of H, and maps $x + u$ into $x - u$ for $x \in H$, then $\Phi K = K$, and hence this Φ is a symmetry of B^n, as well, by the uniqness of the John ellipsoid (cf. Proposition 6.1.5). It follows that $H = u^\perp$, and Φ is the reflection through $u^\perp$. In particular, K is symmetric through $u^\perp$ for all $u \in S^{n-1}$, which in turn yields that $\|x\|\, S^{n-1} \subset K$ for any $x \in K$; therefore, K is a centered ball. ■

We do not provide the involved argument for the other two statements. We recall that a compact convex set C is centrally symmetric if there exists a $z \in Z$ satisfying $-(C - z) = C - z$ (and naturally, the empty set is centrally symmetric).

Theorem 6.2.2 (Brunn). *Given $2 \le m < n$, a convex body $K \subset \mathbb{R}^n$ is an ellipsoid if and only if any intersection with an affine m-space is centrally symmetric.*

We cannot resist to state the following beautiful strengthening of Theorem 6.2.2 due to Larman [392]:

Theorem 6.2.3 (False Center Theorem). *Given $2 \le m < n$, $p \in \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$, if the intersection of K with any affine m-space containing p is centrally symmetric, then either K is symmetric through p, or K is an ellipsoid.*

6.3 Reverse isoperimetric inequality

Remark. For the box $X_\varepsilon = [-\varepsilon^{-(n-1)}, \varepsilon^{-(n-1)}] \times [-\varepsilon, \varepsilon]^{n-1}$, $V(X_\varepsilon) = 2^n$ but $S(X_\varepsilon) > 1/\varepsilon$ (the area of a "long" facet); therefore, the isoperimetric quotient $S(X_\varepsilon)^n/V(X_\varepsilon)^{n-1}$ can be arbitrary large in general. The "Reverse isoperimetric inequality" says that each convex body has a linear image whose isoperimetric quotient is at most as bad as of a regular simplex, and hence "simplices have the worst isoperimetric quotient" up to linear transforms (cf. Theorem 6.3.1). For origin symmetric convex bodies, "cubes have the worst isoperimetric quotient" up to linear transforms (cf. Theorem 6.3.2).

Let Δ^n denote the regular simplex circumscribed around B^n, and hence each facet touches B^n, and let $W^n = [-1, 1]^n$ be the cube of edge length 2. Theorems 6.3.1 and inverse-iso-cube are due to Keith Ball [36].

Theorem 6.3.1 (Keith Ball). *For any convex body K in $\mathbb{R}^n$, there exists $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that*

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \le \frac{S(\Delta^n)^n}{|\Delta^n|^{n-1}} = \frac{n^{3n/2}(n+1)^{(n+1)/2}}{n!},$$

where strict inequality can be attained unless K is a simplex.

Theorem 6.3.2 (Keith Ball). *For any o-symmetric convex body K in $\mathbb{R}^n$, there exists $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that*

$$\frac{S(\Phi K)^n}{|\Phi K|^{n-1}} \le \frac{S(W^n)^n}{|W^n|^{n-1}} = 2^n n^n,$$

where strict inequality can be attained unless K is a parallopiped (linear image of a cube).

A polytope P is circumscribed around B^n if each facet of P touches B^n.

Lemma 6.3.3. *If $rB^n \subset K$ for a convex body K in $\mathbb{R}^n$ and $r > 0$, then $S(K) \le \frac{n}{r}|K|$, and equality holds if K is a polytope circumscribed around rB^n.*

Proof. The inequality $S(K) \leq \frac{n}{r}|K|$ follows from

$$S(K) = \lim_{\varrho\to 0^+} \frac{|K+\varrho B^n| - |K|}{\varrho} \leq \lim_{\varrho\to 0^+} \frac{|K+\frac{\varrho}{r}K| - |K|}{\varrho} = \frac{n}{r}|K|.$$

If K is a polytope circumscribed around rB^n, then considering the bounded "cones" with apex o and of height r over the facets shows that $|K| = \frac{r}{n}S(P)$ in this case. ■

The proof of the Reverse Isoperimetric inequality both in the o-symmetric and non-symmetric cases is based on the rank one Brascamp-Lieb inequality Theorem 6.3.4 in harmonic analysis. We sketch a proof using optimal transpoint in Section 6.B.

Theorem 6.3.4 (Brascamp-Lieb, Keith Ball). *If $u_1,\dots,u_k \in S^{n-1}$ and $c_1,\dots,c_k > 0$ satisfy*

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n, \tag{6.5}$$

and $f_1,\dots,f_k \in L^1(\mathbb{R})$ are non-negative, then

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x,u_i\rangle)^{c_i}\,dx \leq \prod_{i=1}^{k}\left(\int_{\mathbb{R}} f_i\right)^{c_i}. \tag{6.6}$$

Remarks.

(i) If $n = 1$, then the Brascamp-Lieb inequality (6.6) is just the Hölder inequality.

(ii) Inequality (6.6) is optimal, and provide two types of examples:

- If $u_1,\dots,u_k \in S^{n-1}$ and $c_1,\dots,c_k > 0$ satisfy (6.5), and $f_i(t) = e^{-\pi t^2}$ for $i = 1,\dots,k$, then each $\int_{\mathbb{R}} f_i = 1$, and

$$\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle x,u_i\rangle)^{c_i}\,dx = \int_{\mathbb{R}^n} e^{-\pi\sum_{i=1}^{k} c_i\langle x,u_i\rangle^2}\,dx = \int_{\mathbb{R}^n} e^{-\pi\langle x,x\rangle^2}\,dx = 1.$$

- If $u_1,\dots,u_n$ is an orthonormal basis and $c_1 = \dots = c_k = 1$, and hence (6.5) holds, and $f_1,\dots,f_n \in L^1(\mathbb{R})$ any functions, then the Fubini Theorem yields

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} f_i(\langle x,u_i\rangle)^{c_i}\,dx = \prod_{i=1}^{n}\left(\int_{\mathbb{R}} f_i\right)^{c_i}.$$

More precisely, Theorem 6.3.4 is the so-called Geometric form of the rank one Brascamp-Lieb inequality discovered by Keith Ball (see the Comments Section 6.10), which, as Keith Ball realized it, matches nicely the form of John's theorem as in Theorem 6.1.2.

Equality in Theorem 6.3.4 has been characterized by Barthe [50]. It is more involved; therefore, we only quote the special case that we need.

Theorem 6.3.5 (Barthe). *Let $\int_{\mathbb{R}} f_i > 0$ for $i = 1, \dots, k$, such that none of the f_is is Gaussian in Theorem 6.3.4, and equality holds in (6.6). Then there exists an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$ such that $\{u_1, \dots, u_k\} \subset \{\pm e_1, \dots, \pm e_n\}$ and $\sum_{u_i \in \mathbb{R}e_p} c_i = 1$ for each e_p, and if $u_i = -u_j$, then $f_i(t) = \lambda_{ij} f_j(-t)$ for $\lambda_{ij} > 0$.*

It is a natural question how well an inscribed ellipsoid can approximate a convex body in terms of volume. This question was answered by Keith Ball [35, 36], see Theorem 6.3.6 for the origin symmetric case, and Theorem 6.3.7 in general.

Theorem 6.3.6 (Volume Ratio in the origin symmetric case, Keith Ball). *For any o-symmetric convex body K in $\mathbb{R}^n$, the maximal volume John ellipsoid $E \subset K$ satisfies*

$$\frac{|K|}{|E|} \le \frac{|W^n|}{|B^n|} = \frac{2^n}{\omega_n},$$

where strict inequality is attained unless K is a parallopiped.

Proof. We may assume after a linear transformation that $E = B^n$. According to John's Theorem 6.1.2, there exists symmetric set $u_1, \dots, u_{2k} \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_{2k} > 0$ with $u_{k+i} = -u_i$ and $c_{k+i} = c_i$, $i = 1, \dots, k$, such that

$$\sum_{i=1}^{2k} c_i u_i \otimes u_i = \mathrm{I}_n.$$

For $i = 1, \dots, 2k$ and $t \in \mathbb{R}$, let $f_i = \mathbf{1}_{[-1,1]}$. Now $K \subset P$ for the polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le 1, i = 1, \dots, 2k\}$ according to the Remarks after John's Theorem 6.1.2 where $\mathbf{1}_P(x) = \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle) = \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle)^{c_i}$. It follows from the Brascamp-Lieb inequality (6.6) and $\sum_{i=1}^{2k} c_i = n$ that

$$|K| \le |P| = \int_{\mathbb{R}^n} \prod_{i=1}^{2k} f_i(\langle x, u_i \rangle)^{c_i}\, dx \le \prod_{i=1}^{2k} \left(\int_{\mathbb{R}} f_i \right)^{c_i} = 2^{\sum_{i=1}^{2k} c_i} = 2^n = |W^n|.$$

If $|K| = |W^n|$, then $|K| = |P|$, and Theorem 6.3.5 yields that $k = n$ and $u_1, \dots, u_n$ is an orthonormal basis of $\mathbb{R}^n$; therefore, K is a cube. ■

Concerning the volume ratio of general convex bodies, we only sketch the argument because it involves a somewhat technical calculation.

Theorem 6.3.7 (Volume Ratio, Keith Ball). *For any convex body K in $\mathbb{R}^n$, the maximal volume John ellipsoid $E \subset K$ satisfies*

$$\frac{|K|}{|E|} \le \frac{|\Delta^n|}{|B^n|} = \frac{n^{n/2}(n+1)^{(n+1)/2}}{n!\,\omega_n},$$

where strict inequality is attained unless K is a simplex.

Sketch of the proof of Theorem 6.3.7. We may assume that B^n is the John ellipsoid of K, and let $c_1, \dots, c_k > 0$ be the coefficients and $u_1, \dots, u_k \in S^{n-1} \cap \partial K$ be the contact points satisfying (6.1) and (6.2) in John's Theorem 6.1.2; namely,

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = \mathrm{I}_n \quad \text{and} \quad \sum_{i=1}^{k} c_i u_i = o. \tag{6.7}$$

Again, $K \subset P$ for the polytope $P = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \le 1, i = 1, \dots, k\}$ according to the Remarks after John's Theorem 6.1.2. We suspect that equality holds in Theorem 6.3.7 when $k = n+1$ and $K = P$ is a simplex, and hence, as implied by Barthe's characterization Theorem 6.3.5 of the equality case, we employ the Brascamp-Lieb inequality in $\mathbb{R}^{n+1}$ following an idea of Keith Ball. Therefore, we identify $\mathbb{R}^n$ with $w^\perp$ for a fixed $w \in S^n \subset \mathbb{R}^{n+1}$, and define $\tilde{u}_i = -\sqrt{\frac{n}{n+1}} \cdot u_i + \sqrt{\frac{1}{n+1}} \cdot w$ and $\tilde{c}_i = \frac{n+1}{n} \cdot c_i$ for $i = 1, \dots, k$, and hence $\sum_{i=1}^{k} \tilde{c}_i \tilde{u}_i \otimes \tilde{u}_i = \mathrm{I}_{n+1}$ follows from (6.7). For $i = 1, \dots, k$, we consider the probability density

$$f_i(t) = \begin{cases} e^{-t} & \text{if } t \ge 0; \\ 0 & \text{if } t < 0 \end{cases}$$

on $\mathbb{R}$ where some not too complicated calculations show that

$$\int_{\mathbb{R}^{n+1}} \prod_{i=1}^{k} f_i(\langle x, \tilde{u}_i \rangle)^{\tilde{c}_i} = \frac{|P|}{|\Delta^n|}.$$

We conclude from the Brascamp-Lieb inequality (6.6) that $|K| \ge |P| \ge |\Delta^n|$.

If $|K| = |\Delta^n|$, then $K = P$ and equality holds in the Brascamp-Lieb inequality. Therefore, Theorem 6.3.5 provides an orthonormal basis $e_1, \dots, e_{n+1}$ of $\mathbb{R}^{n+1}$ such that $\{\tilde{u}_1, \dots, \tilde{u}_k\} \subset \{\pm e_1, \dots, \pm e_{n+1}\}$. Since $\langle w, \tilde{u}_i \rangle = \sqrt{\frac{1}{n+1}}$ for $i = 1, \dots, k$, we conlude that $k = n+1$, $\tilde{u}_1, \dots, \tilde{u}_{n+1}$ is an an orthonormal basis of $\mathbb{R}^{n+1}$, and hence P is congruent to Δ^n. ∎

Proof of the Reverse Isoperimetric Inequality Theorem 6.3.1 and Theorem 6.3.2: After applying an affine transformation, we may assume that the John ellipsoid of K is B^n both in Theorem 6.3.1 and Theorem 6.3.2.

For Theorem 6.3.1, Theorem 6.3.7 yields that $|K| \le |\Delta^n|$, thus we deduce from Lemma 6.3.3 that

$$\frac{S(K)^n}{|K|^{n-1}} \le \frac{n^n |K|^n}{|K|^{n-1}} = n^n |K| \le n^n |\Delta^n| = \frac{S(\Delta^n)^n}{|\Delta^n|^{n-1}}.$$

If equality holds in Theorem 6.3.1, then the equality case of Theorem 6.3.7 yields that K is congruent to Δ^n.

For Theorem 6.3.2, we use the same argument, only with Theorem 6.3.6 in place of Theorem 6.3.7.

Petty's condition in Theorem 6.C.1 and the symmetries of the regular simplex and the cube ensure that both the regular simplex and the cube are in minimal surface area position. ■

6.4 Ellipsoid of inertia, Isotropic position of a convex body

6.4.1 Ellipsoid of inertia, Isotropic constant

We have seen already seen how useful some notions of physics are in studies, like the notions of centroid and Wulff shape in Section 1.11 and Section 4.4. Now we consider the so-called ellipsoid of inertia or Legendre ellipsoid that has the same inertia as the corresponding convex body.

Definition 6.4.1. Let $K \subset \mathbb{R}^n$ be a centered convex body; namely, $\sigma_K = o$ holds for the centroid.

Matrix M_K of inertia: $\langle M_K u, v\rangle = \int_K \langle u, x\rangle\langle v, x\rangle\, dx$ for $u, v \in \mathbb{R}^n$ for the positive definite symmetrix $n \times n$ matrix M_K; or in other words, $\langle M_K u, u\rangle = \int_K \langle u, x\rangle^2\, dx$ for $u \in S^{n-1}$.

Ellipsoid E_K of inertia or *Legendre ellipsoid:* $\int_{E_K} \langle u,x\rangle^2\, dx = \int_K \langle u,x\rangle^2\, dx$ for $u \in \mathbb{R}^n$.

Remarks.

(i) $E_{\Phi K} = \Phi E_K$ and $M_{\Phi K} = |\det \Phi| \cdot \Phi M_K \Phi^t$ for $\Phi \in \mathrm{GL}(n)$, and hence the existence of E_K can be seen by by transforming M_K into a diagonal matrix.

(ii) If $|K| = 1$ and $d\mu = \mathbf{1}_K d\mathcal{H}^n$, then $M_K = \mathrm{Cov}(\mu)$ for the log-concave measure μ (cf. Section 4.7).

In this section, we focus on convex bodies in quasi-isotropic position; namely, when the ellipsoid of inertia is a ball.

Definition 6.4.2 (Isotropic position). Let $K \subset \mathbb{R}^n$ be a centered convex body.

Isotropic constant: $L_K = (\det M_K)^{\frac{1}{2n}} / |K|^{\frac{n+2}{2n}}$.

Quasi-isotropic position: K is in quasi-isotropic position if E_K is ball; or equivalently, if $M_K = L_K^2 |K|^{\frac{n+2}{n}} I_n$, which is turn equivalent to saying that

$$\int_K \langle u, x\rangle\langle v, x\rangle\, dx = L_K^2 |K|^{\frac{n+2}{n}} \langle u, v\rangle \quad \text{for } u, v \in \mathbb{R}^n.$$

Isotropic position: K is in isotropic position if $|K| = 1$ and K is in quasi-isotropic position, and hence $M_K = L_K^2 I_n$ and

$$\int_K \langle u, x\rangle^2\, dx = L_K^2 \quad \text{for } u \in S^{n-1}.$$

Remarks.

(i) $L_{\Phi K} = L_K$ for $\Phi \in \mathrm{GL}(n)$ and centered convex body K.

(ii) It follows from the Remarks after Definition 6.4.1 that every centered convex body K has a linear image that is in isotropic position.

It might be confusing but in this book we are considering two different types of notions of isotropy. The point is that if $K \subset \mathbb{R}^n$ is centered convex body in isotropic position, and hence $|K| = 1$, then $d\mu = \mathbf{1}_K d\mathcal{H}^n$ is a log-concave probability measure with mean zero, but μ is typically not an isotropic log-concave measure in the sense of Definition 4.7.10 as $\mathrm{Cov}(\mu) = L_K^2 I_n$.

Lemma 6.4.3. *If $K \subset \mathbb{R}^n$ is a centered convex body in isotropic position, and hence $|K| = 1$, then the log-concave probability measure $d\mu = |\widetilde{K}|^{-1}\mathbf{1}_{\widetilde{K}} d\mathcal{H}^n$ has zero mean and is isotropic for $\widetilde{K} = L_K^{-1}K$.*

Proof. On the one hand, $\mathrm{Cov}(\mu) = \lambda\, I_n$ for $\lambda > 0$ because K is in isotropic position. On the other hand,

$$\mathrm{tr}\,\mathrm{Cov}(\mu) = |\widetilde{K}|^{-1} \int_{\widetilde{K}} \|x\|^2\, dx = L_K^{-2} \int_K \|x\|^2\, dx = L_K^{-2}\, \mathrm{tr}\, M_K = n,$$

and hence $\mathrm{Cov}(\mu) = I_n$. ∎

In order to verify the nice characterization Lemma 6.4.5 of the isotropic constant and quasi-isotropic position, we need the following simple consequence of the AM-GM inequality applied to the eigenvalues of a positive definite matrix:

Lemma 6.4.4. *If A is an $n \times n$ positive definite matrix, then $\mathrm{tr}\, A \geq n(\det A)^{\frac{1}{n}}$, with equality if and only if $A = \lambda I_n$ for $\lambda > 0$.*

Lemma 6.4.5. *If $K \subset \mathbb{R}^n$ is a centered convex body, then*

$$nL_K^2 |K|^{\frac{n+2}{n}} = \min_{\Phi \in \mathrm{SL}(n)} \int_{\Phi K} \|x\|^2\, dx = \min_{\Phi \in \mathrm{SL}(n)} \int_K \|\Phi x\|^2\, dx,$$

and $nL_K^2 |K|^{\frac{n+2}{n}} = \int_K \|x\|^2\, dx = \int_{E_K} \|x\|^2\, dx$ if and only if K is in quasi-isotropic position.

Proof. We may assume that $|K| = 1$ and K is in isotropic position, and hence $M_K = L_K^2 I_n$. For any $\Phi \in \mathrm{SL}(n)$, given orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$,

$$\int_{\Phi K} \|x\|^2\, dx = \sum_{i=1}^n \int_{\Phi K} \langle e_i, x\rangle^2\, dx = \mathrm{tr}\, M_{\Phi K} = \mathrm{tr}\,(\Phi M_K \Phi^t) = L_K^2 \mathrm{tr}\,(\Phi\Phi^t) \geq nL_K^2$$

by Lemma 6.4.4 applied to $A = \Phi\Phi^t$. If $\int_{\Phi K} \|x\|^2\, dx = nL_K^2$, then the equality case of Lemma 6.4.4 yields that $\Phi\Phi^t = I_n$, and hence $\Phi \in SO(n)$. ∎

Now we show that the isotropic constant L_K is minimized by ellipsoids, and hence it is at least a positive absolute constant for any centered convex body K in any dimension.

Proposition 6.4.6. *If $K \subset \mathbb{R}^n$ is a centered convex body, then $L_K \geq L_{B^n} > \frac{1}{\sqrt{2e\pi}}$ where $L_K = L_{B^n}$ if and only if K is an ellipsoid.*

Remark. $L_{B^n} \sim \frac{1}{\sqrt{2e\pi}}$ as n tends to infinity as $\Gamma(t+1) \sim (\frac{t}{e})^t \sqrt{2\pi t}$.

Proof. We may assume that $|K| = |B^n|$ and K is in quasi-isotropic position. It follows that $|K \backslash B^n| = |B^n \backslash K|$ and $\|x\| > 1 \geq \|y\|$ holds for $x \in K \backslash B^n$ and $y \in B^n \backslash K$; therefore, Lemma 6.4.5 yields

$$nL_K^2 |B^n|^{\frac{n+2}{n}} = \int_K \|x\|^2 \, dx \geq \int_{B^n} \|x\|^2 \, dx = nL_{B^n}^2 |B^n|^{\frac{n+2}{n}},$$

with equality if and only if $K = B^n$.

To estimate L_{B^n}, we use polar coordinates (1.26), and that $\omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ where $\Gamma(t+1) \geq (\frac{t}{e})^t \sqrt{2\pi t}$ for $t \geq 1$. It follows that

$$L_{B^n}^2 = \frac{\frac{1}{n} \int_{B^n} \|x\|^2 \, dx}{\omega_n^{\frac{n+2}{n}}} = \frac{\frac{1}{n+2} \omega_n}{\omega_n^{\frac{n+2}{n}}} = \frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{(n+2)\pi} > \frac{n}{(n+2)2\pi e} \cdot (\pi n)^{\frac{1}{n}} > \frac{1}{2\pi e}.$$

■

A nice geometric consequence of Proposition 6.4.6 is that the volume of the ellipsoid of inertia is almost at least the volume of the corresponding convex body.

Corollary 6.4.7. *If $K \subset \mathbb{R}^n$ is a centered convex body, then $|E_K| \geq |K|$, with equality if and only if K is an ellipsoid.*

Proof. We may assume that $E_K = B^n$, and hence Lemma 6.4.5 yields that

$$nL_K^2 |K|^{\frac{n+2}{n}} = \int_K \|x\|^2 \, dx = \int_{B^n} \|x\|^2 \, dx = nL_{B^n}^2 |B^n|^{\frac{n+2}{n}}.$$

Since $L_K \geq L_{B^n}$ by Proposition 6.4.6, we deduce $|K| \leq |B^n|$, together with the characterization of equality. ■

According to Paouris [481], the convex body K is in isotropic position, the constant L_K also controls volume concentration:

Theorem 6.4.8 (Pauoris). *If the convex body $K \subset \mathbb{R}^n$ is in isotropic position and $t \geq 1$, then for an absolute constant $c > 0$, we have*

$$\left|\{x \in K : \|x\| \geq ct\sqrt{n}\, L_K\}\right| \leq e^{-\sqrt{n}\,t}.$$

The fundamental paper Kannan, Lovász, Simonovits [361], stating the KLS Conjecture 4.7.11, also proved that the ellipsoid of inertia approximates just as well the corresponding centered convex body (cf. (6.8)) as any ellipsoid can do.

Proposition 6.4.9 (Kannan, Lovász, Simonovits). *If a convex body $K \subset \mathbb{R}^n$ is in isotropic position, then*

$$L_K^{\frac{-2}{n+2}} \sqrt{\frac{n+2}{n}}\, B^n \subset K \subset L_K^{\frac{-2}{n+2}} \sqrt{(n+2)n}\, B^n.$$

Remark. In particular, if $K \subset \mathbb{R}^n$ is a centered convex body and $\alpha_n = \sqrt{\frac{n+2}{n}} \cdot (\frac{\omega_n}{n+2})^{\frac{1}{n+2}}$ where $\alpha \sim \sqrt{2e\pi/n}$ as n tends to infinity, then

$$\alpha_n \cdot E_K \subset K \subset n\alpha_n \cdot E_K. \tag{6.8}$$

6.4.2 The Slicing Conjecture

For the "slices" of a convex body (intersections by hyperplanes), Fradelizi [239] (cf. (4.44)) prove the following estimates:

Lemma 6.4.10. *Let $K \subset \mathbb{R}^n$ be a centered convex body with $|K| = 1$.*

- $\frac{1}{\sqrt{12}\sqrt{\|M_K u\|}} \le \mathcal{H}^{n-1}(u^\perp \cap K) \le \frac{1}{\sqrt{2}\sqrt{\|M_K u\|}}$ *for any $u \in S^{n-1}$;*
- *there exists $u \in S^{n-1}$ with $\mathcal{H}^{n-1}(u^\perp \cap K) \ge \frac{1}{\sqrt{12}L_K}$;*
- *if K is in isotropic position, then $\frac{1}{\sqrt{12}L_K} \le \mathcal{H}^{n-1}(u^\perp \cap K) \le \frac{1}{\sqrt{2}L_K}$ for any $u \in S^{n-1}$.*

Proof. Apply (4.44) to $d\mu = \mathbf{1}_K d\mathcal{H}^n$ where the minimal eigenvalue of M_K is at least L_K^2 by $\det M_K = L_K^{2n}$ (cf. Definition 6.4.2). ■

Independently Bourgain [121] and Keith Ball [34] posed the following fundamental conjecture in 1986:

Conjecture 6.4.11 (Slicing Conjecture). *If $K \subset \mathbb{R}^n$ convex body with $|K| = 1$, then there exists a hyperplane H with $\mathcal{H}^{n-1}(H \cap K) \ge c$ for an absolute constant $c > 0$.*

The importance of the Slicing Conjecture is exhibited by the fact how many fundamental equivalent formulations it has (see for example Klartag, V. Milman [375] and Brazitikos, Giannopoulos, Valettas, Vritsiou [125]). Here we list just some of them.

Remark 6.4.12 (Some equivalent formulations of the Slicing Conjecture). Let $K \subset \mathbb{R}^n$ be a centered convex body .

- $L_K < c$ for an absolute constant $c > 1$ (cf. Lemma 6.4.10).
- There exists a centered ellipsoid E such that $|E| \le c^n |K|$ and $|K \cap E| \ge \frac{1}{2}|K|$ for an absolute constant $c > 1$.

- The expected volume of the convex hull of $n+1$ independent, random points in K according to the uniform distribution is between $c^{-n}n^{-n/2}$ and $c^n n^{-n/2}$ for an absolute constant $c > 1$.

  We note that for any centered convex body $K \subset \mathbb{R}^n$, we have

$$\frac{1}{\sqrt{2e\pi}} < L_K < c\sqrt{\log n}, \tag{6.9}$$

for an absolute constant $c > 1$ where the lower bound is in Proposition 6.4.6, and the upper bound is due to Klartag [373].

The KLS conjecture (see Section 4.7) yields the Slicing Conjecture, as it was observed by Keith Ball around 2003 (see Ball, Nguyen [42]). More precisely, Eldan, Klartag [200] proved that

$$L_K \leq c \cdot \sup_{\mu \in \mathcal{M}^n} C_{\mathrm{Che}}(\mu)$$

for any centred convex body $K \subset \mathbb{R}^n$ and for an absolute constant $c > 1$ where $\mathcal{M}^n$ is the family of all log-concave isotropic measures on $\mathbb{R}^n$.

6.4.3 Cheeger constant for a convex body and a Poincaré-type inequality

We prove the essentially optimal Poincaré-type inequalities (6.14) and (6.20) due to Kolesnikov, E. Milman [381] involving the Cheeger constant assigned to the uniform log-concave probability measure $\mu = \mu_K$ on a convex body $K \subset \mathbb{R}^n$; namely, $d\mu = |K|^{-1}\mathbf{1}_K\, d\mathcal{H}^n$. Abusing the notation introduced in Section 4.7, we set $C_{\mathrm{Che}}(K) = C_{\mathrm{Che}}(\mu) > 0$ for the Cheeger constant; namely, $C_{\mathrm{Che}}(K) > 0$ is minimal such that for every closed $X \subset K$ with locally Lipschitz boundary, we have

$$C_{\mathrm{Che}}(K) \cdot \mathcal{H}^{n-1}\left((\partial X) \cap \operatorname{int} K\right) \geq \min\{|X|, |K| - |X|\}. \tag{6.10}$$

It follows from the definition that if $\lambda > 0$, then

$$C_{\mathrm{Che}}(\lambda K) = \lambda\, C_{\mathrm{Che}}(K). \tag{6.11}$$

Remark 6.4.13. (i) It is enough to consider the case when $|X| = \frac{1}{2}|K|$ in (6.10) according to E. Milman [458] (actually, this case was already dealt with by Sternbergand, Zumbrun [543]).

(ii) According to (4.34), we have

$$C_{\mathrm{Che}}(B^n) = \frac{\omega_n}{2\omega_{n-1}} < \sqrt{\frac{\pi}{2n}}. \tag{6.12}$$

(iii) If $|K| = 1$, then the minimal eigenvalue of M_K is at least L_K^2 by $\det M_K = L_K^{2n}$ (cf. Definition 6.4.2), and hence (4.44) yields that

$$C_{\mathrm{Che}}(K) \geq \frac{\sqrt{2}}{e} L_K.$$

For a convex body $K \subset \mathbb{R}^n$ and a Lipschitz function $f : K \to \mathbb{R}$, we consider the median m_f with respect to $\mu = \mu_K$; namely,

$$\left|\{f > m_f\}\right| \leq |K|/2 \quad \text{and} \quad \left|\{f < m_f\}\right| \leq |K|/2.$$

In particular, we deduce from Proposition 4.7.6 that

$$\int_K \left|f - m_f\right| \, d\mathcal{H}^n \leq C_{\mathrm{Che}}(K) \cdot \int_K \|Df\| \, d\mathcal{H}^n. \tag{6.13}$$

We recall that $\partial' K$ denotes the family of regular points of ∂K; namely, the family of $\mathcal{H}^{n-1}$ a.e. points $x \in \partial K$ where there exists a unique exterior unit normal $\nu_K(x)$ (see Section 1.5).

Lemma 6.4.14 (Kolesnikov-Milman). *If $rB^n \subset K \subset RB^n$ holds for a convex body $K \subset \mathbb{R}^n$ and $R \geq r > 0$, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} \left|f - m_f\right| \, d\mathcal{H}^{n-1} \leq \frac{nC_{\mathrm{Che}}(K) + R}{r} \cdot \int_K \|Df\| \, d\mathcal{H}^n. \tag{6.14}$$

Remark. Figalli, Maggi, Pratelli [224] proved a version of (6.14) with explicit factor $\frac{\sqrt{2}\,nR}{\log 2 \cdot r}$ instead of $\frac{nC_{\mathrm{Che}}(K)+R}{r}$. The advantage of (6.14) is that K has an affine image $\widetilde{K}$ such that $C_{\mathrm{Che}}(\widetilde{K}) \leq c(\log n)^{\frac{1}{2}}$ and $\tilde{r}B^n \subset \widetilde{E} \subset \tilde{R}B^n$ with $\tilde{r} \geq c'(\log n)^{-\frac{1}{2}}$ and $\tilde{R} \leq n\tilde{r}$ for absolute constants $c, c' > 0$ (see (6.17) and (6.18)).

Proof. Since $\langle x, \nu_K(x)\rangle = h_K(\nu_K(x)) \geq r$ for $x \in \partial' K$, we deduce by applying the Divergence Theorem 2.1.4 to the Lipschitz function $x \mapsto |f(x) - m_f| \cdot x$ and by integration by parts that

$$\begin{aligned}
\int_{\partial K} \left|f - m_f\right| \, d\mathcal{H}^{n-1} &\leq \frac{1}{r} \int_{\partial K} \left|f(x) - m_f\right| \langle x, \nu_K(x)\rangle \, d\mathcal{H}^{n-1}(x) \\
&= \frac{1}{r} \int_K \operatorname{div}\left(\left|f(x) - m_f\right| \cdot x\right) \, dx \\
&= \frac{1}{r} \int_K n\left|f(x) - m_f\right| \, dx + \frac{1}{r} \int_K \langle D|f(x) - m_f|, x\rangle \, dx.
\end{aligned}$$

Here the first term is at most $nC_{\mathrm{Che}}(K)/r$ by (6.13). For the second term, we observe that $D|f(x) - m_f| = 0$ if $f(x) - m_f = 0$ and $f(x) - m_f$ is differentiable at $x \in \mathrm{int}K$. On the other hand, if $f(x) - m_f \neq 0$ and $f(x) - m_f$ is differentiable at an $x \in \mathrm{int}K$, then $\|D|f(x) - m_f|\| = \|Df(x)\|$. Since $\|x\| \leq R$ for $x \in K$, we conclude (6.14). ∎

We deduce from (6.12), (6.14) and $n\sqrt{\frac{\pi}{2n}} + 1 < 2\sqrt{n}$ for $n \geq 2$ the following inequality for balls:

Corollary 6.4.15. *If $f : B^n \to \mathbb{R}$ is Lipschitz, then*

$$\int_{S^{n-1}} \left|f - m_f\right| \, d\mathcal{H}^{n-1} \leq 2\sqrt{n} \cdot \int_{B^n} \|Df\| \, d\mathcal{H}^n. \tag{6.15}$$

Remark. There exists a Lipschitz function $f : B^n \to \mathbb{R}$ such that

$$\int_{S^{n-1}} |f - m_f| \, d\mathcal{H}^{n-1} \geq \sqrt{n} \cdot \int_{B^n} \|Df\| \, d\mathcal{H}^n, \tag{6.16}$$

and the optimal factor in (6.15) is most probably $\frac{n\omega_n}{2\omega_{n-1}}$. Fix $u \in S^{n-1}$, and consider the Lipschitz function $f_\varrho : B^n \to \mathbb{R}$ for small $\varrho > 0$ such that if $x \in B^n$, then $f_\varrho(x) = 1$ if $\langle x, u\rangle \geq \varrho$, $f_\varrho(x) - 1$ if $\langle x, u\rangle \leq -\varrho$, and $f_\varrho(x) = \frac{\langle x,u\rangle}{\varrho}$ if $|\langle x, u\rangle| \leq \varrho$. It follows that $m_{f_\varrho} = 0$. Since $\frac{n\omega_n}{2\omega_{n-1}} > \sqrt{n}\sqrt{\frac{\pi n}{2(n+1)}} > \sqrt{n}$ according to (10.1), we conclude (6.16) for $f = f_\varrho$ and small enough $\varrho > 0$.

According Lemma 6.4.3, for any convex body $K \subset \mathbb{R}^n$, there exists a $\Phi \in \mathrm{GL}(n)$ such that the (log-concave) uniform probability measure $\tilde{\mu}$ on $\widetilde{K} = \Phi(K - \sigma_K)$ is isotropic; namely, $d\tilde{\mu} = |\widetilde{K}|^{-1}\mathbf{1}_{\widetilde{K}} \, d\mathcal{H}^n$, $\int_{\mathbb{R}^n} x \, d\tilde{\mu}(x) = o$ and $\operatorname{cov} \tilde{\mu} = I_n$ (see also Section 4.7). In this case, Klartag [373] proves (cf. Theorem 4.7.12) that

$$C_{\mathrm{Che}}(\widetilde{K}) = C_{\mathrm{Che}}(\tilde{\mu}) \leq c(\log n)^{\frac{1}{2}} \tag{6.17}$$

for an absolute constant $c > 0$. In addition, we deduce from combining Lemma 6.4.3 and Proposition 6.4.9 about the Kannan, Lovász, Simonovits ellipsoid that

$$\tilde{r}B^n \subset \widetilde{K} \subset \tilde{R}B^n \text{ where } \tilde{r} \geq L_K^{-\frac{n+4}{n+2}} \text{ and } \tilde{r} \leq \tilde{R} \leq n\tilde{r}. \tag{6.18}$$

Here (6.9) due to Klartag [373] implies that

$$\tilde{r} \geq c'(\log n)^{-\frac{1}{2}} \tag{6.19}$$

for some absolute constant $c' > 0$.

Proposition 6.4.16 (Kolesnikov-Milman). *If $K \subset \mathbb{R}^n$ is a centered convex body in quasi-isotropic position, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} |f - m_f| \, d\mathcal{H}^{n-1} \leq cn \log n \cdot \int_K \|Df\| \, d\mathcal{H}^n \tag{6.20}$$

for an absolute constant $c > 0$.

Remark. Assuming the KLS conjecture (see Section 4.7), which in turn yields the Slicing conjecture above, the argument below yields (6.20) without the logarithmic factor, which would be the optimal estimate (see Example 6.4.17).

Proof. It follows from the discussion above that we may assume that the log-concave uniform probability measure μ on K with $d\mu = |K|^{-1}\mathbf{1}_K \, d\mathcal{H}^n$ is isotropic. In this case, combining Lemma 6.4.14, (6.17), (6.18) and (6.19) yields (6.20). ■

Example 6.4.17. If $K \subset \mathbb{R}^n$ is the regular simplex circumscribed around B^n (and hence K is a centered convex body in quasi-isotropic position), then there exists a Lipschitz function $f : K \to \mathbb{R}$ such that

$$\int_{\partial K} |f - m_f| \, d\mathcal{H}^{n-1} \geq \frac{n}{2} \cdot \int_K \|Df\| \, d\mathcal{H}^n. \tag{6.21}$$

We fix an exterior unit normal u to a facet F of K, and let $t = 1 - (n+1)(1 - 2^{\frac{-1}{n}}) > 0$, and hence $|\{x \in K : \langle x, u\rangle \leq t\}| = \frac{1}{2}|K|$. For small $\varrho > 0$, we consider the Lipschitz function $f_\varrho : K \to \mathbb{R}$ such that if $x \in K$, then

$$f_\varrho(x) = \begin{cases} 1 & \text{if } \langle x, u\rangle \geq t + \varrho; \\ \frac{\langle x,u\rangle - t}{\varrho} & \text{if } t - \varrho \leq \langle x, u\rangle \leq t + \varrho; \\ -1 & \text{if } \langle x, u\rangle \leq t - \varrho. \end{cases}$$

It follows that $m_{f_\varrho} = 0$, the set $\{x \in K : \langle x, u\rangle = t\}$ has $\mathcal{H}^{n-1}$-measure $2^{\frac{1-n}{n}}\mathcal{H}^{n-1}(F)$ and $\|Df_\varrho(x)\| = \varrho^{-1}$ if $x \in \operatorname{int} K$ and $t - \varrho < \langle x, u\rangle < t + \varrho$; therefore,

$$\lim_{\varrho \to 0^+} \int_{\partial K} |f_\varrho - m_{f_\varrho}| \, d\mathcal{H}^{n-1} = \mathcal{H}^{n-1}(\partial K) = (n+1)\mathcal{H}^{n-1}(F);$$
$$\lim_{\varrho \to 0^+} \int_K \|Df_\varrho\| \, d\mathcal{H}^n = 2 \cdot 2^{\frac{1-n}{n}}\mathcal{H}^{n-1}(F) = 2^{\frac{1}{n}}\mathcal{H}^{n-1}(F).$$

Finally, we improve on (6.20) if the convex body K is close to be a ball.

Proposition 6.4.18. *If $K \subset \mathbb{R}^n$ is a convex body with $|K| = |B^n|$ and $|K \Delta B^n| \leq (4n)^{-2n}|B^n|$, and $f : K \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial K} |f - m_f| \, d\mathcal{H}^{n-1} \leq 8\sqrt{n} \cdot \int_K \|Df\| \, d\mathcal{H}^n. \tag{6.22}$$

Proof. We may assume that $K \neq B^n$, and observe that $|K \backslash B^n| = |B^n \backslash K| \leq$. Let $r \in (0, 1)$ be maximal with the property that $rB^n \subset K$; therefore, $B^n \backslash K$ contains a circular cone having height $1 - r$ and radius $\sqrt{1-r}$ of the base. We deduce that $|B^n \backslash K| \geq \frac{\omega_{n-1}}{n}(1-r)^{\frac{n+1}{2}}$, and hence $r \geq 1 - \frac{1}{4n}$, which in turn yields that $K \subset 2B^n$.

According to Lemma 6.4.14, Proposition 6.4.18 follows from the estimate

$$C_{\text{Che}}(K) \leq \frac{4}{\sqrt{n}}. \tag{6.23}$$

To estimate $C_{\text{Che}}(K)$, it is sufficient to consider any closed set $X \subset K$ with $|X| = \frac{1}{2}|B^n|$ and locally Lipschitz boundary according to Remark 6.4.13. For $X_0 = X \cap rB^n$, $\frac{1}{4}|rB^n| < |X_0| < \frac{3}{4}|rB^n|$ follows from $|Y \backslash (rB^n)| \leq |K \backslash (rB^n)| = (1 - r^n)|B^n|$ for $Y = X$

and $Y = K\backslash X$; therefore, $C_{\mathrm{Che}}(rB^n) < r\sqrt{\frac{\pi}{2n}}$ (cf. (6.11) and (6.12)) and (6.10) yield that

$$\mathcal{H}^{n-1}(\partial X \cap \operatorname{int} K) \geq \mathcal{H}^{n-1}(\partial X_0 \cap \operatorname{int}(rB^n)) \geq r^{-1}\sqrt{\frac{2n}{\pi}} \cdot |rB^n| > \frac{\sqrt{n}}{2}\,|B^n|.$$

In turn, we conclude (6.23). ■

Remark 6.4.19 (Isoperimetric problem in an open bounded convex set). Given a convex body $K \subset \mathbb{R}^n$ with C^2 boundary and $0 < t < |K|$, the theory of minimal surfaces asks for the subset $X \subset \operatorname{int} K$ of finite perimeter minimizing $\mathcal{H}^{n-1}((\partial X) \cap \operatorname{int} K)$ under the condition $|X| = t$. According to classical results, see the surveys Ros [499] and Cozzi, Figalli [181], there exists a minimizing $X \subset \operatorname{int} K$. Moreover; except for a relatively closed singular set of Hausdorff dimension at most $n - 8$, $(\partial X) \cap \operatorname{int} K$ is a smooth embedded hypersurface with constant mean curvature meeting ∂K orthogonally. For example, when $K = B^n$, and the minimizing X is the intersection with a ball if $t < \omega_n/2$, and is a half ball (intersection with a half space) if $t = \omega_n/2$, which facts lead to (6.12) (see (4.34) for more details).

6.5 Blaschke-Santaló inequality for the polar body

This section discusses the fundamental Blaschke-Santaló inequality (6.25) for a convex body and its polar. The main reason why this linearly invariant inequality is discussed in this chapter about associated ellipsoids is because the characterization of the equality case is greatly simplified by using the ellipsoids of inertia from Section 6.4.

We recall from Section 1.9 the definition of the polar body. If $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int}K$, then

$$K^* = \{x \in \mathbb{R}^n : \langle x, y\rangle \leq 1 \text{ for } y \in \mathbb{R}^n\}.$$

As we have seen in Section 1.9, K^* convex body with $o \in \operatorname{int}K^*$, and it satisfies $(K^*)^* = K$, $\varrho_{K^*}(u) = \frac{1}{h_K(u)}$ for $u \in S^{n-1}$, and $(\Phi K)^* = \Phi^{-t}K^*$ for $\Phi \in \mathrm{GL}(n)$. In particular, $|K| \cdot |K^*|$ is called the volume product as $|\Phi K| \cdot |(\Phi K)^*| = |K| \cdot |K^*|$ for $\Phi \in \mathrm{GL}(n)$.

Lemma 6.5.1 (Santaló point). *For a convex body $K \subset \mathbb{R}^n$, there exists a unique so-called Santaló point $\aleph_K \in \operatorname{int}K$ minimizing $\xi \mapsto |(K - \xi)^*|$ for $\xi \in \operatorname{int}K$, and*

$$\int_{S^{n-1}} \frac{u}{(h_K(u) - \langle \aleph_K, u\rangle)^{n+1}}\, du = 0. \tag{6.24}$$

In particular, $\aleph_K = o$ if and only if $\sigma_{K^*} = o$ for the centroid of the polar.

Remark. The Santaló point is affine invariant, and hence it is the origin if K o-symmetric.

Proof. Since $h_{K-\xi}(u) = h_K(u) - \langle \xi, u\rangle$ for $\xi \in \mathrm{int}K$, Lemma 1.11.6 yields

$$g(\xi) = |(K-\xi)^*| = \frac{1}{n}\int_{S^{n-1}} \varrho_{(K-\xi)^*}(u)^n \, du = \frac{1}{n}\int_{S^{n-1}} (h_K(u) - \langle \xi, u\rangle)^{-n} \, du.$$

Since $g(\xi)$ strictly convex (as $t \mapsto t^{-n}$ strictly convex), and $\lim_{\xi\to\partial K} g(\xi) = \infty$, there exists a unique $\aleph_K \in \mathrm{int}K$ minimizing it, and hence (6.24) follows from $Dg(\aleph_K) = o$. Finally, if $\aleph_K = o$, then (6.24) is equivalent to $\sigma_{K^*} = o$ by $\varrho_{K^*} = 1/h_K$ and Lemma 1.11.6. ■

The main statement of this section is the following inequality

Theorem 6.5.2 (Blaschke-Santaló inequality). *If the centroid or the Santaló point of a convex body $K \subset \mathbb{R}^n$ is the origin, then*

$$|K| \cdot |K^*| \le |B^n|^2, \tag{6.25}$$

with equality if and only if K is an ellipsoid centered at o.

As $\varrho_{K^*} = h_K^{-1}$ according to (1.17), and $|K^*| = \int_{S^{n-1}} \varrho_{K^*}^n \, d\mathcal{H}^{n-1}$ follows from using polar coordinates, a useful equivalent form of the Blaschke-Santaló inequality (6.25) is that if $K \subset \mathbb{R}^n$ is a centered convex body, then

$$\int_{S^{n-1}} h_K^{-n} \, d\mathcal{H}^{n-1} \le \frac{n|B^n|^2}{|K|}. \tag{6.26}$$

We only give a full proof of the Blaschke-Santaló inequality for o-symmetric convex bodies, and indicate how to handle the general case. Actually, the inequality itself is a direct consequence of the fact that the volume of the polar of a centered convex body is not decreased by the Steiner symmetrization (see Proposition 6.5.3 in the origin symmetric, and we state but not prove Theorem 6.5.4 in the general case).

Let us recall the definition of the Steiner symmetrization. For $u \in S^{n-1}$, and convex body $K \subset \mathbb{R}^n$, the Steiner symmetrial $\Theta_{u^\perp}K$ of K is (see Section 1.10), note that $|\Theta_{u^\perp}K| = |K|$)

$$\Theta_{u^\perp}K = \left\{x + \frac{t-s}{2}\cdot u : x \in K|H \;\&\; x + tu \in K \;\&\; x + su \in K\right\}.$$

We also need the Brunn-Minkowski inequality (1.32) that yields that

$$|\tfrac{1}{2}\,C + \tfrac{1}{2}\,(-C)| \ge |C| \tag{6.27}$$

for any convex body $C \subset \mathbb{R}^n$ with equality if and only if C and $-C$ are translates; or equivalently, if and only if C is centrally symmetric.

Proposition 6.5.3 (Keith Ball). *If $K \subset \mathbb{R}^n$ o-symmetric convex body and $u \in S^{n-1}$, then $|(\Theta_{u^\perp}K)^*| \ge |K^*|$. where $|(\Theta_{u^\perp}K)^*| = |K^*|$ implies that any section of K^* by a hyperplane parallel to $u^\perp$ is centrally symmetric.*

Proof. Let $\widetilde{K} = \Theta_{u^\perp} K$, and for o-symmetric convex body C and $t \in \mathbb{R}$, let

$$C_t^* = \{x \in u^\perp : x + tu \in C^*\}.$$

The core observation is the claim that if $t \in \mathbb{R}$, then

$$\frac{1}{2}(K_t^* + K_{-t}^*) \subset \widetilde{K}_t^*, \tag{6.28}$$

which is equivalent to saying that if $x + tu \in K^*$, $y - tu \in K^*$ and $z + su \in \widetilde{K}$ for $x, yz \in u^\perp$ and $s \in \mathbb{R}$, then

$$\left\langle \frac{x+y}{2} + tu, z + su \right\rangle \leq 1. \tag{6.29}$$

We have $s = \frac{s_1 - s_2}{2}$ where $z + s_1 u \in K$, $z + s_2 u \in K$ by the definition of Steiner symmetrization, and hence $\langle x + tu, z + s_1 u \rangle \leq 1$ and $\langle y - tu, z + s_2 u \rangle \leq 1$; or in other words, $\langle x, z \rangle + ts_1 \leq 1$ and $\langle y, z \rangle - ts_2 \leq 1$. We deduce (6.29), and in turn (6.28).

Now $K = -K$ yields $K^* = -K^*$, and hence $K_{-t}^* = -K_t^*$ for $t \in \mathbb{R}$. It follows from (6.28) and Brunn-Minkowski inequality (6.27) that if $t \in \mathbb{R}$, then

$$\mathcal{H}^{n-1}(\widetilde{K}_t^*) \geq \left(\frac{1}{2} \mathcal{H}^{n-1}(K_t^*)^{\frac{1}{n-1}} + \frac{1}{2} \mathcal{H}^{n-1}(K_{-t}^*)^{\frac{1}{n-1}} \right)^{n-1} = \mathcal{H}^{n-1}(K_t^*). \tag{6.30}$$

We conclude from the Fubini theorem that $|(\Theta_{u^\perp} K)^*| = |\widetilde{K}^*| \geq |K^*|$.

If $|(\Theta_{u^\perp} K)^*| = |K^*|$, then equality in (6.30) and the equality condition in the Brunn-Minkowski inequality (6.27) imply that K_t^* is centrally symmetric for $t \in \mathbb{R}$. ■

Proof the Blaschke-Santaló inequality Theorem 6.5.2 for o-symmetric bodies: We may assume that $|K| = |B^n|$ for an o-symmetric convex body K. Since iterated Steiner symmetrisations applied to K may lead to a centered ball of the same volume according to Theorem 1.10.7, Proposition 6.5.3 yields that $|K^*| \leq |(B^n)^*| = |B^n|$.

If $|K^*| = |(B^n)^*|$, then we deduce from Proposition 6.5.3 that any hyperplane section parallel to $u^\perp$ of K^* is centrally symmetric for any $u \in S^{n-1}$, and hence Theorem 6.2.2 yields that K^* is a centered ellipsoid, which in turn implies that K is a centered ellipsoid. ■

The proof of Brunn's Theorem 6.2.2 - that is used in order to characterize equality in the Blaschke-Santaló inequality Theorem 6.5.2, - is rather technical, and we do not provide it in this book. Instead, we present a simpler argument to characterize equality in the Blaschke-Santaló inequality using the ellipsoid of inertia in Section 6.E.

Meyer, Pajor [451] generalized Ball's Proposition 6.5.3 to any centered convex body, which we state without proof in this book.

Theorem 6.5.4 (Meyer, Pajor). *If $K \subset \mathbb{R}^n$ is a centered convex body and $u \in S^{n-1}$, then $|(\Theta_{u^\perp} K)^*| \geq |K^*|$.*

Remark. For $v \in S^{n-1}$, using iterated Steiner symmetrizations through $u^\perp$ with $u \in S^{n-1} \cap v^\perp$ (cf. Lemma 1.10.13), we obtain the estimate $|(\Theta_{\mathbb{R}v}K)^*| \geq |K^*|$ for the Schwarz symmetrization.

Proof of the Blaschke-Santaló inequality (6.25) *without equality.* We may assume that K is centered and $|K| = |B^n|$. Since iterated Steiner symmetrisations applied to a centered convex body K may lead to a centered ball of the same volume according to Theorem 1.10.7, Theorem 6.5.4 yields that $|K^*| \leq |(B^n)^*| = |B^n|$. ■

In Section 6.E, we describe how to characterize equality in the Blaschke-Santaló inequality based on Theorem 6.5.4. Our method might be applied to other equi-affine invariant inequalities, as we prove (see Theorem 6.E.2) that if $K \subset \mathbb{R}^n$, $n \geq 2$, is in quasi-isotropic position and is not an ellipsoid, then there exists a sequence of Steiner symmetrizations leading to an o-symmetric convex body $\widetilde{K}$ with axial rotational symmetry that is still not an ellipsoid.

6.6 Reverse Blaschke-Santaló inequality and Mahler's conjecture

We recall that according to the Blaschke-Santaló inequality (6.25), if the centroid or the Santaló point of a convex body $K \subset \mathbb{R}^n$ is the origin, then $|K| \cdot |K^*| \leq |B^n|^2$. For a reverse inequality for any convex body $K \subset \mathbb{R}^n$ with $o \in rmintK$, , let $\aleph_K \in rmintK$ be the Santaló points (cf. Lemma 6.5.1), and hence the orgin is the centroid of $(K - \aleph_K)^*$. We deduce the existence of a KLS-ellipsoid $E \subset (K - \aleph_K)^*$ such that $(K - \aleph_K)^* \subset nE$ (cf. (6.8)), and hence $\frac{1}{n} E^* \subset K - \aleph_K$ and

$$|K| \cdot |K^*| \geq |K - \aleph_K| \cdot |(K - \aleph_K)^*| \geq n^{-n}|E^*| \cdot |E| = n^{-n}|B^n|^2 \tag{6.31}$$

by the linear invariance of the volume product (cf. Proposition 1.9.3). In this section, improvements and possible improvements of (6.31) are discussed, mostly without the rather involved proofs.

Any inequality of the form $|K| \cdot |K^*| \geq c^n V(B^n)^2$, where $c \in (0, 1)$ is an absolute constant, holding for any convex body $K \subset \mathbb{R}^n$ with $o \in \text{int}\, K$ (or at least for any o-symmetric K) is called a Reverse Blaschke-Santaló inequality. The first such inequality is due to Bourgain, V. Milman [123] in 1987, and the the best estimate is the following statement proved by G. Kuperberg [389] (see the arxiv version for the case of non-symmetric convex bodies):

Theorem 6.6.1 (Reverse Blaschke-Santaló inequality). *If $K \subset \mathbb{R}^n$ convex body with $o \in \text{int}K$, then*

$$|K| \cdot |K^*| > 4^{-n}|B^n|^2, \tag{6.32}$$

and even $|K| \cdot |K^*| > \frac{\pi^n}{n!} > 2^{-n}|B^n|^2$ provided $K = -K$.

The exact minimum of the volume product in $\mathbb{R}^n$ is a classical conjecture attributed to Mahler. More precisely, Mahler [441] from 1939 only stated Conjecture 6.6.2 in the o-symmetric case. But since Mahler did settle the 2-dimensional case concerning the extremality of simplices in his 1938 paper [440], he must have been aware of the possibility that centered simplices are extremal in higher dimensional spaces, as well.

Conjecture 6.6.2 (Mahler, 1939). *Let $K \subset \mathbb{R}^n$ be convex body with $o \in \mathrm{int}K$.*

- $|K| \cdot |K^*| \geq \frac{(n+1)^{n+1}}{(n!)^2}$, *with equality if and only K is a centered simplex.*
- *If in addition $K = -K$, then $|K| \cdot |K^*| \geq \frac{4^n}{n!}$, with equality if and only if K is a Hanner polytope (see below, for example, $K = [-1, 1]^n$ is a Hanner polytope).*

Remark 6.6.3 (Hanner polytope). A Hanner polytope is an o-symmetric convex polytope defined by induction on $n \geq 1$. If $n = 1$, then any o-symmetric segment is a Hanner polytope. If L and L' are complementary linear subspaces of $\mathbb{R}^n$, $n \geq 2$, and $P \subset L$ and $P' \subset L'$ are Hanner polytopes, then $P + P'$ and $\mathrm{conv}\{P, P'\}$ are Hanner polytopes in $\mathbb{R}^n$. For example, the cube $W_n = [-1, 1]^n$, and its polar, the crosspolytope $C_n = \mathrm{conv}\{\pm e_1, \dots, \pm e_n\}$ are Hanner polytopes, and K^* is also a Hanner polytope and $|K| \cdot |K^*| = |W_n| \cdot |W_n^*|$ for a Hanner polytope K. However, if $n = 4$, then a cylinder over the octahedron is a Hanner polytope that is different from the linear images of W_n or C_n.

The Mahler Conjecture is still open after more than 80 years of intensive research.

Remark 6.6.4 (Some known cases of the Mahler conjecture).

- $n = 2$ (Mahler [440] in 1938)
- $n = 3$ and $K = -K$ (Iriyeh, Shibata [344] in 2020)
- K unconditional (Saint-Raymond [506] in 1980, see Theorem 6.6.5).
  Equality holds in this case if and only if K is a Hanner polytope.
- K zonoid (cf. Example 1.6.3) (Reisner [496] in 1985). Equality holds in this case if and only K parallelopiped (linear image of $[-1, 1]^n$).
- K has n independent hyperplane symmetries (Barthe, Fradelizi [53] in 2013).

The paper Reisner, Schütt Werner [497] attemts to understand the boundary structutre of a minimizer of the volume product. Finally, we present the simple argument due to Meyer [450] verifying the Mahler conjecture for unconditional convex bodies, originally proved by Saint-Raymond [506].

Theorem 6.6.5 (Mahler Conjecture for unconditional bodies). *If $K \subset \mathbb{R}^n$ is an unconditional convex body, then $|K| \cdot |K^*| \geq \frac{4^n}{n!}$.*

Proof. Induction on $n \geq 1$ where the case $n = 1$ trivial.

For the corresponding orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$, let $C = K \cap [0, \infty)^n$, $C^\circ = K^* \cap [0, \infty)^n$, and for $i = 1, \ldots, n$, let $C_i = K \cap [0, \infty)^n \cap e_i^\perp$ and $C_i^\circ = K^* \cap [0, \infty)^n \cap e_i^\perp$. It follows that K ($K \cap e_i^\perp$) can be dissected into 2^n (2^{n-1}) congruent copies of C (C_i), and

$$\begin{aligned} C^\circ &= \{x \in [0, \infty)^n : \langle x, y\rangle \leq 1 \ \text{ for } y \in C\} \\ C_i^\circ &= \left\{x \in [0, \infty)^n \cap e_i^\perp : \langle x, y\rangle \leq 1 \ \text{ for } y \in C_i\right\}. \end{aligned}$$

In particular, Theorem 6.6.5 is equivalent with the claim

$$|C| \cdot |C^\circ| \geq \frac{1}{n!}, \tag{6.33}$$

and we deduce from the induction hypothesis that if $i = 1, \ldots, n$, then

$$\mathcal{H}^{n-1}(C_i) \cdot \mathcal{H}^{n-1}(C_i^\circ) \geq \frac{1}{(n-1)!}. \tag{6.34}$$

For $x = (x_1, \ldots, x_n) \in (0, \infty)^n$, the interiors of the sets $\operatorname{conv}\{x, C_i\}$ are pairwise disjoint for $i = 1, \ldots, n$, thus

$$|C| \geq \frac{1}{n} \sum_{i=1}^n x_i \mathcal{H}^{n-1}(C_i) \ \text{ for } x = (x_1, \ldots, x_n) \in C \tag{6.35}$$

$$|C^\circ| \geq \frac{1}{n} \sum_{i=1}^n y_i \mathcal{H}^{n-1}(C_i^\circ) \ \text{ for } y = (y_1, \ldots, y_n) \in C^\circ. \tag{6.36}$$

We conclude from (6.35) that $p = \left(\frac{\mathcal{H}^{n-1}(C_1)}{n|C|}, \ldots, \frac{\mathcal{H}^{n-1}(C_n)}{n|C|}\right) \in C^o$.

Applying (6.36) to p, and using (6.34) imply that

$$|C| \cdot |C^\circ| \geq |C| \cdot \frac{1}{n} \sum_{i=1}^n \frac{\mathcal{H}^{n-1}(C_i)}{n|C|} \cdot \mathcal{H}^{n-1}(C_i^\circ) \geq \frac{1}{n} \sum_{i=1}^n \frac{1}{n!} = \frac{1}{n!},$$

which in turn yields (6.33). ■

6.7 Functional Santaló inequality and reverse form

In this section, we review the functional versions of the Blaschke-Santaló inequality (6.25) and its Reverse form (6.32). Since most of the arguments are more involved, we only provide the simple proof of the functional Santaló inequality for even functions due to Keith Ball. First we we define what we mean by the polar of a non-negative function.

Definition 6.7.1. Let $\varphi : \mathbb{R}^n \to (-\infty, \infty]$ and $f : \mathbb{R}^n \to [0, \infty)$ be measurable.

Legendre transform(convex conjugate): $\mathcal{L}(\varphi)(x) = \sup_{y\in\mathbb{R}^n}\{\langle x,y\rangle - \varphi(y)\}$;

Polar: $f^\circ(x) = \inf_{y\in\mathbb{R}^n}\left\{\frac{e^{-\langle x,y\rangle}}{f(y)}\right\}$, and hence if $f = e^{-\varphi}$, then $f^\circ = e^{-\mathcal{L}(\varphi)}$.

Remarks. $\mathcal{L}(\varphi)$ is convex and lower semicontinuous, and f° is log-concave and upper semicontinuous. In addition, if φ is convex and lower semicontinuous, then $\mathcal{L}\mathcal{L}(\varphi) = \varphi$, and if f is log-concave and upper semicontinuous, then $(f^\circ)^\circ = f$.

Example 6.7.2. Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \mathrm{int}K$.

(i) $(\mathbf{1}_K)^\circ(x) = e^{-\|x\|_{K^*}}$;

(ii) $\left(e^{-\frac{1}{2}\|x\|_K^2}\right)^\circ = e^{-\frac{1}{2}\|x\|_{K^*}^2}$, and hence $\left(e^{-\frac{1}{2}\|x\|^2}\right)^\circ = e^{-\frac{1}{2}\|x\|^2}$;

if $g(x) = a\,f(\Phi x)$ for $\Phi \in \mathrm{GL}(n)$ and $a > 0$, then $g^\circ(x) = \frac{1}{a} f^\circ(\Phi^{-t}x)$.

For the exponential expressions in Lemma 6.7.2, we have the following integral formulas.

Lemma 6.7.3. *$K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, then*

(i) $\displaystyle\int_{\mathbb{R}^n} e^{-\frac{1}{2}\|x\|_K^2}\,dx = \frac{(2\pi)^{\frac{n}{2}}|K|}{|B^n|}$;

(ii) $\displaystyle\int_{\mathbb{R}^n} e^{-\|x\|_K}\,dx = |K|$.

Proof. (i) and (ii) follow from the formula $\int_{\mathbb{R}^n}\psi(\|x\|_K)\,dx = n|K|\int_0^\infty \psi(r)r^{n-1}\,dr$ for continuous $\psi : [0,\infty) \to [0,\infty)$, which holds as $\frac{\partial}{\partial r}\int_{rK}\psi(\|x\|_K)\,dx = nr^{n-1}|K|\varphi(r)$. ∎

Artstein-Avidan, Klartag, V. Milman [26] proved the functional version (6.37) of the Blaschke-Santaló inequality, together with the characterization of equality extending Keith Ball's earlier result about even functions (without the characterization of equality) in his PhD thesis [34]. Here we only provide Keith Ball's simple argument in the even case, which started off the quest for functional versions of inequalities on convex bodies.

Theorem 6.7.4 (Functional Santaló). *If $0 < \int_{\mathbb{R}^n} f < \infty$ for $f : \mathbb{R}^n \to [0,\infty)$ and $\int_{\mathbb{R}^n} x f(x)\,dx = o$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \leq (2\pi)^n. \tag{6.37}$$

Equality holds if and only if $f(x) = ae^{-\frac{1}{2}\|\Phi x\|^2}$ for $a > 0$ and $\Phi \in \mathrm{GL}(n)$.

Proof of the functional Santaló inequality (6.37) *if f is even:* The sets $X_r = \{f > e^{-r}\}$ and $K_s = \{f^\circ > e^{-s}\}$ are o-symmetric and K_s is convex, and if $x \in X_r$ and $y \in K_s$, then $e^{-r}e^{-s} \leq e^{-\langle x,y\rangle}$, and hence $\langle x,y\rangle \leq r + s$. It follows by the definition of the polar of a convex body that $X_r \subset (r+s)(K_s)^*$, thus $|X_r| \cdot |K_s| \le (r+s)^n \omega_n^2$ by the Blaschke-Santaló inequlity (6.25).

We now apply the Prékopa-Leindler inequality (3.5) to the functions $\varphi(r) = e^{-r}|\{f > e^{-r}\}|$, $\psi(s) = e^{-s}|\{f^\circ > e^{-s}\}|$ and $h(t) = \omega_n e^{-t}(2t)^{\frac{n}{2}}$ for $t \ge 0$ and $h(t) = 0$ if $t < 0$ that satisfy $\sqrt{\varphi(r)\psi(s)} \le h(\frac{r+s}{2})$, and conclude that

$$\begin{aligned}\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ &= \int_{\mathbb{R}} e^{-r}|\{f > e^{-r}\}|\, dr \cdot \int_{\mathbb{R}} e^{-s}|\{f^\circ > e^{-s}\}|\, ds \\ &\le \left(\int_0^\infty \omega_n e^{-t}(2t)^{\frac{n}{2}}\, dt\right)^2 = \left(\frac{(2\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)} \int_0^\infty e^{-s} s^{\frac{n}{2}}\, ds\right)^2 = (2\pi)^n.\end{aligned}$$

■

Unsurprisingly, the Blaschke-Santaló inequality (6.25) does play a significant role in the proof in the functional Santaló inequality (6.37), and Example 6.7.2 (i) and Lemma 6.7.3 (i) show that in turn the functional Santaló inequality implies the Blaschke-Santaló inequality for convex bodies. We note that Lehec [396, 397] provided direct proofs of the functional Santaló inequality (6.37) without the characterization of equality not using (and hence yielding) the Blaschke-Santaló inequality for convex bodies. For a reversed inequality, Berndtsson [63] proved a Reverse Functional Santaló inequality for even log-concave functions using complex analysis that was inspried by G. Kuperberg's Reverse Blaschke-Santaló inequality (6.32), even if Berndtsson's does not actually use G. Kuperberg's method or result in [389]. We note that in a Reverse Functional Santaló inequality, we have to assume that the function is log-concave.

Theorem 6.7.5 (Even Reverse Functional Santaló). *If $f : \mathbb{R}^n \to [0,\infty)$ is even and log-concave with $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \ge \pi^n. \tag{6.38}$$

Fradelizi [240] extended (6.38) to possibly non-even log-concave functions with a slightly smaller factor. More precisely, Fradelizi [240] provided the strategy for how to obtain the factor $(c/2)^n$ in the Reverse Functional Santaló inequality for any log-concave function if the lower bound c^n, $c > 0$ absolute constant, is known in the even case.

Theorem 6.7.6 (Reverse Functional Santaló). *If $f : \mathbb{R}^n \to [0,\infty)$ is log-concave with $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \ge \left(\frac{\pi}{2}\right)^n. \tag{6.39}$$

Fradelizi, Meyer [243] proposed the following conjecture about the optimal constant in the functional Santaló inequality:

Conjecture 6.7.7 (Functional Mahler conjecture). *If $f : \mathbb{R}^n \to [0, \infty)$ is log-concave with $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geq e^n \tag{6.40}$$

where equality holds if $f(x_1, \ldots, x_n) = e^{-\sum_{i=1}^n x_i} \mathbf{1}_{[-1,\infty)^n}(x_1, \ldots, x_n)$.

If in addition f is even, then

$$\int_{\mathbb{R}^n} f \cdot \int_{\mathbb{R}^n} f^\circ \geq 4^n \tag{6.41}$$

where equality holds if $f(x_1, \ldots, x_n) = e^{-\sum_{i=1}^n |x_i|}$.

Remark. Fradelizi, Meyer [243] proved (6.40) if $n = 1$, and Fradelizi, Meyer [242, 243] verified (6.41) for unconditional functions, and Fradelizi, Gordon, Meyer, Reisner [244] even managed to characterize the equality in (6.41) for unconditional functions (which case is actually related to Hanner polytopes).

6.8 Volume approximation by polytopes

The reason why we discuss volume approximation by polytopes in this chapter is that the Blaschke-Santaló inequality (6.25) and the Reverse Blaschke-Santaló inequality (6.32) are used to verify that for an ellipsoid, volume approximation of by inscribed polytopes of given (not too high) number of vertices is essentially equivalent to volume approximation by circumscribed polytopes of the same number of facets (cf. Theorem 6.8.3). In turn, Theorem 6.D.1 due to Sas [512] on volume approximation in the plane can be used to settle the equality case in the Blaschke-Santaló inequality for non-symmetric convex bodies in any dimension (cf. Theorem 6.E.3).

One of our main results is Theorem 6.8.1 due to Macbeath [438] claiming that ellipsoids are the worst approximable convex bodies by inscribed polytopes in the sense of volume. Here the key tool is Steiner symmetrization, and hence we recall that for $u \in S^{n-1}$ and convex body $K \subset \mathbb{R}^n$, the Steiner symmetrial $\Theta_{u^\perp} K$ of K is

$$\Theta_{u^\perp} K = \left\{ x + \frac{t-s}{2} \cdot u : x \in K|u^\perp \;\&\; x + tu \in K \;\&\; x + su \in K \right\}.$$

The properties of Steiner symmetrization that we need here are that $\Theta_{u^\perp} K$ is also a convex body with $|\Theta_{u^\perp} K| = |K|$ (cf. Proposition 1.10.3), and there is a sequence of iterated Steiner symmetrizations whose results tend to a ball (cf. (Theorem 1.10.7)).

Theorem 6.8.1 (Macbeath). *If $C \subset \mathbb{R}^n$ is a convex body with $|C| = |rB^n|$ for $r > 0$, $k \geq n + 1$, then for any polytope $P \subset rB^n$ with at most k vertices, there exists a polytope $Q \subset C$ with at most k vertices such that $|Q| \geq |P|$.*

Remark. In particular, ellipsoids are worst approximable by inscribed polytopes of given number of vertices, but it is known only if in the planar case that ellipsoids are the only extremal bodies (see the elegant argument presented in Theorem 6.D.1 due to Sas [512]). The fact for triangles in $\mathbb{R}^2$ and for tetrahedra in $\mathbb{R}^3$, the ellipses or ellipsoids, respectively are the only extremizers have been verified earlier by Gross [274] and Blaschke [74], Section 72.

Proof. Since Steiner symmetrization preserves volume, and there is a sequence of iterated Steiner symmetrizations such that the image tends to a ball (see Theorem 1.10.7), enough to prove the following statement:

If $C \subset \mathbb{R}^n$ convex body and $u \in S^{n-1}$, then for any polytope $P \subset \Theta_{u^\perp} C$ with at most k vertices, there exists a polytope $Q \subset C$ with at most k vertices such that $|Q| \geq |P|$.

Let $x_i + \frac{1}{2}(t_i - s_i)u, i = 1, \dots, m, m \leq k$ be the vertices of P for $x_i \in C|u^\perp, t_i, s_i \in \mathbb{R}$ with $x_i + t_i u, x_i + s_i u \in C$. For $Q^+ = \mathrm{conv}\{x_i + t_i u : i = 1, \dots, m\}$ and $Q^- = \mathrm{conv}\{x_i + s_i u : i = 1, \dots, m\}$, we claim that

$$\frac{\mathcal{H}^1(\ell \cap Q^+) + \mathcal{H}^1(\ell \cap Q^-)}{2} \geq \mathcal{H}^1(\ell \cap P) \text{ for any } z \in P|u^\perp \text{ and } \ell = z + \mathbb{R}u. \quad (6.42)$$

To prove Macbeathsecant, let $\ell \cap P = \mathrm{conv}\{p, q\}$ for $p, q \in \Theta_{u^\perp} C$ with $p - q = \mathcal{H}^1(\ell \cap P) \cdot u$. We deduce the existence of $\alpha_i, \beta_i \geq 0$ for $i = 1, \dots, m$ such that $p = z + \frac{1}{2}\sum_{i=1}^m \alpha_i (t_i - s_i)u$, $q = z + \frac{1}{2}\sum_{i=1}^m \beta_i (t_i - s_i)u$ and $\sum_{i=1}^m \alpha_i = \sum_{i=1}^m \beta_i = 1$, and hence

$$\mathcal{H}^1(\ell \cap P) = \frac{1}{2}\sum_{i=1}^m (\alpha_i - \beta_i)(t_i - s_i)$$

$$\mathcal{H}^1(\ell \cap Q^+) \geq \sum_{i=1}^m (\alpha_i - \beta_i)t_i \Longleftarrow z + \textstyle\sum_{i=1}^m \alpha_i t_i u, z + \sum_{i=1}^m \beta_i t_i u \in Q^+$$

$$\mathcal{H}^1(\ell \cap Q^-) \geq \sum_{i=1}^m (\beta_i - \alpha_i)s_i \Longleftarrow z + \textstyle\sum_{i=1}^m \alpha_i s_i u, z + \sum_{i=1}^m \beta_i s_i u \in Q^-.$$

Now the last three estimates yield (6.42).

It follows from (6.42) that $\frac{1}{2}(|Q^+| + |Q^-|) \geq |P|$; therefore, we may choose either $Q = Q^+$ or $Q = Q^-$. ■

We deduce from Theorem 6.8.1 that upper bounds in the case of volume approximation of balls by inscribed ellipsoids lead to upper bounds in the case of volume approximation of any convex body. Therefore, the rest of the section is dedicated to volume approximation of Euclidean balls. We start with an elementary estimate due to G. Elekes [201] that shows that "Computing the volume is difficult"; namely, polytopes with at most polynomial many (in n) vertices provide bad approximation of the ball.

Theorem 6.8.2 (Elekes). *If $P \subset B^n$ a polytope with at most k vertices, then* $\frac{|P|}{|B^n|)} \leq \frac{k}{2^n}$.

Remark. As Bárány, Füredi [46] explains, the simple estimate of Theorem 6.8.2 is a quite reasonable one if k is around $2^{n/2}$ (see the improved estimate (**??**) below if k is subexponential in n).

Proof. We may assume that the vertices $v_1, \dots, v_k$ of P lie on S^{n-1}.

For any $x \in P \setminus \{o\}$, there exists a $v_i \notin \{z \in \mathbb{R}^n : \langle z, x \rangle \leq 0\}$, and hence $x \in \frac{1}{2} v_i + \frac{1}{2} B^n$. We deduce that $P \subset \cup_{i=1}^k (\frac{1}{2} v_i + \frac{1}{2} B^n)$, thus $|P| \leq \frac{k}{2^n} |B^n|$. ∎

The following statement provides the true order of volume approximation of a ball if at most exponential many vertices (in the inscribed case) or at most exponential many facets (in the circumscribed case) are used.

Theorem 6.8.3. *For an absolute constant $c > 1$ and $2n \leq k \leq 2^n$, if $P_k \subset B^n$ polytope of maximal volume with k vertices and $P_{(k)} \supset B^n$ polytope of minimal volume with k facets, then*

$$c^{-1} \cdot \sqrt{\frac{\log \frac{k}{n}}{n}} \leq \sqrt[n]{\frac{|P_k|}{|B^n|}} \leq c \cdot \sqrt{\frac{\log \frac{k}{n}}{n}}; \tag{6.43}$$

$$c^{-1} \cdot \sqrt{\frac{n}{\log \frac{k}{n}}} \leq \sqrt[n]{\frac{|P_{(k)}|}{|B^n|}} \leq c \cdot \sqrt{\frac{n}{\log \frac{k}{n}}}. \tag{6.44}$$

Remarks.

(i) As we may asssume allowing $2k$ vertices/facets that the polytopes are o-symmetric, the lower bound in (6.43) and the upper bound in (6.44) are equivalent according to the Blaschke-Santaló inequality (6.25) and the Reverse Blaschke-Santaló inequality (6.32), and similarly, the upper bound in (6.43) and the lower bound in (6.44) are equivalent.

(ii) Comparing (6.43) and (1.40) shows that if $2n \leq k \leq 2^n$, then for a polytope $P_k \subset B^n$ with k vertices, maximizing the inradius or the volume are equivalent.

(iii) The upper bound in (6.43) is due to Bárány, Füredi [46], and the equivalent lower bound in (6.44) are proved by Carl, Pajor [143] and Gluskin [264], all three papers are from 1988. The argument of Bárány, Füredi [46] is retold in Section 6 of Böröczky, Wintsche [120], while the argument of Gluskin [264] is presented in the survey Ball [39] (see Galicer, Litvak, Merzbacher, Pinasco [261] for a recent approach).

(iv) The lower bound in (6.43) follows from Theorem 1.13.6 due to Böröczky, Wintsche [120]. For the equivalent upper bound in (6.44), Assaf Naor gave the following construction: Assuming n is large, we may assume that $k > 16n$, and we choose integers $m \geq 2$ such that $\frac{k}{4n} \leq \frac{2^m}{m} \leq \frac{k}{2n}$ (and hence $m < n$), $r \geq 1$ maximal such that $rm \leq n$, and $0 \leq \ell \leq m-1$ such that $rm + \ell = n$. The approximating polytope $\widetilde{P}_{(k)}$ is the direct sum of r copies (if $\ell = 0$) or $r+1$ copies (if $\ell > 0$) of centered regular crosspolytopes of inradius 1 in pairwise orthogonal linear subspaces, where we use r crosspolytopes of dimension m, and if $\ell > 0$, then we also use an extra crosspolytope of dimension ℓ. Thus the number of facets of $\widetilde{P}_{(k)}$ is $r2^m$ if $\ell = 0$, or $r2^m + 2^\ell$ if $\ell > 0$, and hence lies between $k/8$ and k. According to the bound $m! > (m/e)^m$ by the Stirling formula, the volume of the m-dimensional regular crosspolytopes $\widetilde{C}_m$ of inradius 1 can be estimated by

$$\sqrt[m]{\mathcal{H}^m(\widetilde{C}_m)} = \sqrt[m]{\frac{\sqrt{m}^m \cdot 2^m}{m!}} < \frac{2e}{\sqrt{m}}$$

thus elementary calculations show that $\widetilde{P}_{(k)}$ satisfies the upper bound in (6.44).

(v) (6.43) and (6.44) are applied in the isomorphic reverse isoperimetric inequality Theorem 6.8.4 for polytopes with at most k vertices and in the isomorphic isoperimetric inequality Theorem 7.7.9 for polytopes with at most k facets in the form

$$c^{-1} \cdot n^{-1}\sqrt{\log\frac{k}{n}} \leq |P_k|^{\frac{1}{n}} \leq c \cdot n^{-1}\sqrt{\log\frac{k}{n}}; \tag{6.45}$$

$$c^{-1}\left(\log\frac{k}{n}\right)^{\frac{-1}{2}} \leq |P_{(k)}|^{\frac{1}{n}} \leq c\left(\log\frac{k}{n}\right)^{\frac{-1}{2}}. \tag{6.46}$$

According to the Reverse Isoperimetric inequality Theorem 6.3.2 for origin symmetric convex bodies, if $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then there exists $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that the isoperimetric quotient satisfies

$$\frac{S(\Phi K)}{|\Phi K|^{\frac{n-1}{n}}} \leq 2n, \tag{6.47}$$

with equality for cubes. However, cubes have rather high number; namely, 2^n vertices. Next we present an improvement on the bound of (6.47) if n is large and an o-symmetric polytope has significantly fewer than 2^n vertices.

Theorem 6.8.4. *If $P \subset \mathbb{R}^n$ is an o-symmetric polytope with at most k vertices where $2n \leq k \leq 2^n$, then*

$$\frac{S(\Phi P)}{|\Phi P|^{\frac{n-1}{n}}} \leq c\sqrt{n} \cdot \sqrt{\log\frac{k}{n}} \tag{6.48}$$

for an absolute constant $c > 0$.

Proof. According to John's Theorem 6.1.1, there exists a $\Phi \in \mathrm{GL}(n, \mathbb{R})$ such that

$$\tfrac{1}{\sqrt{n}} B^n \subset K \subset B^n.$$

We apply first Lemma 6.3.3 and then (6.45) to conclude that

$$\frac{S(\Phi P)}{|\Phi P|^{\frac{n-1}{n}}} \leq n\sqrt{n}|\Phi P|^{\frac{1}{n}} \leq c\sqrt{n} \cdot \sqrt{\log \frac{k}{n}}.$$

■

For large k, it follows from Theorem 1.13.3 and Theorem 6.8.1 that if $k \geq 2^n$ and $C \subset \mathbb{R}^n$ is a convex body, then there exists a polytope $P_k \subset C$ with at most k vertices such that

$$\frac{|C \backslash P_k|}{|C|} \leq \frac{c \cdot n}{k^{\frac{2}{n-1}}}. \tag{6.49}$$

where $c > 1$ is an absolute constant. This estimate is optimal even considering the factor because Prochno, Schütt, Werner [494] proved the existence absolute constants $a, b > 0$ such that if $P_k \subset B^n$ polytope of maximal volume with at most k vertices, then

$$\frac{|B^n \backslash P_k|}{|B^n|} \geq \frac{a \cdot n}{k^{\frac{2}{n-1}}} \quad \text{provided } k \geq b \cdot n^{\frac{n-1}{2}}. \tag{6.50}$$

If ∂C has C^2 boundary and $P_k \subset C$ is a polytope of maximal volume with at most k vertices and $P_{(k)} \supset C$ is a polytope of minimal volume with at most k facets, then the limits $\lim_{k\to\infty} k^{\frac{2}{n-1}} |C \backslash P_k|$ and $\lim_{k\to\infty} k^{\frac{2}{n-1}} |P_{(k)} \backslash C|$ exist and are positive (see Section 8.10 and Böröczky [90]).

Concerning both random and best approximation by polytopes in terms of volume difference, the earlier history is discussed by Gruber [275,276], and more recent developments are reviewed by Prochno, Schütt, Werner [494].

6.9 The M-ellipsoid and the Reverse Brunn-Minkowski inequality

As introduced by V. Milman, Pajor [453], for a centered convex body $K \subset \mathbb{R}^n$, an "M-ellipsoid" E is just any ellipsoid satisfying either of the properties (6.51), (6.52) or Theorem 6.9.4 up to a factor c^n where c is some absolute constant where the three properties are equivalent according to V.Milman, Pajor [453]. The main goal of this section to show how the mere existence of the M-ellipsoid leads to the Reverse Brunn-Minkowski inequality Theorem 6.9.5

Given a centered convex body $K \subset \mathbb{R}^n$ of diameter D, the volume of its intersection with an ellipsoid of inradius r is at most $2rD^{n-1}\omega_{n-1}$, and hence the Blaschke Selection Theorem 1.7.3 yields the existence of an o-symmetric ellipsoid $E \subset \mathbb{R}^n$ with $|E| = |K|$ maximizing $|K \cap E|$. Now V. Milman, Pajor [453] prove that this E is an M ellipsoid.

Theorem 6.9.1 (Existence of an M-ellipsoid). *For a centered convex body $K \subset \mathbb{R}^n$, if E is an o-symmetric ellipsoid with $|E| = |K|$ maximizing $|K \cap E|$, then*

$$|K \cap E| \geq c^n |K| \ \ \textit{and}\ \ |K^* \cap E^*| \geq c^n |E^*| \geq c^n |K^*|. \tag{6.51}$$

for an absolute constant $c \in (0, 1)$.

Remarks.

- Any ellipsoid E satisfying (6.51) for an absolute constant $c \in (0, 1)$ is called an M-ellipsoid for K. If $\Phi \in \mathrm{GL}(n)$, then ΦE is an M-ellipsoid of ΦK.
- $|E^*| \geq |K^*|$ in (6.51) follows from the Blaschke-Santaló inequality Theorem 6.5.2.
- $|K \cap E| \cdot |K + E| < 4^n |K| \cdot |E|$ according to (1.29), and hence for the absolute constant $C = 4/c$ where c comes from (6.51), the M ellipsoid E satisfies

$$|K + E| \leq C^n |K| \text{ and } |K^* + E^*| \leq C^n |K^*|. \tag{6.52}$$

- Assuming $\sigma_K = o$ for the convex body $K \subset \mathbb{R}^n$ and $|K \cap E| \geq \tilde{c}^n |K|$ for $\tilde{c} > 0$ and an o-symmetric ellipsoid $E \subset \mathbb{R}^n$ with $|E| = |K|$, G. Kuperberg's Reverse Blaschke-Santaló inequality Theorem 6.6.1 yields

$$|K^* \cap E^*| = \left|(\mathrm{conv}\{K, E\})^*\right| \geq \left|(K + E)^*\right| \geq \frac{4^{-n}|E| \cdot |E^*|}{|K + E|} \geq \left(\frac{\tilde{c}}{16}\right)^n |E^*|.$$

  Therefore, the crucial part of (6.51) in the definition of the M ellipsoid is the lower bound for $|K \cap E|$. We note that most arguments establishing the existence of the M-ellipsoid use the Reverse Blaschke Santaló ineuqality, see, for example, Brazitikos, Giannopoulos, Valettas, Vritsiou [125].

Besides using the intersections as in (6.51), or Minkowski sums as in (6.52), an M ellipsoid can be defined *via* covering numbers as Theorem 6.9.4 shows.

Definition 6.9.2 (Covering number). For convex bodies $K, L \subset \mathbb{R}^n$, the covering number $N(K, L)$ is the minimal $N \geq 1$ s.t. $K \subset x_1 + L, \ldots, x_N + L$ for some $x_1, \ldots, x_N \in \mathbb{R}^n$.

Remark. $N(K, L) = N(\Phi K, \Phi L)$ for $\Phi \in \mathrm{GL}(n)$.

Erdős, Rogers [202] construct a covering of $\mathbb{R}^n$ by translates of L such that any point of $\mathbb{R}^n$ is covered at most $4n \ln n$ times. If a translate $x + L$ intersects K, then $x \in K - L$, and hence $x + L \subset K - L + L$; therefore, we deduce the following estimate for the covering number.

Theorem 6.9.3 (Rogers bound on the covering number). *For convex bodies $K, L \subset \mathbb{R}^n$,*

$$N(K, L) \leq 4n \log n \cdot \frac{|K + L - L|}{|L|}. \tag{6.53}$$

Remark. The bound (6.53) is close to be optimal in general, as if $L = B^n$ and $nB^n \subset K$, then $N(K, B^n) \geq cn \cdot \frac{|K+2B^n|}{|B^n|}$ for an absolute constant $c > 0$ as any covering of $\mathbb{R}^n$ by unit balls has density at least $c_0 n$ for an absolute constant $c_0 > 0$ according to Coxeter, Few, Rogers [180] (see also Böröczky [91], Theorems 8.2.1 and 9.5.2).

The following estimate is a consequence of (6.52) and Theorem 6.53:

Theorem 6.9.4. *If $K \subset \mathbb{R}^n$ is convex body and E is an M-ellipsoid, then $N(K, E) \leq c^n$ for an absolute constant $c > 0$.*

Remark. Assuming $\sigma_K = o$, also $N(K^*, E^*) \leq c^n$.

We recall that according to the Brunn-Minkowski inequality, if $K, L \subset \mathbb{R}^n$ are convex bodies, then $|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}$. The following reverse form of the Brunn-Minkowski inequality, indicated by V. Milman, Pajor [453], is a generalization of V. Milman's Reverse Brunn-Minkowski inequality for o-symmetric convex bodies in [452], and was seemingly first stated in this form only much later by Brazitikos, Giannopoulos, Valettas, Vritsiou [125].

Theorem 6.9.5 (Reverse Brunn-Minkowski inequality). *There exists an absolute constant $C > 1$ with the following property: For convex bodies $K, L \subset \mathbb{R}^n$, one finds an $\Omega \in \mathrm{SL}(n)$ such that if $\alpha, \beta > 0$, then*

$$|\alpha\,\Omega K + \beta L|^{\frac{1}{n}} \leq C(\alpha|K|^{\frac{1}{n}} + \beta|L|^{\frac{1}{n}}).$$

Remark. Assuming that $\sigma_K = \sigma_L = o$, we also have

$$|\alpha\,(\Omega K)^* + \beta L^*|^{\frac{1}{n}} \leq C(\alpha|K^*|^{\frac{1}{n}} + \beta|L^*|^{\frac{1}{n}}).$$

Proof. We may assume that $\sigma_K = \sigma_L = o$, and $|K| = |rB^n|$ and $|L| = |\varrho B^n|$ for $r, \varrho > 0$.

Theorem 6.9.4 yields the existence of $\Phi, \Psi \in \mathrm{SL}(n)$ such that $N(\Phi K, rB^n) \leq C_0^n$ and $N(\Psi L, \varrho B^n) \leq C_0^n$, and hence $\alpha\Phi K \subset X_K + \alpha r B^n$ and $\beta\Psi L \subset X_L + \beta\varrho B^n$ where $\#X_K, \#X_L \leq C_0^n$. We deduce that $\Phi K + \Psi L \subset X_K + X_L + (\alpha r + \beta\varrho)B^n$ and $\#(X_K + X_L) \leq \#X_K \cdot \#X_L \leq C_0^{2n}$, thus

$$|\alpha\Phi K + \beta\Psi L| \leq C_0^{2n} \cdot |(\alpha r + \beta\varrho)B^n| = C_0^{2n}\left(\alpha|K|^{\frac{1}{n}} + \beta|L|^{\frac{1}{n}}\right)^n.$$

Therefore, we can choose $C = C_0^2$ and $\Omega = \Psi^{-1}\Phi$. ∎

6.10 Comments to Chapter 6

The subject of Chapter 6 is discussed in depth by Artstein-Avidan, Giannopoulos, V. Milman [28, 29], and see also the survey Ball [38] for various aspects of this topic.

For a convex body $K \subset \mathbb{R}^n$, the properties of the inscribed ellipsoid of maximal volume ("John ellipsoid") and circumscribed ellipsoid of minimal volume ("Löwner ellipsoid") were essentially established by John [359] in 1948, and put it into the right context by Keith Ball [37], who also established that "John's conditions" are sufficient for the extremality of the unit ball (see also Gruber, Schuster [278] for a streamlined argument). According to Busemann, Löwner was independently aware of many properties of the extremal ellipsoids in the middle of the 20th century, hence the naming (see Henk [305] on the history of the problem). Note that around 1908, Voronoi [560] established an analogous property of the convex hull of the minimal vectors of an extremal lattice to John's conditions on a ball being the Löwner ellipsoid (see perfect forms in Schürmann [526]).

Hug, Schneider [341] prove that for any convex body $K \subset \mathbb{R}^n$, the ratio of the volumes of the Löwner ellipsoid over the John ellipsoid of K is at most n^n, and equality occurs if and only if K is a simplex. Hug, Schneider [341] even verify a stability version of this statement.

There are numerous widely used characterizatizations of ellipsoids like the one in Brunn's [131] Theorem 6.2.1. For a comprehensive surveys about characterizations of ellipsoids among convex bodies, see Petty [486] and Soltan [536].

The reverse isoperimetric inequality (cf. Theorems 6.3.1 and 6.3.2) is due to Ball [36]. We note that Livshyts [419] proves an analogous statement for unimodule functions (non-negative function on $\mathbb{R}^n$ whose level sets are convex).

The proof of the reverse isoperimetric inequality is based on the Geometric form of the rank one Brascamp-Lieb inequality due to Keith Ball [36], and the general Brascamp-Lieb inequality was proved by Brascamp, Lieb [124] (see Bennett, Carbery, Christ, Tao [59] for a comprehensive study of the Brascamp-Lieb inequality). Equality in the Geometric the rank one Brascamp-Lieb inequality was clarified by Barthe [50].

The general form of the Brascamp-Lieb inequality [124] (see Barthe [50] for an elegant proof) is as follows. Let $B_i : \mathbb{R}^n \to H_i$ be surjective linear maps where H_i is n_i-dimensional Euclidean space, $n_i \geq 1$, for $i = 1, \dots, k$, and let $c_1, \dots, c_k > 0$ satisfy $\sum_{i=1}^k c_i n_i = n$. For non-negative $f_i \in L_1(H_i)$, we have

$$\int_{\mathbb{R}^n} \prod_{i=1}^k f_i(B_i x)^{c_i}\, dx \leq C \prod_{i=1}^k \left(\int_{H_i} f_i \right)^{c_i} \tag{6.54}$$

where C is determined by choosing centered Gaussians $f_i(x) = e^{-\langle A_i x, x\rangle}$, A_i positive definite. The Brascamp-Lieb inequality (6.54) yields for example Young's inequality and the Hölder inequality.

The so-called Geometric form of the Brascamp-Lieb inequality (6.54) is when H_i is a linear subspace of $\mathbb{R}^n$, $B_i = \Pi_{H_i}$ and $\sum_{i=1}^k c_i \Pi_{H_i} = I_n$. In this case, the optimal factor $C = 1$, as it was verified by Ball [36] in the rank one case (each $n_i = 1$), and by Barthe [50] in general.

Naor [470] stats the following Isomorphic Reverse Isoperimetry Conjecture: *For any o-symmetric convex body $K \subset \mathbb{R}^n$, there exists $\varphi \in \mathrm{SL}(n)$ and an o-symmetric convex body $C \subset \Phi K$ such that $|C| \geq c^{-n}|K|$ and $S(C) \leq c\sqrt{n}|C|^{\frac{n-1}{n}}$ for an absolute constant $c > 1$.*
In particular, Isomorphic Reverse Isoperimetry Conjecture says that while $C \subset \Phi K$ has essentially the same volume as ΦK, and the isoperimetric ratio for C is essentially the same as for a ball where $S(B^n)/|B^n|^{\frac{n-1}{n}} \sim \sqrt{2\pi e}\sqrt{n}$ as n tends to infinity. For example, the isoperimetric ratio of a cube is much larger as $S([-1,1]^n)/|[-1,1]^n|^{\frac{n-1}{n}} = 2n$.

Many properties of a convex body related to its centroid can be found in Artstein-Avidan, Giannopoulos, V. Milman [28, 29] and Brazitikos, Giannopoulos, Valettas, Vritsiou [125] where the slicing conjecture is also discussed (see Klartag [373] for more recent developments). We note that Kannan, Lovász, Simonovits [361] uses another normalization to define isotropicity, they say a convex body K is in isotropic position if and only if $\sigma_K = o$ and $M_K = I_n$.

For a brief history of the slicing conjecture (there exists an absolute constant c such that $L_K \leq c$ for any convex body $K \subset \mathbb{R}^n$), it was posed independtly posed by Bourgain [121] and in the PhD thesis Ball [34] in 1986, and Bourgain [121] proved $L_K \leq c\, n^{\frac{1}{4}} \log n$ in [122] in 1991. In spite of serious efforts, for 30 years, the only improvement on the bound was $L_K \leq c\, n^{\frac{1}{4}}$ by Paouris [480] and Klartag [371] in 2006. A breakthrough $L_K \leq n^{o(1)}$ has been achieved by Yuansi Chen [158] in 2021, whose estimate was improved to $L_K \leq \sqrt{\log n}$ by Klartag [373].

The Blaschke-Santaló inequality in all dimensions about the maximum of the volume product of a centered convex body is due Santaló [505] in 1949, whose proof is based on relating it to the Affine Isoperimetric Inequality (see Section 8.9). The equality case was only characterized by Petty [487] in 1985. A direct proof of the Blaschke-Santaló inequality *via* Steiner symmetrization in the o-symmetric case was provided in Ball's PhD thesis [34], whose argument was extended to all convex bodies by Meyer, Pajor [451]. A Fourier analytic proof of the Blaschke-Santaló inequality in the o-symmetric case is provided by Bianchi, Kelly [73]. Stability versions of the Blaschke-Santló inequality are due to Böröczky [92] and Ball, Böröczky [41].

Lehec [396,397] provided simple proofs of the functional Santaló inequality (6.37) without the characterization of equality not using (and hence yielding) the Blaschke-Santaló inequality for convex bodies. Fradelizi, Meyer [241] proved a generalized version of the functional Santaló inequality, which was further generalized to more functions by Kolesnikov, Werner [382] and Kalantzopoulos, Saroglou [360]. Stability versions of the functional Santaló inequality (6.37), and in general, of the inequalities proved by Fradelizi, Meyer [241] are provided by Barthe, Boroczky, Fradelizi [52].

The Mahler conjecture in $\mathbb{R}^n$; namely, that the volume product of is minimized by centered simplices among all convex bodies and by cubes among all o-symmetric convex bodies was stated by Mahler [441] in 1939. It has been verified in the plane earlier by Mahler himself in [440], but it had been open in general in any dimension

$n \geq 3$ until the Mahler conjecture was proved by Iriyeh, Shibata [344] in 2020 at least for o-symmetric convex bodies in $\mathbb{R}^3$ (and the argument is simplified by Fradelizi, Hubard, Meyer, Roldán-Pensado, Zvavitch [245]). In addition, Mahler's conjecture has been verified in various special cases like

- for zonoids by Reisner [496] (see Gordon, Meyer, Reisner [267] for a simpler proof), and Böröczky, Hug [106] provided stability version;
- for unconditional convex bodies by Saint-Raymond [506] (see Meyer [450] or Theorem 6.6.5 for a simple proof), and Kim, Zvavitch [367] even provided a stability version;
- for convex bodies with n independent hyperplane symmetries (by rank n action of a Coxeter group) by Barthe, Fradelizi [53].

Some other cases when the Mahler conjecture is known include sections and projections of Hanner polytopes (see Karasev [362] using symplectic technics), and convex bodies in $\mathbb{R}^3$ invariant under the rotational symmetries of a centered regular tetrahedron (see Iriyeh, Shibata [345]).

Kim, Reisner [366] proved that centered simplices are local minimums for the volume product, providing even a stability estimate, and Kim [365] proved the analogous results for the Hanner polytopes among o-symmetric convex bodies (extending earlier work by Nazarov, Petrov, Ryabogin, Zvavitch [475])

Concerning the order of the minimum of the volume product, the ground breaking work Bourgain, V. Milman [123] proved the Reverse Blaschke-Santaló inequality $|K| \cdot |K^*| \geq c^n V(B^n)^2$ for any o-symmetric convex body $K \subset \mathbb{R}^n$ and an absolute constant $c \in (0, 1)$ (see Giannopoulos, Paouris, Vritsiou [259] for a simpler proof). Explicit estimates for the value of c was obtained, for example, by Nazarov [474], but the best estimate is due to G. Kuperberg [389] (see the arxiv version for the case of non-symmetric convex bodies).

The functional Santaló inequality was proved by Ball [34] for even functions in his PhD thesis, and was extended to centered functions with positive integral by Artstein-Avidan, Klartag, V. Milman [26], also characterizing equality. Versions of the functional Santaló inequality were provided by Fradelizi, Meyer [241] and Lehec [396, 397]. The reverse functional Santaló inequality for log-concave functions with a lower bound c^n for an unknown absolute constant $c > 0$ is due to Klartag, V. Milman [374], and simpler arguments have been provided by Fradelizi, Meyer [242] and Giannopoulos, Paouris, Vritsiou [259]. The beautiful argument by Berndtsson [63] uses complex analysis to verify the lower bound π^n in the case even log-concave functions, and is inspired by ideas in G. Kuperberg [389]. For the case of any log-concave functions, our bound $(\pi/2)^n$ might have not appeared in print, and is based on Fradelizi's [240] strategy to obtain the lower bound $(c/2)^n$ in the Reverse Functional Santaló inequality if the lower bound c^n, $c > 0$ absolute constant, is known in the even case.

For o-symmetric convex bodies, M-ellipsoid was defined in V. Milman [452], where the Reverse Brunn-Minkowski was established. The definition of the M-ellipsoid and the Reverse Brunn-Minkowski was extended to the non-symmetric case by V. Milman, Pajor [453] (see Giannopoulos, Paouris, Vritsiou [259] for a nice survey). The M-ellipsoid is not unique by its definition, and how close two M-ellipsoids are is dicussed by V. Milman, Pajor [454].

For a survey about volume approximation of the unit ball with polytopes of low complexity, see for example Ball [39], Section 6 of Böröczky, Wintsche [120], and Galicer, Litvak, Merzbacher, Pinasco [261]. Randomized algorithms proved to be effective in estimating the volume of a polytope of few vertices within a convex body (see Lee, Vempala [395]).

According to the Dvoretzky theorem, any convex body K in $\mathbb{R}^n$ has a section that is essentially a ball. This line of research was initiated by Groethendick, whose question was answered by Dvoretzky and later V. Milman, and finally Gordon [265] proved the most precise statement: For large n, $\varepsilon \in (0, 1)$, and integer $m \geq 2$ with $m \leq c\varepsilon^2 \log n$ for an absolute constant $c > 1$, if K is a convex body in $\mathbb{R}^n$ with $o \in \operatorname{int} K$, then there exists a linear subspace L of dimension m and $r > 0$ such that

$$rB^n \cap L \subset K \cap L \subset (1 + \varepsilon)(rB^n \cap L).$$

Actually, if K is origin symmetric and m is small enough, then the majority of sections by a linear m-plane is almost spherical (see Mendelson [449]).

6.A Supplement: The John condition on the inscribed maximal volume ellipsoid

The prove the classical properties in Theorem 6.A.2 of John's maximal volume ellipsoid contained in a given convex body, we use various properties of matrices, summarized below.

Remark 6.A.1 (Some properties of Positive Semidefinite Matrices).

(i) $u \otimes u = uu^t$ for $u \in \mathbb{R}^n \backslash \{o\}$ is a rank one symmetric $n \times n$ matrix, $\operatorname{tr} u \otimes u = \|u\|^2$.

(ii) Writing I_n to denote the $n \times n$ identity matrix, if A is any $n \times n$ matrix, then

$$\frac{d}{dt} \det(I_n + tA)\bigg|_{t=0} = \operatorname{tr} A.$$

(iii) $E \subset \mathbb{R}^n$ is an ellipsoid if and only if $E = \Phi\, B^n + v$ for $\Phi \in \mathrm{GL}(n)$ and $v \in \mathbb{R}^n$. According to the Left Polar Decomposition, $\Phi = AQ$ for a positive definite symmetric matrix A and an orthogonal matrix Q where $A^2 = \Phi\Phi^t$, and hence $E = AB^n + v$. Note that if $u_1, \dots, u_n$ is an orthonormal basis representing the principal directions of E, then $Au_i = a_i u_i$ where $a_i > 0$ is the principle axis corresponding to u_i.

It follows from Remark 6.A.1 (iii) that the space of o-symmetric ellipsoids can be identified with the space of positive definite matrices.

Theorem 6.A.2 (John). *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) There exists a unique so-called John ellipsoid $E \subset K$ of maximal volume.

(ii) If B^n is the John ellipsoid of $K \subset \mathbb{R}^n$, then $B^n \subset K$ and there exsits $c_1, \dots, c_k > 0$ and $u_1, \dots, u_k \in S^{n-1} \cap \partial K$, $k \le \frac{n(n+3)}{2}$, such that

$$\sum_{i=1}^{k} c_i u_i \otimes u_i = I_n, \tag{6.55}$$

$$\sum_{i=1}^{k} c_i u_i = o. \tag{6.56}$$

In addition, if K is o-symmetric, then (6.55) is sufficient.

Remarks.

(a) The uniqueness yields that ΦE is the John ellipsoid of ΦK for $\Phi \in \mathrm{GL}(n)$, for example, E is o-symmetric if K is o-symmetric.

(b) $\sum_{i=1}^{k} c_i = n$ by equating traces in (6.55).

(c) $\|x\|^2 = \sum_{i=1}^{k} c_i \langle u_i, x\rangle^2$ for $x \in \mathbb{R}^n$ by (6.55).

(d) $\langle x, u_i\rangle \le 1$ for $x \in K$ and $i = 1, \dots, k$ as K and B^n share the same supporting hyperplanes at $u_1, \dots, u_k$.

We recall (3.5.2), that says that if A, B are symmetric positive definite $n \times n$ matrices and $\lambda \in (0, 1)$, then

$$\det((1-\lambda)A + \lambda B) \ge (\det A)^{1-\lambda} (\det B)^{\lambda} \tag{6.57}$$

where equality holds if and only if $A = B$.

Proof of Theorem 6.A.2.

Step 1 There exists a maximal volume ellipsoid $E \subset K$.

Fix an ellipsoid $\widetilde{E} \subset K$. The family of pairs (A, v) where A is a positive definite matrix and $v \in \mathbb{R}^n$ with $A B^n + v \subset K$ and $|A B^n + v| \ge |\widetilde{E}|$ is compact, as the largest eigenvalue of A is at most $\operatorname{diam} K$.

Step 2 There is a unique ellipsoid of maximal volume contained in K.

Indirekt. We suppose that B^n is an ellipsoid of maximal volume contained in K, and $B^n \ne A B^n + v \subset K$ for a positive definite matrix A and $v \in \mathbb{R}^n$ with $\det A = 1$. It follows that $(\frac{1}{2} I + \frac{1}{2} A)B^n + \frac{1}{2} v \subset K$ as K convex where $\det(\frac{1}{2} I_n + \frac{1}{2} A) > 1$ unless $A = I_n$ according to (6.57); therefore, $A = I_n$ and $v \ne o$. We deduce that $E \subset K$ for the ellipsoid $E = \Phi B^n + \frac{1}{2} v$ where $\Phi v = (1 + \frac{1}{2}\|v\|)v$ and $\Phi w = w$ for $w \in v^{\perp}$, and hence

$\det \Phi = 1 + \frac{1}{2}\|v\| > 1$. This contradiction verifies the uniqueness of the John ellipsoid.

Step 3 If $K = -K$ and B^n is the John ellipsoid, then 6.55 and $k \le d$ hold.

We identify the vector space of $n \times n$ matrices with $\mathbb{R}^{n^2}$, and for any $n \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$, we define their scalar product to be $\langle A, B\rangle = \sum_{i,j=1,\dots,n} a_{ij}b_{ij}$. Let $V \subset \mathbb{R}^{n^2}$ be the linear subspace of symmetric $n \times n$ matrices with $d = \dim V = \frac{n(n+1)}{2}$; namely, $V = \{A = [a_{ij}] : a_{ij} = a_{ji}\ \forall i \ne j\}$. In particular, V can be identified with R^d.

For the compact set $U = \partial K \cap S^{n-1}$, we consider the compact set $\widetilde{U} = \{u \otimes u : u \in U\} \subset V$. Since $\langle I_n, u \otimes u\rangle = \operatorname{tr} u \otimes u = 1$ for any $u \in U$, Lemma 1.3.8 yields that $C = \operatorname{pos} \widetilde{U} \subset V$ is a closed convex cone. As every $u \times u \in \widetilde{U}$ is positive semidefinite, each $M \in C$ is positive semidefinite, as well, and hence $C \cap (-C) = \{o_V\}$.

Proof of (6.55) is indirekt, we suppose that $I_n \notin C$, and seek a contradiction. Lemma 1.3.5 implies the existence of an $A \in V$ with

$$\langle A, I_n\rangle > 0 \text{ and } \langle A, u \otimes u\rangle < 0 \ \text{ for } u \in U, \tag{6.58}$$

while $\operatorname{tr} A > 0$ and Remark 6.A.1 (ii) yield

$$\det(I_n + tA) > 1 \ \text{ for small } t > 0. \tag{6.59}$$

It follows from (6.58) that there exists a $\delta > 0$ and open neighbourhood $\mathcal{N} \subset S^{n-1}$ of U such that $\langle Au, u\rangle = \langle A, u \otimes u\rangle < -\delta$ for $u \in \mathcal{N}$, and hence if $t > 0$ is small, then $(I_n + tA)u \in B^n$ for $u \in \mathcal{N}$. As compact set $S^{n-1}\backslash\mathcal{N}$ lies in $\operatorname{int} K$, we deduce that $(I_n + tA)B^n \subset K$ for small $t > 0$, which is a contradiction by (6.59).

The bound $k \le d$ follows from Lemma 1.3.7.

Step 4 If (6.55) holds for an o-symmetric convex K with $B^n \subset K$, then B^n is the John ellipsoid.

Let $U = \partial K \cap S^{n-1}$. Our argument for Step 5 is indirekt, we suppose that $(I_n + A)B^n \subset K$ for symmetric $n \times n$ matrix A with $\det(I_n + A) > 1$. It follows that $\langle (I_n + A)u, u\rangle \le 1$ for $u \in U$, and $f'(t) > 0$ for the function $f(t) = \log\det(I_n + tA)$ as f is concave by (6.57) and $f(1) > f(0)$, and hence $\langle A, u \otimes u\rangle = \langle Au, u\rangle \le 0$ holds for $u \in U$ by $\langle (I_n + A)u, u\rangle \le 1$ and $\langle u, u\rangle \le 1$, while Remark 6.A.1 (ii) and $f'(t) > 0$ yield that $\langle A, I_n\rangle = \operatorname{tr} A = f'(t) > 0$. We deduce that $I_n \notin \operatorname{pos}\{u \otimes u : u \in U\} \subset \mathbb{R}^d$, that contradicts (6.55).

Step 5 If $K \subset \mathbb{R}^n$ is a convex body and B^n is the John ellipsoid, then (6.55) and 6.56 hold.

We consider the space $\mathbb{R}^{n^2} \oplus \mathbb{R}^n$ of space of pairs (A, v) where A is an $n \times n$ matrix and $v \in \mathbb{R}^n$, and for two such pairs (A, v) and (B, w), their scalar product is $\langle (A, v), (B, w)\rangle = \langle A, B\rangle + \langle v, w\rangle$. Let $W \subset \mathbb{R}^{n^2} \oplus \mathbb{R}^n$ be the linear subspace of pairs of the form (A,v) where A is a symmetric $n\times n$ matrix and $v\in\mathbb{R}^n$, and let $d_0=\dim W=\frac{n(n+3)}{2}$.

For $U=\partial K\cap S^{n-1}$, the set $\widetilde{U}_0=\{(u\otimes u,u):u\in U\}\subset W$ compact. Since $\langle (I_n,o),(u\otimes u,u)\rangle=\operatorname{tr} u\otimes u=1$ for any $u\in U$, Lemma 1.3.8 yields that $C_0=\operatorname{pos}\widetilde{U}\subset W$ is a closed convex cone. We have $C_0\cap(-C_0)=o_W$, because if $\sum_{i=1}^d\lambda_i(u_i\otimes u_i,u_i)=-\sum_{i=1}^d\tilde{\lambda}_i(\tilde{u}_i\otimes\tilde{u}_i,\tilde{u}_i)$ for $\lambda_i,\tilde{\lambda}_i\geq 0$ and $u_i,\tilde{u}_i\in U$, then $u\otimes u$ being positive semidefinite for $u\in U$ yields that each $\lambda_i=\tilde{\lambda}_i=0$.

The proof of the properties (6.55) and (6.56) is indirekt, we suppose that $(I_n,o)\notin C_0$, and seek a contradiction. Lemma 1.3.5 implies the existence of $(A,v)\in W$ for symmetric matrix A and $v\in\mathbb{R}^n$ with $\langle (A,v),(I_n,o)\rangle>0$ and $\langle (A,v),(u\otimes u,u)\rangle<0$ for $u\in U$, and hence

$$\det(I_n+tA)>1 \text{ for small } t>0 \text{ by Remark 6.A.1 (ii) and } \operatorname{tr} A=\langle (A,v),(I_n,o)\rangle>0; \tag{6.60}$$

$$\langle Au+v,u\rangle=\langle (A,v),(u\otimes u,u)\rangle<0 \text{ for } u\in U. \tag{6.61}$$

We deduce from (6.61) that there exists $\delta>0$ and open neighbourhood $\mathcal{N}\subset S^{n-1}$ of U such that $\langle Au+v,u\rangle<-\delta$ for $u\in\mathcal{N}$, and hence if $t>0$ is small, then $(I_n+tA)u+tv\in B^n$ for $u\in\mathcal{N}$. Therefore, $(I_n+tA)B^n+tv\subset K$ holds for small $t>0$, which contradicts (6.60).

Step 6 If (6.55) and (6.56) hold for a convex K with $B^n\subset K$, then B^n is the John ellipsoid.

This statement can be proved as in Step 4. ■

6.B Supplement: The rank one Geometric Brascamp-Lieb inequality

In this section, we sketch Barthe's argument - presented in [50] and that using optimal transport, - of the form (6.63) of the Brascampl-Lieb inequality due to Keith Ball .

Theorem 6.B.1. *If $u_1,\ldots,u_k\in S^{n-1}$ and $c_1,\ldots,c_k>0$ satisfy*

$$\sum_{i=1}^k c_iu_i\otimes u_i=I_n, \tag{6.62}$$

and $f_1,\ldots,f_k\in L^1(\mathbb{R})$ are non-negative, then

$$\int_{\mathbb{R}^n}\prod_{i=1}^k f_i(\langle x,u_i\rangle)^{c_i}\,dx\leq\prod_{i=1}^k\left(\int_{\mathbb{R}}f_i\right)^{c_i}. \tag{6.63}$$

Remark. For any $u_1,\ldots,u_k\in S^{n-1}$ and $c_1,\ldots,c_k>0$ satisfying (6.5), we have equality if we choose each f_i to be the same Gaussian probability density; for example, if

$f_i(t) = e^{-\pi t^2}$ for $i = 1, \dots, k$.

In the rest of the section, we assume that $u_1, \dots, u_k \in S^{n-1}$ and $c_1, \dots, c_k > 0$ satisfy (6.62), thus for any $z \in \mathbb{R}^n$, we have

$$z = \sum_{i=1}^{k} c_i \langle u_i, z\rangle u_i \text{ and } \|z\|^2 = \sum_{i=1}^{k} c_i \langle u_i, z\rangle^2.$$

Before proving the Brascampl-Lieb inequality, we verify two auxiliary estimates due to Keith Ball [34].

Lemma 6.B.2. *If* $t_1, \dots, t_k > 0$, *then*

$$\det\left(\sum_{i=1}^{k} t_i c_i u_i \otimes u_i\right) \geq \prod_{i=1}^{k} t_i^{c_i}.$$

Proof. Let $v_i = \sqrt{c_i}\, u_i$ for $i = 1, \dots, k$, and hence $\sum_{i=1}^k t_i c_i u_i \otimes u_i = \sum_{i=1}^k t_i v_i \otimes v_i$, $\langle v_i, v_i\rangle = c_i$ for $i = 1, \dots, k$ and $\sum_{i=1}^k v_i \otimes v_i = I_n$. In this argument, J is always an element of the family Θ_n^k of all n element subsets of $\{1, \dots, k\}$, and for $J = \{i_1, \dots, i_n\} \in \Theta_n^k$, we define

$$d_J = \det[v_{i_1}, \dots, v_{i_n}]^2 \text{ and } t_J = t_{i_1} \cdots t_{i_n}.$$

Applying the Cauchy-Binet formula to the $n \times k$ matrix $U = [\sqrt{t_1}\, v_1, \dots, \sqrt{t_k}\, v_k]$ yields

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) = \det\left(UU^\top\right) = \sum_{J \in \Theta_n^k} t_J d_J, \tag{6.64}$$

and hence $\sum_{J \in \Theta_n^k} d_J = \det\left(\sum_{i=1}^k v_i \otimes v_i\right) = 1$ follows from (6.64) with $t_i = 1$. Therefore, applying the AM-GM inequality in (6.64) leads to

$$\det\left(\sum_{i=1}^{k} t_i v_i \otimes v_i\right) \geq \prod_{J \in \Theta_n^k} t_J^{d_J}. \tag{6.65}$$

For a fixed $i \in \{1, \dots, k\}$, the factor t_i occurs exactly $\sum_{J, i \in J} d_J$ time in $\prod_J t_J^{d_J}$. Moreover, (6.64)) applied to the vectors $v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k$ implies

$$\begin{aligned}\sum_{J, i \in J} d_I &= \sum_J d_J - \sum_{J, i \notin J} d_J = 1 - \det\left(\sum_{j \neq i} v_j \otimes v_j\right)\\ &= 1 - \det\left(\mathrm{Id}_n - v_i \otimes v_i\right) = \langle v_i, v_i\rangle.\end{aligned}$$

Substituting this into (6.65) yields the lemma. ∎

Lemma 6.B.3. *If* $z = \sum_{i=1}^k c_i\theta_i u_i$ *for* $\theta_1, \dots, \theta_k \in \mathbb{R}$, *then*

$$\|z\|^2 \le \sum_{i=1}^k c_i\theta_i^2. \tag{6.66}$$

Proof. The condition $z = \sum_{i=1}^k c_i\theta_i u_i$ and the Cauchy-Schwarz inequality yield that

$$\|z\|^2 = \left\langle z, \sum_{i=1}^k c_i\theta_i u_i \right\rangle = \sum_{i=1}^k c_i\theta_i\langle z, u_i\rangle \le \sqrt{\sum_{i=1}^k c_i\theta_i^2}\sqrt{\sum_{i=1}^k c_i\langle u_i, z\rangle^2},$$

and hence the lemma follows from $\|z\|^2 = \sum_{i=1}^k c_i\langle u_i, z\rangle^2$. ■

Following Barthe [50], we prove the Brascamp-Lieb inequality (6.63) via optimal transport based on the Gaussian density $g(t) = e^{-\pi t^2}$. Approximating f_i first by piecewise linear functions with compact support, and then in turn these functions by continuous functions, we may assume that each f_i is a positive continuous probabiity density function. For $i = 1, \dots, k$, the transport map $T_i : \mathbb{R} \to \mathbb{R}$ is defined by

$$\int_{-\infty}^t f_i(s)\, ds = \int_{-\infty}^{T_i(t)} g(s)\, ds.$$

It follows that each T_i is differentiable, monotone inscreasing and

$$f_i(t) = g(T_i(t)) \cdot T_i'(t) \text{ holds for } t \in \mathbb{R}. \tag{6.67}$$

Now we consider the differentiable map $\Theta : \mathbb{R}^n \to \mathbb{R}^n$,

$$\Theta(x) := \sum_{i=1}^k c_i T_i(\langle u_i, x\rangle)\, u_i, \qquad x \in \mathbb{R}^n,$$

which satisfies

$$D\Theta(x) = \sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i.$$

We claim that Θ is injective; namely,

$$\Theta(x_2) \ne \Theta(x_1) \text{ for } x_2 \ne x_1.$$

To prove the claim, we observe that if $v \in \mathbb{R}^n\backslash\{o\}$ and $x \in \mathbb{R}^n$, then

$$\langle v, D\Theta(x)v\rangle = v^t \left(\sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, u_i u_i^t\right) v = \sum_{i=1}^k c_i T_i'(\langle u_i, x\rangle)\, \langle u_i, v\rangle^2 > 0,$$

thus $D\Theta$ is positive definite. We deduce that $\varphi'(t) > 0$ for $t \in [0, 1]$ for the function $\varphi(t) = \langle x_2 - x_1, \Theta((1-t)x_1 + tx_2)\rangle$, and hence $\langle x_2 - x_1, \Theta(x_2)\rangle = \varphi(1) > \varphi(0) = \langle x_2 - x_1, \Theta(x_1)\rangle$.

Proof of the Brascampl-Lieb inequality (6.63). Using first $f_i(t) = g(T_i(t)) \cdot T_i'(t)$ for $g(t) = e^{-\pi t^2}$, then Lemmas 6.B.2 and 6.B.3, and finally the injectivity of Θ, we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n} \prod_{i=1}^{k} f_i(\langle u_i, x\rangle)^{c_i}\, dx &= \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} g(T_i(\langle u_i, x\rangle))^{c_i}\right)\left(\prod_{i=1}^{k} T_i'(\langle u_i, x\rangle)^{c_i}\right) dx \\
&\le \int_{\mathbb{R}^n} \left(\prod_{i=1}^{k} e^{-\pi c_i T_i(\langle u_i, x\rangle)^2}\right) \det\left(\sum_{i=1}^{k} c_i T_i'(\langle u_i, x\rangle)\, u_i \otimes u_i\right) dx \\
&\le \int_{\mathbb{R}^n} e^{-\pi \|\Theta(x)\|^2} \det\left(D\Theta(x)\right)\, dx \\
&\le \int_{\mathbb{R}^n} e^{-\pi \|y\|^2}\, dy = 1.
\end{aligned}$$

■

6.C Supplement: Minimal surface area position - isotropic surface area measure

If $X = \mathbb{R}^n$ or $X = S^{n-1}$, we say that a measure on X is quasi isotropic (sometimes simply called isotropic) if the integral of the positive seminite rank one $n \times n$ matrix $u \mapsto u \otimes u$ is a multiple of I_n. In this section, we prove Petty's isotopicity condition in Petty [483] on when a convex body is in a minimal surface area position (see also Giannopoulos, Papadimitrakis [260], while the paper Giannopoulos, V. Milman [258] considers an extension for the mean projections defined in Theorem 7.1.1):

Theorem 6.C.1 (Petty). *For any convex body $K \subset \mathbb{R}^n$, $S(\Phi K)$ for $\Phi \in \mathrm{SL}(\mathbb{R}, n)$ attains its minimum at some $K_0 = \Phi_0 K$ for some $\Phi_0 \in \mathrm{SL}(\mathbb{R}, n)$ where K_0 is unique up to orthogonal transformations, and is characterized by the property that*

$$\int_{S^{n-1}} u \otimes u\, dS_{K_0} = \frac{S(K_0)}{n} \cdot I_n. \tag{6.68}$$

In order to prepare for the proof of Theorem 6.C.1, first we verify a statement showing that given the volume of a convex body, the surface area is large if the body is elongated.

Lemma 6.C.2. *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$S(K) \ge \alpha_n (\operatorname{diam} K)^{\frac{1}{n-1}} |K|^{\frac{n-2}{n-1}} \text{ for } \alpha_n > 0 \text{ depending on } n.$$

Proof. If $n = 2$, then $S(K) \ge 2\operatorname{diam} K$ as K contains a segment of length $\operatorname{diam} K$. If $n \ge 3$, then acccording to John's theorem Theorem 6.1.1, we may assume that $E \subset K \subset nE$ for a centered ellipsoid E. Let $a_1 \ge \ldots \ge a_n > 0$ be the semi axes of E, and hence E contains an $(n-1)$-dimensional ellipsoid of surface area $2\omega_{n-1}a_1 \dots a_{n-1}$, and K satisfies $\operatorname{diam} K \leq 2a_1$ and $|K| \leq n^n \omega_n a_1 \dots a_n$. Since $(a_2 \dots a_{n-1})^{\frac{1}{n-2}} \geq a_n$, we deduce that

$$S(K) \geq 2\omega_{n-1}a_1 \dots a_{n-1} \geq 2\omega_{n-1}a_1^{\frac{1}{n-1}}(a_1 \dots a_n)^{\frac{n-2}{n-1}} \geq \alpha_n \cdot (\operatorname{diam} K)^{\frac{1}{n-1}}|K|^{\frac{n-2}{n-1}}$$

for $\alpha_n = 2\omega_{n-1}(2n)^{-\frac{1}{n-1}}(n^n\omega_n)^{-\frac{n-2}{n-1}}$. ■

Next we need a formula for the surface area of a linear image.

Lemma 6.C.3. *If $K \subset \mathbb{R}^n$ is a convex body and $\Phi \in \mathrm{SL}(n)$, then*

$$S(\Phi^{-t}K) = \int_{S^{n-1}} \|\Phi(u)\| \, dS_K(u).$$

Proof. As in Definition 2.6.13, let $\widetilde{\Phi} : S^{n-1} \to S^{n-1}$ be defined by $\widetilde{\Phi}(u) = \frac{\Phi(u)}{\|\Phi(u)\|}$. As $V_{\Phi^{-t}K} = \widetilde{\Phi}_* V_K$ by the linear equivariance Proposition 2.6.15 of the cone volume measure $dV_K = \frac{1}{n} h_K \, dS_K$, and $h_{\Phi^{-t}K}(\widetilde{\Phi}_* u) = h_K(u)/\|\Phi(u)\|$ by Lemma 2.6.14, we deduce that

$$S(\Phi^{-t}K) = n\int_{S^{n-1}} \frac{1}{h_{\Phi^{-t}K}} \, dV_{\Phi^{-t}K} = n\int_{S^{n-1}} \frac{\|\Phi(u)\|}{h_K(u)} \, dV_K(u) = \int_{S^{n-1}} \|\Phi(u)\| \, dS_K(u).$$

■

We also need an algebraic condition when the surface area measure is quasi isotropic.

Lemma 6.C.4. *For any convex body $K \subset \mathbb{R}^n$,*

$$\int_{S^{n-1}} u \otimes u \, dS_K = \frac{S(K)}{n} \cdot I_n \tag{6.69}$$

if and only if any $n \times n$ matrix Ψ satisfies

$$\int_{S^{n-1}} \langle u, \Psi u\rangle \, dS_K(u) = \frac{\operatorname{tr}\Psi}{n} \cdot S(K_0). \tag{6.70}$$

Proof. We fix an orthonormal basis $e_1, \dots, e_n$ of $\mathbb{R}^n$. If (6.70) holds, then choosing Ψ in (6.71) in a way such that one entry is 1 and the rest is zero, we deduce that

$$\int_{S^{n-1}} \langle u, e_i\rangle\langle u, e_j\rangle \, dS_K(u) = \begin{cases} 0 & \text{if } i \neq j \\ \frac{S(K_0)}{n} & \text{if } i = j, \end{cases}$$

which is equivalent to (6.69)

On the other hand, if (6.69) holds, then the previous argument yields that (6.70) holds whenever one entry of Ψ is 1 and the rest is zero. Any $n \times n$ matrix is the linear combination of such matrices; therefore, we conclude (6.70) . ■

Proof of Theorem 6.C.1. The existence of a $\Phi_0 \in \mathrm{SL}(\mathbb{R}, n)$ such that $S(\Phi_0 K) \le S(\Phi K)$ for any $\Phi \in \mathrm{SL}(\mathbb{R}, n)$ follows from Lemma 6.C.2. Let $K_0 = \Phi_0 K$. To prove (6.68), we consider an $n \times n$ matrix Ψ, small $\varepsilon > 0$ and $\Omega = (I_n + \varepsilon \Psi)/\det(I_n + \varepsilon \Psi)^{\frac{1}{n}} \in \mathrm{SL}(\mathbb{R}, n)$, and hence $S(\Omega^{-t} K_0) \ge S(K_0)$ and Lemma 6.C.3 yield that

$$\int_{S^{n-1}} \|u + \varepsilon \Psi u\| \, dS_{K_0}(u) \ge \det(I_n + \varepsilon \Psi)^{\frac{1}{n}} S(K_0).$$

Letting $\varepsilon > 0$ tend to zero, we deduce that

$$\int_{S^{n-1}} \langle u, \Psi u \rangle \, dS_{K_0}(u) \ge \frac{\operatorname{tr} \Psi}{n} \cdot S(K_0), \tag{6.71}$$

and as (6.71) holds for $-\Psi$, as well, it follows that

$$\int_{S^{n-1}} \langle u, \Psi u \rangle \, dS_{K_0}(u) = \frac{\operatorname{tr} \Psi}{n} \cdot S(K_0). \tag{6.72}$$

We conlude (6.68) from (6.72) and Lemma 6.C.4.

Next we show that if S_K is quasi isotropic; namely, it satifies (6.68), then K is in minimal surface area position, and such position is unique up to orthogonal transformations. In order to verify $S(\Phi^{-t} K) \ge S(K)$ for $\Phi \in \mathrm{SL}(\mathbb{R}, n)$, using the polar decomposition, we write $\Phi = U\Omega$ for $U \in O(n)$ and positive semidefinite symmetric Ω with $\det \Omega = 1$, and hence Lemma 6.C.3, Lemma 6.C.4 and the the AM-GM inequality applied to the eigenvalues of Ω (cf. Lemma 3.5.2) yield that

$$\begin{aligned} S(\Phi^{-t} K) = S(U^{-t} \Omega^{-t} K) = S(\Omega^{-t} K) &= \int_{S^{n-1}} \|\Omega u\| \, dS_K(u) \ge \int_{S^{n-1}} \langle u, \Omega u \rangle \, dS_K(u) \\ &= \frac{\operatorname{tr} \Omega}{n} \cdot S(K) \ge (\det \Omega)^{\frac{1}{n}} \cdot S(K) = S(K). \end{aligned} \tag{6.73}$$

If $S(\Phi^{-t} K) = S(K)$, then $\frac{\operatorname{tr} \Omega}{n} = (\det \Omega)^{\frac{1}{n}}$ in (6.73), thus the equality conditions in the AM-GM inequality (or in Lemma 3.5.2) imply that $\Omega = I_n$. Therefore, $\Phi^{-t} = U^{-t} \in O(n)$. ■

6.D Supplement: Maximal area of inscribed polygons and the extremality of ellipses

We have seen in Macbeath' Theorem 6.8.1 that ellipsoids are the worst approximable convex bodies in terms of volume approximation by inscribed polytopes of given number of vertices. In this section, we provide the elegant argument due to Sas [512] verifying that ellipses are the only extremizers in the plane.

Theorem 6.D.1 (Sas). *If P_k is a polygon with at most k vertices of maximal area contained in a convex body $C \subset \mathbb{R}^2$, then*

$$|P_k| \geq |C| \cdot \frac{k}{2\pi} \sin \frac{2\pi}{k},$$

with equality if and only if C is an ellipsoid.

Proof. We may asssume that $(-1,0),(1,0) \in \partial C$ are the endpoints of a diameter of C, and hence $(x, f(x)),(x,-g(x)) \in \partial C$ for the concave functions $f, g : [-1,1] \to [0,\infty)$ where we may assume that $f(0) > 0$. It follows that

$$\begin{array}{rcll} f(x), g(x) & \leq & 2\sqrt{1-x} & \text{for } x \in [0,1]; \\ f(x), g(x) & \leq & 2\sqrt{x+1} & \text{for } x \in [-1,0]. \end{array} \tag{6.74}$$

Parametrize ∂C as $(x(t), y(t)) = (\cos t, \varphi(t)\sin t)$ for $t \in R$ where $\varphi(t)$ is 2π-periodic, is continuous on $(0,\pi)$ and $(\pi, 2\pi)$, and $\varphi \leq 2$ on $[0,2\pi]$ by (6.74), and hence $y(t) = f(\cos t)$ if $t \in [0,\pi]$ and $y(t) = -g(\cos t)$ if $t \in [\pi, 2\pi]$.

For $t \in \mathbb{R}$, let $Q_t = \operatorname{conv}\{p_i(t_i)\}_{i=0,\dots,k-1}$ where $p_i(t) = (x(t_i), y(t_i))$ and $t_i = t + \frac{i2\pi}{k}$, $i \in \mathbb{Z}$, therefore,

$$\begin{aligned} |Q_t| &= \frac{1}{2}\sum_{i=0}^{k-1} x(t_i)y(t_{i+1}) - x(t_{i+1})y(t_i) = \frac{1}{2}\sum_{i=0}^{k-1} \varphi(t_i)\sin t_i(\cos t_{i+1} - (\cos t_{i-1}) \\ &= \sin\frac{2\pi}{k}\sum_{i=0}^{k-1}\varphi\left(t + \frac{i2\pi}{k}\right)\sin^2\left(t + \frac{i2\pi}{k}\right). \end{aligned}$$

Using substitution $x = \cos t$, the mean value of $|Q_t|$ is

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} |Q_t|\,dt &= \frac{k}{2\pi}\sin\frac{2\pi}{k}\int_0^{2\pi}\varphi(t)\sin^2 t\,dt = \frac{k}{2\pi}\sin\frac{2\pi}{k}\left(\int_1^{-1} -f(x)\,dx + \int_{-1}^{1} g(x)\,dx\right) \\ &= \frac{k}{2\pi}\sin\frac{2\pi}{k}\cdot|C|, \end{aligned}$$

and hence there exists a $t \in [0, 2\pi]$ with $|Q_t| \geq \frac{k}{2\pi}\sin\frac{2\pi}{k}\cdot|C|$.

Equality in Theorem 6.D.1 yields that $|Q_t| = |P_k|$ holds for $t \in [0,2\pi]$, thus for any $t \in [0, 2\pi]$, the line through $p_0(t)$ and parallel to $p_1(t) - p_{-1}(t) \neq o$ is a supporting line to C. As $p_1(t) - p_{-1}(t)$ is continuous and C is convex, it follows that ∂C is C^1, and hence $y(t)$ differentiable on $(0,\pi)$ and on $(\pi, 2\pi)$, and satisfies

$$y'(t) = \frac{y(t+\frac{2\pi}{k}) - y(t-\frac{2\pi}{k})}{2\sin\frac{2\pi}{k}}. \tag{6.75}$$

We deduce from (6.75) that $\lim_{t\to 0} y'(t) = y(\frac{2\pi}{k}) - y(-\frac{2\pi}{k})$ and $\lim_{t\to\pi} y'(t) = y(\pi + \frac{2\pi}{k}) - y(\pi - \frac{2\pi}{k})$; therefore, $y(t)$ is continuous, 2π periodic and piecewise smooth, which in turn yields that it has a Fourier series

$$y(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt) \text{ and } y'(t) = \sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt).$$

It follows from (6.75) that

$$\sum_{n=1}^{\infty}(nb_n \cos nt - na_n \sin nt) = \sum_{n=1}^{\infty} \frac{\sin \frac{n2\pi}{k}}{\sin \frac{2\pi}{k}}(b_n \cos nt - a_n \sin nt).$$

Comparing coefficients leads to $a_n = b_n = 0$ for $n \geq 2$; or in other words, $y(t) = a_0 + a_1 \cos t + b_1 \sin t$. It follows that ∂C is parametrized as $(\cos t, a_0 + a_1 \cos t + b_1 \sin t)$, and hence C is an ellipse. ■

6.E Supplement: Equality in the Blaschke-Santaló inequality Theorem 6.5.2

The main goal of this section is to prove Theorem 6.E.2 on the one hand, and to show how it leads to the characterization of equality in the Blaschke-Santaló inequality Theorem 6.5.2. First, we present the simple proof of the Blaschke-Santaló inequality Theorem 6.5.2 for unconditional convex bodies in Keith Ball's PhD thesis [34]:

Proposition 6.E.1 (Blaschke-Santaló inequality if K unconditional). *If $K \subset \mathbb{R}^n$ is an uncoditional convex body, then $|K| \cdot |K^*| \leq |B^n|^2$, with equality if and only if K is a centered ellipsoid.*

Proof. We observe that the definition of a polar body implies that $K^{\frac{1}{2}} \cdot (K^*)^{\frac{1}{2}} = B^n$, and hence the Uhrin-Bollobas-Leader inequality (3.15) for coordinatewise product yields

$$|B^n| \geq |K|^{\frac{1}{2}}|K^*|^{\frac{1}{2}}. \tag{6.76}$$

For equality, if $e_1, \ldots, e_n$ are the orthonormal basis, then we may assume that $e_i \in \partial K$, $i = 1, \ldots, n$; as the volume product $|K| \cdot |K^*|$ is linear invariant, and hence $\varrho_{K^*}(e_i) = h_K(e_i)^{-1} = 1$ and $e_i \in \partial K^*$. Equality holds in Blaschke-Santaló inequality (6.76) if and only if equality holds in the Uhrin-Bollobas-Leader inequality (3.15); therefore, $K^* = \Phi K$ for positive definite diagonal Φ. Since $e_i \in \partial K \cap \partial K^*$ for $i = 1, \ldots, n$, it follows that $\Phi = I_n$, and hence $K = K^*$, which in turn yields

$$\varrho_K(u) = \varrho_{K^*}(u) = \frac{1}{h_K(u)} \text{ for } u \in S^{n-1}. \tag{6.77}$$

For the $v, w \in S^{n-1}$ with $\varrho_K(v) = \min_{u \in S^{n1}} \varrho_K(u)$ and $\varrho_K(w) = \max_{u \in S^{n1}} \varrho_K(u)$, we have $h_K(v) = \varrho_K(v)$ and $h_K(w) = \varrho_K(w)$ as $\varrho_K(v)B^n \subset K \subset \varrho_K(w)B^n$. We deduce from (6.77) that $\varrho_K(v) = \varrho_K(w) = 1$, and hence $K = B^n$. ■

According to Section 6.4, for any centered convex body $K \subset \mathbb{R}^n$ ($\sigma_K = o$), there exists $\Phi \in \mathrm{SL}(n)$ such that ΦK is in quasi-isotropic position; namely,

$$\int_{\Phi K} \langle u_n, x\rangle^2 \, dx = L_K^2 |K|^{\frac{n+2}{n}} \quad \text{for } u \in S^{n-1}$$

for some $L_K > 0$. Here $L_{\Phi K} = L_K$ for $\Phi \in \mathrm{GL}(n)$, and Proposition 6.4.6 says that

$$L_K \geq L_{B^n}. \tag{6.78}$$

The following statement can be applied to characterize equality cases of certain affine invariant inequality:

Proposition 6.E.2. *If $K \subset \mathbb{R}^n$, $n \geq 2$, is in quasi-isotropic position and is not an ellipsoid, then there exists an o-symmetric convex body $\widetilde{K}$ with axial rotational symmetry that is not an ellipsoid, and obtained from K by applying Steiner symmerization and taking limit.*

Proof. We may assume that $|K| = |B^n|$.
Case 1 K is o-symmetric.

Let r be maximal with the property $rB^n \subset K$. Since K is not a ball, we have $r < 1$. For a $u \in S^{n-1}$ with $ru \in \partial K$, let $C = \Theta_{\mathbb{R}u} K$.

We suppose C is an ellipsoid, and seek a contradiction. As if $t \in [-r, r]$, then

$$\mathcal{H}^{n-1}(K \cap (tu + u^\perp)) = \mathcal{H}^{n-1}(C \cap (tu_n + u_n^\perp)) = \left(1 - \frac{t^2}{r^2}\right)^{\frac{n-1}{2}} \omega_{n-1},$$

and hence (6.78) and K being in in quasi-isotropic position yield that

$$\begin{aligned} L_{B^n}^2 |B^n|^{\frac{n+2}{n}} \leq & L_K^2 |K|^{\frac{n+2}{n}} = \int_K \langle u_n, x\rangle^2 \, dx = \int_C \langle u, x\rangle^2 \, dx \\ = & r \int_{B^n} \langle u, x\rangle^2 \, dx = r L_{B^n}^2 |B^n|^{\frac{n+2}{n}}, \end{aligned}$$

which is a contradiction with $r < 1$.
Case 2 K is not o-symmetric.

There exists $v \in S^{n-1}$ such that $h_K(v) \neq h_K(-v)$. It follows that the Schwarz rounding $\widetilde{K} = \Theta_{\mathbb{R}v} K$ is not o-symmetric, but has rotational symmetry aroun $\mathbb{R}v$ and satisfies $\sigma_{\widetilde{K}} = o$ and $|\widetilde{K}| = |B^n|$ by Corollary 1.10.14, and $|\widetilde{K}^*| \geq |K^*|$ by Theorem 6.5.4. Since $h_{\widetilde{K}}(v) = h_K(v)$ and $h_{\widetilde{K}}(-v) = h_K(-v)$, we deduce that $\widetilde{K}$ not o-symmetric. Combining this fact with $\sigma_{\widetilde{K}} = o$ implies that $\widetilde{K}$ is not an ellipsoid.

We choose a linear 2-subspace L with $v \in L$, and hence $\widetilde{K} \cap L$ is not an ellipse. It follows from Sas's Theorem 6.D.1 that there exists $w_1, w_2, w_3, w_4 \in \partial\widetilde{K} \cap L$ such that $A = V_2(\mathrm{conv}\{w_1, w_2, w_3, w_4\}) > \frac{2}{\pi} V_2(\widetilde{K} \cap L)$ where $\frac{2}{\pi} V_2(\widetilde{K} \cap L)$ is the maximal area of a quadrilateral contained in an ellips of area $V_2(\widetilde{K} \cap L)$. Now $\bar{K} = \Theta_{u_2^\perp} \Theta_{u_1^\perp} \widetilde{K}$ where

$u_1 = (w_1 - w_3)/\|w_1 - w_3\|$ and $u_2 \in S^{n-1} \cap \cap u_1^{\perp}$ satisfies that $\bar{K}$ is unconditional for an orthonormal basis $u_1, u_2, \ldots$ of $\mathbb{R}^n$ where $u_i \in L^{\perp}$ for $i > 2$, and $\bar{K}$ not an ellipsoid because $\bar{K} \cap L$ contains a quadlilateral of area A that is larger than $\frac{2}{\pi} V_2(\widetilde{K} \cap L)$. As $\bar{K}$ is o symmetric and not and ellipsoid, we now apply Case 1. ■

Let us restate and characterize the equality case of the Blaschke-Santaló inequality Theorem 6.5.2.

Theorem 6.E.3 (The Blaschke-Santaló inequality with equality). *If the origin is the Santaló point or the centroid of a convex body $K \subset \mathbb{R}^n$, then*

$$|K| \cdot |K^*| \leq |B^n|^2, \tag{6.79}$$

with equality if and only if K is an ellipsoid centered at o.

Proof. According to Theorem 6.5.4, Steiner symmetrization does not decrease the volume of the polar of a centered convex body. Since iterated Steiner symmetrisations applied to a centered convex body K may lead to a centered ball of the same volume according to Theorem 1.10.7, we deduce (6.79).

If K is centered but not an ellipsoid, then we may assume that K is in quasi-isotropic position and $|K| = |B^n|$. It follows from Proposition 6.E.2 that there exists an o-symmetric convex body $\widetilde{K}$ with axial rotational symmetry that is not an ellipsoid, and obtained from K by applying Steiner symmerization and taking limit. Therefore $|\widetilde{K}^*| \geq |K^*|$ by Theorem 6.5.4, and $|\widetilde{K}^*| < |B^n|$ by Proposition 6.E.1, and hence $|K| \cdot |K^*| < |B^n|^2$. ■

Chapter 7

Steiner formula and Mixed volumes

This chapter discusses one of the most striking features of convex geometry - that the volume of a non-negative linear combination of given compact convex sets is polynomial in the coefficents as Steiner in a special case around 1840, and Minkowski in general around 1900 proved it. In turn, many of the coefficents of these polynomiasl - the so-called mixed volumes - have deep geometric meaning (for example, mean projections, combinatorial quantities related to poset extensions, intersection numbers of algebraic hypersurfaces, etc), and satisfy some fundamental inequalities, like Minkowski's inequality, and its generalization, the Aleksandrov-Fenchel Inequality.

7.1 Steiner formula, Intrinsic volumes and Mean projections

For a compact convex set $K \subset \mathbb{R}^n$ (which we always assume to be non-empty in this book), its parallel domain of radius $r > 0$ (the set of points of distance at most r from K) is $K + rB^n$. One of the early results of convex geometry that as Steiner [541] observed already around 1840, the volume of $K + rB^n$ is a polynomial in r with coefficients with deep geometric meaning. To prove this statement, we need some properties of cones and polytopes discussed in Section 1.4. We recall that a polyhedral cone C is the intersection of finitely many half spaces that have the orgin on their boundary; in particular, $\lambda x \in C$ for any $x \in C$ and $\lambda \geq 0$. The normalized angle of C is (cf. (1.5))

$$\beta(C) = \frac{\mathcal{H}^m(C \cap B^n)}{\mathcal{H}^m(B^m)} = \frac{\mathcal{H}^{m-1}(C \cap S^{n-1})}{\mathcal{H}^{m-1}(S^{n-1} \cap \mathrm{lin}C)} = \int_C e^{-\pi \|x\|^2}\, d\mathcal{H}^m(x).$$

In addition, a polytope $P \subset \mathbb{R}^n$ is the convex hull finitely many vertices, which has finitely many faces (intersection with a supporting hyperplane). According to Lemma 1.4.10, we can assign a polyhedral cone N_F (the so-called normal cone) of dimension $n - d$ to a face F of P of dimension d in a way such that $z \in (\mathrm{relint}\, F) + N_P(F)$ if and only if $\Pi_P(z) \in \mathrm{relint}\, F$ for the closest point $\Pi_P(z)$ of P to z. In addition, writing $\sqcup$ to denote disjoint union, Lemma 1.4.10 says that if $\varrho > 0$, then

$$P + \varrho B^n = \mathrm{int}\, P \sqcup \bigsqcup_{F \text{ face of } P} ((N_F \cap \varrho B^n) + \mathrm{relint}\, F)\,; \tag{7.1}$$

$$|(N_F \cap \varrho B^n) + \mathrm{relint}\, F| = \varrho^{n-d}\omega_{n-d}\beta(N_F)\,\mathcal{H}^d(F) \quad \text{if } d = \mathrm{dim}F. \tag{7.2}$$

Theorem 7.1.1 (Steiner formula). *If $K \subset \mathbb{R}^n$ is a convex convex compact set and $r \geq 0$, then there exists unique $V_i(K) \in \mathbb{R}$ called intrinsic i-volume for $i = 0, \dots, n$ such that*

$$|K + rB^n| = \sum_{i=0}^{n} V_i(K)\omega_{n-i}r^{n-i}. \tag{7.3}$$

In addition, these intrinsic i volumes satisfy the following properties:

(i) Isometry Invariance: $V_i(\Phi K + z) = V_i(K)$ *for* $\Phi \in O(n)$, $z \in \mathbb{R}^n$ *and* $i = 0, \dots, n$;

(ii) $V_n(K) = |K|$, $V_{n-1} = \frac{1}{2} S(K)$, $V_0(K) = 1$;

(iii) $V_i(K) > 0$ *if* $i \leq \dim K$, *and* $V_i(K) = 0$ *if* $i > \dim K$;

(iv) Dimension Invariance: *If $K \subset L$ for a linear d-subspace $L \subset \mathbb{R}^n$, then $V_i(K)$ with respect to L and $\mathbb{R}^n$ coincide. In particular,* $V_i(K) = \mathcal{H}^i(K)$ *if* $\dim K = i$;

(v) $V_i(K)$ *is continuous in K for* $i = 0, \dots, n$;

(vi) If P is a polytope, $\dim P \geq i$ *and $\mathcal{F}^i(P)$ denotes the family of i-dimensional faces of P, then*

$$V_i(P) = \sum_{F \in \mathcal{F}^i(P)} \mathcal{H}^i(F) \cdot \beta(N_F(P)); \tag{7.4}$$

(vii) $V_i(B^n) = \frac{\binom{n}{i}\omega_n}{\omega_{n-i}}$ *for* $i = 0, \dots, n$;

(viii) Mean projections (Kubota formula): *If $i = 1, \dots, n-1$, $\mathcal{L}^{i,n}$ is the space of linear i-planes (Grassmanian), and $\varrho_{i,n}$ is the Haar probability measure on $\mathcal{L}^{i,n}$ (invariant under $O(n)$), then*

$$V_i(K) = \frac{\binom{n}{i}\omega_n}{\omega_i\omega_{n-i}} \cdot \int_{\mathcal{L}^{i,n}} \mathcal{H}^i(K|L)\, d\varrho_{i,n}(L); \tag{7.5}$$

(ix) $V_i(K) \leq V_i(C)$ *if $K \subset C$ for compact convex sets $K, C \subset \mathbb{R}^n$, and even* $V_i(K) < V_i(C)$ *if* $\dim C \geq i$ *and* $K \neq C$.

Remark. (Mean curvatures and mean width)

Mean curvature: If ∂K is C^2_+ and $i = 1, \dots, n-1$, then

$$V_i(K) = \frac{1}{(n-i)\omega_{n-i}} \int_{\partial K} \sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))\, dx$$

where $\sigma_{n-1-i}(\kappa_1(x), \dots, \kappa_{n-1}(x))$ is the $(n-1-i)th$ symmetric function of the principal curvatures at an $x \in \partial K$ (see (8.27) in Theorem 8.3.4).

Mean width: If $K \subset \mathbb{R}^n$ is compact convex, then the width in the direction $u \in S^{n-1}$ is $\mathcal{H}^1(K|(\mathbb{R}u)) = h_K(u) + h_K(-u)$, and hence the Kubota formula (7.5) yields

$$V_1(K) = \frac{1}{2\omega_{n-1}} \int_{S^{n-1}} h_K(u) + h_K(-u)\, du = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} h_K\, d\mathcal{H}^{n-1}. \tag{7.6}$$

In particular, if $P = \mathrm{conv}\{v_1, \dots, v_m\} \subset \mathbb{R}^n$ is a polytope (for example, $v_1, \dots, v_m$ are the vertices), then

$$V_1(P) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \max_{i=1,\dots,k} \langle v_i, u \rangle \, du. \tag{7.7}$$

Proof of Theorem 7.1.1. First we observe that if C is any family of compact convex sets such that the Steiner formula (7.3) holds for any $K \in C$, then using Vandermonde matrix shows that there exist $\alpha_{n,i,j} \in \mathbb{R}$ depending on $n, i, j \in \mathbb{N}$ with $V_i(K) = \sum_{j=0}^n \alpha_{n,i,j} |K + jB^n|$ for any $K \in C$; therefore, $V_i(K)$ is unique and continuous for $K \in C$ (see Lemma 1.7.4 for the continuity of the volume).

First we prove (7.3) and (i)-(vi) for a polytope $P \subset \mathbb{R}^n$. For $K = P$, the Steiner formula (7.3) with coefficients as in (7.4) follows from (7.1) and (7.2), proving (i)-(vi) where we note that for $d = \dim P$, if $d < n$, then $\beta(N_P(P)) = 1$, and if $d = n$ and $F_1, \dots, F_k$ are the facets, then $\beta(N_P(F_i)) = \frac{1}{2}$.

If $P \subset RB^n$ for a polytope P and $R > 0$, then $|P + B^n| \le (R+1)^n B^n$, and hence the Steiner formula (7.3) yields that $V_i(P) \le \frac{(R+1)^n \omega_n}{\omega_{n-i}}$ for $i = 1, \dots, n$. We deduce *via* approximating by polytopes (*cf* (1.13)) and the continuity of the volume that (7.3) and the properties (i)-(v) hold for any compact convex set $K \subset \mathbb{R}^n$.

Next, $V_i(B^n) = \frac{\binom{n}{i}\omega_n}{\omega_{n-i}}$ for $i = 0, \dots, n$ in (vii) follows from (7.3).

Kubota formula (7.5) in (viii) is proved by induction on $n \ge 2$. If $n = 2$, or $n \ge 3$ and $i = n - 1$, then (7.5) is just the Cauchy formula (2.9), therefore, we may assume that $i \le n - 2$. Derivating the Steiner formula (7.3) with respect to r (cf. (2.7)) shows that

$$S(K + rB^n) = \sum_{i=0}^{n-1} (n-i) V_i(K) \omega_{n-i} r^{n-1-i}.$$

We deduce from the Cauchy formula (2.9) and Lemma 2.3.7 and by induction that

$$\begin{aligned}\sum_{i=0}^{n-1} (n-i) V_i(K) \omega_{n-i} r^{n-1-i} &= S(K + rB^n) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \mathcal{H}^{n-1}((K + rB^n)|u^\perp) \, du \\ &= \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \sum_{i=0}^{n-1} V_i(K|u^\perp) \omega_{n-1-i} r^{n-1-i} \, du.\end{aligned}$$

Therefore, equating coefficients of r^{n-1-i} for $i = 1, \dots, n-2$ leads to the formula

$$V_i(K) = \frac{\omega_{n-1-i}}{(n-i)\omega_{n-i}\omega_{n-1}} \cdot \int_{S^{n-1}} V_i(K|u^\perp) \, du.$$

Applying the Kubota formula in each $u^\perp$, $u \in S^{n-1}$ by induction, and using the uniqueness of the $O(n)$ invariant Haar probability measure on $\mathcal{L}^{i,n}$ yields that there exists $c_{n,i} > 0$ depending on n and i with

$$V_i(K) = c_{n,i} \cdot \int_{\mathcal{L}^{i,n}} \mathcal{H}^i(K|L) \, d\varrho_{i,n}(L). \tag{7.8}$$

Substituting $K = B^n$ in (7.8) implies that $c_{n,i} = \frac{\binom{n}{i}\omega_n}{\omega_i \omega_{n-i}}$. Monotonicity property (ix) follows from the Kubota formula (7.5). ■

7.2 Isoperimetric inequality for Mean Projections

According to the Isoperimetric Inequality for convex bodies (see Section 2.4), given the volume, the surface area is minimized by the ball. Here we extend this property to any Mean Projection; namely, to any intrinsic volume $V_i(K)$ for $i = 1, \ldots, n-1$.

Theorem 7.2.1 (Isoperimetric inequality for Mean Projections). *If $K \in \mathcal{K}^n$, $n \geq 2$ with $|K| = |rB^n|$ for $r > 0$ and $i = 1, \ldots, n-1$, then $V_i(K) \geq V_i(rB^n)$ with equality if and only if K is a ball. In particular,*

$$V_i(K) \geq c_{i,n}|K|^{\frac{i}{n}}$$

for $c_{i,n} = \binom{n}{i}\omega_n^{\frac{n-i}{n}}/\omega_{n-i}$ with equality if and only if K is a ball.

The main method to verify Theorem 7.2.1 is Steiner symmetrization originally designed to prove the Isoperimetric inequality, and applied by Hadwiger [295] in our context. We recall that if $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then the Steiner symmetrial $\Theta_{u^\perp}K$ (cf. Definition 1.10.1) of K is

$$\Theta_{u^\perp}K = \left\{x + \frac{t-s}{2} \cdot u \,:\, x \in K|u^\perp \,\&\, x + tu \in K \,\&\, x + su \in K\right\}.$$

Proposition 7.2.2. *If $K \subset \mathbb{R}^n$ convex body, $n \geq 2$, $i = 1, \ldots, n-1$ and $u \in S^{n-1}$, then*

$$V_i(\Theta_{u^\perp}K) \leq V_i(K), \tag{7.9}$$

with equality if and only if K has a hyperplane of symmetry parallel to $u^\perp$.

Proof. Let $K_0 = \Theta_{u^\perp}K$ be the sSteiner symmetrial and $\widetilde{K} = \xi_{u^\perp}K = \{x - tu \,:\, x \in K|u^\perp \,\&\, x + tu \in K\}$ be the reflected image over $u^\perp$.

According to the Kubota formula (7.5), to prove (7.9), it is enough to verify that if $L \in \mathcal{L}^{i,n}$ such that $L \not\subset u^\perp$, then

$$V_i(K_0|L) \leq \frac{1}{2}\Big(V_i(K|L) + V_i(\widetilde{K}|L)\Big) \tag{7.10}$$

Writing $L_0 = L \cap u^\perp$ and $X' = X|L$ for any $X \subset \mathbb{R}^n$, (7.10) yields for any line $\ell = z + \mathbb{R}u'$, $z \in K_0|L_0 = K|L_0$, we claim that

$$V_1(\ell \cap K_0') \leq \frac{1}{2}\left(V_1(\ell \cap K') + V_1(\ell \cap \widetilde{K}')\right). \tag{7.11}$$

To prove (7.11), we consider $v = u'/\|u'\|$, and observe that there exist $p_1, p_2 \in \partial K_0$ with $p_1', p_2' \in \ell$ and $V_1(\ell \cap K_0') = \langle v, p_2' - p_1' \rangle$. Let $p_j = a_j + \frac{1}{2}(t_j - s_j)u$ for $a_j + t_j u, a_j + s_j u \in K$ where $a_j \in K|u^\perp$, $t_j, s_j \in \mathbb{R}$, $j = 1, 2$, and hence $a_1', a_2' \in \ell$ and

$$
\begin{aligned}
V_1(\ell \cap K_0') &= \left\langle v, a_2' - a_1' + \tfrac{1}{2}(t_2 - s_2 - t_1 + s_1)u' \right\rangle ;\\
V_1(\ell \cap K') &= \max\{\langle v, x - y \rangle : x, y \in \ell \cap K'\} \geq \langle v, a_2' + t_2 u' - (a_1' + t_1 u') \rangle;\\
V_1(\ell \cap \widetilde{K}') &= \max\{\langle v, x - y \rangle : x, y \in \ell \cap \widetilde{K}'\} \geq \langle v, a_2' - s_2 u' - (a_1' - s_1 u') \rangle.
\end{aligned}
$$

These last estimates yield the claim (7.11), which then implies (7.10), and in turn (7.9).

Equality in (7.9) yields that for $L \in \mathcal{L}^{i,n}$ such that $L \not\subset u^\perp$, and any line $\ell \subset L$ orthogonal to $L_0 = L \cap u^\perp$ and intersecting $K|L$ (and hence $u \in \ell + L^\perp$) and any $z \in K|L_0$, using the notation as above, we have

- $a_2 + t_2 u, a_1 + t_1 u \in \mathrm{relbd}(K \cap (\ell + L^\perp))$, and a tangent hyperplane in $\ell + L^\perp$ at $a_2 + t_2 u$ and at $a_1 + t_1 u$ is parallel to $L^\perp$;
- $a_2 - s_2 u, a_1 - s_1 u \in \mathrm{relbd}(\widetilde{K} \cap (\ell + L^\perp))$, and a tangent hyperplane in $\ell + L^\perp$ at $a_2 - s_2 u$ and at $a_1 - s_1 u$ is parallel to $L^\perp$.

Therefore, fixing an affine $(n-i+1)$-subspace A with $A \cap \mathrm{int}K \neq \emptyset$ and parallel to u, if $L \in \mathcal{L}^{i,n}$, $L \not\subset u^\perp$, such that $L^\perp$ is parallel to A, and $a + tu, a + su \in \mathrm{relbd}(K \cap A)$ for $t > s$ and $a \in K|L$ satisfies that a tangent hyperplane in A at $a + tu$ is parallel to $L^\perp$, then a tangent hyperplane in A at $a + su$ is parallel to $\xi_{u^\perp} L^\perp$. It follows that if $f : (K \cap A)|u^\perp \to \mathbb{R}$ is the convex function and $g : (K \cap A)|u^\perp \to \mathbb{R}$ is the concave function satisfying that

$$
K \cap A = \left\{ x + t \cdot u : x \in (K \cap A)|u^\perp \text{ and } f(x) \leq t \leq g(x) \right\},
$$

then $-Df(x) = Dg(x)$ for $\mathcal{H}^i$ a.e $x \in (K \cap A)|u^\perp$. Thus $f(x) + g(x) = c$ for $x \in (K \cap A)|u^\perp$ where $c \in \mathbb{R}$ is a a constant, and hence $g(x) - \frac{c}{2} = \frac{c}{2} - f(x)$ for $x \in (K \cap A)|u^\perp$. We conclude that $A \cap K$ is symmetric through the hyperplane $u^\perp + \frac{c}{2}\, u$ (where $c \in \mathbb{R}$ depends on A).

Fix $a \in \mathrm{int}K|u^\perp$, and let $a + tu, a + su \in \partial K$ for $t > s$. The previous argument yields that $A \cap K$ is symmetric through the hyperplane $u^\perp + \frac{t+s}{2}\, u$ for any $(n-i+1)$-subspace A with $a + tu, a + su \in A$; therefore, K is symmetric through the hyperplane $u^\perp + \frac{t+s}{2}\, u$. ■

Proof of Theorem 7.2.1. Proposition 7.2.2 and the use of iterated Steiner symmetrisations leading to a ball (cf. Lemma 1.10.4) yield (7.9). If $V_i(K) = V_i(rB^n)$, then we may asume that the centroid of K is the origin o. It follows Proposition 7.2.2 that $u^\perp$ is a hyperplane of symmetry for each $u \in S^{n-1}$, and hence $K = rB^n$. ■

7.3 Mixed volumes

We have already seen in the case of the Steiner formula (7.3) that a non-negative linear combination of a compact convex set and a ball is a polynomial of degree n in the coefficents. Following Minkowski's ideas in [464, 465], we extend this result to non-negative linear combination of any number of compact convex sets. For the notion of the surface area measure S_K on S^{n-1} of a compact convex set $K \subset \mathbb{R}^n$, see Section 2.5.

Theorem 7.3.1 (Mixed volumes). *If $K_1, \ldots, K_m, C_1, \ldots, C_l \subset \mathbb{R}^n$ are compact convex sets, then*

$$\left|\sum_{i=1}^{m} \lambda_i K_i\right| = \sum_{\substack{\alpha_1+\ldots+\alpha_m=n\\ \alpha_1,\ldots,\alpha_m\in\mathbb{N}}} \frac{n!}{\alpha_1! \cdot \ldots \cdot \alpha_m!} \cdot V(K_1, \alpha_1; \ldots; K_m, \alpha_m) \cdot \prod_{i=1}^{m} \lambda_i^{\alpha_i} \tag{7.12}$$

is a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_m \geq 0$.

(i) $V(K_1, \alpha_1; \ldots; K_m, \alpha_m) \geq 0$, it is uniquely determined, has the natural symmetry, and depends only on the positive α_i. We set $V(K_1, \ldots, K_n) = V(K_1, 1; \ldots; K_n, 1)$.

(ii) $\displaystyle\sum_{j=1}^{l} \varrho_j V(K_1, \ldots, K_{n-1}, C_j) = V\left(K_1, \ldots, K_{n-1}, \sum_{j=1}^{l} \varrho_j C_j\right)$ *for $\varrho_j \geq 0$, $j = 1, \ldots, l$.*

(iii) $\displaystyle n!V(K_1, \ldots, K_n) = \sum_{i=1}^{n} (-1)^{n-i} \sum_{1\leq j_1<\ldots<j_i\leq n} |K_{j_1} + \ldots + K_{j_i}|$*; or in other words,*

$$n!V(K_1, \ldots, K_n) = \left|\sum_{i=1}^{n} K_i\right| - \sum_{j=1}^{n} \left|\sum_{i\neq j} K_i\right| + \ldots.$$

(iv) $V(K, \ldots, K) = |K|$.

(v) $V(K_1 + z_1, \ldots, K_n + z_n) = V(K_1, \ldots, K_n)$ for $z_i \in \mathbb{R}^n$, $i = 1, \ldots, n$.

(vi) $V(\Phi K_1, \ldots, \Phi K_n) = V(K_1, \ldots, K_n)$ for $\Phi \in \mathrm{SL}(n)$.

(vii) Setting $V(K_1, n - i; K_2, i) = V(K_1, K_2; i)$, we have

$$|\lambda_1 K_1 + \lambda_2 K_2| = \sum_{i=0}^{n} \binom{n}{i} V(K_1, K_2; i)\lambda_1^{n-i}\lambda_2^{i}. \tag{7.13}$$

(viii) $V(K_1, \ldots, K_n)$ is continuous in $K_1, \ldots, K_n$.

(ix) If $K \subset \mathbb{R}^n$ convex body and $C \subset \mathbb{R}^n$ is a compact convex set, then

$$V(K, C; 1) = \frac{1}{n}\int_{S^{n-1}} h_C \, dS_K = \frac{1}{n}\int_{\partial' K} h_C \circ \nu_K \, d\mathcal{H}^{n-1} \tag{7.14}$$

where the first inequality holds also if K is a compact convex set.

(x) $V(C_1, \ldots, C_n) \leq V(K_1, \ldots, K_n)$ if $C_i \subset K_i$.

Let us show that some funcdamental geometric quantities, like intrinsic volumes and the area of an orthonal projection, occur as mixed volumes.

Example 7.3.2.

Intrinsic volumes: For a compact convex set $K \in \mathcal{K}^n$, comparing the Steiner formula (7.3) and (7.13) shows that

$$S(K) = nV(K, B^n; 1); \tag{7.15}$$

$$V_i(K) = \frac{\binom{n}{i}}{\omega_{n-i}} V(B^n, K; i) \quad \text{for } i = 0, \ldots, n. \tag{7.16}$$

Orthogonal projection: For a compact convex set $K \in \mathcal{K}^n$, $x \in \mathbb{R}^n \backslash \{o\}$ and segment $s = \mathrm{conv}\{0, x\}$, (7.13) and calculating $|K + \lambda s|$ using integration over $x^\perp$ and the Fubini theorem yield that

$$nV(K, s; 1) = \|x\| \cdot \mathcal{H}^{n-1}(K|x^\perp) = \mathcal{H}^1(s) \cdot \mathcal{H}^{n-1}(K|x^\perp). \tag{7.17}$$

Let us review various properties of convex bodies that we need to prove Theorem 7.3.1. If $K \subset \mathbb{R}^n$ convex body and $C \subset \mathbb{R}^n$ is a compact convex set, then Proposition 2.5.9 yields

$$\lim_{r \to 0^+} \frac{|K + rC| - |K|}{r} = \int_{S^{n-1}} h_C \, dS_K = \int_{\partial K} h_C \circ \nu_K \, d\mathcal{H}^{n-1}, \tag{7.18}$$

where the first inequality holds also if K is a compact convex set. For an n-polytope $P \subset R^n$ and $u \in S^{n-1}$, u is an exterior normal the face $F_P(u) = \{x \in P : \langle x, u \rangle = h_P(u)\}$ of P (see Section 1.4 for properties of polytopes). We observe that if $P = \sum_{i=1}^m \lambda_i P_i$ for polytopes $P_1, \ldots, P_m \subset \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \geq 0$, then

$$F_P(u) = \sum_{i=1}^m \lambda_i F_{P_i}(u). \tag{7.19}$$

In addition, if $F_1, , \ldots, F_k$ are the facets of P with exterior unit vectors $u_1, \ldots, u_k$, then (2.4) says

$$|P| = \frac{1}{n} \sum_{i=1}^k h_P(u_i)\, |F_i|. \tag{7.20}$$

We need the fact that the (discrete) support of the surface area measure of positive linear combination of polytopes does not depend on the coefficients. This property follows from Lemma 1.6.9 about the normal cones of a Minkowski sum of polytopes, but we also provide a direct argument.

Lemma 7.3.3. *If $P_1, \ldots, P_n \subset \mathbb{R}^n$ are n-dimensional polytopes and $\lambda_1, \ldots, \lambda_m > 0$, then* $\mathrm{supp}\, S_{\sum_{i=1}^m P_i} = \mathrm{supp}\, S_{\sum_{i=1}^m \lambda_i P_i}$.

Proof. If $0 < a \le \lambda_1, \dots, \lambda_m \le b$ and $u \in S^{n-1}$, then

$$a^{n-1}\mathcal{H}^{n-1}\left(\sum_{i=1}^{m} F_{P_i}(u)\right) \le \mathcal{H}^{n-1}\left(\sum_{i=1}^{m} \lambda_i F_{P_i}(u)\right) \le b^{n-1}\mathcal{H}^{n-1}\left(\sum_{i=1}^{m} F_{P_i}(u)\right).$$

■

Proof of Theorem 7.3.1. First we consider the case of n-dimensional polytopes, and then verify the general case of compact convex sets *via* polytopal approximation.

Case 1 $K_i = P_i$ and $C_j = Q_j$ are n-dimensional polytopes We may assume $o \in \operatorname{int} P_i, \operatorname{int} Q_j$, and hence $h_{P_i}(u) > 0$, $h_{Q_j}(u) > 0$ for $u \in S^{n-1}$.

We prove Case 1 by induction on $n \ge 1$ where the case $n = 1$ is trivial as $V(s, 1) = \mathcal{H}^1(s)$ for a segment s. If $n \ge 2$, $u \in S^{n-1}$, $\alpha_1, \dots, \alpha_m \in \mathbb{N}$ with $\sum_{i=1}^{m} \alpha_i = n - 1$ and $F_i \subset u^\perp + t_i u$ are compact convex for $t_i \in \mathbb{R}$, $i = 1, \dots, m$, then let $v(F_1, \alpha_1; \dots; F_m, \alpha_m)$ be the $(n-1)$-dimensional mixed-volume.

For $P = \sum_{i=1}^{m} \lambda_i P_i$, $\lambda_i > 0$, and $\mathcal{U} = \operatorname{supp} S_{P_1 + \dots + P_m}$, (7.19), (7.20) and the induction hypothesis imply

$$\begin{aligned}
|P| =& \frac{1}{n}\sum_{u\in\mathcal{U}} h_P(u) F_P(u) = \frac{1}{n}\sum_{u\in\mathcal{U}}\left(\sum_{u\in\mathcal{U}} \lambda_i h_{P_i}(u)\right)\cdot \mathcal{H}^{n-1}\left(\sum_{i=1}^{m}\lambda_i F_{P_i}(u)\right)\\
=& \frac{1}{n}\sum_{u\in\mathcal{U}}\left(\sum_{u\in\mathcal{U}} \lambda_i h_{P_i}(u)\right)\times \qquad (7.21)\\
&\times\left(\sum_{\substack{\alpha_1+\dots+\alpha_m=n-1\\ \alpha_1,\dots,\alpha_m\in\mathbb{N}}} \frac{(n-1)!}{\alpha_1!\cdot\ldots\cdot\alpha_m!}\cdot v(F_{P_1}(u),\alpha_1;\dots;F_{P_m},\alpha_m)\cdot\prod_{i=1}^{m}\lambda_i^{\alpha_i}\right).
\end{aligned}$$

We conclude (7.12) with coefficients $V(P_1, \alpha_1; \dots; P_m, \alpha_m) \ge 0$ where $V(P_1, \alpha_1; \dots; P_m, \alpha_m)$ is symmetric in its variables (P_i, α_i), and if $P_{m-1} = P_m$, then $V(P_1, \alpha_1; \dots; P_{m-1}, \alpha_{m-1}; P_m, \alpha_m) = V(P_1, \alpha_1; \dots; P_{m-1}, \alpha_{m-1} + \alpha_m)$. Uniqueness in (i) of $V(P_1, \alpha_1; \dots; P_m, \alpha_m)$ follows from taking mixed partial derivatives at $\lambda_i = 0$ in (7.12). In particular, we deduce the properties (i) to (vii). As the volume and the Minkowski sum are continuous, (iii) yields the continuity of the mixed volume in (viii), and representation (7.14) of the mixed volume using the surface area measire in (ix) follows from (7.13) in (vii) and (7.18).

Finally, to prove the monotonicity property (x), we deduce from (ii) that if $P = \sum_{i=1}^{n-1} \lambda_i P_i$ for n-polytopes P_i, Q, Q_1, Q_2, and $\lambda_i \ge 0$ for $i = 1, \dots, n-1$, then

$$V(P,\dots,P,Q) = \sum_{\substack{\alpha_1+\dots+\alpha_m=n-1\\ \alpha_1,\dots,\alpha_m\in\mathbb{N}}} \frac{(n-1)!}{\alpha_1!\cdot\ldots\cdot\alpha_m!}\cdot V(P_1,\alpha_1;\dots;P_{n-1},\alpha_{n-1};Q,1)\cdot\prod_{i=1}^{m}\lambda_i^{\alpha_i},$$

and hence for $\mathcal{U} = \mathrm{supp}\, S_{P_1+\ldots+P_{n-1}}$, we use (7.14) in (ix) and induction to deduce that

$$V(P_1, \ldots, P_{n-1}, Q) = \frac{1}{n} \sum_{u \in \mathcal{U}} h_Q(u) \cdot v(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)). \tag{7.22}$$

In turn, (7.22) implies that $V(P_1, \ldots, P_{n-1}, Q_1) \leq V(P_1, \ldots, P_{n-1}, Q_2)$ if $Q_1 \subset Q_2$ as $h_{Q_1} \leq h_{Q_2}$, and we finally conlude (x).

Case 2 $K_1, \ldots, K_m, C_1, \ldots, C_l \subset \mathbb{R}^n$ are general compact convex sets

We choose $a > 0$ such that $K_1, \ldots, K_m, C_1, \ldots, C_l \subset \mathrm{int}\, W$ for $W = [-a, a]^n$, and n-dimensional polytopes $P_1^{(k)}, \ldots, P_m^{(k)}, Q_1^{(k)}, \ldots, Q_l^{(k)} \subset W$ such that $P_i^{(k)} \to K_i$, $Q_j^{(k)} \to C_i$ (cf. (1.14)), and hence (iv) and (x) yield that each mixed volume built from these polytopes them is at most $|W|$. Except for the uniqueness in (i), we deduce (7.12) and the properties (i) to (vii) for any compact convex sets by approximation and by the by now known properties of mixed volumes of polytopes. Uniqueness in (i) of $V(K_1, \alpha_1; \ldots; K_m, \alpha_m)$ follows from taking mixed partial derivatives at $\lambda_i = 0$ in (7.12). Again, as the volume and the Minkowski sum are continuous, (iii) yields the continuity of the mixed volume in (viii), and (7.14) in (ix) follows from (7.13) in (vii) and (7.18). Finally, for the monotonicity property $V(C_1, \ldots, C_n) \leq V(K_1, \ldots, K_n)$ if $C_i \subset K_i$ in (x), we may assume that $Q_i^{(k)} \subset P_i^{(k)}$ for each i and k. ■

Remark 7.3.4 (Mixed area measure for n-polytopes). The formula (7.22) in the argument above and approximating a compact convex set C by n-dimensional potopes yields that if $P_1, \ldots, P_{n-1} \subset \mathbb{R}^n$ are n-polytopes, then

$$V(P_1, \ldots, P_{n-1}, C) = \frac{1}{n} \sum_{u \in \mathrm{supp}\, S_{P_1+\ldots+P_{n-1}}} h_C(u) \cdot v(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)). \tag{7.23}$$

Therefore, there exists a discrete measure $S_{P_1,\ldots,P_{n-1}}$ on S^{n-1} with $\mathrm{supp}\, S_{P_1,\ldots,P_{n-1}} \subset \mathrm{supp}\, S_{P_1+\ldots+P_{n-1}}$ such that for any convex compact set $C \subset \mathbb{R}^n$, we have

$$V(P_1, \ldots, P_{n-1}, C) = \frac{1}{n} \int_{S^{n-1}} h_C \, dS_{P_1,\ldots,P_{n-1}}. \tag{7.24}$$

We will show the existence of mixed area measure $S_{K_1,\ldots,K_{n-1}}$ on S^{n-1} for any compact convex sets $K_1, \ldots, K_{n-1} \subset \mathbb{R}^n$ in Theorem 8.3.5.

We have seen in Theorem 7.3.1 (x) that the mixed volumes are monotone increasing in each of their variables. Now we exhibit an example showing that this monotonicity may not be strict

Example 7.3.5 (Monotonicity of mixed volumes may not be strict). We observe that $B^n \subset W^n = [-1, 1]^n$, $B^n \neq W^n$, but for any exterior unit normal u to a facet of W^n, we have $h_{W^n}(u) = 1 = h_{B^n}(u)$; therefore, (7.14) in Theorem 7.3.1 (ix) yields that $V(W^n, B^n; 1) = V(W^n, W^n; 1)$.

Proposition 7.3.6. *For convex compact $K_1,\dots,K_n \subset \mathbb{R}^n$, $V(K_1,\dots,K_n) > 0$ if and only $\exists x_i, y_i \in K_i$ such that $x_1 - y_1,\dots,x_n - y_n$ are independent.*

Proof. If $x_1 - y_, \dots, x_n - y_n$ are independent, then for $s_i = [x_i, y_i]$, we have

$$V(K_1,\dots,K_n) \geq V(s_1,\dots,s_n) = |\det[x_1 - y_1,\dots,x_n - y_n]|/n! > 0.$$

If there exist no suitable $x_i, y_i \in K_i$, then after possibly translating and reindexing, there exist $1 \leq m \leq n$ and linear $(m-1)$-plane L such that $K_1,\dots,K_m \subset L$. Thus there exist compact convex sets $C \subset L$ and $M \subset L^{\perp}$ such that $K_i \subset C$ if $i \leq m$ and $K_j \subset C + M$ if $j > m$. We deduce that

$$V(K_1,\dots,K_n) \leq V(C,m;C+M,n-m) = \sum_{j=0}^{n-m} \binom{n-m}{j} V(C,m+j;M,n-m-j) = 0$$

as $|\alpha\, C + \beta\, M| = \alpha^{m-1}\beta^{n-m+1}$ for $\alpha,\beta > 0$ shows that the only positive mixed volume involving C and M is $V(C,m-1;M,n-m+1)$. ∎

7.4 The Minkowski inequality and the Aleksandrov-Fenchel inequality

The Minkowski inequality (7.26) is an equivalent form of the Brunn-Minkowski inequality below (cf. Remark 7.4.4), and the Aleksandrov-Fenchel inequality (7.30) is a fast reaching generalization. We recall that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\alpha,\beta > 0$, then the Brunn-Minkowski inequality says that

$$|\alpha K + \beta C|^{\frac{1}{n}} \geq \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \tag{7.25}$$

with equality if and only if $K = \gamma\, C + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$; namely, K and C are homothetic. The following statement was also included in Lemma 1.12.2 about equivalent forms of the Brunn-Minkowski inequality, but we provide the simple argument for the reader's convenience:

Lemma 7.4.1. *If $K, C \in \mathcal{K}^n$ are compact and convex, then*

$$f(\lambda) = |(1-\lambda)K + \lambda\, C|^{\frac{1}{n}}$$

is a concave function of $\lambda \in [0,1]$.

If $K + C$ is a convex body, then f is linear if and only if K and C are homothetic convex bodies.

Proof. For $t \in [0,1]$, let $M_t = (1-t)K + tC$, and hence $M_{\frac{1}{2}t+\frac{1}{2}s} = \frac{1}{2}M_t + \frac{1}{2}M_s$ because K and C are convex; therefore, the Brunn-Minkowski inequality yields that f is concave.

If $K + C$ convex body and f is linear, then $0 < f(\frac{1}{2}) = \frac{1}{2} f(0) + \frac{1}{2} f(1)$ as f is concave, thus K and C are homothetic convex bodiest by the equality case of the Brunn-Minkowski inequality. ■

Theorem 7.4.2 (Minkowski inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$V(K, C; 1)^n \geq |K|^{n-1}|C| \tag{7.26}$$

where equality holds if and only if K and C are homothetic.

Proof. We may assume that $|K| = |C|$. Since $f(\lambda) = |(1-\lambda)K + \lambda\, C|^{\frac{1}{n}}$ is concave where $f(\lambda) = \left(|K|(1-\lambda)^n + nV(K, C, 1)(1-\lambda)^{n-1}\lambda + g(\lambda)\right)^{\frac{1}{n}}$ and and the degree of each term in $g(\lambda)$ is at least two, we have

$$0 \leq f'(0) = \frac{|K|^{\frac{-(n-1)}{n}}}{n} \cdot (-n|K| + nV(K, C; 1));$$

therefore, $V(K, C; 1) \geq |K|$.

Equality in the Minkowski inequality yields that $f'(0) = 0$, and hence f, being concave, is linear. Therefore, K and C are translates by Lemma 7.4.1. ■

Using the representation (7.14) of certain mixed volumes *via* the integration over some surface area measure, we deduce the following useful forms of the Minkowski inequality:

Corollary 7.4.3 (Minkowski inequality with surface area measure).

(i) If $K, C \subset \mathbb{R}^n$ are convex bodies, then

$$\int_{S^{n-1}} h_C\, dS_K \geq n|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{7.27}$$

where equality holds if and only if K and C are homothetic.

(ii) If in addition, $|K| = |C|$, then

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K \tag{7.28}$$

where equality holds if and only if K and C are translates.

Remark 7.4.4 (The Minkowski and the Brunn-Minkowski inequalities are equivalent). The argument in Theorem 7.4.2 shows that the Brunn-Minkowski inequality (7.25) yields the Minkowski inequality. For the reverse implication, let us assume the Minkowski inequality for any pair of convex bodies. If $K, C \subset \mathbb{R}^n$ are convex bodies and $\lambda \in (0, 1)$, then the linearity of the mixed volume in each variable and the

Minkowski inequality (7.26) yields for $M_\lambda = (1-\lambda)K + \lambda C$ that

$$\begin{aligned}|M_\lambda| &= V(M_\lambda, (1-\lambda)K + \lambda C; 1) = (1-\lambda)V(M_\lambda, K; 1) + \lambda V(M_\lambda, C; 1)\\ &\geq (1-\lambda)|M_\lambda|^{\frac{n-1}{n}}|K|^{\frac{1}{n}} + \lambda |M_\lambda|^{\frac{n-1}{n}}|C|^{\frac{1}{n}}.\end{aligned}$$

In turn, we conclude the Brunn-Minkowski inequality $|M_\lambda|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}} + \lambda |C|^{\frac{1}{n}}$.

The Minkowski inequality directly yields for example the Isoperimetric and Urysohn Inequalities (see Theorem 7.2.1). For $K \subset \mathbb{R}^n$ convex body with $|K| = |rB^n|$, $r > 0$, (7.27) implies that

$$S(K) = \int_{S^{n-1}} h_{B^n}\, dS_K \geq n|B^n|^{\frac{1}{n}}|K|^{\frac{n-1}{n}} = S(rB^n)$$

with equality if and only if K is a ball. Similarly, for the Urysohn Inequality, (7.26) yields

$$V_1(K) = \frac{n}{\omega_{n-1}} V(B^n, K; 1) \geq \frac{n}{\omega_{n-1}} |B^n|^{\frac{n-1}{n}}|K|^{\frac{1}{n}} = V_1(rB^n)$$

with equality if and only if K is a ball.

Remark 7.4.5 (Anisotropic perimeter). Given a convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$, if $K \subset \mathbb{R}^n$ compact convex, $n \geq 2$, then the Anisoperimetric Perimeter of K is

$$P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho} = nV(K, C; 1),$$

and hence $K \mapsto P_C(K)$ is continuous and monotone increasing in K. If K is a convex body, then

$$P_C(K) = \int_{\partial K} h_C(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{\partial K} \|\nu_K(x)\|_*\, d\mathcal{H}^{n-1}(x).$$

In turn, the Minkowski inequality (7.26) is equivalent to the Anisoperimetric Isoperimetric Inequality Theorem 2.4.4 stating that

$$P_C(K) \geq n|C|^{\frac{1}{n}}|K|^{\frac{n-1}{n}},$$

with equality if and only if C and K are homothetic.

We have proved the Minkowski inequality from the Brunn-Minkowski inequality (7.25) using the first derivative of an associated concave function. Using that the second derivative of a concave function is non-positive leads to the second inequality of Minkowski [464, 465]:

Theorem 7.4.6 (Minkowski's second inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$V(K, \dots, K, C)^2 \geq |K| \cdot V(K, \dots, K, C, C). \tag{7.29}$$

Proof. For $g(\lambda) = |K + \lambda C| = \sum_{i=0}^{n} \binom{n}{i} V(K, C; i)\lambda^i$, the function $\lambda \mapsto g(\lambda)^{\frac{1}{n}}$ is concave for $\lambda \geq 0$ by the Brunn-Minkowski inequality (7.25), and hence

$$
\begin{aligned}
0 \geq \frac{\partial^2}{\partial \lambda^2} g(\lambda)^{\frac{1}{n}}\Big|_{\lambda=0} &= \tfrac{1}{n} g''(0) g(0)^{\frac{1}{n}-1} - \tfrac{n-1}{n^2} (g'(0))^2 g(0)^{\frac{1}{n}-2} \\
&= (n-1)|K|^{\frac{1}{n}-2} \left(V(K, C; 2)|K| - V(K, C; 1)^2 \right).
\end{aligned}
$$

■

A far reaching generalization of Minkowski's first inequality (7.26) and second inequality (7.29) is the Aleksandrov-Fenchel inequality (7.30) below (proved actually only by Aleksandrov [3, 5, 7], see the Comments in Section 7.8). In this monograph, we provide two arguments, one using strongly isomorphic polytopes in Section 7.A, and one using the theory of elliptic operators in Section 8.5.2, both are based on Aleksandrov's original ideas as developed further by van Handel, Shenfeld [300].

Theorem 7.4.7 (Aleksandrov-Fenchel Inequality). *If $n \geq 3$ and $K_1, K_2, C_3, \ldots, C_n$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K_1, K_2, C_3, \ldots, C_n)^2 \geq V(K_1, K_1, C_3, \ldots, C_n) V(K_2, K_2, C_3, \ldots, C_n). \tag{7.30}$$

Unlike in the case of the Minkowski inequality, equality may occur in the Aleksandrov-Fenchel inequality (7.30) even if the bodies are not pairwise homotopic, and the cases of equality are not fully understood.

Remark 7.4.8 (Known equality cases of the Aleksandrov-Fenchel inequality (7.30)).

- K_1, K_2, C_i are convex bodies with C^1 boundary (see Schneider [519], equality if and only if the bodies pairwise homothetic, and even a stability estimate is proved in this case by Schneider [520]);
- K_1, K_2, C_i are zonoids (see Schneider [518] for the result, and Example 1.6.3 for the notion of a zonoid);
- K_1, K_2, C_i are polytopes (see van Handel, Shenfeld [302]). This case is especially important because the Aleksandrov-Fenchel inequality for polytopes appears in Algebraic Geometry and in Combinatorics (see the Comments Section 7.8);
- $C_3 = \ldots = C_n$ (Minkowski's second inequality) where the characterization of the equality case had been open for 100 years (see van Handel, Shenfeld [301]).

Let us state some consequences of the Aleksandrov-Fenchel inequality (7.30). "Algebraic manipulation" starting with (7.30) lead to the following version:

Corollary 7.4.9 (Aleksandrov-Fenchel Inequality - general form).
If $2 \leq m \leq n-1$ *and* $K_1, \ldots, K_m, C_{m+1}, \ldots, C_n \subset \mathbb{R}^n$ *are compact convex sets, then*

$$V(K_1, \ldots, K_m, C_{m+1}, \ldots, C_n)^m \geq \prod_{i=1}^{m} V(K_i, \ldots, K_i, C_{m+1}, \ldots, C_n).$$

If $m = n$, then Corollary 7.4.9 reads as

$$V(K_1, \ldots, K_n)^n \geq \prod_{i=1}^{n} V(K_i). \tag{7.31}$$

Alesker, Dar, V. Milman [12] give a simpler proof of (7.31) using optimal transport, and even manage to verify that equality holds in (7.31) for convex bodies $K_1, \ldots, K_n$ if and only if $K_1, \ldots, K_n$ are pairwise homothetic. The Isoperimetric inequality

$$V(B^n, K; i)^n \geq |B^n|^{n-i} |K|^i$$

for the mean i-dimensional projections of a convex body K (cf. Theorem 7.2.1) directly follows from the consquence (7.31) of the Aleksandrov-Fenchel inequality including the characterization of equality.

In combinatorics, the consequence of Aleksandrov-Fenchel inequality (7.30) that certain sequence of mixed volumes is log-concave (and hence uni-modal) is frequently used:

Remark 7.4.10 (Log-concavity of mixed volumes). For $i, j \geq 1$ and $i + j \leq n$, and compact, convex sets $K_1, K_2, C_{i+j+1}, \ldots, C_n \subset \mathbb{R}^n$, if $C = C_{i+j+1}, \ldots, C_n$ (there is no C if $i + j = n$), then

$$V(K_1, i; K_2, j; C)^2 \geq V(K_1, i-1; K_2, j+1; C) V(K_1, i+1; K_2, j-1; C). \tag{7.32}$$

7.5 Aleksandrov's lemma for Wulff shapes

The main goal of this section is to present Aleksandrov's result that the surface area measure is the first variation of volume in the sense of Theorem 7.5.2 below. We note that for a convex body $K \subset \mathbb{R}^n$, a closed $\Omega \subset S^{n-1}$ and $\varphi : \Omega \to \mathbb{R}$, if $K = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq \varphi(u) \text{ for } x \in \Omega\}$, then Lemma 2.5.6 yields that Ω is not contained in any closed hemisphere and

$$\{\nu_K(x) : x \in \partial' K\} \subset \operatorname{supp} S_K \subset \Omega. \tag{7.33}$$

Here we say that φ is lower semicontinuous if $\lim_{m\to\infty} x_m = x$ for $x_m, x \in \Omega$ yields that $\liminf_{m\to\infty} \varphi(x_m) \geq \varphi(x)$, and $\inf_{u\in\Omega} \varphi > 0$ in this case. For Wulff shapes - that have been introduced in Section 4.4 - we consider a more general notion where the integrals in (7.34) make sense by (7.33):

Lemma 7.5.1 (Aleksandrov). *Let $\Omega \subset S^{n-1}$ be closed and not contained in any closed hemisphere, and let $\varphi : \Omega \to (0, \infty)$ be bounded and lower semicontinous. Then the Wulff shape*

$$W = \{x \in \mathbb{R}^n : \langle x, v\rangle \leq \varphi(v) \text{ for } v \in \Omega\}$$

is a convex body with $o \in \operatorname{int} W$, $h_W(\nu_W(x)) = \varphi(\nu_W(x))$ for $x \in \partial' W$ and

$$\int_{S^{n-1}} \varphi\, dS_W = \int_{S^{n-1}} h_W\, dS_W = n|W|. \tag{7.34}$$

Remark. The definition of S_W yields (cf. (2.15)) that (7.34) is equivalent with

$$\left(\int_{S^{n-1}} \varphi\, dS_W =\right) \int_{\partial' W} \varphi(\nu_W)\, d\mathcal{H}^{n-1} = \int_{\partial W} h_W(\nu_W)\, d\mathcal{H}^{n-1}. \tag{7.35}$$

Proof. W is readily a closed convex set with $o \in W$. Since Ω is closed and not contained in any closed hemisphere, the function $f(u) = \max_{v\in\Omega}\langle u, v\rangle$ of $u \in S^{n-1}$ is positive and continuous, thus there exists $\delta \in (0, 1)$ such that $f(u) \geq \delta$ for any $u \in S^{n-1}$. In addition, as φ is bounded and lower semicontinous, there exists $\theta \in (0, 1)$ such that $\theta \leq \varphi(v) \leq \theta^{-1}$ for $v \in \Omega$. If $x \in \theta B^n$ and $v \in \Omega$, then $\langle x, v\rangle \leq \theta \leq \varphi(v)$; therefore, $\theta B^n \subset W$. On the other hand, if $x = ru \in W$ for $r \geq 0$ and $u \in S^{n-1}$, then there exists $v \in \Omega$ with $\langle u, v\rangle \geq \delta$, and hence $\theta^{-1} \geq \varphi(v) \geq \langle x, v\rangle = r\langle u, v\rangle \geq r\delta$. We deduce that $r \leq (\delta\theta)^{-1}$, thus W is a convex body with $o \in \operatorname{int} W$.

Turning to (7.34), according to (7.33) and (7.35), it is sufficient to prove that

$$\varphi(\nu_W(x)) = h_W(\nu_W(x)) \quad \text{for } x \in \partial' W. \tag{7.36}$$

If $x \in \partial' W$, then there exists $x_k \notin W$ with $x_k \to x$, and hence there exists $v_k \in S^{n-1}$ satisfying $\langle v_k, x_k\rangle > \varphi(v_k)$ by $x_k \notin W$. We may assume that $v_k \to v \in \Omega$, thus the lower semicontinuity of φ implies that

$$\varphi(v) \leq \liminf_{k\to\infty} \varphi(v_k) \leq \liminf_{k\to\infty} \langle v_k, x_k\rangle = \langle v, x\rangle$$

where $x \in W$ yields $\langle v, x\rangle \leq h_W(v) \leq \varphi(v)$. We deduce that v is exterior normal at the $x \in \partial' W$ with $h_W(v) = \varphi(v)$, and in turn (7.36) follows. ■

The typical cases in Aleksandrov's Theorem 7.5.2 - essentially due to Aleksandrov [7], and in this form to Schneider [522] - are when Ω is discrete (and hence the Wulff shapes are polytopes) or $\Omega = S^{n-1}$.

Theorem 7.5.2 (Aleksandrov Lemma for Wulff shapes). *Let $\Omega \subset S^{n-1}$ be closed set not contained in a closed hemisphere, let $\varphi : \Omega \times (-t_0, t_0) \to (0, \infty)$, $t_0 > 0$, be continuous such that*

$$\lim_{t\to 0} \frac{\varphi(u, t) - \varphi(u, 0)}{t} = \partial_t \varphi(u, 0)$$

exists uniformly in $u \in \Omega$ where $\partial_t \varphi(u,0)$ is continuous on Ω. For $t \in (-t_0, t_0)$, the Wulff shape $K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \le \varphi(u,t)$ for $u \in S^{n-1}\}$ satisfies that

$$\lim_{t\to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0)\, dS_{K_0}(u).$$

Remarks.

- If $\varphi : \Omega \times [0, t_0) \to (0, \infty)$, then

$$\lim_{t\to 0^+} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0)\, dS_{K_0}(u). \tag{7.37}$$

- If $\varphi(u,t) = h_{K_0}(u) + tg(u)$ for some continuous $g : S^{n-1} \to \mathbb{R}$ and $t \in (-t_0, t_0)$, then

$$\lim_{t\to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} g\, dS_{K_0}. \tag{7.38}$$

Proof of Theorem 7.5.2. For any $\varepsilon \in (0,1)$, since $\frac{\varphi(u,t)-\varphi(u,0)}{t}$ tends uniformly in $u \in \Omega$ to $\partial_t \varphi(u,0)$ as $t \to 0$, we have $(1-\varepsilon)\varphi(u,0) < \varphi(u,t) < (1+\varepsilon)\varphi(u,0)$ for $u \in \Omega$ if $|t|$ is small. We deduce that $\lim_{t\to 0} K_t = K_0$, and hence S_{K_t} tends to S_{K_0} weakly according to Proposition 2.6.12. Using again the uniform convergence of $\frac{\varphi(u,t)-\varphi(u,0)}{t}$, it follows that

$$\lim_{t\to 0^+} \int_{S^{n-1}} \frac{\varphi(u,t) - \varphi(u,0)}{t}\, dS_{K_t}(u) = \int_{S^{n-1}} \partial_t \varphi(u,0)\, dS_{K_0}(u) = I. \tag{7.39}$$

On the one hand, using first (7.39), then (7.34) and $\varphi(u,0) \ge h_{K_0}(u)$ and later the Minkowski inequality and $K_t \to K_0$, we deduce that

$$\begin{aligned}
I &= \lim_{t\to 0} \frac{1}{t} \left(\int_{S^{n-1}} \varphi(u,t)\, dS_{K_t}(u) - \int_{S^{n-1}} \varphi(u,0)\, dS_{K_t}(u) \right) \\
&\le \liminf_{t\to 0} \frac{1}{t} \left(\int_{S^{n-1}} h_{K_t}(u)\, dS_{K_t}(u) - \int_{S^{n-1}} h_{K_0}(u)\, dS_{K_t}(u) \right) \\
&= \liminf_{t\to 0} \frac{n}{t}\, (V(K_t) - V(K_t; K_0, 1)) = \liminf_{t\to 0} \frac{n}{t} \cdot \frac{V(K_t)^n - V(K_t; K_0, 1)^n}{\sum_{i=0}^{n-1} V(K_t)^i V(K_t; K_0, 1)^{n-1-i}} \\
&\le \liminf_{t\to 0} \frac{n}{t} \cdot \frac{V(K_t)^n - V(K_t)^{n-1} V(K_0)}{\sum_{i=0}^{n-1} V(K_t)^i V(K_t; K_0, 1)^{n-1-i}} = \liminf_{t\to 0} \frac{V(K_t) - V(K_0)}{t}.
\end{aligned}$$

On the other hand, using first (7.39), then (7.34) and $\varphi(u,t) \geq h_{K_t}(u)$ and later the Minkowski inequality and $K_t \to K_0$, we deduce that

$$\begin{aligned}
I &= \lim_{t\to 0} \frac{1}{t}\left(\int_{S^{n-1}} \varphi(u,t)\, dS_{K_0}(u) - \int_{S^{n-1}} \varphi(u,0)\, dS_{K_0}(u)\right)\\
&\geq \limsup_{t\to 0^+} \frac{1}{t}\left(\int_{S^{n-1}} h_{K_t}(u)\, dS_{K_0}(u) - \int_{S^{n-1}} h_{K_0}(u)\, dS_{K_0}(u)\right)\\
&= \limsup_{t\to 0} \frac{n}{t}\,\left(V(K_0;K_t,1) - V(K_0)\right) = \limsup_{t\to 0} \frac{n}{t}\cdot \frac{V(K_0;K_t,1)^n - V(K_0)^n}{\sum_{i=0}^{n-1} V(K_0;K_t,1)^{n-1-i}V(K_t)^i}\\
&\geq \limsup_{t\to 0} \frac{n}{t}\cdot \frac{V(K_0)^{n-1}V(K_t) - V(K_0)^n}{\sum_{i=0}^{n-1} V(K_0;K_t,1)^{n-1-i}V(K_t)^i} = \limsup_{t\to 0} \frac{V(K_t)-V(K_0)}{t}.
\end{aligned}$$

In turn, we conclude the Aleksandrov Lemma Theorem 7.5.2. ■

7.6 The L_p Brunn-Minkowski inequality for $p \geq 1$

In 1962, Firey [231] generalized the Minkowski addition of convex body - when support functions are added - into L_p addition for $p \geq 1$, and verified the corresponding Brunn-Minkowski type inequality. We recall that if $K, C \subset \mathbb{R}^n$ are convex bodies, and $\lambda \in (0,1)\alpha, \beta > 0$, then the Brunn-Minkowski inequality (cf. Lemma 1.12.2) says that

$$\begin{aligned}
|\alpha K + \beta C|^{\frac{1}{n}} &\geq \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \text{ with equality iff } K, C \text{ are homothetic;}\\
|(1-\lambda)\, K + \lambda\, C| &\geq |K|^{1-\lambda}|C|^{\lambda} \text{ with equality iff } K, C \text{ are translates.}
\end{aligned} \tag{7.40}$$

In order to study the notion L_p addition for $p > 1$, we recall some basic inequalities. As t^p is a strictly convex function of $t \geq 0$ (since the second derivative is positive), if $t, s \geq 0$ and $\lambda \in (0,1)$, then

$$((1-\lambda)t + \lambda\, s)^p \leq (1-\lambda)t^p + \lambda\, s^p \text{ with equality if and only if } t = s. \tag{7.41}$$

In addition, we need the Minkowski inequality (cf. (10.6)) for integrals in the form that if $p > 1$, $a_i, b_i \geq 0$ and $\mu_i > 0$ for $i = 1, \ldots, k$, then

$$\left(\sum_{i=1}^{k} \mu_i (a_i + b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{k} \mu_i a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{k} \mu_i b_i^p\right)^{\frac{1}{p}} \tag{7.42}$$

with equality if and only if there exists $\lambda > 0$ such that $a_i = \lambda b_i$, $i = 1, \ldots, k$.

Lemma 7.6.1 (L_p linear combination of convex bodies, $p > 1$). *Let $p \geq 1$. For convex bodies $K, C \subset \mathbb{R}^n$ with $o \in K, C$ and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, there exists a convex body $\alpha \cdot K +_p \beta \cdot C$ containing the origin whose support function is*

$$h_{\alpha\cdot K +_p \beta\cdot C} = \left(\alpha\, h_K^p + \beta\, h_C^p\right)^{\frac{1}{p}}. \tag{7.43}$$

In particular,

$$\alpha \cdot K +_p \beta \cdot C = \left\{x \in \mathbb{R}^n : \langle x, u\rangle^p \leq \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in S^{n-1}\right\}. \tag{7.44}$$

Remarks.

- If $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$, then $o \in \operatorname{int}(\alpha \cdot K +_p \beta \cdot C)$.
- The case $p = 1$ is the classical Minkowski combination; namely, $\alpha \cdot K +_1 \beta \cdot C = \alpha K + \beta C$.

Proof. We may assume that $\alpha, \beta > 0$. Now $h_{\alpha K +_p \beta C}$ defined as in (7.43) is readily homogeneous, continuous, and it is also convex as for $z = \frac{1}{2}x + \frac{1}{2}y$, $x, y \in \mathbb{R}^n$, applying the convexity of h_K, h_C and the Minkowski inequality (7.42) with $\mu_1 = \alpha$, $\mu_2 = \beta$, etc, yields

$$\begin{aligned}(\alpha\, h_K(z)^p + \beta\, h_C(z)^p)^{\frac{1}{p}} &\leq \left(\alpha \cdot \left(\tfrac{1}{2}\, h_K(x) + \tfrac{1}{2}\, h_K(y)\right)^p + \beta \cdot \left(\tfrac{1}{2}\, h_C(x) + \tfrac{1}{2}\, h_C(y)\right)^p\right)^{\frac{1}{p}} \\ &\leq \frac{1}{2}\,(\alpha\, h_K(x)^p + \beta\, h_C(x)^p)^{\frac{1}{p}} + \frac{1}{2}\,(\alpha\, h_K(y)^p + \beta\, h_C(y)^p)^{\frac{1}{p}}.\end{aligned}$$

Therefore, the non-negative function $\left(\alpha\, h_K^p + \beta\, h_C^p\right)^{\frac{1}{p}}$ is the convex hull of a compact convex set $M \subset \mathbb{R}^n$ with $o \in M$. Since K and C being convex bodies yields $h_K(u) + h_K(-u) > 0$ and $h_C(u) + h_C(-u) > 0$ for any $u \in S^{n-1}$, we have $h_M(u) + h_M(-u) > 0$, and hence $M = \alpha \cdot K +_p \beta \cdot C$ is a convex body. ■

We observe that the L_p linear combination is equivariant with respect linear transformations.

Lemma 7.6.2. *If $p \geq 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in K, C$, $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, and $\Phi \in \mathrm{GL}(n)$, then whose support function is*

$$\alpha \cdot (\Phi K) +_p \beta \cdot (\Phi C) = \Phi(\alpha \cdot K +_p \beta \cdot C). \tag{7.45}$$

Proof. Since $\alpha \cdot K +_p \beta \cdot C = \{x \in \mathbb{R}^n : \langle x, u\rangle^p \leq \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in \mathbb{R}^n\}$ according to (7.44), we deduce(7.45) from the fact that $h_{\Phi M}(u) = h_M(\Phi^t u)$ holds for any compact convex set $M \subset \mathbb{R}^n$ and $u \in \mathbb{R}^n$. ■

Firey [231] proved the following simple, but very useful variant of the Brunn-Minkowski inequality:

Theorem 7.6.3 (Firey's L_p Brunn-Minkowski inequality, $p > 1$). *Let $K, C \subset \mathbb{R}^n$ be convex bodies with $o \in K, C$, and let $p > 1$.*

(i) $|\alpha \cdot K +_p \beta \cdot C|^{\frac{p}{n}} \geq \alpha\, |K|^{\frac{p}{n}} + \beta\, |C|^{\frac{p}{n}}$ for $\alpha, \beta > 0$ with equality if and only if K and C are dilates.

(ii) $|(1-\lambda)\cdot K +_p \lambda \cdot C| \geq |K|^{1-\lambda}|C|^{\lambda}$ *for* $\lambda \in (0,1)$ *with equality if and only if* $K = C$.

Proof. For (ii), (7.41) yields that $h_{(1-\lambda)\cdot K+_p\lambda\cdot C}(u) \geq (1-\lambda)h_K(u) + \lambda\, h_C(u)$ for any $u \in S^{n-1}$ with equality if and only if $h_K(u) = h_C(u)$. Therefore, the Brunn-Minkowski inequality (7.40) implies

$$|(1-\lambda)\cdot K +_p \lambda \cdot C| \geq |(1-\lambda)\, K + \lambda\, C| \geq |K|^{1-\lambda}|C|^{\lambda} \tag{7.46}$$

where equality yields that $K = C$ by the equality case in (7.41).

Let us turn to (i). We set $\alpha_0 = |K|^{\frac{-1}{n}}$, $\beta_0 = |C|^{\frac{-1}{n}}$, $K = \alpha_0^{-1}K$ and $C = \beta_0^{-1}C$, and hence $|K_0| = |C_0| = 1$. For $\lambda = \frac{\beta\beta_0^p}{\alpha\alpha_0^p+\beta\beta_0^p}$, we deduce that

$$h_{\alpha\cdot K+_p\beta\cdot C} = (\alpha\alpha_0^p + \beta\beta_0^p)^{\frac{1}{p}}\left((1-\lambda)h_{K_0}^p + \lambda\, h_{C_0}^p\right)^{\frac{1}{p}}.$$

Since $|(1-\lambda)\cdot K_0 +_p \lambda \cdot C_0| \geq 1$ by (ii), we conclude the inequality in (i). If equality holds in (i), then $K_0 = C_0$ by $|(1-\lambda)\cdot K_0 +_p \lambda\cdot C_0| = 1$, and hence K and C are dilates. ■

Similarly to the case of Brunn-Minkowski inequality (cf. Lemma 1.12.2), we deduce the following equivalent formulation of the L_p Brunn-Minkowski inequality:

Corollary 7.6.4. *If* $p > 1$ *and* $K, C \subset \mathbb{R}^n$ *are convex bodies with* $o \in K, C$, *then* $\lambda \mapsto |(1-\lambda)\cdot K +_p \lambda \cdot C|^{\frac{p}{n}}$ *is a concave function of* $\lambda \in [0,1]$, *which is a linear function if and only if K and C are dilates.*

Proof. We set $M_t = (1-t)\cdot K +_p t\cdot C$ for $t \in [0,1]$. We deduce from (7.41) that $\frac{1}{2}\cdot M_t + p\frac{1}{2}\cdot M_s \subset M_{\frac{1}{2}t+\frac{1}{2}s}$; therefore, Theorem 7.6.3 yields Corollary 7.6.4. ■

Next we discuss the L_p analugue of the Minkowski inequality due to Lutwak [433]. The classical Minkowski inequality says that if $K, C \subset \mathbb{R}^n$ are convex bodies, then

$$\int_{S^{n-1}} h_C\, dS_K \geq n|K|^{\frac{n-1}{n}}|C|^{\frac{1}{n}} \tag{7.47}$$

where equality holds if and only if K and C are homothetic. For the L_p analogue, we need the notion of cone volume measure (cf. Definition 2.6.1); namely, V_K is the Borel measure on S^{n-1} satisfying $dV_K = \frac{1}{n}h_K\, dS_K$ for a convex body $K \subset \mathbb{R}^n$ with $o \in K$, and hence $V_K(S^{n-1}) = |K|$. As S_K is a finite measure, we observe that $V_K(\{h_K = 0\}) = 0$ even if $o \in \partial K$ where

$$\{h_K = 0\} = \{u \in S^{n-1} : h_K(u) = 0\} = N_K(o) \cap S^{n-1}$$

for the exterior normal cone $N_K(o)$ at $o \in \partial K$. On the other hand, possibly $S_K(\{h_K = 0\}) = S_K(N_K(o) \cap S^{n-1}) > 0$ if $o \in \partial K$; for example, when K is an n-polytope.

We also need the following form of Jensen's inequlity (cf. (10.4)): If $p > 1$, μ is a Borel probability measure on the topological space X, and $f \in L_1(X, \mu)$ is non-negative, then

$$\int_X f \, d\mu \leq \left(\int_X f^p \, d\mu \right)^{\frac{1}{p}} \tag{7.48}$$

with equality if and only if f is μ a.e. constant.

Theorem 7.6.5 (Lutwak's L_p Minkowski inequality, $p > 1$). *If $p > 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in K, C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then*

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geq |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}} \tag{7.49}$$

with equality if and only if K and C are dilates.

Remark. If $o \in \partial K$, then a condition like $S_K(\{h_K = 0\}) = 0$ is necessary (see Example 7.6.6). In addition, the left hand side of (7.49) might be infinity.

Proof. We may assume that $|K| = |C| = 1$. We deduce from the Jensen inequality (7.48), the fact that $S_K(N_K(o) \cap S^{n-1}) = 0$ if $o \in \partial K$, and from the Minkowski inequality (7.47) that that

$$\left(\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \right)^{\frac{1}{p}} \geq \int_{S^{n-1}} \frac{h_C}{h_K} \, dV_K = \frac{1}{n} \int_{S^{n-1}} h_C \, dS_K \geq 1.$$

If equality holds then K and C are translates (by the equality case in the Minkowski inequality) and there exists $t > 0$ such that $h_C = t\, h_K$ (by the equality case of the Jensen inequality (7.48)); therefore, $t = 1$ and $h_C = h_K$. ■

Example 7.6.6. The L_p Minkowski inequality (7.49) may not hold if $o \in \partial K$ and $S_K(N_K(o) \cap S^{n-1}) > 0$. Let $K \subset \mathbb{R}^n$ be a regular simplex of volume 1 such that the exterior unit normals are $u_0, \ldots, u_n \in S^{n-1}$, the centroid of the facet with exterior unit normal u_0 is o, and hence the opposite vertex is $-t_0 u_0$ for some $t_0 > 0$. Let $C = K + s u_0$ for $s \in (0, t_0)$, and hence there exists $r \in (0, 1)$ such that $h_C(u_i) = r h_K(u_i)$ for $i = 1, \ldots, n$. Since $\operatorname{supp} V_K = \{u_1, \ldots, u_n\}$ and $h_C(u_0) > 0$, it follows that

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K = \left(\int_{S^{n-1}} \frac{h_C}{h_K} \, dV_K \right)^p < \left(\frac{1}{n} \int_{S^{n-1}} h_C \, dS_K \right)^p = 1 = |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}.$$

Lutwak [433] extended the notion of surface area measure to the L_p case (see also Section 9.3 and Section 9.4):

Definition 7.6.7 (L_p surface area measure for $p > 1$). For $p > 1$, if $K \subset \mathbb{R}^n$ is a convex body with either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then its L_p surface area measure on S^{n-1} is

$$dS_{K,p} = h_K^{1-p}\, dS_K = nh_K^{-p}\, dV_K.$$

Remark. $S_{K,p}$ is a Borel measure. Possibly $S_{K,p}(S^{n-1}) = \infty$ if $o \in \partial K$, but even in that case, $S_{K,p}(X) < \infty$ for any compact $X \subset \{h_K > 0\} = S^{n-1} \backslash N_K(o)$.

Lutwak [433] considered the left hand side of (7.49) as the L_p mixed volume; namely, if $p \geq 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$, then

$$V_p(K, C) = \int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K = \frac{1}{n} \int_{S^{n-1}} h_C^p h_K^{1-p}\, dS_K = \frac{1}{n} \int_{S^{n-1}} h_C^p dS_{K,p}. \qquad (7.50)$$

In particular, $S_{K,1} = S_K$ and $V_1(K, C) = V(K, C; 1)$.

Let us summarize the main properties of these notions; for example, that $V_p(K, C)$ is invariant under $\mathrm{SL}(n)$:

Proposition 7.6.8. *Let $p > 1$, $K, C \subset \mathbb{R}^n$ be convex bodies with $o \in C$, and either $o \in \operatorname{int} K$, or $o \in \partial K$ and $S_K(\{h_K = 0\}) = 0$.*

(i) $V_p(K, K) = |K|$.

(ii) $V_p(K, C) = \frac{1}{n} \int_{S^{n-1}} h_C^p dS_{K,p} \geq |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}$ *with equality if and only if K and C are dilates.*

(iii) If $\Phi \in \mathrm{GL}(n)$ with $|\det \Phi| = 1$, then

$$V_p(\Phi K, \Phi C) = V_p(K, C). \qquad (7.51)$$

(iv) If, in addition, $p \neq n$ and either $o \in \operatorname{int} C$, or $o \in \partial C$ and $S_C(\{h_C = 0\}) = 0$, then $S_{K,p} = S_{C,p}$ implies $K = C$.

Proof. (i) follows directly from the definition, and (ii) is just (7.49).

For (iii), (7.51) is a consequence of Proposition 2.6.15 and Lemma 2.6.14 as

$$\int_{S^{n-1}} \frac{h_{\Phi C}^p}{h_{\Phi K}^p}\, dV_{\Phi K} = \int_{S^{n-1}} \frac{h_{\Phi C}\, (\Phi^{-t} u / \|\Phi^{-t} u\|)^p}{h_{\Phi K}\, (\Phi^{-t} u / \|\Phi^{-t} u\|)^p}\, dV_K(u)$$
$$= \int_{S^{n-1}} \frac{h_{\Phi C}\, (\Phi^{-t} u)^p}{h_{\Phi K}\, (\Phi^{-t} u)^p}\, dV_K(u) = \int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K.$$

For (iv), we deduce from (7.50), $S_{K,p} = S_{C,p}$ and L_p the Minkowski inequality as (i) that

$$|K| = \int_{S^{n-1}} \frac{h_K^p}{h_K^p}\, dV_K = \frac{1}{n} \int_{S^{n-1}} h_K^p dS_{K,p} = \frac{1}{n} \int_{S^{n-1}} h_K^p dS_{C,p} \geq |C|^{\frac{n-p}{n}} |K|^{\frac{p}{n}}; \quad (7.52)$$

therefore, $|K|^{\frac{n-p}{n}} \geq |C|^{\frac{n-p}{n}}$. Reversing the role of K and C in (2.25) implies $|C|^{\frac{n-p}{n}} \geq |K|^{\frac{n-p}{n}}$, and hence $|C| = |K|$ by $p \neq n$. In turn, equality in (7.52) implies equality in (i), which fact combined with $|C| = |K|$ yields that $K = C$. ∎

Finally, the form (7.37) of the Aleksandrov lemma and (7.50) yield the following variational characterization of the L_p surface area due to Lutwak [433]:

Lemma 7.6.9 (Lutwak). *If $p > 1$, $K, C \subset \mathbb{R}^n$ are convex bodies with $o \in C$ and $o \in \operatorname{int} K$, then*

$$\lim_{t \to 0^+} \frac{|K +_p t \cdot C|^{\frac{p}{n}} - |K|^{\frac{p}{n}}}{t} = nV_p(K, C) = \int_{S^{n-1}} h_C^p \, dS_{K,p}.$$

7.7 Isoperimetric type problems for polytopes

Concerning the isoperimetric problem for polygons, Zenodorus (circa 200 BC - 140 BC) already suggested that among polygons of given number of vertices (sides), the regular ones have the minimal perimeter, but this was only proved rigorously by Weierstrass at the end of the 19th century (see for example [561]).

Theorem 7.7.1. *For $k \geq 3$, among convex k-gons of given area in $\mathbb{R}^2$, the regular ones have minimal perimeter.*

Next we discuss the isoperimetric inequlity for polytopes in higher dimension, also in terms of the mean projection. The following estimate, together with the Blaschke Selection Theorem 1.7.3, ensures the existence of exteremal bodies.

Lemma 7.7.2. *For a convex body $K \subset \mathbb{R}^n$ and $1 \leq i \leq n-1$, $n \geq 2$, we have*

$$V_i(K) \geq \gamma \cdot |K|^{\frac{i-1}{n-1}} (\operatorname{diam} K)^{\frac{n-i}{n-1}} \text{ for } \gamma(n,i) > 0 \text{ depending on } n, i.$$

Proof. If $i = 1$, then $V_1(K) \geq \operatorname{diam} K$ as K contains a segment of length $\operatorname{diam} K$.

If $i \geq 2$, then the Aleksandrov-Fenchel inequality Corollary 7.4.9 yields that

$$V(B^n, K; i)^{n-1} = V(B^n, n-i; K, i-1; K, 1)^{n-1} \geq V(B^n, K; 1)^{n-i} |K|^{i-1},$$

where $V(B^n, K; i) = \binom{n}{i}^{-1} \omega_{n-i} V_i(K)$ and $V(B^n, K; 1) = \frac{\omega_{n-1}}{n} V_1(K) \geq \frac{\omega_{n-1}}{n} \operatorname{diam} K$. ∎

The isoperimetric type inequalities for simplices in any dimensions was proved by Steiner [542] alreasy in 1842 in the case of the perimeter, and much later by Hadwiger [295] in the middle of the 20th century in the case of other mean projections.

Theorem 7.7.3 (Isoperimetric Inequality for simplices). *For $i = 1, \ldots, n-1$, $n \geq 2$, if C is a simplex and T is a regular simplex with $|C| = |T|$, then*

$$V_i(C) \geq V_i(T),$$

with equality if and only if C is a regular simplex.

Proof. There exists a simplex C_0 with $|C_0| = |T|$ and minimizing $V_i(C_0)$ according to Lemma 7.7.2 and Blaschke Selection Theorem!1.7.3, so all we need to prove is that C_0 is a regular simplex. We suppose that C_0 has two edges of different length, and seek a contradiction. It follows that C_0 has two intersecting edges of different length, thus $C_0 = \text{conv}\{v_0, \ldots, v_n\}$ where $\|v_2 - v_0\| \neq \|v_2 - v_0\|$.

For $u = (v_1 - v_0)/\|v_1 - v_0\| \in S^{n-1}$, we have

$$C_0 = \{x + tu : x \in C_0|u^\perp \text{ and } g(x) \leq t \leq f(x)\}$$

where f, g are linear functions on $u^\perp$. We deduce from Lemma 1.10.11 that the Steiner symmetrial $\Theta_{u^\perp}C_0$ (cf. Definition 1.10.1) is also a simplex of diameter at most D (cf. Proposition 1.10.3), and $V_i(\Theta_{u^\perp}C_0) < V_i(C_0)$ by Proposition 7.2.2 as $\|v_2 - v_0\| \neq \|v_2 - v_0\|$ yields that there exists no hyperplane of symmetry for C_0 parallel to $u^\perp$. This contradiction verifies Theorem 7.7.3. ■

Remark. Given the number of vertices and the volume of a polytope in $\mathbb{R}^d$, the use of Steiner symmetrization and some calculations yield the solution of the isoperimetric problem in the following cases. For $1 \leq i < n$ and $a, b > 0$, let $Q(i; a, b)$ be a polytope that is the convex hull of an i-dimensional regular simplex T of edge length a and an $(n-i)$-dimensional regular simplex T' of edge length b where the centers of T and T' coincide, and their affine hulls are orthogonal.

- $Q(\lfloor n/2 \rfloor; a, b)$ minimizes the surface area among polytopes of $n+2$ vertices in $\mathbb{R}^n$ where $a/b = \sqrt{(\lfloor n/2 \rfloor + 1)/(\lceil n/2 \rceil + 1)}$ (cf. Böröczky, Böröczky [93]);
- $Q(2; a, a)$ minimizes the ith intrinsice volume $i = 1, 2, 3$ among polytopes of 6 vertices in $\mathbb{R}^4$ (cf. Böröczky, Böröczky [93]);
- $Q(2; a, b)$ minimizes the surface area among polytopes of 5 vertices in $\mathbb{R}^3$ where $a/b \approx 0.9451$ (cf. Böröczky, Böröczky [93]);
- the regular octahedron minimizes the surface area among polytopes of 6 vertices in $\mathbb{R}^3$ (cf. Böröczky, Kovács [108]).

Definition 7.7.4 (Circumscribed polytope). We say that polytope $P \subset \mathbb{R}^n$ is circumscribed around a ball B if $B \subset P$ and each facet of P touches B.

Remark. If $B = B^n$, then $|P| = \frac{1}{n} \cdot S(P)$, and hence the isoperimetric quotient is $S(P)/|P|^{\frac{n-1}{n}} = n\,|P|^{\frac{1}{n}}$; therefore, minimizing the isoperimetric quotient among polytopes circumscribed around B^n is equivalent to minimizing volume.

According to Lindelöf [411] from 1869, when minimizing the isoperimetric quotient $S(P)/|P|^{\frac{n-1}{n}}$ of a polytope P of given exterior normals to the facets, we may assume that the polytope is circumscribed around a ball.

Theorem 7.7.5 (Lindelöf). *Given $u_1, \dots, u_k \in S^{n-1}$ not contained in a closed hemisphere, the minimum of the isoperimetric quotient among polytopes whose facet normals are among $u_1, \dots, u_k$ is attained and only attained by polytopes with k facets circumscribed around some ball.*

Proof. Let $P_0 = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1\}$ be the polytope whose facets touch B^n at $u_1, \dots, u_k$. If P is any polytope whose unit facet unit normals are among $u_1, \dots, u_k$, then $h_{P_0}(u) = 1$ for any unit exterior normal u at a facet of P. It follows form the form (7.27) of the Minkowski inequality that $S(P) = \int_{S^{n-1}} h_{P_0}\, dS_P \geq n|P|^{\frac{n-1}{n}}|P_0|^{\frac{1}{n}}$; therefore, $S(P)/|P|^{\frac{n-1}{n}} \geq n|P_0|^{\frac{1}{n}} = S(P_0)/|P_0|^{\frac{n-1}{n}}$. If equality holds, then equality holds in the Minkowski inequality, and hence P and P_0 are homothetic. ■

For the Platonic solids that are simple polytopes (three faces meet at each vertex), beside Lindelöf's Theorem 7.7.5, L. Fejes Tóth's elegant argument verifying Theorem 7.7.8 depends on the Moment Theorem 7.7.6 proved also by L. Fejes Tóth [213,216,217]. We only quote this theorem, noting that $\mathcal{H}^2(S^2) = 4\pi$, and for an edge to edge tiling of S^2 by k spherically convex tiles, Euler's theorem yields that the mosaic has at most $3k-6$ edges.

Theorem 7.7.6 (Moment Theorem). *Let $x_1, \dots, x_k \in S^2$, $k \geq 4$, be not contained in any open hemispere, and let the spherical triangle $T = \mathrm{conv}_{S^2}\{a, p, q\}$ have angle $\frac{k\cdot 2\pi}{6k-12}$ at p satisfy that $d_{S^2}(a, p) = d_{S^2}(a, q)$ and $\mathcal{H}^2(T) = \frac{2\pi}{3k-6}$.*

(i) If $f : (0, \frac{\pi}{2}) \to [0, \infty)$ is monotone increasing, then

$$\int_{S^2} \min_{x_i : \langle u, x_i \rangle > 0} f\left(d_{S^2}(u, x_i)\right)\, du \geq (6k-12) \int_T f\left(d_{S^2}(u, a)\right)\, du. \tag{7.53}$$

(ii) If $g : (0, \frac{\pi}{2}) \to [0, \infty)$ is monotone decreasing, then

$$\int_{S^2} \max_{x_i : \langle u, x_i \rangle > 0} g\left(d_{S^2}(u, x_i)\right)\, du \leq (6k-12) \int_T g\left(d_{S^2}(u, a)\right)\, du. \tag{7.54}$$

Assuming that f and g are strictly monotone, equality holds if $x_1, \dots, x_k$ are vertices of an inscribed regular tetrahedron ($k = 4$), or octahedron ($k = 6$) or icosahedron ($k = 12$).

Remark 7.7.7. (i) A stability version of Theorem 7.7.6 is verified by Böröczky, G. Fejes Tóth [97].

(ii) The integrals in (7.53) and in (7.54) make sense even if $x_1,\dots,x_k \in S^2$ are contained in a closed hemispere; namely, even if there exists a $u \in S^2$ with $d_{S^2}(u,x_i) \geq \frac{\pi}{2}\ \forall\, x_i$. The point is that $o \in \mathrm{conv}\{x_1,\dots,x_k\} = Q \subset \mathbb{R}^3$ as $x_1,\dots,x_k$ are not contained in any open hemispere, and hence $o \in \partial Q$ and $u \in N_Q(o)$. However, $\dim N_Q(o) \leq 2$ as o is not a vertex (it is not among $x_1,\dots,x_k$); therefore, $N_K(o) \cap S^2 = \{v \in S^2 : d_{S^2}(v,x_i) \geq \frac{\pi}{2}\ \forall\, x_i\}$ satisfies that $\mathcal{H}^2(N_K(o) \cap S^2) = 0$.

We have already seen in Steiner's Theorem 7.7.3 that regular tetrahedra are optimal for the isoperimetric quotient. L. Fejes Tóth [213, 216, 217] proved that the other two Platonic solids that are simple polytopes are also extremal.

Theorem 7.7.8 (The Isoperimetric property of the 3-cube and the Dodecahedron). *For $k = 6, 12$, among 3-polytopes of given volume and having at most k faces, the ones with minimal surface area are the cube and dodecahedron, respectively.*

Proof. According to Lindelöf's Theorem 7.7.5, any polytope $P_0 \subset \mathbb{R}^3$ with at most k faces minimizing the isoperimetric ratio $S(P_0)/|P_0|^{\frac{2}{3}}$ among polytopes with at most k facets is circumscribed around some ball B and has k faces. We may assume that $B = B^3$, and hence $S(P_0)/|P_0|^{\frac{2}{3}} = 3|P_0|^{\frac{1}{3}}$.

Let the faces of P_0 touch S^2 in exterior unit normals $u_1,\dots,u_k \in S^2$; therefore, the radial function of P is $\varrho_{P_0}(u) = \min\{\langle u,u_i\rangle^{-1} : \langle u,u_i\rangle > 0\}$ for $u \in S^2$, thus Lemma 1.11.6 yields

$$|P_0| = \frac{1}{3}\int_{S^2} \varrho_{P_0}(u)^3\,du = \frac{1}{3}\int_{S^2} \min_{u_i:\langle u,u_i\rangle>0} \langle u,u_i\rangle^{-3}\,du$$
$$= \frac{1}{3}\int_{S^2} \min_{u_i:\langle u,u_i\rangle>0} \cos^{-3} d_{S^2}(u,u_i)\,du.$$

In turn, Theorem 7.7.8 follows from the Moment Theorem (7.53) with $f(t) = \cos^{-3} t$. ■

Actually even a stability version of Theorem 7.7.8 is known, due to Böröczky, G. Fejes Tóth [97].

We recall that according to the Isoperimetric Inequality (cf. Theorem 2.4.1 or Theorem 4.1.5), if $K \subset \mathbb{R}^n$ is a convex body, then

$$\frac{S(K)}{|K|^{\frac{n-1}{n}}} \geq \frac{S(B^n)}{|B^n|^{\frac{n-1}{n}}} = n\omega_n^{\frac{1}{n}} \sim \sqrt{2e\pi}\cdot\sqrt{n}$$

where $f(n) \sim g(n)$ for positive $f(n), g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$. We provide a bound on the isoperimetric quotient of convex bodies with given number of facets. The argument uses the bounds in (6.46); namely, if $P_{(k)} \supset B^n$ polytope of minimal volume with k facets, then

$$c_0^{-1}\left(\log\frac{k}{n}\right)^{\frac{-1}{2}} \leq |P_{(k)}|^{\frac{1}{n}} \leq c_0\left(\log\frac{k}{n}\right)^{\frac{-1}{2}} \tag{7.55}$$

for an absolute constant $c_0 > 1$.

Theorem 7.7.9. *For an absolute constant $c > 0$, if $2n \leq k \leq 2^n$ and $P_{(k)} \subset \mathbb{R}^n$ is a polytope with at most k facets, then*

$$\frac{S(P_{(k)})}{|P_{(k)}|^{\frac{n-1}{n}}} \geq \frac{cn}{\sqrt{\log \frac{k}{n}}}. \tag{7.56}$$

Remark. The estimate (7.56) is sharp, as there exist an absolute constant $\tilde{c} > 0$ and a polytope $\widetilde{P}_{(k)} \subset \mathbb{R}^n$ with at most k facets such that

$$\frac{S(\widetilde{P}_{(k)})}{|\widetilde{P}_{(k)}|^{\frac{n-1}{n}}} \leq \frac{\tilde{c}n}{\sqrt{\log \frac{k}{n}}}. \tag{7.57}$$

Proof. It is equivalent to prove that the estimates of (7.56) and (7.57) hold for a polytope $P_{(k)}$ of at most k facets minimizing the isoperimetric quotient $S(P_{(k)})/|P_{(k)}|^{\frac{n-1}{n}}$. According to Lindelöf's Theorem 7.7.5, we may assume that $P_{(k)}$ is circumscribed around B^n, and hence $S(P_{(k)})/|P_{(k)}|^{\frac{n-1}{n}} = n|P_{(k)}|^{\frac{1}{n}}$. In particular, the lower bound in (7.55) verifies (7.56), and the optimality of (7.56) follows from the upper bound in (7.55). ■

7.8 Comments to Section 7

The fact that non-negative linear combination of compact convex sets is a homogeneous polynomial in the coefficients, and the basic properties of mixed volumes were established by Minkowski [464,465] around 1900, extending the ideas by Steiner [541] (cf. Theorem 7.1.1) about the volume of the parallel body around 1840. The intrinsic volume $V_i(K)$ of a compact convex set $K \subset \mathbb{R}^n$, the practical, dimension invariant renormalization of the mixed volumes of form $V(B^n, K; i)$ (traditionally called the quermassintegral $W_{n-i}(K)$), was introduced by McMullen [446] only in 1991. The integral formula in Theorem 7.1.1 representing the intrinsic volumes as mean projections is due to Kubota [388] around 1925.

Minkowski's inequality is due to Minkowski [464,465] around 1900. Aleksandrov [3, 5, 7] already provided two proofs of the Aleksandrov-Fenchel Inequality around 1937-38 (for additional arguments still based on Aleksandrov's ideas, see also van Handel, Shenfeld [300], Schneider [522] and Cordero-Erausquin, Klartag, Merigot, Santambrogio [176]). Fenchel only stated the inequality, never actually provided a proof. Both arguments in this monograph, the one using strongly isomorphic polytopes in Section 7.A, and the one using the theory of elliptic operators in Section 8.5.2, are based on Aleksandrov's original ideas as developed further by van Handel, Shenfeld [300].

Bernstein [65] and Kouchnirenko [386] proved a very interesting connection between the mixed volumes and the intersection number of complex algebraic hypersurfaces. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, we write $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$ for $x = (x_1, \ldots, x_n) \in (\mathbb{C}\backslash\{o\})^n$. à laurent polynomial is of the form $p(x) = \sum_{\alpha\in\mathbb{Z}^n} c_\alpha x^\alpha$ where there are only finitely many $c_\alpha \neq 0$, and the associated Newton polytope of p is $Q(p) = \mathrm{conv}\{\alpha \in \mathbb{Z}^n : c_\alpha \neq 0\}$, and the associated complex hypsurface is $X(p) = \{x \in (\mathbb{C}\backslash\{o\})^n : p(x) = 0\}$. Given lattice n-dimensional polytopes $Q_1, \ldots, Q_n$ (convex hull of finitely many points of $\mathbb{Z}^n$), the Bernstein-Kouchnirenko theorem says that for generic Laurent polynomial $p_1, \ldots, p_n$ with $Q(p_i) = Q_i$, we have

$$n!V(Q_1, \ldots, Q_n) = \#(X(p_1) \cap \ldots \cap X(p_n)). \tag{7.58}$$

Here the condition "generic" ensures that $X(p_1), \ldots, X(p_n)$ are smooth hypersurfaces in $(\mathbb{C}\backslash\{o\})^n$ that intersect transversally in finitely many points.

The Bézout theorem stating that given integers $d_1, \ldots, d_n \geq 1$, the number of common roots of generic polynomials p_i of degree d_i, $i = 1, \ldots, n$, is $d_1 \cdot \ldots \cdot d_n$ is a special case of the Bernstein-Kouchnirenko theorem (7.58). For $\Delta = \mathrm{conv}\{o, e_1, \ldots, e_n\}$ where $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{Z}^n$, we take $Q_i = d_i\Delta$ in (7.58).

Another connection between the mixed volumes and the intersection number in algebraic geometry is via ample divisors on projective toric varieties (see Cox, Little, Schenck [179], Ewald [207], Fulton [250] and Oda [477]). In this setting, the Aleksandrov-Fenchel inequality corresponds to the Hodge index theorem. In turn, the relations between the Aleksandrov-Fenchel inequality and the Hodge-Riemann theorem has been recently investigated in terms of valuations of convex bodies by Kotrbatý [383] and Kotrbatý, Wannerer [384, 385].

The Aleksandrov-Fenchel inequality for mixed volumes also has important applications in combinatorics, for example concerning counting the number of linear extensions of a poset (see for example Chan, Pak [148], Kahn, Saks [364] and Stanley [540]). Here we sketch Stanley's result in [540] about linear extensions of a k element poset (partially ordered set) $P = \{p_0, \ldots, p_{k-1}\}$ for $k \geq 3$. A linear extension of P is an order preserving bijection $\pi : P \to \{1, \ldots, k\}$, and for $m = 1, \ldots, k$, let N_m be the number of linear extensions of P where $\pi(p_0) = m$. For the polytopes

$$\begin{aligned} Q_0 &= \{(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} : x_i \leq x_j \text{ if } p_i \prec p_j \text{ and } x_i = 0 \text{ if } p_0 \prec p_i\}, \\ Q_1 &= \{(x_1, \ldots, x_{k-1}) \in \mathbb{R}^{k-1} : x_i \leq x_j \text{ if } p_i \prec p_j \text{ and } x_i = 1 \text{ if } p_0 \succ p_i\}, \end{aligned}$$

Stanley [540] verifies that N_m can be represenseted as a $(k-1)$-dimensional mixed volume

$$N_m = V(Q_1, i-1; Q_0, k-m),$$

and applies the Aleksandrov-Fenchel inequality (7.32) to prove that

$$N_m^2 \geq N_{m-1}N_{m+1}$$

for $m = 2, \dots, k-1$; or in other words, the sequence $N_1, \dots, N_k$ is log-concave. According to van Handel, Shenfeld [302] characterizing the equality case of the Aleksandrov-Fenchel inequality for polytopes, if $N_m > 0$ and $N_m^2 = N_{m-1}N_{m+1}$, then $N_m = N_{m-1} = N_{m+1}$.

Schneider [524] - clarifying earlier work by Esterov [205], who used representation of mixed volumes using toric varieties - expressed the mixed volumes of polytopes in terms of the family of their normal cones and the product of their support functions. For fixed n-dimensional polytopes $\widetilde{P}_1, \dots, \widetilde{P}_n \subset \mathbb{R}^n$, let $\mathcal{P}$ denote the family of n-tuples $(P_1, \dots, P_n)$ of n-polytopes such that the family of normal cones of P_i coincide with the family of normal cones of $\widetilde{P}_i$. In this case, Schneider [524] prove that the mixed volumes of the n-tuples in $\mathcal{P}$ depend only on the product of the n support functions. A typical example when this theorem applies is when all polytopes are strongly isomorphic (cf. Section 1.8).

Concerning the isoperimetric problem in the spherical and hyperbolic spaces (see Section 4.A for their fundamental properties), the isoperimetric inequality for polygons of given number of vertices (sides) was extended to the spherical and the hyperbolic space by L. Fejes Tóth [216]. In addition, L. Fejes Tóth [215] proved that among tetrahedra of given volume in H^3, the regular one has the minimal surface area. Isoperimetric-type problems for polytopes in two and three-dimensional spaces of constant curvature are discussed in L. Fejes Tóth [216], L. Fejes Tóth, G. Fejes Tóth, W. Kuperberg [217], Florian [237], and in higher dimensional spaces by Basit, Láng [58].

Even if mixed volumes of arbitrary convex bodies have no analogues in the spherical and hyperbolic space, mean projections and mean curvatures do have analogues, and Aleksandrov-Fenchel-type inequalities are proved for them by for example Andrews, Hu, Li [24], Hu, Li [327] and de Lima, Girão [410].

7.A Supplement: Proof of the Aleksandrov-Fenchel inequality based on polytopes

This section proves of the Aleksandrov-Fenchel inequality for the mixed volumes based on the argument by Aleksandrov [4], as it was simplified by Shenfeld, van Handel [300].

Theorem 7.A.1 (Aleksandrov-Fenchel Inequality). *If $n \geq 3$ and $C_1, \dots, C_{n-2}, K, L$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K, L, C_1, \dots, C_{n-2})^2 \geq V(K, K, C_1, \dots, C_{n-2})V(L, L, C_1, \dots, C_{n-2}). \quad (7.59)$$

7.A.1 Some properties of symmetric matrices

If $\mathcal{E}$ is a symmetric $d \times d$ matrix for $d \geq 2$, then there exist eigenvectors $x_1, \dots, x_d$ forming an orthonormal basis of $\mathbb{R}^d$ and eigenvalues $\lambda_1 \geq \dots \geq \lambda_d$ such that $\mathcal{E}x_i = \lambda_i x_i$ and if $j = 1, \dots, d-1$ and $x \in \mathbb{R}^d$ satisfy that $\langle x_i, x \rangle = 0$ for $i = 1, \dots, j$, then

$$\langle \mathcal{E}x, x \rangle \leq \lambda_{j+1} \langle x, x \rangle. \tag{7.60}$$

We also need the version Theorem 7.A.2 of the Perron-Frobenius theorem (for a proof, see Theorem 10.8.2 in the Appendix). Let $\mathcal{E} = [e_{ij}]$ be a $d \times d$ matrix. We say that $\mathcal{E}$ is non-negative, if $e_{ij} \geq 0$ for any $i, j = 1, \dots, d$, and we say that the off-diagonal entries of $\mathcal{E}$ are non-negative, if $e_{ij} \geq 0$ when $i \neq j$. Assuming that the off-diagonal entries of $\mathcal{E}$ are non-negative, we say that $\mathcal{E}$ is irreducible, if for any $i \neq j$ there exist pairwise different $i_0, \dots, i_k \in \{1, \dots, d\}$ such that $i_0 = i$, $i_k = j$, and $e_{i_{m-1}, i_m} > 0$ for $m = 1, \dots, k$.

Theorem 7.A.2 (Perron-Frobenius Theorem for irreducible symmetric matrices). *If $\mathcal{E}$ is symmetric non-negative irreducible $d \times d$ matrix with positive entries on the diagonal, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $\mathcal{E}x_1 = \lambda_1 x_1$;*
- *any eigenvector x of $\mathcal{E}$ whose coordinates are all non-negative satisfy $x = r\, x_1$ for $r > 0$.*

If all we know about a $d \times d$ matrix $\mathcal{E}$ that its off-diagonal entries of $\mathcal{E}$ are non-negative and $\mathcal{E}$ is irreducible, then there exists $\gamma > 0$ such that $\mathcal{E} + \gamma\, I_d$ is a non-negative irreducible $d \times d$ matrix with positive entries on the diagonal. Since λ is an eigenvalue of $\mathcal{E}$ with eigenvector x if and only if x is an eigenvector for $\mathcal{E} + \gamma\, I_d$ with eigenvalue $\lambda + \gamma$, we deduce the following consequence of the Perron-Frobenius Theorem 7.A.2:

Corollary 7.A.3. *If the off-diagonal entries of a symmetric $d \times d$ matrix $\mathcal{E}$ are non-negative and $\mathcal{E}$ is irreducible, and λ_1 is the largest eigenvalue, then*

- *λ_1 is a simple eigenvalue;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $\mathcal{E}x_1 = \lambda_1 x_1$;*
- *any eigenvector x of $\mathcal{E}$ whose coordinates are all non-negative satisfy $x = r\, x_1$ for $r > 0$.*

If $\mathcal{E}$ is a symmetric matrix, its positive eigenspace is the subspace spanned by the eigeinvectors corresponding to positive eigenvalues.

Lemma 7.A.4 (Hyperbolic Quadratic Forms). *For a symmetric $d \times d$ matrix $\mathcal{E}$, $d \geq 2$, the following conditions are equivalent for $x, y \in \mathbb{R}^d$.*

(i) $\langle x, \mathcal{E}y \rangle^2 \geq \langle x, \mathcal{E}x \rangle \langle y, \mathcal{E}y \rangle$ *if* $\langle y, \mathcal{E}y \rangle \geq 0$.

(ii) The dimension of the positive eigenspace of $\mathcal{E}$ is at most one.

Proof. Let $\lambda_1 \geq \ldots \geq \lambda_d$ be the eigenvectors of $\mathcal{E}$, and let $x_1, \ldots, x_d$ be the eigenvectors forming an orthonormal basis of $\mathbb{R}^d$ such that $\mathcal{E}x_i = \lambda_i x_i$. We may assume that $\lambda_1 > 0$ because otherwise $\langle x, \mathcal{E}x\rangle \leq 0$ for any $x \in \mathbb{R}^d$, and hence Lemma 7.A.4 readily holds.

Assuming (i), $0 \geq \lambda_1\lambda_2\langle x_1, x_1\rangle\langle x_2, x_2\rangle$ follows by applying (i) to $x = x_1$ and $y = x_2$; therefore, $\lambda_i \leq \lambda_2 \leq 0$ for $i \geq 2$.

Assuming (ii), we may also assume that $\langle y, \mathcal{E}y\rangle > 0$. Since $\lambda_i \leq \lambda_2 \leq 0$ by (ii), we deduce that $\langle z, \mathcal{E}z\rangle \leq 0$ holds for any $z \in \mathbb{R}^d$ satisfying $\langle z, \mathcal{E}x_1\rangle = 0$. In particular, $\langle y, \mathcal{E}x_1\rangle \neq 0$ follows from $\langle y, \mathcal{E}y\rangle > 0$. Since $\langle z, \mathcal{E}x_1\rangle = 0$ for $\alpha = \langle x, \mathcal{E}w\rangle/\langle y, \mathcal{E}w\rangle$ and $z = x - \alpha y$, it also follows that the condition that $\mathcal{E}$ is symmetric yields

$$\begin{aligned}0 &\geq \langle z, \mathcal{E}z\rangle = \langle x, \mathcal{E}x\rangle - 2\alpha\langle x, \mathcal{E}y\rangle + \alpha^2\langle y, \mathcal{E}y\rangle \\ &= \langle x, \mathcal{E}x\rangle - \frac{\langle x, \mathcal{E}y\rangle^2}{\langle y, \mathcal{E}y\rangle} + \langle y, \mathcal{E}y\rangle\left(\alpha - \frac{\langle x, \mathcal{E}y\rangle}{\langle y, \mathcal{E}y\rangle}\right)^2 \geq \langle x, \mathcal{E}x\rangle - \frac{\langle x, \mathcal{E}y\rangle^2}{\langle y, \mathcal{E}y\rangle}.\end{aligned}$$

In turn, we conclude $\langle x, \mathcal{E}y\rangle^2 \geq \langle x, \mathcal{E}x\rangle\langle y, \mathcal{E}y\rangle$. ∎

7.A.2 Mixed volumes of strongly isomorphic simple polytopes and multilinear forms

The sole goal of this section is to prove Theorem 7.A.7, which yields that mixed volumes of strongly isomorphic simple n-polytopes in $\mathbb{R}^n$ can be represented by symmetric multilinear forms of the vectors made up from the values of the support functions at the unit exterior normals of the facets. First, we recall some facts about simple and strongly isomorphic polytopes (cf. Section 1.8). For $d \geq 1$ and d-dimensional strongly isomorphic $M_1, \ldots, M_d \subset \mathbb{R}^n$ such that their affine hulls are parallel to a linear d-space $L \subset \mathbb{R}^n$, we write $V(M_1, \ldots, M_d)$ to denote their d-dimensional mixed volume (the dimension is determined by the number of variables). Writing $v_1, \ldots, v_m \in L \cap S^{n-1}$ to denote the common normals to the $(d-1)$-dimensional faces, (7.23) says that

$$V(M_1, \ldots, M_d) = \frac{1}{d}\sum_{i=1}^{m} h_{M_d}(v_i) \cdot V(F_{M_1}(v_i), \ldots, F_{M_{d-1}}(v_i)) \tag{7.61}$$

where we set $V(F_{M_1}(v_i), \ldots, F_{M_{d-1}}(v_i)) = 1$ if $d = 1$.

The polytopes $P_1, \ldots, P_k \subset \mathbb{R}^n$ are strongly isomorphic if for any $1 \leq i < j \leq n$, P_i and P_j have exactly the same set of normal cones; or equivalently, $\dim F_{P_i}(u) = \dim F_{P_j}(u)$ for any $u \in S^{n-1}$. In this case, $\dim P_i = \dim P_j$ and the affine hulls of P_i and P_j are parallel if $\dim P_i < n$. A d-dimensional polytope $P \subset \mathbb{R}^n$ is simple, $1 \leq d \leq n$, if exactly d edges meet at any vertex of P. We note that any at least one-dimensional face of a simple polytope is simple. According to Lemma 1.8.6, a family of strongly isomorphic polytopes can be obtained by deforming a simple polytope.

Lemma 7.A.5. *For a simple n-polytope $P \subset \mathbb{R}^n$, there exists $\varepsilon_0 > 0$, such that if $u_1, \ldots, u_n$ are the unit exterior normals of the facets of P, and $|t_i - h_P(u_i)| < \varepsilon_0$ for $i = 1, \ldots, k$, then $P' = \{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq t_i\}$ is a simple n-polytope strongly isomorphic to P.*

For the rest of the section, we fix a simple n-polytope $K \subset \mathbb{R}^n$, $n \geq 2$, and write $\mathcal{P}_K$ to denote the family of simple polytopes in $\mathbb{R}^n$ strongly ismorphic to K, $\mathcal{F}_K^d$ to denote the family of d-dimensional faces K, $d = 0, \ldots, n-1$, and $\mathcal{U}_K$ to denote the family of exterior unit normals to the facets $F \in \mathcal{F}_K^{n-1}$ of K. Any $P \in \mathcal{P}_K$ can be encoded by the vector $\bar{h}_P \in \mathbb{R}^{\mathcal{U}(K)}$ such that the coordinate of $\bar{h}_P$ corresponding to $u \in \mathcal{U}_K$ is $\bar{h}_P^{(u)} = h_P(u)$. Next, for any face $F \in \mathcal{F}_K^d$, $d = 1, \ldots, n-1$, let $L_F = \mathrm{lin}(F - F)$ be the pararllel linear d-subspace, let $\mathcal{P}_F$ be the family of simple d-polytopes in $\mathbb{R}^n$ strongly ismorphic to F, and let $\mathcal{U}_F \subset L_F \cap S^{n-1}$ be the family of exterior unit normals to the $(d-1)$-faces of F. Similarly as above, any polytope $P \in \mathcal{P}_F$ can be encoded by the vector $\bar{h}_P \in \mathbb{R}^{\mathcal{U}(F)}$ such that the coordinate of $\bar{h}_P$ corresponding to $u \in \mathcal{U}(F)$ is $\bar{h}_P^{(u)} = h_P(u)$. Let us show how these vectors of the form $\bar{h}_P$, $P \in \mathcal{P}_F$ generate $\mathbb{R}^{\mathcal{U}(F)}$.

Claim 7.A.6. *Let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \ldots, n-1\}$. For any $x \in \mathbb{R}^{\mathcal{U}(G)}$ and $P \in \mathcal{P}_G$, there exist $a > 0$ and $Q \in \mathcal{P}_G$ such that $x = \bar{h}_Q - a\bar{h}_P$.*

Proof. Choose $a > 0$ such $a^{-1}\|x\| < \varepsilon_0$ for the $\varepsilon_0 > 0$ in Lemma 7.A.5. Writing $x^{(u)}$ to denote the coordinate of x corresponding to a $u \in \mathcal{U}(G)$, Lemma 7.A.5 provides a $Q' \in \mathcal{P}$ such that if $u \in \mathcal{U}(G)$, then $h_{Q'}(u) = h_P(u) + a^{-1}x^{(u)}$, and hence $x = \bar{h}_Q - a\bar{h}_P$ for $Q = aQ'$. ■

For any $F \in \mathcal{F}_K^d$, $d = 1, \ldots, n-1$, and strongly isomorphic $P_1, \ldots, P_d \in \mathcal{P}_F$, we write $V(P_1, \ldots, P_d)$ to denote their mixed volume (the common dimension of the polytopes is determined by the number of variables). For the irreducibility of a symmetric matrix, see the discussion in front of Theorem 7.A.2

Theorem 7.A.7. *Using the notation as above, there exists a symmetric (invariant under the permutations of the variables) real multilinear form $V(x_1, \ldots, x_n)$ of $x_1, \ldots, x_n \in \mathbb{R}^{\mathcal{U}(K)}$, and for any $G \in \mathcal{F}_K^d$ for $d \in \{1, \ldots, n-1\}$, there exists a symmetric real multilinear form $V(x_1, \ldots, x_d)$ of $x_1, \ldots, x_d \in \mathbb{R}^{\mathcal{U}(G)}$ with the following properties:*

(i) $V(\bar{h}_{P_1}, \ldots, \bar{h}_{P_n}) = V(P_1, \ldots, P_n)$ for $P_1, \ldots, P_n \in \mathcal{P}_K$, and if $F \in \mathcal{F}_K^d$ for $d \in \{1, \ldots, n-1\}$, then $V(\bar{h}_{P_1}, \ldots, \bar{h}_{P_d}) = V(P_1, \ldots, P_d)$ for $P_1, \ldots, P_d \in \mathcal{P}_F$.

(ii) Let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \ldots, n-1\}$. For any $x \in \mathbb{R}^{\mathcal{U}(G)}$ and $u \in \mathcal{U}(G)$, there exists a "$(d-1)$-face" $F_x(u) \in \mathbb{R}^{\mathcal{U}(F_G(u))}$ of x such that

- *for fixed $u \in \mathcal{U}(G)$, $F_x(u)$ is a linear function of $x \in \mathbb{R}^{\mathcal{U}(G)}$;*
- *for fixed $u \in \mathcal{U}(G)$, $F_x(u) = \bar{h}_{F_P(u)}$ if $x = \bar{h}_P$ for $P \in \mathcal{P}_G$;*

- *if* $x_1, \ldots, x_d \in \mathbb{R}^{\mathcal{U}(G)}$ *and* $x_d = (x_d^{(u)})_{u \in \mathcal{U}(G)}$*, then*

$$V(x_1, \ldots, x_d) = \frac{1}{d} \sum_{u \in \mathcal{U}(G)} x_d^{(u)} \cdot V\left(F_{x_1}(u), \ldots, F_{x_{d-1}}(u)\right). \tag{7.62}$$

(iii) Let $n = 2$*, or let* $n \geq 3$ *and* $P_1, \ldots, P_{n-2} \in \mathcal{P}_K$*. Then there exist a symmetric irreducible matrix* $\mathcal{A} : \mathbb{R}^{\mathcal{U}(K)} \to \mathbb{R}^{\mathcal{U}(K)}$ *such that* $\mathcal{A}$ *has non-negative off-diagonal entries, and for* $x, y \in \mathbb{R}^{\mathcal{U}(K)}$*, the coordinate of* $\mathcal{A}x$ *corresponding to* $u \in \mathcal{U}(K)$ *is*

$$(\mathcal{A}x)^{(u)} = \begin{cases} V\left(F_x(u), F_{P_1}(u), \ldots, F_{P_{n-2}}(u)\right) & \text{if } n \geq 3, \\ V(F_x(u)) & \text{if } n = 2, \end{cases} \tag{7.63}$$

and hence

$$\langle x, \mathcal{A}y \rangle = nV(x, y, P_1, \ldots, P_{n-2}) \tag{7.64}$$

where $V(x, y, P_1, \ldots, P_{n-2}) = V\left(x, y, \bar{h}_{P_1}, \ldots, \bar{h}_{P_{n-2}}\right)$*, and* $\langle x, \mathcal{A}y \rangle = 2V(x, y)$ *if* $n = 2$*.*

Proof. We prove (i) and (ii) by induction on $d = 1, \ldots, n$. If $d = 1$ and $F \in \mathcal{F}_K^1$ is an edge, then let $F = [p, q]$, and hence $u = \frac{p-q}{\|p-q\|}$ is the exterior unit normal at p and $\mathcal{U}(F) = \{u, -u\}$. We deduce that $V(F) = \langle u, p - q \rangle = h_F(u) + h_F(-u)$ (cf. (7.61)); therefore, $V(x) = x^{(u)} + x^{(-u)}$ for $x \in \mathbb{R}^{\mathcal{U}(F)}$.

Next let $G = K$, or $G \in \mathcal{F}_K^d$ for $d \in \{2, \ldots, n-1\}$. Any $(d-2)$-dimensional face of G is contained in exactly two $(d-1)$-faces according to Proposition 1.4.3, thus let $Q_G = \{(u, v) \in \mathcal{U}(G) \times \mathcal{U}(G) : \dim(F_G(u) \cap F_G(v) = d - 2\}$ where the empty set has dimension -1. For $(u, v) \in Q_G$ and $P \in \mathcal{P}_G$, let α_{uv} be the angle of u and v, let $F_P(u, v) = F_P(u) \cap F_P(v)$ the common $(d-2)$-face of $F_P(u)$ and $F_P(v)$, and let $w_{u,v} \in \mathcal{U}(F_P(u, v))$ be the exterior unit normal to the $(d-2)$-face $F_P(u, v)$ of $F_P(u)$ in $L_{F_P(u,v)}$ where $w_{u,v} \in \mathcal{U}(F_G(u, v))$ depends only on G, u, v and not on the choice of P. We observe that $\alpha_{uv} = \alpha_{vu}$ and $F_P(u, v) = F_P(v, u)$, but $w_{u,v} \neq w_{v,u}$ for $(u, v) \in Q_G$. For $(u, v) \in Q_G$, it follows that

$$w_{u,v} = \csc \alpha_{uv} \cdot v - \cot \alpha_{uv} \cdot u$$

for $\csc \alpha_{uv} = 1/\sin \alpha_{uv} > 0$, and hence choosing some $p_{uv} \in F_P(u, v)$, we have

$$h_{F_P(u)}(w_{u,v}) = \langle p_{uv}, w_{u,v} \rangle = \csc \alpha_{uv} \cdot h_P(v) - \cot \alpha_{uv} \cdot h_P(u). \tag{7.65}$$

Since $\mathcal{U}(F_G(u)) = \{w_{u,v} : (u, v) \in Q_G\}$ for $u \in \mathcal{U}(G)$, we deduce from (7.61) that if $P_1, \ldots, P_d \in \mathcal{P}_G$, then

$$V(F_{P_1}(u), \ldots, F_{P_{d-1}}(u)) = \tag{7.66}$$
$$\frac{1}{d-1} \sum_{\{v:(u,v) \in Q_G\}} \left(\csc \alpha_{uv} \cdot h_{P_{d-1}}(v) - \cot \alpha_{uv} \cdot h_{P_{d-1}}\right) V(F_{P_1}(u, v), \ldots, F_{P_{d-2}}(u, v))$$

where we set $V(F_{P_1}(u,v),\ldots,F_{P_{d-2}}(u,v)) = 1$ if $d = 2$. Applying again (7.61) yields

$$V(P_1,\ldots,P_d) = \frac{1}{d}\sum_{u\in\mathcal{U}(G)} h_{P_d}(u)\cdot V(F_{P_1}(u),\ldots,F_{P_{d-1}}(u)) = \tag{7.67}$$

$$\frac{1}{d(d-1)}\sum_{(u,v)\in Q_G}\left(\csc\alpha_{uv}\cdot h_{P_d}(u)\cdot h_{P_{d-1}}(v) - \cot\alpha_{uv}\cdot h_{P_d}(u)\cdot h_{P_{d-1}}(u)\right)\cdot$$
$$\cdot V(F_{P_1}(u,v),\ldots,F_{P_{d-2}}(u,v)).$$

Therefore, we define $F_x(u)$ for $x\in\mathbb{R}^{\mathcal{U}(G)}$ and $u\in\mathcal{U}(G)$ in a way such that the coordinate of $F_x(u)$ corresponding to $w_{u,v}$ provided $(u,v)\in Q_G$ is is

$$\csc\alpha_{uv}\cdot x^{(v)} - \cot\alpha_{uv}\cdot x^{(u)}.$$

We conclude the existence of a suitable multilinear function $V(x_1,\ldots,x_d)$ of $x_1,\ldots,x_d\in\mathbb{R}^{\mathcal{U}(G)}$ satisfying (i) and (ii) (*via*) combining (7.65), (7.66), (7.67) and the induction hypothesis. To show that $V(x_1,\ldots,x_d)$ is symmetric in $x_1,\ldots,x_d$, we note that $x_i = \bar{h}_{Q_i} - \bar{h}_{Q_i'}$ for some $Q_i, Q_i'\in\mathcal{P}_G$, $i = 1,\ldots,d$, according to Claim 7.A.6, and we use the symmetry of the mixed volumes in its variables.

Finally, we consider (iii), where either $n = 2$, or $n\geq 3$ and we are given some $P_1,\ldots,P_{n-2}\in\mathcal{P}_K$. We deduce from (7.66) and (7.67) that there exists a symmetric matrix $\mathcal{A}:\mathbb{R}^{\mathcal{U}(K)}\to\mathbb{R}^{\mathcal{U}(K)}$ such that $\mathcal{A}$ has non-negative off-diagonal entries, and $\mathcal{A}$ satisfies (7.63) and (7.64) for $x,y\in\mathbb{R}^{\mathcal{U}(K)}$.

Therefore, all we have to check is the irreducibility of $\mathcal{A}$. What we know is that the entry of $\mathcal{A}$ corresponding to a $(u,v)\in Q_K$ is positive by (7.63). Now let $u,v\in\mathcal{U}(K)$, $u\neq v$ be arbitrary. Choose $p\in\operatorname{relint}F_K(u)$ and $q\in\operatorname{relint}F_K(v)$, and an affine 2-plane A containing p and q such that A does not intersect any $(n-3)$-dimensional face of K. It follows that $A\cap K$ is a polygon whose vertices are the intersections of A with certain $(n-2)$-faces of K, and the edges are intersections with facets of K. Therefore, taking a path on the relative boundary of $A\cap K$ from p to q, the facets of K containing the edges of the path generate an ordered list $u_0,\ldots,u_m\in Q_K$ with $u_0 = u$ and $u_m = v$ where the $(n-2)$-face of K corresponding to $(u_{i-1},u_i)\in Q_K$ generates the ith vertex of the path. In turn, we conclude the irreducibility of $\mathcal{A}$. ■

7.A.3 Proof of the Aleksandrov-Fenchel inequality using strongly isomorphic simple polytopes

The key step of the proof of the Aleksandrov-Fenchel inequality (7.59) is to verify the "reverse Cauchy-Schwarz" inequality in Proposition 7.A.8 for bilinear forms built from strongly isomorphic polytopes. We recall for a simple n-polytope $K\subset\mathbb{R}^n$, $\mathcal{P}_K$ is the family of polytopes strongly isomorphic to K, and $\mathcal{U}(K)$ is the family of exterior unit normals to the facets of K. A $P\in\mathcal{P}_K$ is encoded by the vector $\bar{h}_P\in\mathbb{R}^{\mathcal{U}(K)}$ where the coordinate of $\bar{h}_P$ corresponding to a $u \in \mathcal{U}(K)$ is $h_P(u)$, and we use P and $\bar{h}_P$ interchangeably in the symmetric multilinear form $V(\cdot, \dots, \cdot)$ of Theorem 7.A.7. We also note that according to the Minkowski inequality (7.26), if $K, L \subset \mathbb{R}^2$ are polygons, then

$$V(K,L)^2 \geq V(K,K)V(L,L). \tag{7.68}$$

Proposition 7.A.8. *Let $K \subset \mathbb{R}^n$ be a simple n-polytope, $n \geq 2$, let $P \in \mathcal{P}_K$ and if $n \geq 3$, then let $C_1, \dots, C_{n-2} \in \mathcal{P}_K$ such that $o \in \operatorname{int} C_1$. If $x \in \mathbb{R}^{\mathcal{U}(K)}$, then*

$$V(x, P, C_1, \dots, C_{n-2})^2 \geq V(x, x, C_1, \dots, C_{n-2})\, V(P, P, C_1, \dots, C_{n-2}) \tag{7.69}$$

where the inequality is $V(x,P)^2 \geq V(x,x)V(P,P)$ in the case $n = 2$.

Proof. Let the family of exterior unit normals to the facets of K be $\mathcal{U}(K) = \{u_1, \dots, u_k\} \subset S^{n-1}$, and we identify $\mathbb{R}^{\mathcal{U}(K)}$ with $\mathbb{R}^k$. We write $e_1, \dots, e_k$ to denote the corresponding orthonormalt basis of $\mathbb{R}^k$ where e_i corresponds to u_i. We also note that in the case $n \geq 3$, $o \in \operatorname{int} C_1$ yields that

$$h_{C_1}(u_i) > 0 \ \text{ for } i = 1, \dots, k. \tag{7.70}$$

We prove Proposition 7.A.8 by induction on $n \geq 2$. If $n = 2$, then Claim 7.A.6 provides an $a > 0$ and a $Q \in \mathcal{P}_K$ such that $x + a\bar{h}_P = \bar{h}_Q$. We deduce from the Minkowski Inequlaity (7.68) that

$$V(x + a\bar{h}_P, \bar{h}_P)^2 = V(Q,P)^2 \geq V(Q,Q)\,V(P,P) = V(x + a\bar{h}_P, x + a\bar{h}_P)\,V(\bar{h}_P, \bar{h}_P),$$

which inequality is equivalent with $V(x, \bar{h}_P)^2 \geq V(x,x)V(\bar{h}_P, \bar{h}_P)$ by the linearity and symmetry of the bilinear form $V(x,y)$, $x, y \in \mathbb{R}^k$.

Next let $n \geq 3$. Now Theorem 7.A.7 provides a $k \times k$ irreducible symmetric matrix $\mathcal{A}$ with non-negative off-diagonal entries such that if $x, y \in \mathbb{R}^k$, then

$$\langle x, \mathcal{A}y \rangle = V(x, y, C_1, \dots, C_{n-2}).$$

We alter the definition of $\mathcal{A}$ in order to have better control of the eigenvalues, and also alter the scalar product accordingly: Let $\widetilde{\mathcal{A}} : \mathbb{R}^k \to \mathbb{R}^k$ be the linear transform satisfying that for $x \in \mathbb{R}^k$ and $i = 1, \dots, k$, the ith coordinate of $\widetilde{\mathcal{A}}x$ is (cf. (7.63))

$$\begin{aligned}(\widetilde{\mathcal{A}}x)_i &= \frac{h_{C_1}(u_i) \cdot V\left(F_x(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)}{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)} \\ &= (\mathcal{A}x)_i \cdot \frac{h_{C_1}(u_i)}{V\left(F_{C_1}(u_i), F_{C_1}(u_i), \dots, F_{C_{n-2}}(u_i)\right)},\end{aligned} \tag{7.71}$$

and hence

$$\widetilde{\mathcal{A}}\bar{h}_{C_1} = \bar{h}_{C_1}. \tag{7.72}$$

In addition, let $\langle\cdot,\rangle_0$ be the scalar product on $\mathbb{R}^k$ (cf. (7.70)) such that if $x=(x_1,\ldots,x_k)\in\mathbb{R}^k$ and $y=(y_1,\ldots,y_k)\in\mathbb{R}^k$, then

$$\langle x,y\rangle_0=\sum_{i=1}^k x_iy_i\cdot\frac{V\left(F_{C_1}(u_i),F_{C_1}(u_i),\ldots,F_{C_{n-2}}(u_i)\right)}{h_{C_1}(u_i)},$$

and hence there exist $\theta_1,\ldots,\theta_k>0$ such that $\tilde{e}_i=\theta_ie_i, i=1,\ldots,k$ form an orthonormal basis of $\mathbb{R}^k$ with respect to $\langle\cdot,\rangle_0$, and

$$\langle x,\widetilde{\mathcal{A}}y\rangle_0=\langle x,\mathcal{A}y\rangle=V\left(x,y,C_1,\ldots,C_{n-2}\right). \tag{7.73}$$

We deduce that the $k\times k$ matrix of $\widetilde{\mathcal{A}}$ with respect to the basis $\tilde{e}_1,\ldots,\tilde{e}_k$ is symmetric, irreducible, and has non-negative off-diagonal entries. Therefore (7.72), the positivity of $\bar{h}_{C_1}$ (cf. (7.70)) and the Perron-Frobenius theorem Corollary 7.A.3 yields that 1 is the maximal eigenvalue of $\widetilde{\mathcal{A}}$, and it is a simple eigenvaule.

Next we claim that if $x\in\mathbb{R}^k$, then

$$\langle\widetilde{\mathcal{A}}x,\widetilde{\mathcal{A}}x\rangle_0\geq\langle x,\widetilde{\mathcal{A}}x\rangle_0. \tag{7.74}$$

For $n\geq4$, it follows from first applying (7.71) and the definition of $\langle\cdot,\cdot\rangle_0$, and then the known $(n-1)$-dimensional version of (7.69), and finally (7.62) that

$$\begin{aligned}\langle\widetilde{\mathcal{A}}x,\widetilde{\mathcal{A}}x\rangle_0&=\sum_{i=1}^k\frac{h_{C_1}(u_i)\cdot V\left(F_x(u_i),F_{C_1}(u_i),\ldots,F_{C_{n-2}}(u_i)\right)^2}{V\left(F_{C_1}(u_i),F_{C_1}(u_i),\ldots,F_{C_{n-2}}(u_i)\right)}\\&\geq\sum_{i=1}^k h_{C_1}(u_i)\cdot V\left(F_x(u_i),F_x(u_i),F_{C_2}(u_i),\ldots,F_{C_{n-2}}(u_i)\right)\\&=nV(C_1,x,x,C_2,\ldots,C_n).\end{aligned}$$

Here $\langle x,\widetilde{\mathcal{A}}x\rangle_0=nV(C_1,x,x,C_2,\ldots,C_n)$ follows from the the symmetry of the multilinear form and (7.64), proving (7.74) if $n\geq4$. The argument is the same if $n=3$, only no $C_2,\ldots,C_n$ are involved.

Finally, (7.74) yields that any eigenvalue λ of $\widetilde{\mathcal{A}}$ satisfies $\lambda^2\geq\lambda$. Since we have already seen that 1 is the maximal eigenvaule and it is a simple eigenvalue, we deduce that the positive eigenspace of $\widetilde{\mathcal{A}}$ is one dimensional. Since $\langle\bar{h}_P,\widetilde{\mathcal{A}}\bar{h}_P\rangle_0=V\left(P,P,C_1,\ldots,C_{n-2}\right)>0$ by (7.73), Lemma 7.A.4 implies (7.69). ■

Before starting the argument to prove the Aleksandrov-Fenchel Inequality, let us point out that any finite family of compact convex sets can be simultaniously approximated by strongly isomorphic simple polytopes according to Proposition 1.8.5.

Proposition 7.A.9. *For compact convex sets $C_1,\ldots,C_k\subset\mathbb{R}^n$ and $\varepsilon>0$, there exist strongly isomorphic simple polytopes $P_1,\ldots,P_k\subset\mathbb{R}^n$ such that $\delta_H(C_i,P_i)<\varepsilon$ for $i=1,\ldots,k$.*

Proof of the Aleksandrov-Fenchel Inequality (7.59). It follows from Proposition 7.A.9 that the Aleksandrov-Fenchel Inequality (7.59) is equivalent to the following statement: If $n \geq 3$ and $C_1, \ldots, C_{n-2}, K, L \subset \mathbb{R}^n$ are strongly isomorphic simple n-polytopes containing the origin in their interior, then

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geq V(K, K, C_1, \ldots, C_{n-2}) V(L, L, C_1, \ldots, C_{n-2}). \quad (7.75)$$

In turn, (7.75) follows from Poroposition 7.A.8 by taking $L = P$ and $x = \bar{h}_K$ (cf. Theorem 7.A.7. ∎

7.B Supplement: The Simplex Mean Width conjecture and some related results

In this section, we mostly survey some extremal properties of the mean width of polytopes of bounded complexity. We recall that the first instrinsic volume $V_1(P)$ is proportional to the mean width of the polytope $P = \operatorname{conv}\{v_1, \ldots, v_k\} \subset \mathbb{R}^n$, (cf. (7.6) and (7.7)), and

$$V_1(K) = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \max_{i=1,\ldots,k} \langle v_i, u \rangle \, du. \quad (7.76)$$

Remark 7.B.1 (Some extremal properties of the Platonic solids). Let $k = 4, 6, 12$ and $i = 1, 2, 3$. For a 3-polytope $P \supset B^3$ and having at most k faces, the minimum of $V_i(P)$ is attained exactly if P is a circumscribed regular tetrahedron, cube and dodecahedron, respectively. For volume and surface area, this is due to L. Fejes Tóth [213, 216, 217] (see Theorem 7.7.8), for the mean width (or $V_1(P)$), this is due to Florian [236].

For a 3-polytope $P \subset B^3$ and having at most k vertices, the maximum of $V_i(P)$ is attained exactly if P is an inscribed regular tetrahedron, octahedron and icosahedron, respectively. For volume, this is due to L. Fejes Tóth [214, 216], for the surface area ($V_2(P)$), this due to Linhart [413], and for the mean width ($V_1(P)$), this due to Linhart [412] (see Theorem 7.B.2).

We present the argument for the following theorem due to Linhart [412] based on the Moment Theorem (7.54).

Theorem 7.B.2 (Linhart). *For $k = 4, 6, 12$, among 3-polytopes of at most k vertices contained in B^3, the ones with maximal mean width are the inscribed regular tetrahedron, octahedron and icosahedron, respectively.*

Remark. Stability version is due to Böröczky, G. Fejes Tóth [97].

Proof. Let $P \subset B^3$ be a polytope having maximal mean width assuming that P has at most k vertices, and hence P has k vertices $v_1, \ldots, v_k \in S^{n-1}$, and $o \in P$. Therefore, Theorem 7.B.2 follows from (7.76) and applying (7.54) in the Moment Theorem with $g(t) = \cos t$. ∎

If B is ball in either in $\mathbb{R}^n$, or in the hyperbolic space H^n, or in the sphere S^n, then among simplices contained in B, the regular simplex has the maximal volume. Probably known to Steiner in $\mathbb{R}^n$ (as his symmetrization method yields the statement), due to Böröczky [89] on S^n, using spherical Steiner symmetrization, and due to Peyerimhoff [488] in H^n, using Euclidean Steiner symmetrization in the Beltrami-Cayley-Klein model of H^n (see Section 4.A for the Beltrami-Cayley-Klein model). Actually, Haagerup, Munkholm [291] (see Peyerimhoff [488] for a simpler proof) prove that among all simplices in H^n, $n \geq 3$, the regular ideal simplex has maximal volume (ideal simplex in the Beltrami-Cayley-Klein model of H^n is a Euclidean simplex whose vertices are ideal points, and hence lie on S^{n-1}).

If we are to maximize the mean width and not the volume of a simplex contained in a ball in $\mathbb{R}^n$, the problem becomes much more difficult.

Conjecture 7.B.3 (Inscribed Simplex Meanwidth Conjecture). *Among simplices contained B^n, $n \geq 4$, the inscribed regular simplices, and only them, maximize the mean width.*

Litvak [414] provides a survey about history, equivalent formulations of Conjecture 7.B.3 in terms of random Gaussian processes and coding theory. We note that the conjecture was believed to hold by the imformation theory comunity (see for example Balakrishnan [33] in 1963), and false proof was published for example by Landau, Slepian [409] in 1966 applying similar ideas as in the 3-dimensional case Theorem 7.B.2.

If in oder to verify Conjecture 7.B.3, one tries the mimic the argument in the 3-dimensional case of the Simplex Mean Width Conjecture 7.B.3 using the Momen Theorem (7.54), the missing result is the following one:

Conjecture 7.B.4. *For $p \in S^m$, $m \geq 3$ and $r \in (0, \frac{\pi}{2})$, among spherical simplices $Q \subset S^m$ of given m-volume, the regular simplices centered at p maximize $\mathcal{H}^m(Q \cap B_{S^m}(p, r))$.*

The case $m = 2$ of Conjecture 7.B.4 is due to L. Fejes Tóth [216, 217]. Conjecture 7.B.4 would yield the analogue of the Moment Theorem 7.7.6 on S^{n-1} for $n \geq 4$ and $k = n + 1$, and *via* this statement it would yield Conjecture 7.B.3 (see Litvak [414]).

Following Litvak [414], let us collect some observations concerning the Simplex Mean Width Conjecture 7.B.3 where Δ_n is the regular simplex inscribed into B^n.

- $V_1(\Delta_n) \sim 4\sqrt{\pi}\sqrt{\ln n}$ as n tends to infty;
- if $Q \subset B^n$ is a simplex, then $V_1(Q) \leq (1 + \frac{c \ln \ln n}{\ln n})V_1(\Delta_n)$ for an absolute constant $c > 0$;
- if $n = 2m - 1$, and C_m regular cross polytope inscribed into $\mathbb{R}^m \cap B^n$, then $V_1(\Delta_n) \leq \sqrt{\frac{n+1}{n}}\, V_1(C_m)$. This shows how hard the Simplex Mean Width Conjecture is, as an

inscribed polytope of dimension about $n/2$ has the same number of vertices and essentially the same mean width as the n-dimensional regular simplex.

Chapter 8

Convex bodies and Gaussian curvature

We recall from Section 2.1 that if $K \subset \mathbb{R}^n$ is a convex body, then ∂K is Lipschitz, and $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$ is regular; namely, there exists a unique exterior unit normal vector $\nu_K(y) \in S^{n-1}$ at y, and we write $\partial' K$ to denote the set of regular boundary points. However, ∂K is even twice differentiable at $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$ according to Aleksandrov's theorem. This chapter is concerned with notions and arguments using the Gaussian curvature at these points. These notions are naturally easiest to handle for convex bodies with C^2_+ boundaries; therefore, this is the case where we focus our attention. We show that ∂K being C^2_+ is equivalent to saying that h_K is C^2 with positive defnite Hessian on S^{n-1}. The topics include approximation of any convex body by ones with C^∞_+ boundary, using similar type of approximation to verify the weak convergence of surface area measure, various versions of affine surface, and a close to be optimal stability version of Brunn-Minkowski inequality for convex bodies. While Chapter 7 has already introduced mixed volumes using Minkowski's original ideas based on polytopes, in this chapter, we build the theory independently based on Hilbert's and Aleksandrov's approach based on of mixed discriminants of the Hessians of the support functions of convex bodies with C^2_+ boundary. This other approach connects the Brunn-Minkowski theory to the realm of Minkowski type Monge-Ampère equations, discussed in Chapter 9 and Chapter **??**.

8.1 Second order differentiability of the boundary

According to Aleksandrov's Theorem 10.6.2 (ii) on the second order differentiabily of convex functions, for $\mathcal{H}^{n-1}$ a.e. point on the boundary of a convex body in $\mathbb{R}^n$, the boundary is twice differentiable in the following sense:

Theorem 8.1.1 (Aleksandrov). *If $K \subset \mathbb{R}^n$ be a convex body, then $\mathcal{H}^{n-1}$ a.e. point $y \in \partial K$ satifies that $y \in \partial' K$ is a regular point, and writing $u = -\nu_K(y)$ and $x_0 = y|u^\perp$, for the convex function $\varphi : (\mathrm{int}K)|u^\perp \to \mathbb{R}$ with the properties that $x + \varphi(x)u \in \partial K$, $y = x_0 + \varphi(x_0)u$, there exists a positive semi-definite quadratic form Q_y on $u^\perp$ such that*

$$\varphi(x) - \varphi(x_0) = \frac{1}{2} Q_y(x - x_0) + o(\|x - x_0\|^2)$$

as $x \in u^\perp$ tends to x_0. In this case, we set

$$\kappa(y) = \det Q_y \ \ \textit{(Gaussian curvature).}$$

Moreover, there exists an orthonormal basis $v_1, \dots, v_{n-1} \in u^\perp$ (the "principal directions") such that

$$Q_y\left(\sum_{i=1} t_i v_i\right) = \sum_{i=1} \kappa_i(y) t_i^2$$

for $\kappa_1(y), \dots, \kappa_{n-1}(y) \geq 0$ (the "principal curvatures" at y), and $\kappa(y) = \prod_{i=1}^{n-1} \kappa_i(y)$.

For $\mathcal{H}^{n-1}$ a.e. $x \in (\mathrm{int}K)|u^\perp$, we have writing $z = x + \varphi(x)u$,

$$\kappa(z) = \frac{\det D^2\varphi(x)}{(1 + \|D\varphi\|^2)^{\frac{n+1}{2}}}. \tag{8.1}$$

Remark. ∂K is C^2_+ if ∂K is C^2 and $\kappa(y) > 0$ for $y \in \partial K$. In this case, $\nu_K : \partial K \to S^{n-1}$ is the Gauss map, and as K is strictly convex (∂K contains no segment), we deduce that h_K is C^1 on $\mathbb{R}^n \backslash \{o\}$ and $Dh_K(u) = x$ if $x \in \partial' K$, and $u = t\nu_K(x)$ for $t > 0$ (see Lemma 1.6.7).

Definition 8.1.2. If $1 \leq i \leq d$ and A is a $d \times d$ symmetric matrix, then $\sigma_i A$ is the i^{th} symmetric function of the eigenvalues of A. In particular, $\sigma_i A$ is the sum the determinants of the $i \times i$ principal minors of A.

Example 8.1.3 (Gaussian curvature for balls and polytopes).

Polytopes: If K is an n-polytope, then $\kappa(y) = 0$ for any $y \in \partial' K$.

Balls: Let $v_1, \dots, v_n$ be any orthonormal basis of $\mathbb{R}^n$. For $z = (z_1, \dots, z_n) = \sum_{i=1}^n z_i v_i$ and $R > 0$, we have

$$h_{R\,B^n}(z) = R\|z\| = R\sqrt{z_1^2 + \ldots + z_n^2}.$$

If $z \neq o$, then

$$\begin{aligned}
\frac{d}{dz_i} h_{R\,B^n}(z) &= \frac{Rz_i}{\sqrt{z_1^2 + \ldots + z_n^2}}; \\
\frac{d^2}{dz_i^2} h_{R\,B^n}(z) &= R \cdot \frac{(z_1^2 + \ldots + z_n^2) - z_i^2}{(z_1^2 + \ldots + z_n^2)^{\frac{3}{2}}}; \\
\frac{d^2}{dz_i z_j} h_{R\,B^n}(z) &= \frac{-Rz_i z_j}{(z_1^2 + \ldots + z_n^2)^{\frac{3}{2}}} \quad \text{for } i \neq j.
\end{aligned}$$

Since $v_n = (0, \dots, 0, 1)$, it follows that

$$D^2 h_{RB^n}(v_n) = \mathrm{diag}[R, \dots, R, 0]. \tag{8.2}$$

It follows that for any $x \in \partial RB^n$ and the exterior unit vector $u = x/R$, we have

$$\begin{aligned}\sigma_{n-1}D^2h_{RB^n}(u) &= R^{n-1};\\ \kappa(x) &= R^{-(n-1)};\\ \sigma_i D^2h_{RB^n}(u) &= \binom{n-1}{i}R^i.\end{aligned}$$

Definition 8.1.4 (Curvature function). If ∂K is C^2_+ for a convex body $K \subset \mathbb{R}^n$, then $f_K : S^{n-1} \to (0,\infty)$ is the curvature function where $f_K(\nu_K(y)) = \frac{1}{\kappa(y)}$ for $y \in \partial K$.

Theorem 8.1.5. *If ∂K be C^2_+, then h_K is C^2 and D^2h_K is positive definite on $\mathbb{R}^n \backslash \{o\}$. Let $y \in \partial K$.*

(i) $D^2h_K(\nu_K(y)) = \mathrm{diag}\left[\frac{1}{\kappa_1(y)}, \ldots, \frac{1}{\kappa_{n-1}(y)}, 0\right]$ with respect to the orthonormal basis $v_1, \ldots, v_{n-1}, \nu_K(y)$ of $\mathbb{R}^n$ where $v_1, \ldots, v_{n-1}$ are the principal directions at y, and $\kappa_1(y), \ldots, \kappa_{n-1}(y)$ are the corresponding principal curvatures.

(ii) $\dfrac{1}{\kappa(y)} = \sigma_{n-1}D^2h_K(\nu_K(y)) = f_K(\nu_K(y))$.

(iii) If $g : S^{n-1} \to \mathbb{R}$ is bounded measurable, then

$$\int_{S^{n-1}} g(u)d\mathcal{H}^{n-1}(u) = \int_{\partial K} g(\nu_K(x))\kappa(x)\, d\mathcal{H}^{n-1}(x) \tag{8.3}$$

$$\int_{\partial K} g(\nu_K(x))\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} g(u)\sigma_{n-1}D^2h_K(u)\, d\mathcal{H}^{n-1}(u) \tag{8.4}$$

$$= \int_{S^{n-1}} g(u)f_K(u)\, d\mathcal{H}^{n-1}(u); \tag{8.5}$$

(iv) $|K| = \dfrac{1}{n}\displaystyle\int_{S^{n-1}} h_K f_K\, d\mathcal{H}^{n-1} = \dfrac{1}{n}\int_{S^{n-1}} h_K \cdot \sigma_{n-1}D^2h_K\, d\mathcal{H}^{n-1}$.

Proof. For (i), $\kappa(y) = \det Q_y > 0$ and $\kappa_1(y), \ldots, \kappa_{n-1}(y) > 0$ in Theorem 8.1.1 (iii). Assume $y = x_0 = o$, and use the orthonormal basis $v_1, \ldots, v_{n-1}, v_n$ of $\mathbb{R}^n$ where $v_n = \nu_K(y)$ and $v_1, \ldots, v_{n-1}$ the principal directions at y, thus $Dh_K(tv_n) = y = Dh_K(v_n)$ for $t > 0$ yields

$$\partial_n\partial_i h_K(v_n) = 0 \text{ for } i = 1, \ldots, n. \tag{8.6}$$

Using the notation of Theorem 8.1.1 and $\kappa_i = \kappa_i(y)$, we have

$$\varphi(x_1, \ldots, x_{n-1}) = \frac{1}{2}\sum_{i=1}^{n-1} \kappa_i x_i^2 + o(\|x\|^2)$$

for $x = (x_1, \ldots, x_{n-1}) \in v_n^\perp$ and C^2 function φ, thus

$$D\varphi(x_1, \ldots, x_{n-1}) = (\kappa_1 x_1 + o(\|x\|), \ldots, \kappa_{n-1}x_{n-1} + o(\|x\|)).$$

For $z = (x, \varphi(x)) \in \partial K$, we deduce that $\tilde{v}_z = (D\varphi(x), -1)$ exterior normal at z, and

$$\tilde{v}_z = (\kappa_1 x_1 + o(\|x\|), \dots, \kappa_{n-1} x_{n-1} + o(\|x\|), -1). \tag{8.7}$$

Combining (8.7) and $\kappa_1, \dots, \kappa_{n-1} > 0$ yields a local diffeomorphism $x \mapsto \tilde{v}_z$, $x \in v_n^\perp$, $\tilde{v}_z \in v_n + v_n^\perp$, $z = (x, \varphi(x))$ implying

$$Dh_K(\kappa_1 x_1, \dots, \kappa_{n-1} x_{n-1}, -1) = (x_1 + o(\|x\|), \dots, x_{n-1} + o(\|x\|), o(\|x\|^2)) \in \partial K.$$

It follows that for $w = (w_1, \dots, w_{n-1}) \in v_n^\perp$, we have

$$Dh_K(w_1, \dots, w_{n-1}, -1) = \left(\frac{w_1}{\kappa_1} + o(\|x\|), \dots, \frac{w_{n-1}}{\kappa_{n-1}} + o(\|w\|), o(\|w\|) \right),$$

and hence $D^2 h_K(\nu_K(y)) = \mathrm{diag}\left[\frac{1}{\kappa_1(y)}, \dots, \frac{1}{\kappa_{n-1}(y)}, 0 \right]$ (cf. (8.6)).

Readily, (i) implies (ii). For (iii), (8.7) yields that if $X \subset \partial K$ measurable, then

$$\mathcal{H}^{n-1}(\nu_K(X)) = \int_X \kappa(x)\, d\mathcal{H}^{n-1}(x).$$

We deduce (8.3), and in turn (8.3) yields (8.4) and (8.5).

Finally, (iv) follows from (2.3) taking $g = h_K$ in (8.4) and (8.5). ∎

We deduce from Theorem 8.1.5 that if ∂K is C^2_+ for a convex body, $t > 0$ and $u \in S^{n-1}$, then

$$D^2 h_K(tu) = \frac{1}{t} \cdot D^2 h_K(u). \tag{8.8}$$

Still, we usually consider the restriction of h_K to S^{n-1}. Let us discuss the various related notions of differentials of functions on the sphere:

Definition 8.1.6. For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t > 0$ and $u \in S^{n-1}$, and hence $\tilde{h}, \bar{h} : \mathbb{R}^n \backslash \{o\} \to \mathbb{R}$ are C^2 on $\mathbb{R}^n \backslash \{o\}$. In particular, $\mathbb{R}u$ is an eigenspace (with eigen value zero) of both $D^2\tilde{h}(u)$ and $D^2\bar{h}(u)$, and we define

$$\nabla h(u) = D\bar{h}(u)|_{u^\perp} = D\tilde{h}(u)|_{u^\perp} \tag{8.9}$$

$$\nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp} \tag{8.10}$$

$$\widetilde{D}^2 h(u) = \widetilde{D}^2 \tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}. \tag{8.11}$$

Remark. ∇h is the spherical gradient and $\nabla^2 h$ is the spherical Hessian of h in Definition 8.1.6 with respect to a moving orthogonal frame in the sense of Riemannian geometry (see Schneider [522], Section 2.5) where combining (8.10), (8.11) and $\tilde{h}(x) = \|x\| \cdot \bar{h}(x)$ yield

$$\widetilde{D}^2 h(u) = \widetilde{D}^2 \tilde{h}(u) = \nabla^2 h(u) + h(u) I_{n-1} \tag{8.12}$$

for $u \in S^{n-1}$ on the tangent space $u^\perp$.

We recall that if $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then $\varrho_K(u) \cdot u \in \partial K$ and $\varrho_K(u) = \|u\|_K^{-1}$ for $u \in S^{n-1}$. In particular (cf. Section 2.2),

$$\partial K \text{ is } C^2 \iff \varrho_K \text{ is } C^2 \text{ on } S^{n-1} \iff \|\cdot\|_K \text{ is } C^2 \text{ on } \mathbb{R}^n \backslash \{o\}.$$

In addition, $\|u\|_{K^*} = h_K$ and $\|u\|_K = h_{K*}$ for the polar $K^* = \{x \in \mathbb{R}^n : \langle x, y \rangle,\ y \in K\}$ of K (*cf.* Section 1.9).

Theorem 8.1.7. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) ∂K is C^2_+ if and only if h_K is C^2 and $\widetilde{D}^2 h_K$ positive definite on $\in S^{n-1}$, which in turn holds if and only if $h = h_K|_{S^{n-1}}$ is C^2 and $\nabla^2 h$ is positive.

(ii) Assuming $o \in \operatorname{int} K$, ∂K is C^2_+ if and only if ∂K^* is C^2_+.

Remarks. Similar results hold if C^2_+ is replaced by C^∞_+. For the relation between the Gaussian curvatures at $x \in \partial K$ and $x^* \in \partial K^*$ with $\langle x, x^* \rangle = 1$, see Theorem 8.9.4.

Proof. We may assume $o \in \operatorname{int} K$. (i) and (ii) follow from verifying the following cycle of implications:

$$\partial K \text{ is } C^2_+ \Longrightarrow \widetilde{D}^2 h_K \text{ pos. def.} \Longrightarrow \partial K^* \text{ is } C^2_+ \Longrightarrow \widetilde{D}^2 h_{K^*} \text{ pos. def.} \Longrightarrow \partial K \text{ is } C^2_+.$$

Having Theorem 8.1.5, (i) and (ii) follow by proving that if h_K is C^2 on $\mathbb{R}^n \backslash \{o\}$ and $D^2 h_K(u)|_{u^\perp}$ positive definite $u \in S^{n-1}$, then ∂K^* is C^2_+. As $\|\cdot\|_{K^*} = h_K$, we already know that ∂K^* is C^2. Therefore, all we have to prove is that

$$\kappa_{\partial K^*}(z) > 0 \text{ if } z \in \partial K^*. \tag{8.13}$$

Since this property is invariant under linear transformations, we consider the (unique) $y \in \partial K$ such that $\langle y, z \rangle = 1$, and apply $\Phi \in \mathrm{GL}(n)$ such that $\tilde{y} = \Phi y \in S^{n-1}$, $\nu_{\widetilde{K}} = \tilde{y}$ for $\widetilde{K} = \Phi K$ and each principal curvature at $\tilde{y} \in \partial \widetilde{K}$ is one. It follows that there exists $a \in (0, 1)$ such that for the ellipsoid E with one principal semi axis is $\operatorname{conv}\{0, y\}$, and the other principal semi axes are of length a, the "upper half of E" is contained in K; namely, $\{x \in E : \langle x, \tilde{y} \rangle \geq 0\} \subset K$.

On the other hand, $\widetilde{K}^* = \Phi^{-t} K^*$ satisfies that $\Phi^{-t} z = \tilde{y}$ and $\nu_{\widetilde{K}^*} = \tilde{y}$. Since any $w \in \widetilde{K}^*$ with $\langle w, \tilde{y} \rangle \geq 0$ satisfies $\langle w, x \rangle \leq 1$ for any $x \in E$ with $\langle x, \tilde{y} \rangle \geq 0$, we deduce that $\{w \in \widetilde{K}^* : \langle w, \tilde{y} \rangle \geq 0\} \subset E^*$, which in turn yields (8.13). ∎

Now we characterize C^2 functions on the sphere S^{n-1} that are restrictions of support functions.

Lemma 8.1.8. *For C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ for $t \geq 0$ and $u \in S^{n-1}$, and hence $\tilde{h} : \mathbb{R}^n \to \mathbb{R}$ is C^2 on $\mathbb{R}^n \backslash \{o\}$.*

(i) $h = h_K|_{S^{n-1}}$ *for a convex body* K *if and only if* $D^2\tilde{h}(u)$ *is positive semi-definite for* $u \in S^{n-1}$*;*

(ii) ∂K *in (i) is* C^2_+ *if and only if* $\nabla^2 h = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}$ *is positive definite for* $u \in S^{n-1}$.

Remark.

- The convex body K in (i) is strictly convex (∂K contains no segment), but may not have a C^1 boundary (cf. Lemma 1.9.6).
- If h is C^∞ and $\nabla^2 h$ is positive definite, then ∂K is C^∞_+ in (ii).
- It follows that for any $f \in C^2(S^{n-1})$ ($f \in C^\infty(S^{n-1})$), there exist $R > 0$ and a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary (with C^∞_+ boundary) such that $f = h_K - h_{R\,B^n}$.

Proof. $D^2\tilde{h}(x)$ positive semi-definite for $\mathbb{R}^n\backslash\{o\}$ if and only if $\tilde{h}$ is convex on the half space $\{x \in \mathbb{R}^n : \langle x, u\rangle > 0\}$ for any $u \in S^{n-1}$. As $\tilde{h}$ is homogeneous by definition, the convexity of $\tilde{h}$ is equivalent to the property that $\tilde{h} = h_K$ for a compact convex set (cf. Lemma 1.6.8). In this case, K is a strictly convex body by Lemma 1.6.7.

Finally, (ii) follows from Theorem 8.1.7. ∎

Next we verify that differences of support functions with C^∞_+ boundary are dense among continuous functions on S^{n-1}.

Proposition 8.1.9. *If* $g : S^{n-1} \to \mathbb{R}$ *continuous and* $\varepsilon > 0$*, then there exists convex bodies* $K, C \subset \mathbb{R}^n$ *with* C^∞_+ *boundary such that*

$$\|g - (h_C - h_K)\|_\infty < \varepsilon; \text{ namely, } |g(u) - h_C(u) + h_K(u)| < \varepsilon \text{ for } u \in S^{n-1}.$$

Proof. According to the Stone-Weierstrass theorem Corollary 10.5.2, there exists a polynomial on $\mathbb{R}^n$ such that its restriction h to S^{n-1} satisfies $\|g - h\|_\infty < \varepsilon$. In particular, h is C^∞, and let $\tilde{h}(tu) = t \cdot h(u)$ for $t \geq 0$ and $u \in S^{n-1}$.

Choose $R > 1$ such that for any $u \in S^{n-1}$, each eigenvalue of $D^2\tilde{h}(u)$ is larger than $1 - R$. For $K = RB^n$, each eigenvalue of $D^2 h_K(u)|u^\perp$ is R for $u \in S^{n-1}$, and hence $D^2(\tilde{h} + h_K)(u)|u^\perp$ is positive definite for $u \in S^{n-1}$. It follows from Lemma 8.1.8 that $\tilde{h} + h_K = h_C$ for a convex body $C \subset \mathbb{R}^n$ with C^∞_+ boundary by Lemma 8.1.8. ∎

Firey [234] solved the problem of approximating by convex bodies with C^∞_+ boundary in an elegant manner.

Theorem 8.1.10 (Approximation by smooth convex bodies). *If* $C \subset \mathbb{R}^n$ *is compact convex and* $\varepsilon > 0$*, then there exists a convex body* $K \supset C$ *in* $\mathbb{R}^n$ *with* C^∞_+ *boundary such that* $\delta_H(K, C) < \varepsilon$*, where* K *is o-symmetric if* C *is o-symmetric.*

Remark. It is equivalent to saying that h_K is C^∞ on $\mathbb{R}^n\backslash\{o\}$ such that $D^2 h_K(u)|_{u^\perp}$ is positive definite for $u \in S^{n-1}$ and $\|h_C - h_K\|_\infty < \varepsilon$.

Proof. Let P be an n-polytope such that $C \subset \operatorname{int} P$ and $\delta_H(C,P) < \frac{\varepsilon}{3}$ (cf. (1.14) where P is o-symmetric if C is o-symmetric). It is sufficient to construct a convex body $M \supset C$ such that h_M is C^∞ and $\delta_H(P,M) \leq \frac{\varepsilon}{3}$ because $K = M + \frac{\varepsilon}{3} B^n$ works for Theorem 8.1.10.

Assume that $o \in \operatorname{int} P$ and $v_1, \dots, v_k$ are the vertices of P. For $i = 1, \dots, k$, E_i is an ellipsoid centered at $v_i/2$ such that

$$\operatorname{conv}\{o, v_i\} \subset \operatorname{int} E_i \subset \operatorname{conv}\{o, v_i\} + \frac{\varepsilon}{3} B^n$$

(e.g. E_i has semi axes $\frac{\|v_i\|}{2} + \frac{\varepsilon}{3}, \frac{\varepsilon}{3}, \dots, \frac{\varepsilon}{3}$). As E_i is an ellipsoid, $h_{E_i}(x) = \sqrt{\langle A_i x, x\rangle} + \langle \frac{v_i}{2}, x\rangle$ for a positive definite $n \times n$ matrix A_i, thus $h_{E_i} > 0$ is C^∞ on $\mathbb{R}^n \backslash \{o\}$.

Let M_p be the convex body for $p \in [1, \infty]$ with

$$h_{M_\infty} = \max_{i=1,\dots,k} h_{E_i} \quad \text{and} \quad h_{M_p} = \left(\frac{1}{k}\sum_{i=1}^{k} h_{E_i}^p\right)^{\frac{1}{p}} \quad \text{for } p \in [1,\infty)$$

where h_{M_p} is convex by the Minkowski inequality and is C^∞ on $\mathbb{R}^n \backslash \{o\}$ if $p < \infty$. For large $p \in (1, \infty)$, we have $P \subset M_p \subset M_\infty \subset P + \frac{\varepsilon}{3} B^n$, and hence $\delta_H(P, M_p) \leq \frac{\varepsilon}{3}$.

Finally, let $K = M_p + \frac{\varepsilon}{3} B^n$ for large p. Since $h_{M_p}|_{S^{n-1}}$ is C^∞, it follows that $h_K|_{S^{n-1}}$ is C_+^∞, and $\delta_H(C,K) \leq \varepsilon$. ■

8.2 Surface area measure and the curvature function

Surface area measure has been already introduced in Section 2.5. Here we discuss properties that are related to the second differentiability of the boundary. First, we recall the definition of the surface area measure.

Definition 8.2.1. For a compact convex set $K \subset \mathbb{R}^n$, the surface area measure S_K is the following Borel measure on S^{n-1}.

- If K is convex body, then

$$S_K(\omega) = \mathcal{H}^{n-1}\left(\{x \in \partial' K : \nu_K(x) \in \omega\}\right);$$

  namely, $\int_{S^{n-1}} g dS_K = \int_{\partial K} g \circ \nu_K \, d\mathcal{H}^{n-1}$ for bounded measurable $g : S^{n-1} \to \mathbb{R}$.
- If $\dim K = n-1$ and $K \subset x + u^\perp$ for $u \in S^{n-1}$ and $x \in \mathbb{R}^n$, then $\operatorname{supp} S_K = \{u, -u\}$ and $S_K(\{u\}) = S_K(\{-u\}) = \mathcal{H}^{n-1}(K)$.
- If $\dim K \leq n-2$, then $S_K \equiv 0$.

Example 8.2.2.

- If K is an n-dimensional polytope with facets $F_1, \ldots, F_m$ and exterior unit normals $u_1, \ldots, u_m$, then

$$\operatorname{supp} S_K = \{u_1, \ldots, u_m\} \text{ and } S_K(\{u_i\}) = \mathcal{H}^{n-1}(F_i),\ i = 1, \ldots, m.$$

- If $K = rB^n$ for $r > 0$, then $S_K = r^{n-1} \cdot \mathcal{H}^{n-1}$.
- If ∂K be C^2_+, then $dS_K = f_K d\mathcal{H}^{n-1}$ where $f_K(\nu_K(x)) = 1/\kappa(x)$, for $x \in \partial K$, is the so called curvature function on S^{n-1} according to Theorem 8.1.5. In other words (cf. (8.12)),

$$\begin{aligned} dS_K &= \sigma_{n-1} D^2 h_K \, d\mathcal{H}^{n-1} = \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1} \\ &= \det \left(\nabla^2 h + h I_{n-1}\right) \, d\mathcal{H}^{n-1} \end{aligned} \tag{8.14}$$

  for the C^2 function $h = h_K|_{S^{n-1}}$.

Using Aleksandrov's Theorem 10.6.2 and Theorem 8.1.1 on the second order differentiabily of convex functions and hypersurfaces, the formula (8.14) for convex bodies with C^2_+ boundary can be partially generalized to any convex body.

Remark 8.2.3. For any convex body $K \subset \mathbb{R}^n$, $S_K = S^a_K + S^s_K$ on S^{n-1} where

- $dS^a_K = f_K d\mathcal{H}^{n-1}$ is the absolutely continuous part, and $f_K(u) = \sigma_{n-1} D^2 h_K(u)$ (see Theorem 3.5 in Hug [337]) is the "generalized" curvature function for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$;
- S^s_K is a singular Borel measure (i.e. there exists $X \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(X) = 0$ and $S^s_K(S^{n-1} \backslash X) = 0$) and S^s_K is regular (see Theorem 10.1.3).

Example 8.2.4. Let $K \subset \mathbb{R}^n$ be a convex body.

- If K is a polytope, then $S_K = S^s_K$.
- If ∂K is C^2_+, then $dS_K = dS^a_K = f_K d\mathcal{H}^{n-1}$.

Let us list some fundamental properties of the surface area measure that are discussed in this book.

Basic properties of S_K

- $S_K(S^{n-1}) = S(K)$;
- $S_{\lambda K} = \lambda^{n-1} \cdot S_K$;
- S_K Borel measure, "first variation of the volume" (see Theorem 7.5.2); in particular (cf. Lemma 2.5.7 and Proposition 2.5.9),

$$|K| = \frac{1}{n} \int_{S^{n-1}} h_K \, dS_K \quad \text{and} \quad \lim_{\varrho \to 0^+} \frac{|K + \varrho\, C|}{\varrho} = \int_{S^{n-1}} h_C \, dS_K; \tag{8.15}$$

- $S_K = S_C$ if and only if K and C are translates (cf. Proposition 8.4.3);
- S_K is weakly continuous in K (see Proposition 8.4.1);

- $\int_{S^{n-1}} u\, dS_K(u) = o$ (cf. Lemma 2.5.7) and supp S_K is not contained in a closed hemisphere (cf. Lemma 2.5.6), which properties characterize a surface area measure of convex body (see Theorem 9.2.3).

8.3 Mixed volumes and smooth convex bodies

Mixed volumes have already been considered in Chapter 7, and their study there was based on polytopes. In this section, our discussion of the mixed volumes is independent of Chapter 7, and is based on Theorem 8.1.5 for a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary. Let us summarize the corresponding properties of convex bodies with smooth boundary following Theorem 8.1.5:

Remark 8.3.1. For a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary and $u \in S^{n-1}$, $\widetilde{D}^2 h_K(u) = D^2 h_K(u)|_{u^\perp}$ is a $(n-1)\times(n-1)$ positive definite matrix, and u is an eigenvector of $D^2 h_K(u)$ with eigenvector zero. In particular,

$$\sigma_i(\widetilde{D}^2 h_K(u)) = \sigma_i(D^2 h_K(u)) \tag{8.16}$$

for $i = 1, \ldots, n-1$. If $u = \nu_K(x)$ for $x \in \partial K$, then

$$\det \widetilde{D}^2 h_K(u) = \sigma_{n-1} D^2 h_K(u) = f_K(u) = \kappa(x)^{-1}; \tag{8.17}$$

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1}. \tag{8.18}$$

If $C \subset \mathbb{R}^n$ with C^2_+ boundary and $\alpha, \beta > 0$, then

$$\widetilde{D}^2 h_{\alpha K + \beta C} = \alpha\, \widetilde{D}^2 h_K + \beta\, \widetilde{D}^2 h_C. \tag{8.19}$$

It follows from (8.14) and (8.15) that

$$\lim_{\varrho \to 0^+} \frac{|K + \varrho\, C|}{\varrho} = \int_{S^{n-1}} h_C \cdot \det \widetilde{D}^2 h_K \, d\mathcal{H}^{n-1}. \tag{8.20}$$

If $h = h_K|_{u^\perp}$, then the spherical Hessian $\nabla^2 h$ satisfies (ıcf. (8.12))

$$\widetilde{D}^2 h_K(u) = \nabla^2 h(u) + h(u) I_{n-1} \ \text{ on the tangent space } u^\perp. \tag{8.21}$$

Based on (8.18) and (8.19), the idea is that first we understand the determinant of linear combination of matrices in order to verify that the volume of a positive linear combination of smooth convex bodies is polynomial in the coefficient.

Definition 8.3.2 (Mixed Discriminant). Id $A_1, \ldots, A_d$ are $d \times d$ real matrices, $A_i = [a_1^{(i)}, \ldots, a_d^{(i)}]$ for $a_j^{(i)} \in \mathbb{R}^d$, $d \geq 1$, then

$$\mathcal{D}(A_1, \ldots, A_d) = \frac{1}{d!} \sum_{\pi:\{1,\ldots,d\} \to \{1,\ldots,d\} \text{ bijection}} \det[a_1^{(\pi(1))}, \ldots, a_d^{(\pi(d))}].$$

Remark. $\mathcal{D}(A_1,\dots,A_d)$ is symmetric and linear in its variables $A_1,\dots,A_d$; namely, if $\beta_1,\dots,\beta_k \in \mathbb{R}$ and $B_1,\dots,B_k$ are $d\times d$ real matrices, then

$$\mathcal{D}\left(A_1,\dots,A_{d-1},\sum_{j=1}^{k}\beta_j B_j\right)=\sum_{j=1}^{k}\beta_j\mathcal{D}(A_1,\dots,A_{d-1},B_j), \tag{8.22}$$

and if $A_1,\dots,A_m$ are $d\times d$ real matrices and $\lambda_1,\dots,\lambda_m\in\mathbb{R}$, then

$$\det\left(\sum_{j=1}^{m}\lambda_j A_j\right)=\sum_{i_1,\dots,i_d\in\{1,\dots,m\}}\mathcal{D}(A_{i_1},\dots,A_{i_d})\lambda_{i_1},\dots,\lambda_{i_d}. \tag{8.23}$$

Lemma 8.3.3. *If $A_1,\dots,A_d$ are $d\times d$ positive semi-definite symmetric matrices, $d\ge 2$, then $\mathcal{D}(A_1,\dots,A_d)\ge 0$, and even $\mathcal{D}(A_1,\dots,A_d)>0$ if $A_1,\dots,A_d$ are positive definite.*

Remark. More precisely, given positive semi-definite $A_1,\dots,A_d$, $\mathcal{D}(A_1,\dots,A_d)>0$ if and only if there exist independent $v_1,\dots,v_d$ such that v_i is an eigenvector of A_i corresponding to a positive eigenvalue. In particular, if $A_1\ne 0$ is positive semi-definite, and $A_2,\dots,A_d$ are positive definite, then

$$\mathcal{D}(A_1,\dots,A_d)>0. \tag{8.24}$$

Proof. As any semi-definite matrix is a non-negative linear combination of matrixes of the form $v\,v^t$ for $v\in S^{d-1}$, we may assume that $A_i=v_i\,v_i^t$ for $v_i\in S^{d-1}$.

Let $e_1,\dots,e_d$ form an orthonormal basis, and let $v_i=\Phi e_i$ for $d\times d$ matrix Φ and hence

$$\det\left(\sum_{i=1}^{d}\lambda_i A_i\right)=\det\left[\Phi\left(\sum_{i=1}^{d}\lambda_i e_i e_i^t\right)\Phi^t\right]=(\det\Phi)^2\lambda_i\cdot\ldots\cdot\lambda_d.$$

Therefore, $\mathcal{D}(A_1,\dots,A_d)=\det[v_1,\dots,v_d]^2/d!\ge 0$, and $\mathcal{D}(A_1,\dots,A_d)>0$ if $v_1,\dots,v_n$ independent. ∎

Theorem 8.3.4 (Mixed volumes for smooth convex bodies). *If $K_1,\dots,K_m$ and $C_1,\dots,C_l$ are convex bodies in $\mathbb{R}^n$ with C_+^2 boundaries, then*

$$V\left(\sum_{i=1}^{m}\lambda_i K_i\right)=\sum_{i_1,\dots,i_n\in\{1,\dots,m\}}V(K_{i_1},\dots,K_{i_n})\lambda_{i_1},\dots,\lambda_{i_n}.$$

(i) $\displaystyle\sum_{j=1}^{l}\varrho_j V(K_1,\dots,K_{n-1},C_j)=V\left(K_1,\dots,K_{n-1},\sum_{j=1}^{l}\varrho_j C_j\right)$ *for any* $\varrho_j\ge 0$.

(ii) $n!V(K_1,\ldots,K_n) = \sum_{i=1}^{n}(-1)^{n-i} \sum_{1\le j_1<\ldots<j_i\le n} |K_{j_1}+\ldots+K_{j_i}|.$

(iii) $V(K_1,\ldots,K_n)$ *is continuous and symmetric in* $K_1,\ldots,K_n$.

(iv) $V(K_1+z_1,\ldots,K_n+z_n) = V(K_1,\ldots,K_n)$ *for* $z_1,\ldots,z_n \in \mathbb{R}^n$.

(v) $V(\Phi K_1,\ldots,\Phi K_n) = V(K_1,\ldots,K_n)$ *for* $\Phi \in \mathrm{SL}(n)$.

(vi) Setting $V(\overbrace{K_1,\ldots,K_1}^{n-i},\overbrace{K_2,\ldots,K_2}^{i}) = V(K_1,K_2;i)$, *we have*

$$V(\lambda_1 K_1+\lambda_2 K_2) = \sum_{i=0}^{n}\binom{n}{i}V(K_1,K_2;i)\lambda_1^{n-i}\lambda_2^{i}. \tag{8.25}$$

(vii) For an absolutely continuous so called mixed area measure $S_{K_1,\ldots,K_{n-1}}$ *on* S^{n-1},

$$\begin{aligned} V(K_1,\ldots,K_n) &= \frac{1}{n}\int_{S^{n-1}} h_{K_n}\mathcal{D}\left(\widetilde{D}^2 h_{K_1},\ldots,\widetilde{D}^2 h_{K_{n-1}}\right)\, d\mathcal{H}^{n-1} \\ &= \frac{1}{n}\int_{S^{n-1}} h_{K_n}\, dS_{K_1,\ldots,K_{n-1}}. \end{aligned} \tag{8.26}$$

(viii) $V(K,\ldots,K) = |K|$ *and* $S_{K,\ldots,K} = S_K$.

(ix) $V(K_1,\ldots,K_n) > 0$ *and* $V(K_1,\ldots,K_n) \ge V(C_1,\ldots,C_n)$ *if* $C_i \subset K_i$.

(x) **Mean curvatures:** *if* $i = 1,\ldots,n-1$ *and* ∂K *is* C^2_+, *then*

$$\begin{aligned} n\binom{n-1}{i}\cdot V(B^n,K;i) &= (n-i)\omega_{n-i}\cdot V_i(K) = \int_{S^{n-1}} \sigma_i D^2 h_K(u)\,du \\ &= \int_{\partial K} \sigma_{n-1-i}(\kappa_1(x),\ldots,\kappa_{n-1}(x))\,dx. \end{aligned} \tag{8.27}$$

Proof. We may assume that $o \in \operatorname{int} K_i$, thus $h_{K_i}(u) > 0$ for $u \in S^{n-1}$ and $i = 1,\ldots,m$. According to (8.18) and (8.19), if $K = \sum_{i=1}^m \lambda_i K_i$, then

$$\begin{aligned} V(K) &= \frac{1}{n}\int_{S^{n-1}} h_K(u)\widetilde{D}^2 h_K(u)\,du; \\ h_K &= \sum_{i=1}^{m}\lambda_i h_{K_i}; \\ \widetilde{D}^2 h_K &= \sum_{i=1}^{m}\lambda_i \widetilde{D}^2 h_{K_i}. \end{aligned}$$

Therefore, (8.23) and (8.26) yield (i)-(vi), and Lemma 8.3.3 implies that $V(K_1,\ldots,K_n) > 0$ in (ix). It follows from (8.20) that if $M, C \subset \mathbb{R}^n$ are convex bodies with C^2_+ boundaries, then

$$V(M,\ldots,M,C) = \frac{1}{n}\int_{S^{n-1}} h_C\cdot \det \widetilde{D}^2 h_M\, d\mathcal{H}^{n-1}.$$

Considering $M = \sum_{i=1}^{n-1} \lambda_i K_i$ for $\lambda_1, \dots, \lambda_{n-1} \geq 0$ and the linearity of the mixed volume and the mixed discriminant imply (vii). In turn, (vii) yields (viii) and (ix).

For (x), $\binom{d}{i}\mathcal{D}(\overbrace{I_d, \dots, I_d}^{d-i}, \overbrace{A, \dots, A}^{i}) = \sigma_i(A)$ for any $d \times d$ positive definite matrix A, and hence (8.16) and (vii) imply

$$\begin{aligned} V(B^n, K; i) &= \frac{1}{n} \int_{S^{n-1}} h_{B^n} \cdot \mathcal{D}(\overbrace{\widetilde{D}^2 h_{B^n}, \dots, \widetilde{D}^2 h_{B^n}}^{n-1-i}, \overbrace{\widetilde{D}^2 h_K, \dots, \widetilde{D}^2 h_K}^{i})\, d\mathcal{H}^{n-1} \\ &= \frac{1}{n}\binom{n-1}{i}^{-1} \int_{S^{n-1}} \sigma_i(\widetilde{D}^2 h_K)\, d\mathcal{H}^{n-1}. \end{aligned}$$

■

Theorem 8.3.5. *The mixed volumes can be defined for any convex compact sets in $\mathbb{R}^n$ in a way such that Theorem 8.3.4 (i)-(vi) hold; moreover, $V(K, \dots, K) = |K|$, $V(K_1, \dots, K_n) \geq 0$ and $V(K_1, \dots, K_n) \geq V(C_1, \dots, C_n)$ if $C_i \subset K_i$.*

(a) If $K \subset \mathbb{R}^n$ convex compact and $C \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} C$, then $S(K) = nV(K, B^n; 1)$ and $P_C(K) = nV(K, C; 1)$.

(b) For any convex compact sets $K_1, \dots, K_{n-1} \subset \mathbb{R}^n$, there exists a (unique) finite Borel measure $S_{K_1,\dots,K_{n-1}}$ on S^{n-1} called mixed surface area measure such that for any compact convex $C \subset \mathbb{R}^n$, we have

$$V(K_1, \dots, K_{n-1}, C) = \frac{1}{n} \int_{S^{n-1}} h_C \, dS_{K_1,\dots,K_{n-1}} \tag{8.28}$$

where $S_{K,\dots,K} = S_K$.

Remarks. The symmetry and positive linearity of the mixed volumes and the uniqueness of the mixed surface area measure yield that $S_{K_1,\dots,K_{n-1}}$ is symmetric and positive linear in each of its variables.

Proof. We may assume that all convex compact sets $K_1, \dots, K_m, C_1, \dots, C_m$ are contained in $\operatorname{int} RB^n$ for some $R > 0$. We can approximate them by convex bodies $\widetilde{K}_1, \dots, \widetilde{K}_m, \widetilde{C}_1, \dots, \widetilde{C}_m$ with C^2_+ boundary, and each mixed volume is at most $V(RB^n, \dots, RB^n) = R^n \omega_n$ by the monotonicity property (ix). Therefore each formula holds by approximation.

For (a), use Theorem 8.3.4 (vi) and $P_C(K) = \lim_{\varrho \to 0^+} \frac{|K + \varrho C| - |K|}{\varrho}$.

To construct the mixed surface area measure $S_{K_1,\dots,K_{n-1}}$ for convex compact sets $K_1, \dots, K_{n-1} \subset \mathbb{R}^n$ in (b), the idea is to consider a positive linear operator $L : C(S^{n-1}) \to \mathbb{R}$ such that for any compact convex $M \subset \mathbb{R}^n$,

$$L\left(h_M|_{S^{n-1}}\right) = V(K_1, \dots, K_{n-1}, M). \tag{8.29}$$

Let $\mathcal{V} \subset C(S^{n-1})$ be the real vector vectorspace generated by the restrictions of the support functions of compact convex sets to S^{n-1}. As the mixed volume is positive linear (see Theorem 8.3.4 (i)), the definition of L as in (8.29) extends to a linear operator on $\mathcal{V}$. It follows from the monotonicity of the mixed volumes (see Theorem 8.3.4 (ix)) that L is a positive operator; namely, $L(f) \geq 0$ for non-negative $f \in \mathcal{V}$, and hence also $L(|f|) \leq \|f\|_\infty \cdot L(\mathbf{1})$ where $L(\mathbf{1}) = V(K_1, \ldots, K_{n-1}, B^n)$. Since $\mathcal{V}$ is dense in $C(S^{n-1})$ according to Proposition 8.1.9, L can be extended into a positive linear operator on $C(S^{n-1})$. Then $S_{K_1,\ldots,K_{n-1}}$ is the unique Borel measure on S^{n-1} representing L provided by the Riesz Representation Theorem 10.1.4. ∎

We repeat the statement and proof of Lemma 7.3.6.

Lemma 8.3.6. *For convex compact $K_1, \ldots, K_n \subset \mathbb{R}^n$, $V(K_1, \ldots, K_n) > 0$ if and only $\exists x_i, y_i \in K_i$ such that $x_1 - y_1, \ldots, x_n - y_n$ are independent.*

Proof. If $x_1 - y_, \ldots, x_n - y_n$ are independent, then for $s_i = [x_i, y_i]$, we have

$$V(K_1, \ldots, K_n) \geq V(s_1, \ldots, s_n) = |\det[x_1 - y_1, \ldots, x_n - y_n]|/n! > 0.$$

If there exist no suitable $x_i, y_i \in K_i$, then after possibly translating and reindexing, there exist $1 \leq m \leq n$ and linear $(m-1)$-plane L such that $K_1, \ldots, K_m \subset L$. Thus there exist compact convex sets $C \subset L$ and $M \subset L^\perp$ such that $K_i \subset C$ if $i \leq m$ and $K_j \subset C + M$ if $j > m$. It follows that

$$V(K_1, \ldots, K_n) \leq V(C, m; C + M, n - m) = \sum_{j=0}^{n-m} \binom{n-m}{j} V(C, m + j; M, n - m - j) = 0.$$

∎

Next we show that the centroid of the mixed surface area measure $S_{K_1,\ldots,K_{n-1}}$ on S^{n-1} is the origin in $\mathbb{R}^n$.

Lemma 8.3.7. *If $K_1, \ldots, K_{n-1} \subset \mathbb{R}^n$ are convex and compact, then*

$$\int_{S^{n-1}} u \, dS_{K_1,\ldots,K_{n-1}}(u) = o; \tag{8.30}$$

and if $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary and $i = 1, \ldots, n-1$, then

$$\int_{S^{n-1}} u \cdot \sigma_i(D^2 h_K(u)) \, d\mathcal{H}^{n-1}(u) = 0. \tag{8.31}$$

Proof. For any $z \in \mathbb{R}^n$, $h_{\{z\}}(u) = \langle z, u \rangle$, and hence Proposition 7.3.6 and (8.28) yield

$$0 = V(K_1, \ldots, K_{n-1}, \{z\}) = \frac{1}{n} \int_{S^{n-1}} \langle z, u \rangle \, dS_{K_1,\ldots,K_{n-1}}(u),$$

implying (8.30).

(8.31) is the consequence of (8.30) and the observation that

$$\sigma_i(D^2 h_K)\, d\mathcal{H}^{n-1} = dS_{\underbrace{B^n,\ldots,B^n}_{n-1-i},\underbrace{K,\ldots,K}_{i}}$$

(see the proof of Theorem 8.3.4 (x)). ∎

8.4 Mixed volumes, Minkowski inequality and the Surface area measure

In this section, we use the theory of Mixed Volumes to establish two fundamental properties of the surface area measure; namely, weak continuity and characterization of uniqueness. As a nice application of the continuity of mixed volumes (cf. Theorem 8.3.4) and the fact that differences of support functions with C^2_+ boundary are dense among continuous functions on S^{n-1} (cf. Proposition 8.1.9), first we show that the surface area measure is weakly continuous on the space of compact convex sets.

Proposition 8.4.1. *If compact convex sets K_m tend to K in $\mathbb{R}^n$, then S_{K_m} tends weakly to S_K; namely, $\lim_{m\to\infty}\int_{S^{n-1}} g\, dS_{K_m} = \int_{S^{n-1}} g\, dS_K$ for any continuous $g : S^{n-1} \to \mathbb{R}$.*

Proof. As $K_m \to K$, $\int_{S^{n-1}} h_C\, dS_{K_m} = nV(K_m, C; 1) \to nV(K, C; 1) = \int_{S^{n-1}} h_C\, dS_K$ holds for any $C \in \mathcal{K}^n$ with C^2_+ boundary by continuity of mixed volumes (cf. Theorem 8.3.4), and hence $\int_{S^{n-1}} g\, dS_{K_m} \to \int_{S^{n-1}} g\, dS_K$ for $g \in C(S^{n-1})$ by Proposition 8.1.9. ∎

As we will shortly see, the Minkowski inequality (8.33) follows from the Brunn-Minkowski inequality (cf. Theorem 1.12.3) stating that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\alpha, \beta > 0$, then

$$|\alpha K + \beta C|^{\frac{1}{n}} \geq \alpha |K|^{\frac{1}{n}} + \beta |C|^{\frac{1}{n}} \tag{8.32}$$

with equality if and only if $K = \gamma C + z$ for $\gamma > 0$ and $z \in \mathbb{R}^n$. Here we provide a proof of the Minkowski inequality that is actually shorter than the one already provided in Theorem 7.4.2. We use the formulation of the mixed volumes in terms of surface area measures (cf. (8.28)), and note that the Minkowski inequality (8.33) is actually equivalent with the Brunn-Minkowski inequality according to Remark 7.4.4.

Theorem 8.4.2 (Minkowski inequality). *If $K, C \subset \mathbb{R}^n$ are convex bodies, then*

$$\frac{1}{n}\int_{S^{n-1}} h_C\, dS_K = V(K, C; 1) \geq |K|^{\frac{n-1}{n}} |C|^{\frac{1}{n}} \tag{8.33}$$

with equality if and only if K and C are homothetic.

Proof. We may assume that $|K| = |C|$, and hence (8.33) is equivalent to $V(K, C; 1) \geq |K|$ with equality if and only if K and C are translates.

The function $f(t) = |K + t\,C|^{\frac{1}{n}}$ is concave for $t \in [0, 1]$ by the Brunn-Minkowski inequality (8.32) because $M_t = K + tC$ satisfies $M_{\frac{1}{2}t+\frac{1}{2}s} = \frac{1}{2}M_t + \frac{1}{2}M_s$; therefore, the representation of $|K + t\,C|$ in terms of mixed volumes (cf. Theorem 8.3.4) and the Brunn-Minkowski inequality (8.32) lead to

$$V(K, C; 1) \cdot |K|^{\frac{1-n}{n}} = f'(0) \geq f(1) - f(0) = |K + C|^{\frac{1}{n}} - |K|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}},$$

yielding the Minkowski inequality (8.33).

If equality holds in the Minkowski inequality (8.33), then $|K + C|^{\frac{1}{n}} = 2\,|K|^{\frac{1}{n}}$, and hence K and C are translates. ∎

Finally, we characterize the cases when two surface area measures coincide.

Proposition 8.4.3. *For convex bodies $K, C \subset \mathbb{R}^n$, $S_K = S_C$ if and only if K and C are translates.*

Proof. It follows from $S_K = S_C$ and the Minkowski inequality (8.33) that

$$|K| = \frac{1}{n}\int_{S^{n-1}} h_K\, dS_K = \frac{1}{n}\int_{S^{n-1}} h_K\, dS_C = V(C, K; 1) \geq |C|^{\frac{n-1}{n}}|K|^{\frac{1}{n}}; \tag{8.34}$$

therefore, $|K| \geq |C|$. Reversing the role of K and C in (8.34) yields $|C| \geq |K|$, and hence $|C| = |K|$. In turn, equality in (8.34) implies equality in the Minkowski inequality (8.33), and hence we deduce from $|C| = |K|$ that K and C are translates. ∎

8.5 The Hilbert-Aleksandrov operator and the Aleksandrov-Fenchel inequality

Our main goal is to sketch the poof of the Aleksandrov-Fenchel inequality for the mixed volumes based on the paper Shenfeld, van Handel [300] that puts the classical approach by Hilbert and Aleksandrov on Brunn-Minkowski type inequalities into a new perspective.

Theorem 8.5.1 (Aleksandrov-Fenchel Inequality). *If $n \geq 3$ and $C_1, \ldots, C_{n-2}, K, L$ are compact convex sets in $\mathbb{R}^n$, then*

$$V(K, L, C_1, \ldots, C_{n-2})^2 \geq V(K, K, C_1, \ldots, C_{n-2})V(L, L, C_1, \ldots, C_{n-2}). \tag{8.35}$$

It was David Hilbert who provided a proof of the Brunn-Minkowski inequality based on the theory of elliptic differential operators at the beginning of 20th century (his argument is sketched in the 1934 classic Bonnesen, Fenchel [81]), and Aleksandrov developed further Hilbert's approach in the 1930's. The main idea, is that given convex bodies $C_1, \ldots, C_{n-2} \subset \mathbb{R}^n$ with C_+^∞ boundary and $o \in \operatorname{int} C_i$, one considers the elliptic differential operator (cf. Section 8.5.1)

$$\mathcal{A}f = \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}}), \tag{8.36}$$

for $f \in C^\infty(S^{n-1})$, which, according to (8.26), satifies that

$$\begin{aligned} V(K, L, C_1, \ldots, C_{n-2}) &= \frac{1}{n} \int_{S^{n-1}} h_K \mathcal{D}\left(\widetilde{D}^2 h_L, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1} \\ &= \frac{1}{n} (h_K, \mathcal{A} h_L) \end{aligned} \tag{8.37}$$

for convex bodies $K, L \subset \mathbb{R}^n$ with C_+^2 boundary and $o \in \operatorname{int} K, o \in \operatorname{int} L$. Here for $\varphi, \psi \in L_2(S^{n-1}, \mathcal{H}^{n-1}) \supset C^2(S^{n-1})$, we have

$$(\varphi, \psi) = \int_{S^{n-1}} \varphi\psi \, d\mathcal{H}^{n-1}.$$

The representation (8.37) of the mixed volume suggests that in order to understand the Hilber-Aleksandrov operator (8.36), we need some properties of mixed discriminants. We note that if $A_1 \neq 0$ is a symmetric positive semi-definite matrix, and $A_2, \ldots, A_d$ are symmetric positive definite matrices, then

$$\mathcal{D}(A_1, A_2, \ldots, A_d) > 0 \tag{8.38}$$

according to (8.24). The other key property is an Aleksandrov-Fenchel type inequality, proved in Section 8.A:

Theorem 8.5.2 (Aleksandrov's Mixed Discriminant Inequality). *If $d \geq 2$, A is any symmetric $d \times d$ matrix, and $B, M_1, \ldots, M_{d-2}$ are positive-semidefinite symmetric $d \times d$ matrices, then*

$$\mathcal{D}(A, B, M_1, \ldots, M_{d-2})^2 \geq \mathcal{D}(A, A, M_1, \ldots, M_{d-2})\mathcal{D}(B, B, M_1, \ldots, M_{d-2}) \tag{8.39}$$

where no $M_1, \ldots, M_{d-2}$ occur in the case of $d = 2$.

In the upcoming Section 8.5.1, we collect some fundamental properties of elliptic operators defined on $C^\infty(S^{n-1})$, and we prove the Aleksandrov-Fenchel Inequality (8.35) in Section 8.5.2.

8.5.1 Self-adjoint Elliptic linear operators and Hyperbolic Quadratic Forms

For properties of self-adjoint elliptic linear operators, see Section 10.7 (based on Caffarelli, Cabré [138] and Evans [206], Chapter 6). Following Caffarelli, Cabré [138]).

If $\mathcal{E}$ is a self adjoint elliptic operator as above, then its positive eigenspace is the subspace spanned by the eigeinfunctions corresponding to positive eigenvalues.

Lemma 8.5.3 (Hyperbolic Quadratic Forms). *Let $\mathcal{E}$ be a self adjoint elliptic operator densily defined on $C^2(S^{n-1}) \subset L_2(S^{n-1}, \mu)$ for an absolutely continuous measure μ on S^{n-1} with positive C^∞ density function, and let $(\cdot, \cdot)_\mu$ be the inner product corresponding to μ. Then the following are equivalent:*

(i) $(\varphi, \mathcal{E}\psi)_\mu^2 \geq (\varphi, \mathcal{E}\varphi)_\mu (\psi, \mathcal{E}\psi)_\mu$ if $\varphi, \psi \in C^\infty(S^{n-1})$ and $(\psi, \mathcal{E}\psi)_\mu \geq 0$.

(ii) The dimension of the positive eigenspace of $\mathcal{E}$ is at most one.

Proof. Let $\lambda_1 > \lambda_2 \geq \dots$ be the eigenvectors of $\mathcal{E}$, and let $\varphi_1, \varphi_2, \dots \in C^\infty(S^{n-1})$ form an orthogonal basis of $L_2(S^{n-1}, \mu)$ where $\mathcal{E}x_i = \lambda_i x_i$. We may assume that $\lambda_1 > 0$ because otherwise $(\varphi, \mathcal{E}\varphi)_\mu \leq 0$ for any $\varphi \in C^\infty(S^{n-1})$, and hence Lemma 8.5.3 readily holds.

If (i) holds, then $0 \geq \lambda_1 \lambda_2 (\varphi_1, \varphi_1)_\mu (\varphi_2, \varphi_2)_\mu$ follows by applying (i) to $\varphi = \varphi_1$ and $\psi = \varphi_2$; therefore, $\lambda_i \leq \lambda_2 \leq 0$ for $i \geq 2$.

Assuming (ii), we observe that $(\xi, \mathcal{E}\xi)_\mu \leq 0$ for any $\xi \in C^\infty(S^{n-1})$ with $(\xi, \mathcal{E}\varphi_1)_\mu = 0$ by (10.15). We may assume that $(\psi, \mathcal{E}\psi)_\mu > 0$, and hence $(\psi, \mathcal{E}\varphi_1)_\mu \neq 0$. It follows that $(\xi, \mathcal{E}\varphi_1)_\mu = 0$ for $\alpha = (\varphi, \mathcal{E}\varphi_1)_\mu / (\psi, \mathcal{E}\varphi_1)_\mu$ and $\xi = \varphi - \alpha\psi$; therefore, the condition that $\mathcal{E}$ is symmetric yields

$$
\begin{aligned}
0 \geq (\xi, \mathcal{E}\xi)_\mu &= (\varphi, \mathcal{E}\varphi)_\mu - 2\alpha(\varphi, \mathcal{E}\psi)_\mu + \alpha^2 (\psi, \mathcal{E}\psi)_\mu \\
&= (\varphi, \mathcal{E}\varphi)_\mu - \frac{(\varphi, \mathcal{E}\psi)_\mu^2}{(\psi, \mathcal{E}\psi)_\mu} + (\psi, \mathcal{E}\psi)_\mu \left(\alpha - \frac{(\varphi, \mathcal{E}\psi)_\mu}{(\psi, \mathcal{E}\psi)_\mu} \right)^2 \geq (\varphi, \mathcal{E}\varphi)_\mu - \frac{(\varphi, \mathcal{E}\psi)_\mu^2}{(\psi, \mathcal{E}\psi)_\mu}.
\end{aligned}
$$

In turn, we conclude $(\varphi, \mathcal{E}\psi)_\mu^2 \geq (\varphi, \mathcal{E}\varphi)_\mu (\psi, \mathcal{E}\psi)_\mu$. ∎

8.5.2 The Aleksandrov-Fenchel Inequality via Elliptic operators

In this section, we prove the Aleksandrov-Fenchel Inequality (8.35) using an argument due to Shenfeld, van Handel [300] based on Hilbert's and Aleksandrov's ideas.

Since any compact convex set can be approximated by convex bodies with C^∞_+ boundary (cf. Theorem 8.1.10), we may assume that $C_1, \dots, C_{n-2}, K, L \subset \mathbb{R}^n$ are convex bodies with C^∞_+ boundary and containing the origin in their interior, and hence $h_{C_i}(u) > 0$ and $\widetilde{D}^2 h_{C_i}(u)$ is postive definite for $i = 1, \dots, n-2$ and $u \in S^{n-1}$. We consider the differential operator

$$
\begin{aligned}
\mathcal{A} f &= \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}} \right) \\
&= \mathcal{D}\left(\nabla^2 f, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}} \right) + \mathcal{D}\left(f \cdot I_{n-1}, \widetilde{D}^2 h_{C_1}, \dots, \widetilde{D}^2 h_{C_{n-2}} \right)
\end{aligned}
\tag{8.40}
$$

for $f \in C^\infty(S^{n-1})$.

Next we claim that if $f, g, h \in C^\infty(S^{n-1})$, then

$$\int_{S^{n-1}} f \cdot \mathcal{D}\left(\widetilde{D}^2 g, \widetilde{D}^2 h, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1} \tag{8.41}$$
$$= \int_{S^{n-1}} g \cdot \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1}$$

which reads as $\int_{S^2} f \cdot \mathcal{D}\left(\widetilde{D}^2 g, \widetilde{D}^2 h\right) d\mathcal{H}^2 = \int_{S^2} g \cdot \mathcal{D}\left(\widetilde{D}^2 f, \widetilde{D}^2 h\right) d\mathcal{H}^2$ if $n = 3$. According to Lemma 8.1.8, there exist convex bodies $K, K', L, L', M, M' \subset \mathbb{R}^n$ with C^∞_+ boundary such that $f = h_K - h_{K'}$, $g = h_L - h_{L'}$ and $h = h_M - h_{M'}$. Since for any convex bodies $P, Q, R \subset \mathbb{R}^n$ with C^∞_+ boundary, (8.26) and the symmetry of mixed volumes (cf. Theorem 8.3.4 (iii)) yield

$$\int_{S^{n-1}} h_P \cdot \mathcal{D}\left(\widetilde{D}^2 h_Q, \widetilde{D}^2 h_R, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1}$$
$$= nV(P, Q, R, C_2, \ldots, C_{n-2}) = nV(Q, P, R, C_2, \ldots, C_{n-2})$$
$$= \int_{S^{n-1}} h_Q \cdot \mathcal{D}\left(\widetilde{D}^2 h_P, \widetilde{D}^2 h_R, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}}\right) d\mathcal{H}^{n-1},$$

and the mixed discriminant is linear in each of its variable, we conclude (8.41).

If M is any $d \times d$ positive semidefinite matrix with $\|M\| = 1$ (cf. Section 8.5.1), then $\mathcal{D}(M, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}}) > 0$ by (8.38), and hence $\mathcal{A}$ is a uniformly elliptic differential operator (see (10.13) in Section 10.7). In addition, (8.41) yields that $\mathcal{A}$ is also symmetric with respect to $L_2(S^{n-1}, \mathcal{H}^{n-1})$; namely,

$$(f, \mathcal{A}g) = (f, \mathcal{A}g) \ \text{ for any } f, g \in C^\infty(S^{n-1}). \tag{8.42}$$

The Aleksandrov-Fenchel Inequality (8.35) is equivalent to the inequality $(h_K, \mathcal{A}h_L)^2 \geq (h_K, \mathcal{A}h_K)(h_L, \mathcal{A}h_L)$ (cf. (8.37)); in particular, it is sufficient to verify that the bilinear form $(\cdot, \mathcal{A}\cdot)$ extended to $L_2(S^{n-1}, \mathcal{H}^{n-1})$ is hyperbolic (cf. Lemma 8.5.3 (i)). In order to achieve this goal, we provide another interpretation of this bilinear form on $C^\infty(S^{n-1})$. We consider the absolutely continuous measure

$$d\mu = \frac{\mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}})}{h_{C_1}} \, d\mathcal{H}^{n-1}$$

with positive C^∞ density function, and the differential operator

$$\widetilde{\mathcal{A}} f = \frac{h_{C_1} \cdot \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}})}{\mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 h_{C_1}, \ldots, \widetilde{D}^2 h_{C_{n-2}})}$$

for $f \in C^2(S^{n-1})$. Similarly, as for $\mathcal{A}$, we deduce that $\widetilde{\mathcal{A}}$ is uniformly elliptic. For the inner product $(\varphi, \psi)_\mu = \int_{S^{n-1}} \varphi\psi \, d\mu$ of $\varphi, \psi \in L_2(S^{n-1}, \mu)$, we have

$$(f, \mathcal{A}g) = (f, \widetilde{\mathcal{A}}g)_\mu \ \text{ for any } f, g \in C^\infty(S^{n-1}) \tag{8.43}$$

by definition; therefore, (8.42) yields that $\widetilde{\mathcal{A}}$ is symmetric (and hence self adjoint) with respect to the inner product $(\cdot,\cdot)_\mu$.

The reason to work with $\widetilde{\mathcal{A}}$ instead of $\mathcal{A}$ is that h_{C_1} is obviously an eigenfunction of $\widetilde{\mathcal{A}}$ with eigenvalue 1. As $h_{C_1} > 0$, we are automatically ensured that 1 is the maximal principal eigenvalue of $\widetilde{\mathcal{A}}$ (cf. (**??**)).

Lemma 8.5.4. *If $f \in C^\infty(S^{n-1})$, then using the notation as above, we have*

$$(\widetilde{\mathcal{A}}f, \widetilde{\mathcal{A}}f)_\mu \geq (f, \widetilde{\mathcal{A}}f)_\mu.$$

Proof. It follows from applying first Aleksandrov's Mixed Discriminant Inequality (8.39), then (8.41), and finally (8.43) that

$$\begin{aligned}(\widetilde{\mathcal{A}}f, \widetilde{\mathcal{A}}f)_\mu &\geq \int_{S^{n-1}} h_{C_1} \cdot \mathcal{D}(\widetilde{D}^2 f, \widetilde{D}^2 f, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}})\, d\mathcal{H}^{n-1} \\ &= \int_{S^{n-1}} f \cdot \mathcal{D}(\widetilde{D}^2 h_{C_1}, \widetilde{D}^2 f, \widetilde{D}^2 h_{C_2}, \ldots, \widetilde{D}^2 h_{C_{n-2}})\, d\mathcal{H}^{n-1} = (f, \widetilde{\mathcal{A}}f)_\mu.\end{aligned}$$

■

Proof of the Aleksandrov-Fenchel Inequality (8.35). It follows by approximation (cf. Theorem 8.1.10) that we may assume that $C_1, \ldots, C_{n-2}, K, L \subset \mathbb{R}^n$ are convex bodies with C^∞_+ boundary and containing the origin in their interior, and hence $h_{C_i} > 0$ and $\widetilde{D}^2 h_{C_i}$ is positive definite for $i = 1, \ldots, n-2$. We use the elliptic differential operator $\widetilde{\mathcal{A}}$ as above that is self adjoint with respect to $(\cdot,\cdot)_\mu$, and satifies that

$$(h_P, \widetilde{\mathcal{A}}h_Q)_\mu = nV(P, Q, C_1, \ldots, C_{n-2}) \tag{8.44}$$

for convex bodies $P, Q \subset \mathbb{R}^n$ with C^∞_+ boundary by (8.37) and (8.43).

Let λ be any eigenvalue of $\widetilde{\mathcal{A}}$ with eigenfunction $\varphi \in C^\infty(S^{n-1})$. We deduce from Lemma 8.5.4 that $\lambda^2 \geq \lambda$, and hence either $\lambda \geq 1$ or $\lambda \leq 0$. Now $h_{C_1} > 0$ is an eigenfunction of $\widetilde{\mathcal{A}}$ with eigenvalue 1, thus 1 is the simple maximal principal eigenvalue of $\widetilde{\mathcal{A}}$ (cf. Proposition 10.7.1).

It follows that the positive eigenspace of $\widetilde{\mathcal{A}}$ is one-dimensional; therefore, $(h_K, \widetilde{\mathcal{A}}h_L)^2_\mu \geq (h_K, \widetilde{\mathcal{A}}h_K)_\mu (h_L, \widetilde{\mathcal{A}}h_L)_\mu$ by Lemma 8.5.3, yielding the Aleksandrov-Fenchel Inequality (8.35) by (8.44). ■

8.6 Stability of the (Anistropic) Isoperimetric and the Brunn-Minkowski inequalities for convex bodies

The Anisotropic Perimeter and the Anisotropic Isoperimetric Inequality has been discussed for convex bodies in Section 2.4, for sets with rectifiable (or simply Lipschitz) boundary in Section 4.3, and most generally, for sets of finite perimeter in Section 5.2.1.

In this section, we provide stability versions of the Brunn-Minkowski Inequality and the Anisotropic Isoperimetric Inequality fior convex bodies based on estimates in the case of convex bodies with C^2_+ boundary. In turn, we conlude the stability of the Isoperimetric Inequality for convex bodies.

While we derived the Anisotropic Isoperimetric Inequality from the Brunn-Minkowski inequality in Section 4.3, in this section, we use the reverse path: First we prove the stability of the Anisotropic Isoperimetric Inequality using optimal transport and the divergence theorem for convex bodies with C^2_+ boundary, which in turn yields the case of general convex bodies, and then deduce the corresponding stability results for the Brunn-Minkowski inequality. This section essentially follows the arguments of Figalli, Maggi, Pratelli [224,225], and for the improvements on the factor θ_n, we borrow ideas from Segal [531] and Kolesnikov, E. Milman [381].

Let us recall that if $K, E \subset \mathbb{R}^n$ are convex bodies with $o \in \mathrm{int}K$, then the Anisotropic perimeter of E in terms of K is

$$P_K(E) = \int_{\partial E} h_K(\nu_E)\, d\mathcal{H}^{n-1} = \int_{\partial E} \|\nu_E\|_{K^*}\, d\mathcal{H}^{n-1} = \lim_{\varrho\to 0^+} \frac{|E+\varrho K| - |E|}{\varrho}. \quad (8.45)$$

The last formula yields that $P_K(E) = nV(E, K; 1)$ is continuous in E and K, and

$$P_{\Phi K}(\Phi E) = |\det \Phi| \cdot P_K(E) \quad \text{for } \Phi \in \mathrm{GL}(n). \quad (8.46)$$

In this setting, the natural distance of convex bodies is in terms of the volume of the symmetric difference. To define the "homothetic distance" $A(K, E)$ of convex bodies $K, E \subset \mathbb{R}^n$, let $\alpha = |K|^{\frac{-1}{n}}$ and $\beta = |E|^{\frac{-1}{n}}$, and let

$$A(K, E) = \min\left\{|\alpha K \Delta (x + \beta E)| : x \in \mathbb{R}^n\right\}.$$

$A(\cdot,\cdot)$ is actually a metric on the homothety classes of convex bodies (see Claim 8.6.14). Figalli, Maggi, Pratelli [224,225] proved the following estimate with the optimal exponent in terms of $A(K, E)$, and the best estimate for the factor θ_n has been obtained by Kolesnikov, E. Milman [381].

Theorem 8.6.1. *For $\theta_n = cn^{-5}(\log n)^{-2}$ where $c \in (0,1)$ is an absolute constant, if $K, E \subset \mathbb{R}^n$ are convex bodies with $o \in \mathrm{int}K$, then*

$$P_K(E) \geq n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1 + \theta_n \cdot A(K, E)^2\right]. \quad (8.47)$$

Remark. Here the exponent 2 of $A(K, E)^2$ is optimal, and θ_n can't be larger than $36n^{-2}$ (see Remark 8.6.6).

Theorem 8.6.1 yields directly the corresponding stability version of the Isoperimetric Inequality, due to Fusco, Maggi, Pratelli [251]. We verify a version where the factor θ_n is slightly better than in Theorem 8.6.1 if E is close to be a ball:

Theorem 8.6.2. *If $E \subset \mathbb{R}^n$ is a convex body, then*

$$S(E) \geq n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1+\min\{\theta_n A(B^n,E)^2,\delta_n\}\right] \tag{8.48}$$

where $\theta_n = 2^{-12}n^{-4}$ and $\delta_n > 0$ depends on n and can be explicitly calculated.

To have a stability version of the Isoperimetric Inequality in terms of the Hausdorff metric, we note that if $\delta_H(E,B^n) \geq \delta$ for $\delta \in (0,\frac{1}{2})$, then either there exists $u \in S^{n-1}$ such that $\langle x,u\rangle \leq 1-\delta$ for $x \in E$, and hence $B^n\backslash K$ contains a circular cone of height δ and of base of radius $\sqrt{1-(1-\delta)^2} > \sqrt{\delta}$; or $(1-\delta)B^n \subset K$ and there exists $z \in K$ with $\|z\| \geq 1+\delta$, and hence $K\backslash B^n$ contains a circular cone of height δ and of base of radius $\delta \cdot \frac{1-\delta}{\sqrt{(1+\delta)^2-(1-\delta)^2}} > \frac{1}{4}\sqrt{\delta}$; therefore,

$$|E\Delta B^n| \geq \frac{\omega_{n-1}}{n4^{n-1}} \cdot \delta^{\frac{n+1}{2}}. \tag{8.49}$$

In turn, we deduce the following estimate from Theorem 8.6.2:

Corollary 8.6.3. *For $\delta \in [0,\frac{1}{2})$, $r > 0$ and centered convex body $E \subset \mathbb{R}^n$ with $|E| = |rB^n|$, if $\delta_H(E,rB^n) \geq r\delta$, then*

$$S(E) \geq n\omega_n^{\frac{1}{n}}|E|^{\frac{n-1}{n}}\left[1+\eta_n \cdot \delta^{n+1}\right] \tag{8.50}$$

where $\eta_n > 0$ depends only on n and can be explicitely calculated.

Remark. The optimal exponent of δ in (8.50) is 2 if $n=2$ by Bonnesen [84], $(n+1)/2$ if $n \geq 4$ by Fuglede [249], and δ^4 is replaced by $\delta^2/|\log\delta|$ if $n=3$ by Fuglede [249] (see Groemer [272] for a comprehensive survey).

In order to consider the stability of the Brunn-Minkowski inequality , let

$$\sigma(K,E) = \max\left\{\frac{|E|}{|K|},\frac{|K|}{|E|}\right\} \geq 1$$

for convex bodies $K,E \subset \mathbb{R}^n$.

We note that Figalli, Maggi, Pratelli [225] proved Theorem 8.6.1 with the explicit factor $\theta_n^* = \left(\frac{(2-2^{\frac{n-1}{n}})^{\frac{3}{2}}}{122n^7}\right)^2$ that has somewhat worst order as n tends to infinity. Theorem 8.6.1 yields the following (essentially optimal) estimate:

Theorem 8.6.4. *For $\theta_n = cn^{-5}(\log n)^{-2}$ where $c \in (0,1)$ is an absolute constant, if $K,E \subset \mathbb{R}^n$ are convex bodies, then*

$$|K+E|^{\frac{1}{n}} \geq (|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}})\left[1+\frac{\theta_n}{\sigma(K,E)^{\frac{1}{n}}}\cdot A(K,E)^2\right]. \tag{8.51}$$

In turn, Theorem 8.6.4 can be written in the following form:

Corollary 8.6.5. *For $\theta_n = cn^{-4}(\log n)^{-2}$ where $c \in (0,1)$ is an absolute constant, if $K, E \subset \mathbb{R}^n$ are convex bodies with $|E| = |K|$ and $\tau \le \lambda \le 1-\tau$ for $\tau \in (0, \frac{1}{2}]$, then*

$$|(1-\lambda)K + \lambda E| \ge |K| \left[1 + \theta_n \tau \cdot A(K,E)^2\right]. \tag{8.52}$$

Remark. According to Theorem 8.6.7 due to Figalli, van Hintum, Tiba [223], $A(K,E)$ in (8.52) can be replaced by $|E|^{-1} \min_{z\in\mathbb{R}^n} |M_z \backslash E|$ for $M_z = \mathrm{conv}\{E \cup (K-z)\}$, while in this case, $\theta_n > 0$ depending on n might be much smaller.

Remark 8.6.6 (Constants in the stability results for convex bodies).

- If the absolute constant $c = \tilde{c}$ works in Theorem 8.6.1, then (8.79) in the proof of Theorem 8.6.4 shows that $c = \frac{1}{4}\tilde{c}$ works in Theorem 8.6.4 and Corollary 8.6.5.
- Here the exponent 2 of $A(K,E)^2$ and the exponent 1 of τ are optimal, and θ_n can't be larger than $36n^{-2}$ in (8.47) and (8.51) as the following example by Harutyunyan [303] shows: It is sufficient to verify that θ_n can't be larger than $9n^{-2}$ in (8.51). For small $\varepsilon > 0$, let $m = [n/2]$, $K = [-1,1]^n$ and for $t_\varepsilon = 1+\varepsilon$, let

$$E_\varepsilon = \begin{cases} [-t_\varepsilon, t_\varepsilon]^m \times [-t_\varepsilon^{-1}, t_\varepsilon^{-1}]^m & \text{if } n = 2m \\ [-t_\varepsilon, t_\varepsilon]^m \times [-t_\varepsilon^{-1}, t_\varepsilon^{-1}]^m \times [-1,1] & \text{if } n = 2m+1. \end{cases}$$

  It follows that $|E_\varepsilon| = |K| = 2^n$, and as E, K are o-symmetric, we have

$$A(E_\varepsilon, K) = 2^{-n}|E_\varepsilon \Delta K| = t_\varepsilon^{-m}(t_\varepsilon^m - 1) + (1 - t_\varepsilon^{-m}) \ge m\,\varepsilon \ge (n/3)\,\varepsilon$$

  if $\varepsilon > 0$ is small enough. On the other hand, if $\varepsilon > 0$ is small enough, then

$$\begin{aligned}\frac{|K+E_\varepsilon|^{\frac{1}{n}}}{|K|^{\frac{1}{n}} + |E_\varepsilon|^{\frac{1}{n}}} &= \frac{1}{2}\cdot 2^{\frac{n-2m}{n}} \left(1+\varepsilon+(1+\varepsilon)^{-1}\right)^{\frac{2m}{n}} = \left(\frac{1+\varepsilon+(1+\varepsilon)^{-1}}{2}\right)^{\frac{2m}{n}} \\ &\le 1+\varepsilon^2 \le 1 + 9n^{-2}A(E,K)^2.\end{aligned}$$

- As it was observed by Segal [531], Dar's conjecture (1.43) would imply that one can choose $\theta_n \le c/n^2$ for an absolute constant $c > 0$ in (8.47) and (8.51).

Figalli, van Hintum, Tiba [223] have proved an essentially optimal stability version of the Brunn-Minkowski Inequality for linear combinations of measurable sets:

Theorem 8.6.7 (Stability of the Brunn-Minkowski Inequality for measurable sets, Figalli, van Hintum, Tiba). *For $n \ge 2$ and $t \in (0, 1/2]$, there exist $c_n, d_{n,t} > 0$ depending on n and n,t such that if the measurable sets $X, Y \subset \mathbb{R}^n$ satisfy that*

$$|(1-t)X + tY| \le (1+\delta)|X| \text{ and } |X| = |Y| > 0 \tag{8.53}$$

for $\delta \in (0, d_{n,t})$, then there exists a convex body K such that $X, Y - y \subset K$ for some $y \in \mathbb{R}^n$, and

$$|K\backslash X| = |K\backslash(Y-y)| \le c_n t^{-1/2}\delta^{1/2}|X|. \tag{8.54}$$

Remarks.

- The exponents of t and δ are optimal in (3.3) according to Remark 8.6.6.
- The condition that $\delta < d_{n,t}$ in Theorem 3.1.5 for some $d_{n,t} > 0$ depending on n and t is necessary. For example, take $X = [0,1]^n \cap \{p\}$ and $Y = [0,1]^n$ where $\|p\| > 2n$. Then $|X| = |Y| = 1$ and $|(1-t)X + tY| = (1+t^n)|X|$ but $|\text{conv}\, X|$ can be arbitrarily large, and hence $d_{n,t} \le t^n$. Actually, $d_{n,t} = t^n$ according to van Hintum, Keevash [312].
- Figalli, van Hintum, Tiba [223] verified an even stronger estimate if we compare the X and Y satisfying (3.2) to their respective convex hulls:

$$|\text{conv}X\backslash X| \le c_{n,t}\delta\,|X| \text{ and } |\text{conv}Y\backslash Y| \le c_{n,t}\delta\,|Y|$$

where $c_{n,t} > 0$ depends on n, t.

8.6.1 The Anisotropic Isoperimetric Inequality in the case of C_+^2 boundary

In this section, we provide provide a simple proof of the Anisotropic Isoperimetric Inequality for convex bodies with C_+^2 boundaries using the same idea as in Section 5.2.1 in the case of sets of finite perimeter. The fundamental tool is optimal transport, and we heavily use Theorem 8.6.8 by Caffarelli [137] (see also Villani [558], Theorem 4.14). For a convex body $E \subset \mathbb{R}^n$ and $\alpha \in (0,1)$, we say that a function $F : E \to \mathbb{R}^m$ is $C^{k,\alpha}$ for $k \in \mathbb{N}$ if F is C^k in int_E, and all of its partial derivatiaves up to order k extend to a $C^{0,\alpha}$ function on E.

Theorem 8.6.8 (Caffarelli). *Let $E, K \subset \mathbb{R}^n$ be convex bodies with C_+^2 boundaries, and let $f : E \to (0,\infty)$ and $g : K \to (0,\infty)$ be $C^{0,\alpha}$. Then there exists a $C^{2,\alpha}$ convex function φ on E such that $T = D\varphi : E \to K \subset \mathbb{R}^n$ is a $C^{1,\alpha}$ diffeomorphism satisfying $DT(x)$ is a positive definite symmetric matrix for $x \in E$ and*

$$f(x) = g(T(x)) \cdot \det DT(x). \tag{8.55}$$

Theorem 8.6.9 is a special case of Theorem 5.2.4 for sets of finite perimeter:

Theorem 8.6.9. *If $K, E \subset \mathbb{R}^n$ are convex bodies with C_+^2 boundary and $o \in \text{int}\, K$, then*

$$P_K(E) \ge n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}, \tag{8.56}$$

with equality if and only if K and E are homothetic.

Proof. We may assume that $|E| = |K|$, and then we show that the equality case is characterized by E and K being translates. Applying Theorem 8.6.8 to the functions $f = \mathbf{1}_E$ and $g = \mathbf{1}_K$, there exists a C^1 diffeomorphism $T : E \to K$ such that $DT(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in E$. For $\text{div}\, T = \text{tr}\, DT$, the

AM-GM inequality for the eigenvalues of DT yields that $\operatorname{div} T(x) \geq n(\det DT(x))^{\frac{1}{n}} = n$ for $x \in E$, with equality if and only if each eigenvalue of $DT(x)$ is 1; or equivalently, if $DT(x) = I_n$. We deduce from the Divergence Theorem 2.1.4, and as $T(x) \in K$ yields $\langle T(x), \nu_E(x)\rangle \leq h_K(\nu_E(x))$ for $x \in \partial E$ that

$$P_K(E) = \int_{\partial E} h_K(\nu_E(x))\, d\mathcal{H}^{n-1}(x) \geq \int_{\partial E} \langle T(x), \nu_E(x)\rangle\, d\mathcal{H}^{n-1}(x) \tag{8.57}$$

$$= \int_E \operatorname{div} T(x) dx \geq \int_E n\, dx = n|E| = n|K|^{\frac{1}{n}} |E|^{\frac{n-1}{n}}. \tag{8.58}$$

which proves the anisotropic isoperimetric inequality. If $P_K(E) = n|K|^{\frac{1}{n}}|E|^{\frac{n-1}{n}}$ and $|E| = |K|$, then equality in (8.58) yields that $\operatorname{div} T(x) = n(\det DT(x))^{\frac{1}{n}} = n$ for $x \in \operatorname{int} E$; therefore, $DT \equiv I_n$. We conclude that $T(x) = x + z$ for a $z \in \mathbb{R}^n$, and hence $K = E + z$. ■

8.6.2 Stability of the Anisotropic Isoperimetric inequality for convex bodies

In order to estimate the error in (8.58) withing the proof of the Anisotropic Isoperimetric inequality Theorem 8.6.9, we consider the error in the AM-GM inequality. The argument leading to the stability version Lemma 8.6.10 of the AM-GM inequality is due to Harutyunyan [303]. We note that the optimal factor is $2n$ instead of $4n$ in (8.59) (see Alzer [17]).

Lemma 8.6.10. *If $\lambda_1, \ldots, \lambda_n \geq 0$, $\lambda_A = \frac{1}{n}(\lambda_1 + \ldots + \lambda_n)$ is the arighmetic mean, and $\lambda_G = (\lambda_n \ldots \lambda_1)^{1/n}$ is the geometric mean, then*

$$\sum_{i=1}^{n} (\lambda_i - \lambda_G)^2 \leq 4n\, (\max_{i=1}^{n} \lambda_i) \cdot (\lambda_A - \lambda_G). \tag{8.59}$$

Proof. We may assume that $\lambda_n \geq \ldots \geq \lambda_1 > 0$. As $\frac{1}{n}\sum_{i=1}^{n} \sqrt{\lambda_i} \geq \sqrt{\lambda_G}$ and $\left(\sqrt{\lambda_i} + \sqrt{\lambda_G}\right)^2 \leq 2(\lambda_i + \lambda_G)$ by the AM-GM inequality, we deduce that

$$\lambda_A \geq \lambda_G + \frac{1}{n}\sum_{i=1}^{n}\left(\sqrt{\lambda_i} - \sqrt{\lambda_G}\right)^2 \geq \lambda_G + \frac{1}{n}\sum_{i=1}^{n}\frac{(\lambda_i - \lambda_G)^2}{2(\lambda_i + \lambda_G)},$$

which in turn yields (8.59) as $\lambda_i + \lambda_G \leq 2\lambda_n$. ■

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. For a symmetric $n \times n$ matrix A, its Hilbert-Schmidt norm is $\|A\| = \sqrt{\sum_{i=1}^{n} \|Ae_i\|^2}$. As the Hilbert-Schmidt norm (see Section 10.8) is independent of the orthonomal basis of $\mathbb{R}^n$, we have

$$\|A\| = \sqrt{\sum_{i=1}^{n} \mu_i^2} \tag{8.60}$$

where $\mu_1, \ldots, \mu_n$ are the eigenvalues of A. We deduce from (8.59) (applied to the eigenvalues of Φ) and (8.60) (applied to $\Phi - I_n$) that if Φ is a symmetric positive definite matrix with $\det \Phi = 1$ and eigenvalues $\lambda_1, \ldots, \lambda_n > 0$ (and hence $\max_{i=1}^n \lambda_i < \operatorname{tr} \Phi$, then

$$\|\Phi - I_n\| = \sqrt{\sum_{i=1}^{n} (\lambda_i - 1)^2} \leq 2\sqrt{\operatorname{tr} \Phi}\sqrt{\operatorname{tr} \Phi - n}. \tag{8.61}$$

We also need the Poincaré type inequality (6.20) due to Kolesnikov, E. Milman [381] that we quote as (8.62). We recall that the centroid of a convex body $E \subset \mathbb{R}^n$ is denoted by σ_K, see Section 1.11). According Section 6.4, for any convex body $E \subset \mathbb{R}^n$, one find a $\Phi \in \mathrm{SL}(n)$ such that $\widetilde{E} = \Phi(E - \sigma_E)$ is in quasi-isotropic; namely, there exists $\lambda > 0$ such that $\int_{\widetilde{E}} \langle x, u \rangle^2 \, dx = \lambda$ for any $u \in S^{n-1}$. Here E is in isotropic position if, in addition, $|E| = 1$. We recall Proposition 6.4.16 as follows:

Proposition 8.6.11 (Kolesnikov-Milman). *If $E \subset \mathbb{R}^n$ is a centered convex body in quasi-isotropic position, and $f : E \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial E} \left| f - m_f \right| \, d\mathcal{H}^{n-1} \leq cn \log n \cdot \int_E \|Df\| \, d\mathcal{H}^n \tag{8.62}$$

for an absolute constant $c > 0$.

Finally, we need an estimate (8.63) for the volume of the difference of two convex bodies due to Figalli, Maggi, Pratelli [225]. We recall that for a convex body $K \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$, $\Pi_K(x)$ is the closest point of K to x, and $\|\Pi_K(x) - \Pi_K(y)\| \leq \|x - y\|$ for $x, y \in \mathbb{R}^n$ according to Lemma 1.2.11.

Lemma 8.6.12. *If $E, K \subset \mathbb{R}^n$ are convex bodies and $E \not\subset K$, then*

$$|E \backslash K| \leq \int_{(\partial E) \backslash K} |\Pi_K(x) - x| \, d\mathcal{H}^{n-1}(x). \tag{8.63}$$

Proof. We consider $\mathbb{R}^n$ embedded into $\mathbb{R}^{n+1}$ as $f_0^\perp$ for $f_0 \in S^n \subset \mathbb{R}^{n+1}$, and hence $X = ((\partial E) \backslash K) + (0, 1) f_0 \subset \mathbb{R}^{n+1}$ is an embedded Lipschitz n-manifold (see Remark 10.4.7), whose points we write as $(x, t) \in X$ for $x \in (\partial E) \backslash K$ and $t \in (0, 1)$.

We claim that the function $F : X \to \mathbb{R}^n$ defined by

$$F(x, t) = (1 - t)x + x\Pi_K(x) = x + t(\Pi_K(x) - x) \tag{8.64}$$

is Lipschitz and satisfies $(\operatorname{int} E) \backslash K \subset F(X)$. As Π_K is a contraction, if $x, y \in (\partial E) \backslash K$ and $t \in (0, 1)$, then

$$\|F(x, t) - F(y, t)\| = \|(1 - t)(x - y) + t\,(\Pi_K(x) - \Pi_K(y))\| \leq \|x - y\|, \tag{8.65}$$

thus F is Lipschitz with factor $1 + \max_{x \in E} \|\Pi_K(x) - x\|$. If $z \in (\operatorname{int} E) \backslash K$, then $x = s(z - \Pi_K(z)) \in \partial E$ for some $s > 1$ where $z - \Pi_K(z)$ is an exterior normal at $\Pi_K(z) \in \partial K$, and hence $z = F(x, t)$ for $t = \frac{s-1}{s}$.

Next we estimate $|\det DF(x,t)|$ for $x \in (\partial' E)\backslash K$ and $t \in (0,1)$, and hence X is differentiable at (x,t). We consider an orthonomal basis $f_0, f_1, \dots, f_{n-1}$ of the tangent space $\nu_E(x)^\perp \subset \mathbb{R}^{n+1}$ at (x,t), and write $\partial_i F$ to denote the partial derivative in the direction of $i = 0, \dots, n-1$. It follows from (8.64) that $\partial_0 F(x,t) = \Pi_K(x) - x$, and from (8.65) that $\|\partial_i F(x,t)\| \leq 1$ for $i = 1, \dots, n-1$; therefore, Hadanard's inequality (10.17) yields

$$|\det DF(x,t)| \leq \prod_{i=0}^{n-1} \|\partial_i F(x,t)\| \leq \|\Pi_K(x) - x\|\,.$$

We conclude that

$$|E\backslash K| \leq \int_{(\partial E)\backslash K} \int_0^1 |\det DF(x,t)|\, dt\, d\mathcal{H}^{n-1}(x) \leq \int_{(\partial E)\backslash K} \|\Pi_K(x) - x\|\, d\mathcal{H}^{n-1}(x).$$

■

Proof of Theorem 8.6.1. Since $P_K(E) = nV(E,K;1)$ is continuous in E and K, we may assume that E and K have C^2_+ boundary. According to (8.46), we may also assume that $|E| = |K| = 1$, and E is a centered convex body in isotropic position. In particular, the Anisotropic Isotropic inequality (8.56) is equivalent to $P_K(E) \geq n$. We set

$$\delta(E,K) = \frac{P_K(E)}{n} - 1,$$

and hence Theorem 8.6.1 is equivalent with the existence of a $z \in \mathbb{R}^n$ such that

$$|E\Delta(K - z)| \leq cn^{\frac{5}{2}} \log n \sqrt{\delta(E,K)} \tag{8.66}$$

for an absolute constant $c > 0$. Since $|E\Delta(K - z)| \leq 2$ for any $z \in \mathbb{R}^n$, we may assume that

$$\delta(E,K) \leq 1. \tag{8.67}$$

Applying Theorem 8.6.8 to the functions $f = \mathbf{1}_E$ and $g = \mathbf{1}_K$, there exists a C^1 diffeomorphism $T : E \to K$ such that $DT(x)$ is a positive definite symmetric matrix and $\det DT(x) = 1$ for $x \in E$ where $\operatorname{div} T = \operatorname{tr} DT$. We deduce from (8.58) in the proof of the Anisotropic Isoperimetric inequality that

$$\delta(E,K) \geq \frac{1}{n} \int_E (\operatorname{div} T(x) - n)\; dx \tag{8.68}$$

In addition, the Divergence Theorem, $T(x) \in K$ for $x \in E$ and (8.67) yield that

$$\begin{aligned}\int_E \operatorname{div} T(x)\, dx &= \int_{\partial E} \langle T(x), \nu_E(x)\rangle\, d\mathcal{H}^{n-1}(x) \leq \int_{\partial E} h_K(\nu_E(x)\, d\mathcal{H}^{n-1}(x) \\ &= P_K(E) = n(\delta(E,K) + 1) \leq 2n. \end{aligned} \tag{8.69}$$

As $\det DT(x) = 1$ for $x \in E$ and $DT(x) = 1$ is positive definite, we deduce from (8.61), the Hölder inequality, $\operatorname{tr} DT(x) = \operatorname{div} T(x)$, (8.68) and (8.69) that

$$\begin{aligned}\int_E \|DT(x) - I_n\|\, dx &\leq 2\int_E \sqrt{\operatorname{tr} DT(x)}\sqrt{\operatorname{tr} DT(x) - n}\, dx \\ &\leq 2 \cdot \sqrt{\int_E \operatorname{div} T(x)\, dx} \cdot \sqrt{\int_E (\operatorname{div} T(x) - n)\ dx} \\ &\leq 4n\sqrt{\delta(E,K)}. \end{aligned} \tag{8.70}$$

On the other hand, writing $T = (T_1, \dots, T_n)$ for C^1 functions $T_i : E \to \mathbb{R}$, $i = 1, \dots, n$, we have $\|DT(x) - I_n\| = \sqrt{\sum_{i=1}^n \|T_i(x) - e_i\|^2}$ for $x \in \operatorname{int} E$. To verify (8.66), we translate K in a way such that $m_{f_i} = 0$ for $f_i(x) = T_i(x) - \langle e_i, x\rangle$. We deduce by applying the inequality between the quadratic mean and the arithmetic mean, then Proposition 8.6.11 to $f_i(x) = T_i(x) - \langle e_i, x\rangle$, and finally the triangle inequality that

$$\begin{aligned}\int_E \|DT(x) - I_n\|\, dx &\geq \frac{1}{\sqrt{n}}\int_E \sum_{i=1}^n \|DT_i(x) - e_i\|\, dx \\ &\geq \frac{1}{c_1 n^{\frac{3}{2}} \log n}\int_{\partial E} \sum_{i=1}^n \|T_i(x) - \langle e_i, x\rangle\|\, dx \\ &\geq \frac{1}{c_1 n^{\frac{3}{2}} \log n}\int_{\partial E} \|T(x) - x\|\, dx \end{aligned} \tag{8.71}$$

for some absolute constant $c_1 > 0$. Here $|E\Delta K| = 2|E\backslash K|$ follows from $|E| = |K|$, and hence Lemma 8.6.12, $T(x) \in K$, and combining (8.70) and (8.71) yield

$$\begin{aligned}|E\Delta K| = 2|E\backslash K| &\leq 2\int_{(\partial E)\backslash K} \|\Pi_K(x) - x\|\ d\mathcal{H}^{n-1}(x) \\ &\leq 2\int_{\partial E} \|T(x) - x\|\, dx \leq 8c_1 n^{\frac{5}{2}} \log n\sqrt{\delta(E,K)}. \end{aligned}$$

We conclude (8.66), and in turn Theorem 8.6.1. ■

8.6.3 Improvement in the case of the Isoperimetric inequality for convex bodies

If $K = B^n$ in the argument above and E is close to B^n, then we use the following improvement of Proposition 8.6.11 (cf. Proposition 6.4.18):

Proposition 8.6.13. *If $E \subset \mathbb{R}^n$ is a convex body with $|E| = |B^n|$ and $|E\Delta B^n| \leq (4n)^{-2n}|B^n|$, and $f : E \to \mathbb{R}$ is Lipschitz, then*

$$\int_{\partial E} \left|f - m_f\right|\, d\mathcal{H}^{n-1} \leq 8\sqrt{n} \cdot \int_E \|Df\|\, d\mathcal{H}^n.$$

Proof of Theorem 8.6.2. The argument is based on the proof of Theorem 8.6.1 in the previous section. We set K to be the centered Euclidean ball of volume 1, and assume that $|E| = 1$. It follows from Theorem 8.6.1 that there exists an explicit $\delta_n > 0$ such that if $\delta(E, K) \le \delta_n$, then $|(E - w)\Delta K| \le (4n)^{-2n}$ for a suitable $w \in \mathbb{R}^n$. Therefore, Proposition 8.6.13 yields that

$$\int_{\partial E} \left|f - m_f\right| \, d\mathcal{H}^{n-1} \le 8\sqrt{n} \cdot \int_E \|Df\| \, d\mathcal{H}^n. \tag{8.72}$$

for any Lipschitz function $f : E \to \mathbb{R}$.

Using (8.72) instead of Proposition 8.6.11 in the calculations leading to (8.71) implies

$$\begin{aligned}
\int_E \|DT(x) - I_n\| \, dx &\ge \frac{1}{\sqrt{n}} \int_E \sum_{i=1}^n \|DT_i(x) - e_i\| \, dx \\
&\ge \frac{1}{8n} \int_{\partial E} \sum_{i=1}^n \|T_i(x) - \langle e_i, x\rangle\| \, dx \\
&\ge \frac{1}{8n} \int_{\partial E} \|T(x) - x\| \, dx,
\end{aligned}$$

and hence Lemma 8.6.12, $T(x) \in K$, and (8.70) yield

$$\begin{aligned}
|E\Delta K| = 2|E\backslash K| &\le 2 \int_{(\partial E)\backslash K} \|\Pi_K(x) - x\| \; d\mathcal{H}^{n-1}(x) \\
&\le 2 \int_{\partial E} \|T(x) - x\| \, dx \le 64n^2 \sqrt{\delta(E, K)},
\end{aligned}$$

proving Theorem 8.6.2. ∎

8.6.4 Stability of the Brunn-Minkowski Inequality for convex bodies

We prepare the proof of Theorem 8.6.1 by some simple observations. First (2.3) yields that

$$P_M(M) = \int_{\partial M} h_M(\nu_M(x)) \, d\mathcal{H}^{n-1}(x) = n\,|M|. \tag{8.73}$$

holds for any convex body $M \subset \mathbb{R}^n$. In addition, since, $h_{K+E} = h_K + h_E$ for convex bodies $K, E \subset \mathbb{R}^n$, we have

$$P_{K+E}(M) = P_K(M) + P_E(M). \tag{8.74}$$

We also note that $A(\cdot, \cdot)$ is a metric on the homothety classes of convex bodies because it is obviously symmetric, and satisfies the triangle inequality:

Claim 8.6.14. *If $K, E, M \subset \mathbb{R}^n$ are convex bodies, then*

$$A(K,E) \le A(K,M) + A(E,M).$$

Proof. We may assume that $|K| = |E| = |M| = 1$, when the claim follows from $K\Delta E \subset (K\Delta M) \cup (E\Delta M)$ as if $x \in K\backslash E$, then either $x \in M$, and hence $x \in E\Delta M$, or $x \notin M$, and hence $x \in K\Delta M$. ∎

Proof of Theorem 8.6.4. According to Theorem 8.6.1, there exists and absolute constant $\tilde{c} > 0$ such that for $\tilde{\theta}_n = \tilde{c}n^{-5}(\log n)^{-2}$, we have

$$\begin{aligned} P_K(K+E) &\ge n|K|^{\frac{1}{n}}|E+K|^{\frac{n-1}{n}}\left(1+\tilde{\theta}_n \cdot A(K,K+E)^2\right); \\ P_E(K+E) &\ge n|E|^{\frac{1}{n}}|E+K|^{\frac{n-1}{n}}\left(1+\tilde{\theta}_n \cdot A(E,K+E)^2\right). \end{aligned} \tag{8.75}$$

When adding the two estimates in (8.75), we use first that $P_K(K+E) + P_E(K+E) = n|K+E|$ by (8.73) and (8.74), after that the estimate

$$2\sigma(K,E)^{\frac{1}{n}} \ge \max\left\{\frac{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}}, \frac{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}{|E|^{\frac{1}{n}}}\right\},$$

and finally Claim 8.6.14 and the inequality between the quadratic and arithmetic means to obtain

$$\frac{|K+E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}} \ge \frac{|K|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}\left(1+\tilde{\theta}_n \cdot A(K,K+E)^2\right) \tag{8.76}$$

$$+\frac{|E|^{\frac{1}{n}}}{|K|^{\frac{1}{n}}+|E|^{\frac{1}{n}}}\left(1+\tilde{\theta}_n \cdot A(E,K+E)^2\right) \tag{8.77}$$

$$\ge 1 + \frac{\tilde{\theta}_n \cdot \left(A(K,K+E)^2 + A(E,K+E)^2\right)}{2\sigma(K,E)^{\frac{1}{n}}} \tag{8.78}$$

$$\ge 1 + \frac{\tilde{\theta}_n \cdot A(K,E)^2}{4\sigma(K,E)^{\frac{1}{n}}}. \tag{8.79}$$

∎

8.7 The Logarithmic Brunn-Minkowski and the Logarithmic Minkowski conjectures

This section discusses the Logarithmic (L_0) Brunn-Minkowski Conjecture (8.82) and the Logarithmic Minkowski (L_0) conjecture (8.83) that strengthen the Brunn-Minkowski inequality and the Minkowski inequality. We recall that for $\lambda \in (0,1)$ and convex bodies $K, C \subset \mathbb{R}^n$, the Minkowski linear combination is $(1-\lambda)K + \lambda C = \{x \in \mathbb{R}^n : \langle x, u\rangle \le$

$(1-\lambda)h_K(u)+\lambda\, h_C(u)\ \forall u\in S^{n-1}\}$, and according to the Brunn-Minkowski inequality is (cf. Lemma 1.12.2), we have

$$|(1-\lambda)\,K+\lambda\,C|\geq|K|^{1-\lambda}|C|^{\lambda}, \tag{8.80}$$

with equality if and only if K and C are translates. In addition, the Minkowski's inequality (8.33) can written in the form that if $|K|=|C|$, then

$$\int_{S^{n-1}} h_C\,dS_K\geq\int_{S^{n-1}} h_K\,dS_K, \tag{8.81}$$

with equality if and only if K and C are translates.

For $\lambda\in(0,1)$ and o-symmetric convex bodies $K,C\subset\mathbb{R}^n$, their logarithmic or L_0 combination is

$$(1-\lambda)K+_0\lambda C=\{x\in\mathbb{R}^n:\langle x,u\rangle\leq h_K(u)^{1-\lambda}h_C(u)^{\lambda}\text{ for }u\in S^{n-1}\},$$

and hence $(1-\lambda)K+_0\lambda C\subset(1-\lambda)\,K+\lambda\,C$. The Log Brunn-Minkowski Conjecture is the strengthening of (8.80) for origin symmetric convex bodies, stating that

$$|(1-\lambda)K+_0\lambda C|\geq|K|^{1-\lambda}|C|^{\lambda} \tag{8.82}$$

where assuming that K and C have C^1 boundary, equality holds only if K and C are dilates. The equivalent Logarithmic Minkowski conjecture claims that if $|K|=|C|$, then

$$\int_{S^{n-1}} \log h_C\,dV_K\geq\int_{S^{n-1}} \log h_K\,dV_K \tag{8.83}$$

where assuming that K and C have C^1 boundary, equality holds only if $K=C$.

The importance of these conjectures is exhibited by some equivalent formulations and possible consequences. As it is dicussed in Section 9.4, the Log Minkowski conjecture (8.83), including the characterization of equality case for o-symmetric convex bodies $K,C\subset\mathbb{R}^n$ with C^∞_+ boundary and $|K|=|C|$, is equivalent with the uniqueness, conjectured by Lutwak [433], of the solution of the Monge-Ampère equation

$$h\det(\nabla^2h+h\,\mathrm{Id})=f \tag{8.84}$$

for a given positive even C^∞ function f on S^{n-1}.

According to Saroglou [509], the Log Brunn-Minkowski Conjecture yields the "Strong B-conjecture" stating that if μ is an even log-concave measure on $\mathbb{R}^n$ and $K\subset\mathbb{R}^n$ is an o-symmetric convex body, then $t\mapsto\mu(e^tK)$ is a log-concave function of $t\in\mathbb{R}$. Concerning the Gaussian probability measure γ_n on $\mathbb{R}^n$ with density function $e^{-\pi\|x\|^2}$, it was an earlier celebrated "B-inequality" by Cordero-Erausquin, Fradelizi, Maurey [174], and the case when μ is a rotationally symmetric log-concave even measure has been verified by Cordero-Erausquin, Rotem [178].

As another possible consequence of the Log Brunn-Minkowski Conjecture (cf. Livshyts, Marsiglietti, Nayar, Zvavitch [422]), Colesanti, Livshyts, Marsiglietti [171] conjectured the following generalization of the Gardner-Zvavitch conjecture: If μ is an even log-concave measure on $\mathbb{R}^n$, then

$$\mu((1-\lambda)K+\lambda C)^{\frac{1}{n}} \geq (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\mu(C)^{\frac{1}{n}} \tag{8.85}$$

holds for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. In the case when μ is a Gaussian density (the case of the original Gardner-Zvavitch conjecture), the conjecture was finally verified by Eskenazis, Moschidis [204]. In additon, Cordero-Erausquin, Rotem [178] proved (8.101) if μ is a rotationally symmetric log-concave measure.

8.7.1 Some equivalent formulations of the Logarithmic Brunn-Minkowski and Logarithmic Minkowski conjectures

For $\lambda \in (0,1)$ and convex bodies $K, C \subset \mathbb{R}^n$ containing o in their interior, Böröczky, Lutwak, Yang Zhang [110] define the logarithmic or L_0 combination by the formula

$$(1-\lambda)\cdot K +_0 \lambda \cdot C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \text{ for } u \in S^{n-1}\}.$$

In some sense, the L_0 combination generalizes the coordinatewise product of unconditional convex bodies (see Section 8.7.6).

Let us list some basic properities of the L_0 combination of convex bodies $K, C \subset \mathbb{R}^n$ containing o in their interior (see Section 8.7.5):

- $(1-\lambda)\cdot K +_0 \lambda \cdot C \subset (1-\lambda)\,K + \lambda\, C$ is a convex body containing the origin its interior.
- The L_0 combination is linear invariant; namely, if $\Phi \in \mathrm{GL}(n)$, then

$$(1-\lambda)\cdot \Phi(K) +_0 \lambda \cdot \Phi(C) = \Phi\left((1-\lambda)\cdot K +_0 \lambda \cdot C\right). \tag{8.86}$$

- $(1-\lambda)\cdot(\alpha K) +_0 \lambda \cdot (\beta C) = \alpha^{1-\lambda}\beta^{\lambda}\Big((1-\lambda)\cdot K +_0 \lambda \cdot C\Big)$ for $\alpha, \beta > 0$.
- The L_0 combination of polytopes is a polytope, but the boundary of the L_0 combination of convex bodies with C^2_+ boundaries may not be even C^1.
- $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and containing the origin where $\sum_{i=1}^m \dim K_i = n$ and $C_i = \theta_i K_i$ for $\theta_i > 0$, $i = 1, \ldots, m$, then $(1-\lambda)K +_0 \lambda C = \sum_{i=1}^m \theta_i^{\lambda} K_i$.

Böröczky, Lutwak, Yang, Zhang [110] conjectured the following strengthening of the Brunn-Minkowski inequality for origin symetric convex bodies, and Martin Henk proposed the version with centered convex bodies (see also [107]). We recall from Section 1.11 that a convex compact set is centered if its centroid (with respect to its affine hulll) is the origin.

Conjecture 8.7.1 (Log Brunn-Minkowski conjecture). *If $\lambda \in (0,1)$ and $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$|(1-\lambda)\cdot K +_0 \lambda \cdot C| \geq |K|^{1-\lambda}|C|^{\lambda} \tag{8.87}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

Remark. For the case of equality, see Lemma 8.7.7. In particular, if K strictly convex or ∂K is C^1, then equality in (8.87) yields that K and C are dilates according to Conjecture 8.7.1.

As it was observed by Nayar, Tkocz [472], (8.87) may not hold if $K, C \subset \mathbb{R}^n$ are arbitrary convex bodies with $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$. For example, if $K = [-\frac{1}{2}, \frac{1}{2}]^n$, then $|(1-\lambda)K +_0 \lambda(K-z)| < 1$ if $z \in \operatorname{int} K$ and $z \neq o$.

The following examle shows that $(1-\lambda)K +_0 \lambda C$ might be much smaller than $(1-\lambda)K + \lambda C$:

Example 8.7.2. If $a > 0$ is large, $n = 2$, $K = [\frac{-1}{a}, \frac{1}{a}] \times [-a, a]$ and $C = [-a, a] \times [\frac{-1}{a}, \frac{1}{a}]$, then (cf. Lemma 8.7.7)

$$\begin{array}{rcl} \frac{1}{2}\cdot K +_0 \frac{1}{2}\cdot C & = & [-1,1]^2 \\ \frac{1}{2}\cdot K + \frac{1}{2}\cdot C & = & \left[-\frac{1}{2}(a+\frac{1}{a}), \frac{1}{2}(a+\frac{1}{a})\right]^2. \end{array} \tag{8.88}$$

The importance of the L_0 combination is exhibited by the observation (cf. (8.104)) that if $\lambda \in (0,1)$ and K and C are centered convex bodies in $\mathbb{R}^n$, then

$$4^{-n}|K|^{1-\lambda}|C|^{\lambda} \leq |(1-\lambda)\cdot K +_0 \lambda \cdot C| \leq n^{3n/2}|K|^{1-\lambda}|C|^{\lambda}. \tag{8.89}$$

Naturally, the Log Brunn-Minkowski Conjecture 8.7.1 states that 4^{-n} can be replaced by 1. The upper bound in (8.89) shows that the Log Brunn-Minkowski Conjecture 8.7.1 is significantly stronger than the Brunn-Minkowski inequality as $|(1-\lambda)K + \lambda C|$ might be arbitrary large if $|K| = |C| = 1$ according to Example 8.7.2.

Let us state the Logarithmic Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [110] for origin symmetric bodies, and to Böröczky, Kalantzopoulos [107] for centered convex bodies.

Conjecture 8.7.3 (Log-Minkowski conjecture). *If $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{|K|}{n} \log \frac{|C|}{|K|} \tag{8.90}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

Remarks.

- The conjecture is $\mathrm{GL}(n)$ invariant by the invariance properties of the cone volume measure (cf. Proposition 2.6.15 and Lemma 2.6.14).
- An equivalent form of Conjecture 8.7.3 is that if $|C| = |K|$ for centered C and K, then
$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K \tag{8.91}$$
where the case of equality is like in Conjecture 8.7.3.
- We note that the choice of the right translates of K and C are important in Conjecture 8.7.3 as it is shown by the example - due to Nayar, Tkocz [472] - when $K = [-1, 1]^n$ and $C \neq K$ is a translate of K.

In Lemma 8.7.4, we say that the centered convex bodies $K, C \subset \mathbb{R}^n$ have dilated summands if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ that are of dimension at least one, $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

Lemma 8.7.4. *If $\mathcal{F}$ is a family of centered convex bodies in $\mathbb{R}^n$ that is closed under L_0 combination, then the following are equivalent if they hold for all $K, C \in \mathcal{F}$:*

(i) $|(1-\lambda) \cdot K +_0 \lambda \cdot C| \geq |K|^{1-\lambda}|C|^\lambda$ with equality if and only if K and C have dilated summands;

(ii) $f_{C,K}(\lambda) = |(1-\lambda) \cdot K +_0 \lambda \cdot C|$ is log-concave where $\log f_{C,K}$ is linear if and only if K and C have dilated summands;

(iii) $\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geq \frac{|K|}{n} \log \frac{|C|}{|K|}$ with equality if and only if K and C have dilated summands.

Remark. The typical $\mathcal{F}$ is the family of convex bodies invariant under a subgroup $G \subset \mathrm{GL}(n)$ (cf. (8.86)), like origin symmetric convex bodies (when $G = \{I_n, -I_n\}$).

Proof. (i) $\Longrightarrow$ (ii): Let $M_\lambda = (1-\lambda) \cdot K +_0 \lambda \cdot C$. According to (i), (ii) follows if $t, s, \alpha\beta \in (0,1)$ with $\alpha + \beta = 1$, then $\alpha \cdot M_t +_0 \beta \cdot M_s \subset M_{\alpha t + \beta s}$. In turn, Lemma 2.5.6 yields that it is sufficient to prove that $h_{M_{\alpha t+\beta s}}(\nu_{M_{\alpha t+\beta s}}) \geq h_N(\nu_{M_{\alpha t+\beta s}})$ for $z \in \partial' M_{\alpha t+\beta s}$ and $N = \alpha \cdot M_t +_0 \beta \cdot M_s$, which inequality is the consequence of the fact that if $z \in \partial' M_{\alpha t+\beta s}$ and $u = \nu_{M_{\alpha t+\beta s}}$, then Lemma 7.5.1 yields

$$\begin{aligned} h_{M_{\alpha t+\beta s}}(u) &= h_K(u)^{1-\alpha t-\beta s} h_C(u)^{\alpha t+\beta s} = \left(h_K(u)^{1-t} h_C(u)^t\right)^\alpha \left(h_K(u)^{1-s} h_C(u)^s\right)^\beta \\ &\geq h_{M_t}(u)^\alpha h_{M_s}(u)^\beta \geq h_N(u). \end{aligned}$$

Since $\log f_{K,C}$ concave, it is linear if and only if $f_{K,C}(\frac{1}{2}) = f_{K,C}(0)^{\frac{1}{2}} f_{K,C}(1)^{\frac{1}{2}}$, and hence (ii) has the same equality conditions as (i).

(ii) $\Longrightarrow$ (iii): We may assume that $|K| = |C| = 1$ and $R^{-1}B^n \subset K, C \subset RB^n$ for some $R > 1$, and hence $f_{K,C}(\lambda) \geq 1$ for $\lambda \in (0, 1)$ and $R^{-2} \leq h_C/h_K \leq R^2$. As $e^t = 1 + t + O(t^2)$ provided $|t| \leq \frac{1}{2}$ where the implied constant in $O(\cdot)$ is an absolute cosntant, we deduce that if $0 < \lambda < (4 \log R)^{-1}$ and $u \in S^{n-1}$, then

$$h_K(u)^{1-\lambda} h_C(u)^\lambda = h_K(u) \exp\left(\lambda \cdot \log \frac{h_C}{h_K}\right) = h_K(u) + \lambda \cdot h_K(u) \log \frac{h_C}{h_K} + O(\lambda^2 R \log R).$$

It follows from Aleksandrov's Lemma 7.5.2 for the Wulff shape (cf. (7.37)) that

$$0 \leq f'_{K,C}(0) = \int_{S^{n-1}} h_K \cdot \log \frac{h_C}{h_K}\, dS_K = n \int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K, \tag{8.92}$$

proving (iii). Equality in (8.92) yields $f'_{K,C}(0) = 0$, and hence $\log f_{K,C}(0)$ is linear.

(iii) $\Longrightarrow$ (i): Using again the notation $M_\lambda = (1 - \lambda)K +_0 \lambda C$, we deduce from Lemma 2.5.6 and Lemma 7.5.1 that $h_{M_\lambda}(u) = h_K(u)^{1-\lambda} h_C(u)^\lambda$ for S_{M_λ} a.e. $u \in S^{n-1}$; therefore, (iii) implies

$$\begin{aligned} 0 &= \int_{S^{n-1}} \log \frac{h_K^{1-\lambda} h_C^\lambda}{h_{M_\lambda}}\, dV_{M_\lambda} = (1-\lambda) \int_{S^{n-1}} \log \frac{h_K}{h_{M_\lambda}}\, dV_{M_\lambda} + \lambda \int_{S^{n-1}} \log \frac{h_C}{h_{M_\lambda}}\, dV_{M_\lambda} \\ &\geq \frac{(1-\lambda)|M_\lambda|}{n} \log \frac{|K|}{|M_\lambda|} + \frac{\lambda\, |M_\lambda|}{n} \log \frac{|C|}{|M_\lambda|} = \frac{|M_\lambda|}{n} \log \frac{|K|^{1-\lambda}|C|^\lambda}{|M_\lambda|}. \end{aligned}$$

We conclude that $|M_\lambda| \geq |K|^{1-\lambda}|C|^\lambda$, and equality in (i) yields equality in (iii). ■

Remark 8.7.5 (Equivalent formulations of the Log Minkowski conjecture for o-symmetric convex bodies without the case of equality).

Log Minkowski strengthening Minkowski's second inequality: Kolesnikov, E. Milman [381] and Putterman [495] prove that the Log Minkowski conjecture (8.91) for all o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ is equivalent with the conjectured inequality

$$\frac{V(K, C; 1)^2}{|K|} \geq \frac{n-1}{n} V(K, C; 2) + \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K, \tag{8.93}$$

again for all o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ where the implication that the the Log Minkowski conjecture (8.91) yields (8.93) was proved earlier by Colesanti, Livshyts, Marsiglietti [171]. Here (8.93) is a strengthened form of Minkowski's second inequality (7.29) because $|K| \cdot \frac{1}{n} \int_{S^{n-1}} \frac{h_C^2}{h_K}\, dS_K \geq V(K, C; 1)^2$ by the Hölder's inequality.

Actually, Kolesnikov, E. Milman [381] calculated the second derivative of $\lambda \mapsto |(1 - \lambda)K +_0 \lambda C|$ at $\lambda = 0$ for o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^2_+ boundary to show the equivalence of (8.93) with a local version of the Log Minkowski conjecture, and Putterman [495] extended the result to the global version.

Spectral gap formulation of Log Minkowski: A related equivalent formulation using the Hilbert-Brunn-Minkowski operator is due to Kolesnikov, E. Milman [381], and is discussed in Section 8.8.1 in detail. The elliptic and symmetric Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ for a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary is defined for a $\varphi \in C^2(S^{n-1})$ as

$$\mathcal{L}_K\varphi = \frac{\mathcal{D}(\widetilde{D}^2(\varphi\, h_K), \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \ldots . \widetilde{D}^2 h_K)} - \varphi. \tag{8.94}$$

The normalization is chosen by Kolesnikov, E. Milman [381] in a way such that eigenvalues of $-\mathcal{L}_K$ are non-negative, and the positive eigenvalues are at least 1, which property - in line with Hilbert's original approach - is actually equivalent with the Brunn-Minkowski inequality for convex bodies. If K is o-symmetric, and $\lambda_{1,e}(-\mathcal{L}_K)$ is the minimal positive eigenvalue when we restrict $-\mathcal{L}_K$ to even functions, then Kolesnikov, E. Milman [381] prove that

$$\lambda_{1,e}(-\mathcal{L}_K) \geq \frac{n}{n-1} \tag{8.95}$$

for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary is equivalent to (8.93) for any o-symmetric convex bodies $C, K \subset \mathbb{R}^n$. In turn, (8.95) for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary is equivalent to the Log Brunn-Minkowski Conjecture 8.7.1 for o-symmetric convex bodies $C, K \subset \mathbb{R}^n$. All these properties are discussed in more detail in Section 8.8.1.

Log Minkowski and Monge-Ampère equations: As it is dicussed in Section 9.4, the Log Minkowski Conjecture 8.7.3 for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^∞_+ boundary is equivalent to saying that for any positive even C^∞ function f on S^{n-1}, the Monge-Ampère equation

$$h \det(\nabla^2 h + h\,\mathrm{Id}) = f \tag{8.96}$$

on the sphere S^{n-1} has a unique even solution (cf. Remark 9.4.7). In this case, we deduce the Log Minkowski inequality (8.90) for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ by approximation.

The conjectured "B-property" due to Saroglou [508]*:* For any $N \geq 2$ and o-symmetric convex body $K \subset \mathbb{R}^N$ and $N \times N$ positive definite diagonal matrix Φ, the function $s \mapsto |[-1,1]^n \cap \Phi^s K|$ of $s \in \mathbb{R}$ is log-concave.

The conjectured "strong B-property" due to Nayar, Tkocz [473]*:* For any $N > n$ and n-dimensional linear subspace L of $\mathbb{R}^N$, the n-volume of $L \cap \prod_{i=1}^N [-e^{t_i}, e^{t_i}]$ is a log-concave function of $(t_1, \ldots, t_N) \in \mathbb{R}^N$.

Log Brunn Minkowski for log-concave measures due to Saroglou [509]*:* The even Log Brunn-Minkowski Conjecture 8.7.1 yields that for any even log-concave measure μ on $\mathbb{R}^n$ and $\lambda \in (0,1)$, we have

$$\mu((1-\lambda)\cdot K +_0 \lambda \cdot C) \geq \mu(K)^{1-\lambda}\mu(C)^\lambda \tag{8.97}$$

for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. In turn, if (8.97) holds for one fixed even log-concave measure $d\mu = f\, d\mathcal{H}^{n-1}$ with $f(o) > 0$ and f differentiable at o (for example, for the Gaussian γ_n), then the Log Brunn-Minkowski Conjecture 8.7.1 for origin symmetric convex bodies follows.

Log Brunn Minkowski in terms of optimal transport due to Kolesnikov [377]*:* The paper [377] provides an equivalent formulation of the Log Brunn-Minkowski Conjecture for origin symmetric convex bodies in terms of displacement convexity of certain functional of probability measures on the sphere.

8.7.2 Some known cases of the Log Brunn-Minkowski and the Log Minkowski conjectures

The Log Brunn-Minkowski Conjecture 8.7.1 and the Log Minkowski Conjecture 8.7.3 are still open but have been verified in various cases. In $\mathbb{R}^2$, Conjectures 8.7.1 and 8.7.3 are verified by Böröczky, Lutwak, Yang, Zhang [110] for origin symmetric convex bodies, but it is still open for general centered planar convex bodies.

For unconditional convex bodies, Conjectures 8.7.1 and 8.7.3 have been verified by Saroglou [508] (see Section 8.7.6). This result was extended to convex bodies invariant under reflections through n independent linear hyperplanes by Böröczky, Kalantzopoulos [107], and even a stability version is provided by Böröczky, De [96] in this case. The two conjectures for convex bodies are also verified for complex bodies by Rotem [500].

The Log Brunn-Minkowski Conjecture 8.7.1 and the Log Minkowski Conjecture 8.7.3 also holds for origin symmetric convex bodies in a neighbourhood of a fixed centered ellipsoid E; more precisely, for origin symmetric K and C provided $E \subset K, C \subset (1 + c_n)E$ where $c_n > 0$ depends only on n. In this form, the statement is due to Chen, Huang, Li, Liu [154] extending the local estimate by Kolesnikov, E. Milman [381] (an analogues result holds for linear images of l_q balls for $q > 2$ if the dimension n is high enough according to [381] and the method of [154]). If $\mathbb{R}^3$, some additional partial results are obtained by Chen, Feng, Liu [152], and some earlier partial results are due to Colesanti, Livshyts, Marsiglietti [171].

Xi, Leng [570] considered a version of the Log Brunn-Minkowski Conjecture 8.7.1 where the convex bodies K and C in $\mathbb{R}^n$ are translated by vectors depending in both K and C. We set $r(K, C) = \max\{t > 0 : \exists x,\ x + tC \subset K\}$ and $R(K, C) = \min\{t > 0 : \exists x,\ K \subset x + tC\}$, and say that K and C are in dilated position if $o \in K \cap C$ and $r(K, C)\, C \subset K \subset R(K, C)\, C$. We observe that $r(C, K)\, K \subset C \subset R(C, K)\, K$ in this case. Now for any convex bodies K and C there exist $z \in K$ and $w \in C$ such that $K - z$ and $C - w$ are in dilated position. If $n = 2$ and K and C are in dilated position, then Xi, Leng [570] proved Conjecture 8.7.1 including the characterization of equality.

8.7.3 Some cases when only the Logarithmic Minkowski conjecture is known

There are some cases when the Logarithmic Minkowski Conjecture 8.7.3 is known; namely, given an o-symmetric convex body $K \subset \mathbb{R}^n$,

$$\int_{S^{n-1}} \log h_C \, dV_K \geq \int_{S^{n-1}} \log h_K \, dV_K \tag{8.98}$$

holds for any o-symmetric convex body $C \subset \mathbb{R}^n$ with $|K| = |C|$; however, no corresponding version of the Log Brunn-Minkowski inequality exists because L_0 combinations are not in the family. These results are still very important as (8.98) is the inequality intimately connected to the uniqueness of the solution of the Monge-Ampère equation (8.96) called even Logarithmic Minkowski problem (see Section 9.4).

The most elementary case where Conjecture 8.7.3; or equivalently, (8.98) is known is when K is an o-symmetric ellipsoid, as it is verified by Guan, Ni [285] as follows: We may assume that $K = B^n$ and $|C| = |B^n|$, and hence the Jensen inequality (10.4) and the Blaschke-Santaló inequality (6.26) yield

$$\begin{aligned} \exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{|B^n|} dV_K\right) &= \exp\left(\int_{S^{n-1}} \log h_C \cdot \frac{1}{n|B^n|} \, d\mathcal{H}^{n-1}\right) \\ &\geq \left(\int_{S^{n-1}} h_C^{-n} \cdot \frac{1}{n|B^n|} \, d\mathcal{H}^{n-1}\right)^{\frac{-1}{n}} \geq 1. \end{aligned} \tag{8.99}$$

As generalizations of this observation, (8.98) is verified

- under rather generous explicit curvature pinching bounds for ∂K by E. Milman [460] and Ivaki, E. Milman [351];
- if there exists a centered ellipsoid E such that $E \subset K \subset (1 + c_n)E$ where $c_n > 0$ depends on n according to Chen, Huang, Li, Liu [154] building on earlier local estimates by Kolesnikov, E. Milman [381] (the proof of this result requires much deeper tools than the simple argument in (8.99) for the case $K = E$);
- if K is a zonoid by van Handel [299] (with equality case only clarified when K has C^2_+ boundary).

Let $q > 2$. Then the unit ball $B^n_q \subset \mathbb{R}^n$ of the ℓ_q-norm $\|\cdot\|_q$ is a zonoid, and hence (8.98) holds when $K = B^n_q$. Combining the methods of Kolesnikov, E. Milman [381] and Chen, Huang, Li, Liu [154], one finds a threshold $N_q \geq 2$ and a constant $c_{n,q} > 0$ for $n \geq N_q$ such that (8.98) holds if $\Phi B^n_q \subset K \subset \Phi(1 + c_{n,q})B^n_q$ where $n \geq N_q$ and $\Phi \in \mathrm{GL}(n)$.

While the cases when Logarithmic Minkowski Conjecture 8.7.3 is known seem to be rather spares in the space of all o-symmetric convex bodies, E. Milman [460] proved that for any o-symmetric convex body M, there exists an o-symmetric convex body K with C^∞_+ boundary such that $M \subset K \subset 8M$ and (8.98) holds for any o-symmetric convex body C.

8.7.4 Some consequences and variants of the The Logarithmic Brunn-Minkowski and Logarithmic Minkowski conjectures

The validity of the Log-Minkowski (or Log-Brunn-Minkowski) Conjecture is also supported by the fact that various consequences of it has been verified. For example, the L_p-Minkowski Conjecture has been proved when $p \in (0, 1)$ is close to 1 (see Theorem 8.8.5). It follows from its form (8.97) due to Saroglou [509] that the Log-Brunn-Minkowski) Conjecture yields the "B-conjecture" for any even log-concave measure μ on $\mathbb{R}^n$ stating that if $K \subset \mathbb{R}^n$ is an o-symmetric convex body, then $t \mapsto \mu(e^t K)$ is a log-concave function of $t \in \mathbb{R}$. Concerning the Gaussian probability measure γ_n on $\mathbb{R}^n$ with density function $e^{-\pi\|x\|^2}$, it was an earlier celebrated "B-inequality" by Cordero-Erausquin, Fradelizi, Maurey [174] (see Herscovici, Livshyts, Rotem, Volberg [310] for a stability version), and the case when μ is a rotationally symmetric log-concave even measure has been verified by Cordero-Erausquin, Rotem [178].

Next, the Gardner-Zvavitch conjecture in [257] stated that if K and C are origin symmetric convex bodies in $\mathbb{R}^n$, then

$$\gamma_n((1-\lambda)K + \lambda C)^{\frac{1}{n}} \geq (1-\lambda)\gamma_n(K)^{\frac{1}{n}} + \lambda\gamma_n(C)^{\frac{1}{n}}. \tag{8.100}$$

According to Marsiglietti [442] and Livshyts, Marsiglietti, Nayar, Zvavitch [422], the log-Brunn-Minkowski conjecture would imply the Gardner-Zvavitch conjecture. After various attempts, the conjecture was finally verified by Eskenazis, Moschidis [204] not much before that, Kolesnikov, Livshyts [379] verified that if the exponents $\frac{1}{n}$ in (8.100) are changed into $\frac{1}{2n}$, then this modified Gardner-Zvavitch conjecture holds for any pair of centered convex bodies K and C.

We note that independently of the log-Brunn-Minkowski conjecture, various Brunn-Minkowski type inequalities have been proved and conjectured for the Gaussian measure, the most famous ones being the Ehrhardt inequality and the Gaussian isoperimetric inequality (see Livshyts [421]).

Colesanti, Livshyts, Marsiglietti [171] conjectured the following generalization of the Gardner-Zvavitch conjecture: If μ is an even log-concave measure on $\mathbb{R}^n$, then

$$\mu((1-\lambda)K + \lambda C)^{\frac{1}{n}} \geq (1-\lambda)\mu(K)^{\frac{1}{n}} + \lambda\mu(C)^{\frac{1}{n}} \tag{8.101}$$

holds for any origin symmetric convex bodies K and C. According to Livshyts, Marsiglietti, Nayar, Zvavitch [422], the Log-Brunn-Minkowski Conjecture 8.7.1 would imply the conjecture (8.101). Cordero-Erausquin, Rotem [178] proved (8.101) if μ is a rotationally symmetric log-concave measure. In addition, Livshyts [420] verified that (8.101) holds for any even log-concave measure on $\mathbb{R}^n$ and origin symmetric convex bodies K and C if the exponents $\frac{1}{n}$ in (8.101) are changed into $n^{-4-o(1)}$.

For $\lambda \in (0,1)$ and o-symmetric convex bodies $K_0, K_1 \subset \mathbb{R}^n$, Xi [569] proposes the Reverse log-Brunn-Minkowski Conjecture

$$\left|\widetilde{K}_\lambda\right| \le |K_0|^{1-\lambda}|K_1|^\lambda \tag{8.102}$$

where $dS_{K_i}/d(S_{K_0}+S_{K_1})$ denotes the Radon-Nikodym derivative for $i=1,2$ and

$$dS_{\widetilde{K}_\lambda} = \left(\frac{dS_{K_0}}{d(S_{K_0}+S_{K_1})}\right)^{1-\lambda}\left(\frac{dS_{K_1}}{d(S_{K_0}+S_{K_1})}\right)^{\lambda} d(S_{K_0}+S_{K_1}).$$

Xi [569] prove that the Reverse log-Brunn-Minkowski Conjecture [**?**] is equivalent to the Log Brunn-Minkowski Conjecture 8.7.1 for o-symmetric convex bodies.

Additional local versions of Conjecture 8.7.3 are due to Colesanti, Livshyts, Marsiglietti [171], Kolesnikov, Livshyts [380] and Hosle, Kolesnikov, Livshyts [318].

Saroglou [509] proved the following variant of the Logarithmic Brunn-Minkowski conjecture: If $\lambda \in (0,1)$ and K and C are convex bodies in $\mathbb{R}^n$, then

$$\left|((1-\lambda)\cdot K +_0 \lambda\cdot C)^*\right| \le |K^*|^{1-\lambda}|C^*|^\lambda. \tag{8.103}$$

In turn, V. Milman, Rotem [456] observed that this result leads to the following estimates:

Lemma 8.7.6. *If $\lambda \in (0,1)$ and K and C are centered convex bodies in $\mathbb{R}^n$, then*

$$4^{-n}|K|^{1-\lambda}|C|^\lambda \le |(1-\lambda)\cdot K +_0 \lambda\cdot C| \le n^{3n/2}|K|^{1-\lambda}|C|^\lambda. \tag{8.104}$$

Proof. For the lower bound, G. Kuperberg's Reverse Blaschke-Santaló inequality (6.32), (8.103) and the Blaschke-Santaló inequality (6.25) yield that

$$\begin{aligned}4^{-n}|B^n|^2 &\le |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot\left|((1-\lambda)\cdot K +_0 \lambda\cdot C)^*\right|\\ &\le |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot |K^*|^{1-\lambda}|C^*|^\lambda\\ &\le |(1-\lambda)\cdot K +_0 \lambda\cdot C|\cdot\frac{|B^n|^2}{|K|^{1-\lambda}|C|^\lambda}.\end{aligned}$$

For the upper bound, Lenma 1.11.5 due to Kannan, Lovász, Simonovits [361] yields the existence of centered ellipsoids $E' \subset K$ and $E \subset C$ such that $K \subset nE'$ and $C \subset nE$. After a linear transform, we may assume that $E' = B_2^n$ and E is unconditional. Let $a_1,\dots,a_n$ be the half axes of E, and hence $C \subset \widetilde{C} = \prod_{i=1}^n[-na_i, na_i]$ and $K \subset \widetilde{K} = [-n,n]^n$ with $|\widetilde{C}| \le n^{3n/2}|C|$ and $|\widetilde{K}| \le n^{3n/2}|K|$. Since $|(1-\lambda)\cdot\widetilde{K} +_0 \lambda\cdot\widetilde{C}| = |\widetilde{K}|^{1-\lambda}|\widetilde{C}|^\lambda$, we conclude the upper bound in (8.104). ∎

8.7.5 Some basic properties of the L_0 product

Let $K, C \subset \mathbb{R}^n$ be convex bodies with $o \in \operatorname{int} K, \operatorname{int} C$, and let $\lambda \in (0,1)$. Since there exists $R > 1$ such that $R^{-1}B^n \subset K, C \subset RB^n$, we deduce that $R^{-1}B^n \subset (1-\lambda)K +_0 \lambda C \subset RB^n$, and hence $(1-\lambda)K +_0 \lambda C$ - being convex by definition - is a convex body. Since $h_{\Phi M}(u) = h_M(\Phi^t u)$ holds for any compact convex set $M \subset \mathbb{R}^n$, $\Phi \in \mathrm{GL}(n)$ and $u \in \mathbb{R}^n$, it follows that

$$(1-\lambda)\cdot\Phi(K) +_0 \lambda\cdot\Phi(C) = \Phi\left((1-\lambda)\cdot K +_0 \lambda\cdot C\right). \tag{8.105}$$

We recall that according to Lemma 7.5.1, if $z \in \partial'((1-\lambda)K +_0 \lambda C)$ and $u = \nu_{(1-\lambda)K+_0\lambda C}(z)$, then

$$h_{(1-\lambda)\cdot K+_0\lambda\cdot C}(u) = h_K(u)^{1-\lambda}h_C(u)^{\lambda} \tag{8.106}$$

In addition, Lemma SKsuppBasic yields that $M = \{x \in \mathbb{R}^n : \langle x, \nu_M(y)\rangle \le h_M(\nu_M(y))\ \forall y \in \partial' M\}$ for any convex body $M \subset \mathbb{R}^n$.

Lemma 8.7.7. *If $K = K_1 + \ldots + K_m \subset \mathbb{R}^n$ and $C = C_1 + \ldots + C_m \subset \mathbb{R}^n$ are convex bodies for compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one and containing the origin where $\sum_{i=1}^m \dim K_i = n$ and $C_i = \theta_i K_i$ for $\theta_i > 0$, $i = 1, \ldots, m$, $m \ge 1$, then $(1-\lambda)K +_0 \lambda C = \sum_{i=1}^m \theta_i^{\lambda} K_i$.*

Proof. If $m = 1$, then the lemma readily holds; therefore, we assume that $m \ge 2$. Let $L_i = \operatorname{lin} K_i$, $i = 1, \ldots, m$. According to (8.105), we may assume that $L_1, \ldots, L_m$ are pairwise orthogonal.

Lemma 8.7.7 follows if for any $z \in \partial'((1-\lambda)K +_0 \lambda C)$ with $u = \nu_{(1-\lambda)K+_0\lambda C}(z)$, we have $u \in L_i$ for an $i \in \{1, \ldots, m\}$. Let $x \in \partial K$ such that u is an exterior normal art x. Then $x = x_+ \ldots + x_m$ for $x_i \in K_i$. We may assume that there exists index $p \ge 1$ such that $x_i \in \operatorname{relbd} K_i$ for $i \le p$ and $x_i \in \operatorname{relint} K_i$ for $i > p$. In particular, there exists exterior unit normal $u_i \in L_i$ to K_i at x_i for $i \le p$ such that $u = \sum_{i=1}^p \alpha_i u_i$ for $\alpha_1, \ldots, \alpha_p \ge 0$. We may assume that for some index q with $1 \le q \le p$, $\alpha_i > 0$ if $i \le q$, and $\alpha_i = 0$ if $i > q$. We observe that $u = \sum_{i=1}^q \alpha_i u_i$ is an exterior normal at $y = \sum_{i=1}^m \lambda_i x_i \in \partial C$, $h_K(u_i) = \langle x, u_i\rangle$ and $h_C(u_i) = \langle y, u_i\rangle$ for $i = 1, \ldots, q$. As $z \in \partial'((1-\lambda)\cdot K +_0 \lambda\cdot C)$, (8.106) and the Hölder inequality (10.3) yield that

$$\begin{aligned}
\langle z, u\rangle = h_{(1-\lambda)\cdot K+_0\lambda\cdot C}(u) = h_K(u)^{1-\lambda}h_C(u)^{\lambda} &= \left(\sum_{i=1}^q \alpha_i\langle x, u_i\rangle\right)^{1-\lambda}\left(\sum_{i=1}^q \alpha_i\langle y, u_i\rangle\right)^{\lambda}\\
&\ge \sum_{i=1}^q \alpha_i\langle x, u_i\rangle^{1-\lambda}\langle y, u_i\rangle^{\lambda} = \sum_{i=1}^q \alpha_i h_K(u_i)^{1-\lambda}h_C(u_i)^{\lambda}\\
&\ge \sum_{i=1}^q \alpha_i h_{(1-\lambda)K+_0\lambda C}(u_i) \ge \sum_{i=1}^q \alpha_i\langle z, u_i\rangle = \langle z, u\rangle.
\end{aligned} \tag{8.107}$$

We deduce that $h_{(1-\lambda)K+_0\lambda C}(u_i) = \langle z, u_i\rangle$ for $i = 1, \ldots, q$, and hence $u_1, \ldots, u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conlude that $q = 1$; therefore, $u \in L_1$. ∎

Similar argument yields that the L_0 combination of polytopes is a polytope.

Lemma 8.7.8. *If $K, C \subset \mathbb{R}^n$ are n-polytopes, then $(1-\lambda)K +_0 \lambda C$ is an n-polytope whose facet exterior unit normals are facet exterior unit normals to the Minkowski sum $K + C$, as well.*

Proof. Let $u_1, \ldots, u_m$ be the exterior unit normals to the facets of the polytope $K + C$ (cf. (1.4)). Lemma 8.7.8 follows if whenever $u = \nu_{(1-\lambda)K+_0\lambda C}(z)$ for a $z \in \partial'((1-\lambda)K +_0 \lambda C)$, then $u \in \{u_1, \ldots, u_m\}$. Let w be a vertex of $K + C$ such that u is an exterior normal at w to $K + C$. According to Lemma 1.4.10, the normal cone at w is the positive hull of a subset of $\{u_1, \ldots, u_m\}$; therefore, we may assume that $u = \sum_{i=1}^{q} \alpha_i u_i$ for $\alpha_1, \ldots, \alpha_q \geq 0$, $q \geq 1$, where $u_1, \ldots, u_q$ are exterior normals at w to $K + C$. Now $w = x + y$ for a vertex x of K and a vertex y of C, and hence $u, u_1, \ldots, u_q$ are exterior normals at x to K and at y to C, and $h_K(u_i) = \langle x, u_i\rangle$ and $h_C(u_i) = \langle y, u_i\rangle$ for $i = 1, \ldots, q$. Exactly the same calculations as in (8.107) yield that $h_{(1-\lambda)K+_0\lambda C}(u_i) = \langle z, u_i\rangle$ for $i = 1, \ldots, q$, and hence $u_1, \ldots, u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conclude that $q = 1$; therefore, $u = u_1$. ∎

Example 8.7.9 (The boundary of the L_0 combination of convex bodies with C^2_+ boundaries may not be C^1). Let e_1, e_2 be the orthonormalbasis of $\mathbb{R}^2$. For $a > 2$, and consider the ellipses $E_a = \mathrm{diag}[a, a^{-1}]B^2$ and $\widetilde{E}_a = \mathrm{diag}[a^{-1}, a]B^2$. Since $E_a \subset P_a = [a, -a] \times [\frac{1}{a}, \frac{-1}{a}]$ and $\widetilde{E}_a \subset \widetilde{P}_a = [\frac{1}{a}, \frac{-1}{a}] \times [a, -a]$, we deduce using Lemma 8.7.7 that

$$\tfrac{1}{2} \cdot E_a +_0 \tfrac{1}{2} \cdot \widetilde{E}_a \subset \tfrac{1}{2} \cdot P_a +_0 \tfrac{1}{2} \cdot \widetilde{P}_a = [-1, 1] \times [-1, 1] = W.$$

For $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, we have

$$h_{E_a}(u)^{\frac{1}{2}} h_{\widetilde{E}_a}(u)^{\frac{1}{2}} > \langle ae_1, u\rangle^{\frac{1}{2}} \langle ae_2, u\rangle^{\frac{1}{2}} = a/\sqrt{2} > h_W(u) \geq h_{\frac{1}{2}\cdot E_a +_0 \frac{1}{2}\cdot \widetilde{E}_a}(u);$$

therefore, u is not an exterior normal at a regular boundary point of $\frac{1}{2} \cdot E_a +_0 \frac{1}{2} \cdot \widetilde{E}_a$ by (8.106).

8.7.6 The Logarithmic Brunn-Minkowski and Logarithmic Minkowski conjectures for unconditional convex bodies

This section discusses the L_0 combination of unconditional convex bodies. Here a convex body $K \subset \mathbb{R}^n$ is unconditional if $(\pm x_1, \ldots, \pm x_n) \in K$ for $(x_1, \ldots, x_n) \in K$.

From Section 3.6, we recall the notion of coordinatewise product of unconditional convex bodies: If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then

$$K^{1-\lambda} \cdot C^{\lambda} = \Big\{ \Big(\pm |x_1|^{1-\lambda} |y_1|^{\lambda}, \ldots, \pm |x_n|^{1-\lambda} |y_n|^{\lambda} \Big) : \\ (x_1, \ldots, x_n) \in K \;\&\; (y_1, \ldots, y_n) \in C \Big\}.$$

The coordinatewise product is a classical notion of Functional Analysis, while the L_0 combination was only defined in the 2012 paper Böröczky, Lutwak, Yang, Zhang [110]; still the two notions are closely related as

$$K^{1-\lambda} \cdot C^{\lambda} \subset (1-\lambda) \cdot K +_0 \lambda \cdot C. \tag{8.108}$$

To prove (8.108), it is sufficient to verify that if $x = (x_1, \ldots, x_n) \in K$, $y = (y_1, \ldots, y_n) \in C$ and $u = (u_1, \ldots, u_n) \in S^{n-1}$ with $x_i, y_i, u_i \geq 0$, then $\langle z, u \rangle \leq \langle x, u \rangle^{1-\lambda} \langle y, u \rangle^{\lambda}$ for $z = (x_1 y_1, \ldots, x_n y_n)$. In turn, this property follows from the Hölder inequality using the measure μ on $\{1, \ldots, n$ with $\mu(\{i\}) = u_i$.

According to Theorem 3.6.4, the coordinatewise product satisfies the following Brunn-Minkowski-type inequality:

Theorem 8.7.10 (Uhrin-Bollobas-Leader, equality by Saroglou). *If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$\left| K^{1-\lambda} \cdot C^{\lambda} \right| \geq |K|^{1-\lambda} |C|^{\lambda}. \tag{8.109}$$

Equality holds if and ony if $K = \Phi C$ for a positive definit diagonal matrix Φ.

In turn, (8.108) directly yields the Log Brunn-Minkowski Conjecture 8.7.1 for unconditional convex bodies, as it was observed by Saroglou [508]. The non-trivial part is to characterize equality, which was also done by Saroglou [508] (with a slight correction in [107]).

Theorem 8.7.11. *If $K, C \subset \mathbb{R}^n$ are unconditional convex bodies and $\lambda \in (0,1)$, then*

$$|(1-\lambda) \cdot K +_0 \lambda \cdot C| \geq |K|^{1-\lambda} |C|^{\lambda}, \tag{8.110}$$

with equality if and only if $K = K_1 \oplus \ldots \oplus K_m$ and $L = L_1 \oplus \ldots \oplus L_m$ for unconditional compact convex sets $K_1, \ldots, K_m, L_1, \ldots, L_m$ of dimension at least one, $m \geq 1$ where $\sum_{i=1}^m \dim K_i = n$ and K_i and L_i are dilates, $i = 1, \ldots, m$.

Proof. Combining Theorem 8.7.10 and (8.108) yields (8.110). If K and C are as described after (8.110), then we have equality in (8.110) according to Lemma 8.7.7.

On the other hand, if equality holds in (8.110), then we deduce from Theorem 8.7.10 and (8.108) that $K^{1-\lambda} \cdot C^{\lambda} = (1-\lambda) \cdot K +_0 \lambda \cdot C$ and $|K^{1-\lambda} \cdot C^{\lambda}| = |K|^{1-\lambda} |C|^{\lambda}$. According to Theorem 8.7.10, there exists a positive definite diagonal matrix Φ such that $C =$

ΦK. Let $e_1, \ldots, e_n$ be the corresponding orthonormal basis of $\mathbb{R}^n$, and for $t \in \mathbb{R}$, consider the diagonal matrix Φ^t defined by $\langle \Phi^t e_i, e_i \rangle = \langle \Phi e_i, e_i \rangle^t$ for $i = 1, \ldots, n$. Since the definition of the coordinatewise product yields that $K^{1-\lambda} \cdot (\Phi K)^\lambda = \Phi^\lambda \left(K^{1-\lambda} \cdot K^\lambda\right)$, we have

$$\Phi^\lambda K = (1-\lambda) \cdot K +_0 \lambda \cdot (\Phi K). \tag{8.111}$$

If all eigenvalues of Φ are the same $\theta > 0$, then $C = \theta K$, and hence the characterization of equality follows with $m = 1$. Therefore, we may assume that $L_1, \ldots, L_m$, $m \geq 2$, are the eigenspaces of Φ with eigenvalues $\theta_1, \ldots .\theta_m$ with $\theta_i \neq \theta_j$ for $i \neq j$ Since $K = \{x \in \mathbb{R}^n : \langle x, \nu_K(y) \rangle \leq h_K(\nu_K(y))\ \forall y \in \partial' K\}$ according to Lemma 2.5.6, Theorem 8.7.11 follows from the claim

$$\nu_K(y) \in \cup_{i=1}^m L_i \text{ for } y \in \partial' K. \tag{8.112}$$

We prove (8.112) by contradiction; therefore, we suppose that $\nu_K(y) \notin \cup_{i=1}^m L_i$ for a $y = (y_1, \ldots, y_n) \in \partial' K$, and set $u = (u_1, \ldots, u_n) = \nu_K(y)$. Since K is unconditional, we may assume that each $u_i \geq 0$ and each $y_i \geq 0$. Possibly after reindexing, we may also assume that $e_1 \in L_1$ and $u_1 > 0$, and $e_n \in L_m$ and $u_n > 0$ by the indirect hypothesis. Using the unconditionality of K, $u_1 > 0$, $u_n > 0$ and $y \in \partial' K$, we also deduce that $y_1 > 0$ and $y_n > 0$ As $\langle \Phi^t u, e_1 \rangle = \theta_1^t \langle u, e_1 \rangle > 0$ and $\langle \Phi^t u, e_n \rangle = \theta_m^t \langle u, e_n \rangle > 0$ where $\theta_1 \neq \theta_m$, we deduce that

$$\text{neither } u \text{ nor } \Phi^{-1} u \text{ is parallel to } v = \Phi^{-\lambda} u. \tag{8.113}$$

We observe that $v = \Phi^{-\lambda} u$ is an exterior normal at the regular boundary point $\Phi^\lambda y$ of $\Phi^\lambda K$. Combining this property with (8.111) and Lemma 7.5.1 yields that

$$\langle v, \Phi^\lambda y \rangle = h_K(v)^{1-\lambda} h_{\Phi K}(v)^\lambda. \tag{8.114}$$

On the other hand, since $\nu_K(y) = u$ for $y \in \partial' K$ and $\nu_{\Phi K}(\Phi x) = \Phi^{-1} u$ for $\Phi y \in \partial'(\Phi K)$, we deduce from (8.113) that

$$\langle v, y \rangle < h_K(v) \ \text{ and } \ \langle v, \Phi y \rangle < h_{\Phi K}(v).$$

In particular, the Hölder inequality (10.3) yields that

$$\begin{aligned}\langle v, \Phi^\lambda y \rangle = \sum_{i=1}^n u_i y_i &\leq \left(\sum_{i=1}^n \theta_i^{-\lambda} u_i y_i\right)^{1-\lambda} \left(\sum_{i=1}^n \theta_i^{1-\lambda} u_i y_i\right)^\lambda = \langle v, y \rangle^{1-\lambda} \langle v, \Phi y \rangle^\lambda \\ &< h_K(v)^{1-\lambda} h_{\Phi K}(v)^\lambda.\end{aligned}$$

This contradicts (8.114), and completes the proof of Proposition 8.7.11. ∎

8.8 The L_p Brunn-Minkowski and the L_p Minkowski conjectures and inequalities when $p \in (0, 1)$

This section discusses possible L_p versions of the Brunn-Minkowski inequality and the Minkowski inequality for $p \in (0, 1)$ and centered convex bodies,; namely, the L_p Brunn-Minkowski and the L_p Minkowski conjectures/inequalities. These conjectures and inequalities connect on the one hand, the L_0 (logarithmic) Brunn-Minkowski Conjecture 8.7.1 and the L_0 (logarithmic) Minkowski Conjecture 8.7.3, and on the other hand, the Brunn-Minkowski inequality and the Minkowski inequality (the L_1 case) and their L_p generalizations for $p > 1$ by Firey [231] from 1962 (see Section 7.6). We recall that for $\alpha, \beta > 0$ and convex bodies $K, C \subset \mathbb{R}^n$, the Minkowski linear combination is

$$\alpha \cdot K + \beta \cdot C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \alpha\, h_K(u) + \beta\, h_C(u)\ \forall u \in S^{n-1}\}, \tag{8.115}$$

and according to the Brunn-Minkowski inequality is (cf. Lemma 1.12.2), we have

$$|\alpha\, K + \beta\, C|^{\frac{1}{n}} \geq \alpha\, |K|^{\frac{1}{n}} + \alpha\, |C|^{\frac{1}{n}} \tag{8.116}$$

with equality if and only if K and C are homothetic in (8.116). In addition, Minkowski's inequality (8.33) says that if $o \in \operatorname{int} K$, then

$$\int_{S^{n-1}} \frac{h_C}{h_K}\, dV_K = \frac{1}{n} \int_{S^{n-1}} h_C\, dS_K \geq |K|^{\frac{n-1}{n}} |C|^{\frac{1}{n}}, \tag{8.117}$$

with equality if and only if K and C are homothetic.

On the other hand, for $\lambda \in (0, 1)$ and convex bodies $K, C \subset \mathbb{R}^n$ containing the origin in their interior, the L_0 (logarithmic) combination is

$$(1 - \lambda) \cdot K +_0 \lambda \cdot C = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u)^{1-\lambda} h_C(u)^{\lambda} \text{ for } u \in S^{n-1}\}.$$

As it is discussed in Section 8.7, the L_0 Brunn-Minkowski conjecture and the L_0 Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [110] says that if $K, C \subset \mathbb{R}^n$ are centered convex bodies and $\lambda \in (0, 1)$, then

$$|(1 - \lambda) \cdot K +_0 \lambda \cdot C| \geq |K|^{1-\lambda} |C|^{\lambda}, \tag{8.118}$$

$$\int_{S^{n-1}} \log \frac{h_C}{h_K}\, dV_K \geq \frac{|K|}{n} \log \frac{|C|}{|K|}, \tag{8.119}$$

where equality yields that K and C have dilated summands; namely, $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ that are of dimension at least one, $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

However, our main focus is the case $p \in (0, 1)$ in this section. For $\alpha, \beta \geq 0$, Böröczky, Lutwak, Yang, Zhang [110] defined the L_p combination of the convex bodies $K, C \subset \mathbb{R}^n$ containing the origin in their interior as

$$\alpha \cdot K +_p \beta \cdot C = \left\{x \in \mathbb{R}^n : \langle x, u\rangle^p \leq \alpha\, h_K(u)^p + \beta\, h_C(u)^p,\ u \in S^{n-1}\right\}, \tag{8.120}$$

which is a convex body containing the origin in its interior. This extends the definition of Firey's L_p addition for $p \geq 1$ (see (7.44)). Given $p \in (0,1)$, let us list some basic properities of the L_p combination of convex bodies $K, C \subset \mathbb{R}^n$ containing the origin in their interior where (8.121) follows from the relation $h_{\Phi M}(u) = h_M(\Phi^t u)$ for any convex body M, (8.122) follows from the Jensen inequality (10.4), and the last property is proved in Section 8.8.2:

- The L_p combination is linear invariant; namely, if $\Phi \in \mathrm{GL}(n)$ and $\alpha, \beta > 0$, then

$$\alpha \cdot \Phi(K) +_p \beta \cdot \Phi(C) = \Phi\left(\alpha \cdot K +_p \beta \cdot C\right). \tag{8.121}$$

- If $q > r \geq 0$ and $\lambda \in (0,1)$, then

$$(1-\lambda) \cdot K +_r \lambda \cdot C \subset (1-\lambda) \cdot K +_q \lambda \cdot C. \tag{8.122}$$

- The L_p combination of polytopes is a polytope, but the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be even C^1.

Böröczky, Lutwak, Yang, Zhang [110] conjectured the following strengthening of the Brunn-Minkowski inequality for origin symetric convex bodies, and Martin Henk proposed the version with centered convex bodies (see also [107]).

Conjecture 8.8.1 (L_p Brunn-Minkowski conjecture for $p \in (0,1)$). *If $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$\left|\alpha \cdot K +_p \beta \cdot C\right|^{\frac{p}{n}} \geq \alpha\,|K|^{\frac{p}{n}} + \beta\,|C|^{\frac{p}{n}} \tag{8.123}$$

for $\alpha, \beta > 0$ with equality if and only if K and C are dilates, and

$$\left|(1-\lambda) \cdot K +_p \lambda \cdot C\right| \geq |K|^{1-\lambda}|C|^{\lambda}. \tag{8.124}$$

for $\lambda \in (0,1)$ with equality if and only if $K = C$.

Remark. (8.123) and (8.124) are equivalent where (8.123) yields (8.124) by the AM-GM inequality. On the other hand, writing $K = \alpha_0 K_0$ and $C = \beta_0 C_0$ for $\alpha_0 = |K|^{1/n}$ and $\beta_0 = |C|^{1/n}$, applying (8.124) to K_0, C_0 and $\lambda = \frac{\beta\beta_0}{\alpha\alpha_0 + \beta\beta_0}$ implies (8.123).

As it was observed by Nayar, Tkocz [472], Conjectiure 8.8.1 may not hold if $K, C \subset \mathbb{R}^n$ are arbitrary convex bodies with $o \in \operatorname{int} K$ and $o \in \operatorname{int} C$. For example, if $K = [-\frac{1}{2}, \frac{1}{2}]^n$, then $\left|(1-\lambda) \cdot K +_p \lambda \cdot (K - z)\right| < 1$ if $z \in \operatorname{int} K$ and $z \neq o$.

Next we state the L_p-Minkowski conjecture due to Böröczky, Lutwak, Yang, Zhang [110] for origin symmetric bodies, and to Böröczky, Kalantzopoulos [107] for centered convex bodies.

Conjecture 8.8.2 (L_p-Minkowski conjecture for $p \in (0,1)$). *If $K, C \subset \mathbb{R}^n$ are centered convex bodies, then*

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K \geq |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}, \tag{8.125}$$

with equality if and only if K and C are dilates.

Remarks.

- The conjecture is $\mathrm{GL}(n)$ invariant by the invariance properties of the cone volume measure (cf. Proposition 2.6.15 and Lemma 2.6.14).
- An equivalent form of Conjecture 8.7.3 is that if $|C| = |K|$ for centered C and K, then
$$\int_{S^{n-1}} \left(\frac{h_C}{h_K}\right)^p |K|^{-1} dV_K \geq 1 \tag{8.126}$$
for the probability measure $|K|^{-1} dV_K$ with equality if and only if $K = C$.
- We note that the choice of the right translates of K and C are important in Conjecture 8.7.3 as it is shown by the example - due to Nayar, Tkocz [472] - when $K = [-1, 1]^n$ and $C \neq K$ is a translate of K.

Lemma 8.8.3 shows that the L_p Brunn-Minkowski Conjecture 8.8.1 and the L_p Minkowski Conjecture 8.8.2 for $p \in (0, 1)$ and origin symmetric convex bodies are equivalent.

Lemma 8.8.3. *If $p \in (0, 1)$, and $\mathcal{F}$ is a family of centered convex bodies in $\mathbb{R}^n$ that is closed under L_p combination, then the following are equivalent if they hold for all $K, C \in \mathcal{F}$:*

(i) $\left|(1-\lambda) \cdot K +_p \lambda \cdot C\right| \geq |K|^{1-\lambda} |C|^\lambda$ with equality if and only if K and C have dilated summands;

(ii) The function $\lambda \mapsto \left|(1-\lambda) \cdot K +_p \lambda \cdot C\right|^{\frac{p}{n}}$ is concave on $[0, 1]$, and is linear if and only if K and C are dilates;

(iii) $\int_{S^{n-1}} \frac{h_C^p}{h_K^p} dV_K \geq |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}}$ with equality if and only if K and C are dilates.

Remark. The typical $\mathcal{F}$ is the family of convex bodies invariant under a subgroup $G \subset \mathrm{GL}(n)$ (cf. (8.121)), like origin symmetric convex bodies (when $G = \{I_n, -I_n\}$).

Proof. The proof of (i) $\Longrightarrow$ (ii) and (ii) $\Longrightarrow$ (iii) as essentially the same as in the L_0 case in Lemma 8.7.4; therefore, we not present the argument.

(iii) $\Longrightarrow$ (i): Using the notation $M_\lambda = (1-\lambda) \cdot K +_p \lambda \cdot C$, we deduce from Lemma 2.5.6 and Lemma 7.5.1 that $h_{M_\lambda}(u) = ((1-\lambda) h_K(u)^p + \lambda\, h_C(u)^p)^{\frac{1}{p}}$ for S_{M_λ} (and hence V_{M_λ}) a.e. $u \in S^{n-1}$; therefore, (iii) implies

$$\begin{aligned}
|M_\lambda| &= \int_{S^{n-1}} \frac{(1-\lambda) h_K^p + \lambda\, h_C^p}{h_{M_\lambda}^p} dV_{M_\lambda} = (1-\lambda) \int_{S^{n-1}} \frac{h_K^p}{h_{M_\lambda}^p} dV_{M_\lambda} + \lambda \int_{S^{n-1}} \frac{h_C^p}{h_{M_\lambda}^p} dV_{M_\lambda} \\
&\geq (1-\lambda) |M_\lambda|^{1-\frac{p}{n}} |K|^{\frac{p}{n}} + \lambda |M_\lambda|^{1-\frac{p}{n}} |C|^{\frac{p}{n}}.
\end{aligned}$$

We conclude that $|M_\lambda|^{\frac{p}{n}} \geq (1-\lambda)|K|^{\frac{p}{n}} + \lambda\, |C|^{\frac{p}{n}}$, and equality in (i) yields equality in (iii). ∎

Remark 8.8.4 (The L_p Minkowski conjecture is "monotone in p"). The Jensen inequality (10.4) yields the following where $1 > q > p \geq 0$ and $K, C \subset \mathbb{R}^n$ are centered convex bodies with $|K| = |C| = 1$:

(i) If the inequality in the L_p Brunn-Minkowki inequality holds for K and C; namely, $|(1-\lambda) \cdot K +_p \lambda \cdot C| \geq 1$ (cf. (8.118) and (8.124)), then $|(1-\lambda) \cdot K +_q \lambda \cdot C| \geq 1$.

(ii) If the inequality in the L_p Minkowki inequality holds for K and C; namely, $\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geq 0$ if $p = 0$ (cf. (8.119)) or $\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geq 1$ if $p > 0$ (cf. (8.125)), then

$$\int_{S^{n-1}} \frac{h_C^q}{h_K^q} \, dV_K \geq 1, \tag{8.127}$$

where equality in (8.127) yields $C = K$ provided ∂K is C^1.

For the characterization of equality, what the Jensen inequality (10.4) directly yields is that there exists $t > 0$ such that $h_C(u) = t \cdot h_K(u)$ for S_K a.e. $u \in S^{n-1}$. Since $\operatorname{supp} S_K = S^{n-1}$ if ∂K is C^1, we deduce that $C = t \cdot K$, and hence $|K| = |C|$ implies that $C = K$.

Given "the monotonicity of the L_p Minkowski conjecture" as in Remark 8.8.4, it is not surprising that one of the most major result concerning the L_p Minkowski conjecture is about the case when $p < 1$ is close to 1. This result is due to Chen, Huang, Li, Liu [154], based on the local result by Kolesnikov, E. Milman [381]. Another argument for the local-to-global step is provided by Putterman [495].

Theorem 8.8.5. *If $1 - \frac{c}{n \log n} < p < 1$ for an absolute constant $c > 0$, then the L_p Brunn-Minkowki Conjecture 8.8.1 and the L_p Minkowski Conjecture 8.8.2 hold for any origin symmetric convex bodies K, C and $\lambda \in (0, 1)$.*

According to Remark 8.8.4, the L_p Brunn-Minkowski Conjecture 8.8.1 and the L_p Minkowski Conjecture 8.8.2 follow for $p \in (0, 1)$ in all cases when the L_0 Log Brunn-Minkowski and the Log Minkowski conjectures are known

Remark 8.8.6 (Some known cases of the L_p Brunn-Minkowski (LpBM) and the L_p Minkowski (LpM) conjectures for o-symmetric convex bodies for $p \in [0, 1)$ without characterization of equality).

- $n = 2$ (both (LpBM) and (LpM), Böröczky, Lutwak, Yang, Zhang [110]).
- K, C are unconditional (Saroglou [508]), and more generally, K, C are invariant under reflections through n independent linear hyperplanes (both (LpBM) and (LpM), Böröczky, Kalantzopoulos [107]).
- Complex bodies (both (LpBM) and (LpM), Rotem [500]).

- $E \subset K, C \subset (1 + c_n)E$ where $c_n > 0$ depends only on n and E is a centered ellipsoid (both (LpBM) and (LpM), Chen, Huang, Li, Liu [154] and Kolesnikov, E. Milman [381]).
- K is a zonoid and C is any o-symmetric convex body (only (LpM), van Handel [299]).
- Under rather generous explicit curvature pinching bounds for ∂K by E. Milman [460] and Ivaki, E. Milman [351].

8.8.1 Equivalent formulations of the L_p Minkowski Conjecture 8.8.2 for $p \in (0, 1)$ and o-symmetric convex bodies

If $p \in (0, 1)$, then the following statements are equivalent if they hold for all o-symmetric convex bodies $K, C \subset \mathbb{R}^n$. Here the equivalence of (i), (ii) and (iii) are proved in Lemma 8.8.3, the equivalence of (iii) and (iv) is discussed in Remark 9.4.7. The equivalence of (ii) (without characterization of linearity of the concave function) and (v) follows from taking the second derivative of $\lambda \mapsto |(1-\lambda)K +_p \lambda C|$ at $\lambda = 0^+$ and some additional arguments according to Kolesnikov, E. Milman [381] (see also Putterman [495]), and we discuss the equivalence of (v) and (vi) - due to Kolesnikov, E. Milman [381] - below. The implication that (i) yields (v) was proved earlier by Colesanti, Livshyts, Marsiglietti [171].

(i) $\left|(1-\lambda) \cdot K +_p \lambda \cdot C\right| \geq |K|^{1-\lambda}|C|^{\lambda}$ with equality if and only if K and C are dilates;

(ii) The function $\lambda \mapsto \left|(1-\lambda) \cdot K +_p \lambda \cdot C\right|^{\frac{p}{n}}$ is concave on $[0, 1]$, and is linear if and only if K and C are dilates;

(iii) $\displaystyle\int_{S^{n-1}} \frac{h_C^p}{h_K^p}\, dV_K \geq |K|^{\frac{n-p}{n}}|C|^{\frac{p}{n}}$, with equality if and only if K and C are dilates.

(iv) The inequality in (iii) for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ with C^∞_+ boundary is equivalent to saying that for any positive even C^∞ function f on S^{n-1}, the Monge-Ampère equation

$$h^{1-p} \det(\nabla^2 h + h\,\mathrm{Id}) = f$$

on the sphere S^{n-1} has a unique even solution. The inequality in (iii) for any o-symmetric K, C follows from the smooth case by approximation.

(v) $\displaystyle\frac{V(K, C; 1)^2}{|K|} \geq \frac{n-1}{n-p} V(K, C; 2) + \frac{1-p}{n-p} \int_{S^{n-1}} \frac{h_C^2}{h_K^2}\, dV_K$, which is a strengthened form of Minkowski's second inequality (7.29) because the the Hölder's inequality implies $|K| \cdot \frac{1}{n}\int_{S^{n-1}} \frac{h_C^2}{h_K}\, dS_K \geq V(K, C; 1)^2$.

(vi) The inequalities above without characterization of equality are equivalent to

$$\lambda_{1,e}(-\mathcal{L}_K) \geq \frac{n-p}{n-1}$$

for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary and the Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ (see below for definition).

In the rest of the section, we discuss the equivalent formulation (vi) of the L_p-Minkowski Conjecture 8.8.2 using the Hilbert-Brunn-Minkowski operator introduced by Kolesnikov, E. Milman [381] extending Hilbert's work around 1910. For a convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary, Kolesnikov, E. Milman [381] defines the Hilbert-Brunn-Minkowski operator $\mathcal{L}_K$ as

$$\mathcal{L}_K\varphi = \frac{\mathcal{D}(\widetilde{D}^2(\varphi\, h_K), \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)} - \varphi \tag{8.128}$$

for a $\varphi \in C^\infty(S^{n-1})$ where $\mathcal{D}(\cdot, \ldots, \cdot)$ is the mixed discriminant of $n-1$ matrices of size of $(n-1) \times (n-1)$ (cf. Definition 8.3.2), and we frequently write simply h_K to denote $h_K|_{S^{n-1}}$, as well. As Theorem 8.8.9 below shows, the L_p Minkowski Conjecture 8.8.2 is essentially equivalent with the spectral gap estimate that

$$\lambda_{1,e}(-\mathcal{L}_K) \geq \frac{n-p}{n-1} \tag{8.129}$$

for any o-symmetric convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary where $\lambda_{1,e}(-\mathcal{L}_K)$ is the minimal positive eigenvalue when we restrict $-\mathcal{L}_K$ to even C^∞ functions on S^{n-1}.

First we discuss the elementary properties of Hilbert-Brunn-Minkowski operator. Let us recall (cf. Definition 8.1.6) that for any $\psi \in C^\infty(S^{n-1})$, writing $\tilde{\psi} : \mathbb{R}^n \backslash \{o\}$ and $\bar{\psi} : \mathbb{R}^n \backslash \{o\}$ to denote the functions $\tilde{\psi}(tu) = t \cdot \psi(u)$ and $\bar{\psi}(tu) = \psi(u)$ for $t > 0$ and $u \in S^{n-1}$, we have

$$\nabla^2\psi(u) = D^2\bar{\psi}(u)|_{u^\perp} \text{ and } \widetilde{D}^2\psi(u) = D^2\tilde{\psi}(u)|_{u^\perp} = \nabla^2\psi(u) + \psi(u) \cdot I_{n-1}. \tag{8.130}$$

For basic properties of self-adjoint elliptic operators, see Section 10.7.

Lemma 8.8.7 (Kolesnikov, Milman). *Let $K \subset \mathbb{R}^n$ be a convex body with C^∞_+ boundary and $o \in \text{int}K$.*

(i) $\mathcal{L}_K$ is elliptic.

(ii) $\displaystyle\int_{S^{n-1}} \varphi(\mathcal{L}_K\psi)\, dV_K = \int_{S^{n-1}} (\mathcal{L}_K\varphi)\psi\, dV_K$ for $\varphi, \psi \in C^\infty(S^{n-1})$, and hence $\mathcal{L}_K$ has a self-adjoint extension to $L_2(S^{n-1}, V_K)$.

(iii) $\mathcal{L}_K\varphi \equiv 0$ for any constant function φ.

(iv) $-\mathcal{L}_K\ell_{K,w} = \ell_{K,w}$ for any $w \in \mathbb{R}^n$ where $\ell_{K,w}(u) = \frac{\langle w,u\rangle}{h_K(u)}$.

Proof. For (iv), we have $\ell_{K,w}(u)h_K(u) = \langle w,u\rangle$ is a linear function, and hence $\widetilde{D}^2(\ell_{K,w}h_K)$ is the zero matrix.

To understand $\mathcal{L}_K$, using the notation of (8.130) yields

$$\widetilde{D}^2(\varphi\, h_K)(u) = D^2\,(\bar{\varphi} h_K)(u))\,|_{u^\perp} = \varphi(u)\cdot \widetilde{D}^2 h_K(u) + h_K(u)\cdot \nabla^2\varphi(u) + \mathcal{G}_K\varphi(u)$$

where $\mathcal{G}_K\varphi(u) = [D\bar{\varphi}(u)Dh_K(u)^t]|_{u^\perp} + [Dh_K(u)D\bar{\varphi}(u)^t]|_{u^\perp}$, or in other words, $\mathcal{G}_K\varphi = (\nabla\varphi)(\nabla h_K)^t + (\nabla h_K)(\nabla\varphi)^t$. It follows that

$$\mathcal{L}_K\varphi = h_K \cdot \frac{\mathcal{D}(\nabla^2\varphi, \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)} + \frac{\mathcal{D}(\mathcal{G}_K\varphi, \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}{\mathcal{D}(\widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)}. \tag{8.131}$$

In particular, $\mathcal{L}_K$ is elliptic (cf. (8.24) and (10.13) in Section 10.7).

If φ is a constant, then $\bar{\varphi}$ is also a constant, and hence $\nabla\varphi(u) = D\bar{\varphi}(u)|_{u^\perp}$ and $\nabla^2\varphi(u) = D^2\bar{\varphi}(u)|_{u^\perp}$ are zero. We deduce from (8.131) that $\mathcal{L}_K\varphi \equiv 0$.

By now, all we are left to do is to prove (ii). We recall (cf. (8.40)) the differential operator

$$\mathcal{A}\varphi = \mathcal{D}(\widetilde{D}^2\varphi, \widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K) \tag{8.132}$$

for $\varphi \in C^\infty(S^{n-1})$, which is symmetric with respect to $L_2(S^{n-1}, \mathcal{H}^{n-1})$ according to (8.42); namely,

$$\int_{S^{n-1}} \varphi \cdot \mathcal{A}\psi\, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi \cdot \mathcal{A}\varphi\, d\mathcal{H}^{n-1} \text{ for any } \varphi, \psi \in C^\infty(S^{n-1}). \tag{8.133}$$

For any $f \in C(S^{n-1})$, we have (cf. (8.14))

$$\int_{S^{n-1}} f\, dV_K = \frac{1}{n}\int_{S^{n-1}} f\cdot h_K\, dS_K = \frac{1}{n}\int_{S^{n-1}} f\cdot h_K \cdot \mathcal{D}(\widetilde{D}^2 h_K, \ldots, \widetilde{D}^2 h_K)\, d\mathcal{H}^{n-1}$$

We deduce that

$$\begin{aligned}\int_{S^{n-1}} \varphi(\mathcal{L}_K\psi)\, dV_K &= \frac{1}{n}\int_{S^{n-1}} (\varphi h_K)\cdot \mathcal{A}(\psi h_K)\, d\mathcal{H}^{n-1} \\ &= \frac{1}{n}\int_{S^{n-1}} (\psi h_K)\cdot \mathcal{A}(\varphi h_K)\, d\mathcal{H}^{n-1} = \int_{S^{n-1}} \psi(\mathcal{L}_K\varphi)\, dV_K,\end{aligned}$$

completing the proof of Lemma 8.8.7. ■

The following is Hilbert's result (see [311], Chapter XIX) about the Brunn-Minkowski inequality as reinterpreted by Kolesnikov, E. Milman [381]:

Theorem 8.8.8 (Hilbert). *For a convex body $K \subset \mathbb{R}^n$ with C^∞_+ boundary and $o \in \operatorname{int} K$, all eigenvalues of $-\mathcal{L}_K$ are non-negative, the eigenvalue 0 is simple, the eigenvalue 1 has multiplicity n where the corresponding eigenspace consists of all $\ell_{K,w}$, $w \in \mathbb{R}^n$, and the rest of the eigenvalues are larger than 1.*

Remark. Hilbert's key observation was (in the setting of Kolesnikov, E. Milman [381]) that the positive eigenvalues of $-\mathcal{L}_K$ are at least 1, and this fact is equivalent with the Brunn-Minkowski inequality.

Next let K be an o-symmetric convex body with C_+^∞ boundary, and let $\lambda_{1,e}(-\mathcal{L}_K)$ be the minimal positive eigenvalue when $-\mathcal{L}_K$ is restricted to even C^2 functions on S^{n-1}. Since now the eigenfunctions $\ell_{K,w}$, $w \in \mathbb{R}^n$, are odd, we have $\lambda_{1,e}(-\mathcal{L}_K) > 1$. Kolesnikov, E. Milman [381] proved that any uniform improvement on this spectral gap estimate is equivalent to the L_p Minkowski Conjecture 8.8.2 for a suitable $p \in [0,1)$.

Theorem 8.8.9 (Kolesnikov, E. Milman). *Let $p \in [0,1)$, and let $K \subset \mathbb{R}^n$ be an o-symmetric convex body with C_+^∞ boundary. The estimate $\lambda_{1,e}(-\mathcal{L}_K) \geq \frac{n-p}{n-1}$ is equivalent with (v) above for any o-symmetric convex body $C \subset \mathbb{R}^n$.*

Remark. In particular, for given $p \in [0,1)$, the estimate $\lambda_{1,e}(-\mathcal{L}_K) \geq \frac{n-p}{n-1}$ for any o-symmetric convex body with C_+^∞ boundary is equivalent to saying that the the L_p Minkowski Conjecture 8.8.2 holds for any o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ (without the characterization of the case of equality).

As a spectacular achievement, Kolesnikov, E. Milman [381] managed to provide a uniform positive lower bound for $\lambda_{1,e}(-\mathcal{L}_K) - 1$, which, combined with the bound of Theorem 4.7.12 due to Klartag [373] on the Cheeger constant in the KLS constant, reads as follows.

Theorem 8.8.10 (Kolesnikov, Milman). *If $K \subset \mathbb{R}^n$ is an o-symmetric convex body with C_+^∞ boundary, then*

$$\lambda_{1,e}(-\mathcal{L}_K) \geq 1 + \frac{c}{n^2 \log n}$$

for an absolute constant $c > 0$.

Combining Theorems 8.8.9 and 8.8.10 yields Theorem 8.8.5; namely, if $1 - \frac{c}{n \log n} < p < 1$ for an absolute constant $c > 0$, then the L_p Brunn-Minkowski Conjecture 8.8.1 and the L_p Minkowski Conjecture 8.8.2 hold for any origin symmetric convex bodies K, C and $\lambda \in (0,1)$.

Remark 8.8.11.

- According to E. Milman [460], $\Delta_K = (n-1)\mathcal{L}_K$ is the centro-affine Laplacian for a convex body $K \subset \mathbb{R}^n$ with C_+^∞ boundary and $o \in \operatorname{int} K$.
- While in this section, we have been discussing the case when ∂K is C_+^∞ in order to match the set up of most books on elliptic operators, the theory work for convex bodies with C_+^2 boundary, as well.

8.8.2 L_p combination of polytopes when $p \in (0,1)$

For $p \in (0,1)$, we show that the L_p combination of n-polytopes is a polytope, but the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be even C^1.

Lemma 8.8.12. *If $p \in (0,1)$ and $K, C \subset \mathbb{R}^n$ are n-polytopes, then $(1-\lambda)K +_p \lambda C$ is an n-polytope whose facet exterior unit normals are facet exterior unit normals to the Minkowski sum $K + C$, as well.*

Proof. Let $u_1, \dots, u_m$ be the exterior unit normals to the facets of the polytope $K + C$ (cf. (1.4)). As a convex body is determined by the exterior normals at the regular boundary points (cf. Lemma 2.5.6), Lemma 8.8.12 follows if whenever $u = \nu_{(1-\lambda)K+_p\lambda C}(z)$ for a $z \in \partial'((1-\lambda)K +_p \lambda C)$, then $u \in \{u_1, \dots, u_m\}$. Let w be a vertex of $K + C$ such that u is an exterior normal at w to $K + C$. According to Lemma 1.4.10, the normal cone at w is the positive hull of a subset of $\{u_1, \dots, u_m\}$; therefore, we may assume that $u = \sum_{i=1}^q \alpha_i u_i$ for $\alpha_1, \dots, \alpha_q \geq 0$, $q \geq 1$, where $u_1, \dots, u_q$ are exterior normals at w to $K + C$. Now $w = x + y$ for a vertex x of K and a vertex y of C, and hence $u, u_1, \dots, u_q$ are exterior normals at x to K and at y to C, and $h_K(u_i) = \langle x, u_i \rangle$ and $h_C(u_i) = \langle y, u_i \rangle$ for $i = 1, \dots, q$. It follows from the fact that the function defining the Wulff shape agrees with the support function at the exterior normals at the regular boundary points (cf. Lemma 7.5.1) that

$$
\begin{aligned}
\langle z, u \rangle = h_{(1-\lambda)\cdot K+_p\lambda\cdot C}(u) &= ((1-\lambda)h_K(u)^p + \lambda\, h_C(u)^p)^{\frac{1}{p}} \\
&= \left((1-\lambda)\left(\sum_{i=1}^q \alpha_i \langle x, u_i \rangle\right)^p + \lambda \left(\sum_{i=1}^q \alpha_i \langle y, u_i \rangle\right)^p\right)^{\frac{1}{p}} \\
&= \left((1-\lambda)\left(\sum_{i=1}^q \alpha_i h_K(u_i)\right)^p + \lambda \left(\sum_{i=1}^q \alpha_i h_C(u_i)\right)^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

We apply the Minkowski inequality (10.6) with $f(u_i) = (1-\lambda)h_K(u_i)^p$, $g(u_i) = \lambda\, h_C(u_i)^p$ and $\mu(\{u_i\}) = a_i$ and with parameter $1/p > 1$ instead of p to conclude that

$$
\begin{aligned}
\langle z, u \rangle &\geq \sum_{i=1}^q \alpha_i \left((1-\lambda)h_K(u_i)^p + \lambda\, h_C(u_i)^p\right)^{\frac{1}{p}} \\
&\geq \sum_{i=1}^q \alpha_i h_{(1-\lambda)K+_p\lambda C}(u_i) \geq \sum_{i=1}^q \alpha_i \langle z, u_i \rangle = \langle z, u \rangle.
\end{aligned}
$$

We deduce that $h_{(1-\lambda)K+_0\lambda C}(u_i) = \langle z, u_i \rangle$ for $i = 1, \dots, q$, and hence $u_1, \dots, u_q$ are exterior unit normals at $z \in \partial'((1-\lambda)K +_0 \lambda C)$. We conclude that $q = 1$; therefore, $u = u_1$. ■

Example 8.8.13 (For $p \in (0,1)$, the boundary of the L_p combination of convex bodies with C^2_+ boundaries may not be C^1). Let $p \in (0,1)$, let e_1, e_2 be the orthonormal basis of $\mathbb{R}^2$, and let $a > 2$ be large enough such that $b = \frac{a}{2^{1/p}}(1 + a^{-2p})^{1/p} < \frac{a}{2}$. We consider the ellipses $E_a = \mathrm{diag}[a, a^{-1}]B^2$ and $\widetilde{E}_a = \mathrm{diag}[a^{-1}, a]B^2$. Since $E_a \subset P_a = [a, -a] \times [\frac{1}{a}, \frac{-1}{a}]$ and $\widetilde{E}_a \subset \widetilde{P}_a = [\frac{1}{a}, \frac{-1}{a}] \times [a, -a]$, we deduce using Lemma 8.8.12 that

$$\tfrac{1}{2} \cdot E_a +_p \tfrac{1}{2} \cdot \widetilde{E}_a \subset \tfrac{1}{2} \cdot P_a +_p \tfrac{1}{2} \cdot \widetilde{P}_a = [-b, b] \times [-b, b] = W.$$

On the other hand, for $u = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$, we have

$$\begin{aligned}\left(\frac{1}{2}\, h_{E_a}(u)^p + \frac{1}{2}\, h_{\widetilde{E}_a}(u)^p\right)^{\frac{1}{p}} &> \left(\frac{1}{2}\,\langle ae_1, u\rangle^p + \frac{1}{2}\,\langle ae_2, u\rangle^p\right)^{\frac{1}{p}} \\ &= \frac{a}{\sqrt{2}} > h_W(u) \geq h_{\frac{1}{2}\cdot E_a +_p \frac{1}{2}\cdot \widetilde{E}_a}(u);\end{aligned}$$

therefore, u is not an exterior normal at a regular boundary point of $\frac{1}{2} \cdot E_a +_p \frac{1}{2} \cdot \widetilde{E}_a$ by Lemma 7.5.1.

8.9 Affine surface areas

8.9.1 Affine Surface Area and Centro-Affine Surface Area

Let $K \subset \mathbb{R}^n$ be a convex body. We deduce from Theorem 8.1.1 on the second order differentiabily of ∂K and Aleksandrov's Theorem 10.6.2 on the second order differentiabily of convex functions that the boundary of K are $\mathcal{H}^{n-1}$ and the (restricted) support function $h_K|_{S^{n-1}}$ are a.e. twice differentiable in the Aleksandrov sense. In particular, we can speak about the generalized Gaussian curvature $\kappa(x) \in [0, \infty)$ at $\mathcal{H}^{n-1}$ a.e. $x \in \partial K$, and the *generalized curvature function*

$$f_K(u) = \sigma_{n-1} D^2 h_K(u) = \det\left(D^2 h_K(u)|_{u^\perp}\right)$$

at $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$ where the equality between the two formulations follows from the homogeneity of h_K as a function on $\mathbb{R}^n$. Accoding to Remark 8.2.3, the surface area measure S_K on S^{n-1} can be written as $S_K = S_K^a + S_K^s$ where

- $dS_K^a = f_K d\mathcal{H}^{n-1}$ is the absolutely continuous part and $f_K(u) = \sigma_{n-1} D h_K(u)$ is the generalized curvature function for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$ (see Theorem 3.5 in Hug [337]);
- S_K^s is a singular Borel measure (i.e. there exists $X \subset S^{n-1}$ such that $\mathcal{H}^{n-1}(X) = 0$ and $S_K^s(S^{n-1} \backslash X) = 0$) and S_K^s is regular (see Theorem 10.1.3).

In the following defininition of the two core notions of affine surface area from about 1910, the equivalence of the two types of definitions follows from Theorem 8.1.5 if the convex body K has C^2_+ boundary, and is due to Hug [335] for any convex body.

Definition 8.9.1. Let K be a convex body in $\mathbb{R}^n$.

Affine Surface Area

$$\Omega(K) = \int_{\partial K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} f_K(u)^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1}(u).$$

Centro-Affine Surface Area Assuming that $o \in \mathrm{int}K$,

$$\widetilde{\Omega}(K) = \int_{\partial K} \frac{\kappa_K(x)^{\frac{1}{2}}}{\langle x, \nu_K(x)\rangle^{\frac{n-1}{2}}}\, d\mathcal{H}^{n-1}(x) = \int_{S^{n-1}} \frac{f_K(u)^{\frac{1}{2}}}{h_K(u)^{\frac{n-1}{2}}}\, d\mathcal{H}^{n-1}(u).$$

Remarks.

- $\Omega(K)$ is translation invariant; namely, $\Omega(K) = \Omega(K + w)$ for $w \in \mathbb{R}^n$;
- $\Omega(\lambda K) = \lambda^{\frac{n(n-1)}{n+1}}\, \Omega(K)$ and $\widetilde{\Omega}(\lambda K) = \widetilde{\Omega}(K)$ for $\lambda > 0$;
- If $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary and $o \in \mathrm{int}K$ then $\Omega(K) > 0$ and $\widetilde{\Omega}(K) > 0$;
- If $K \subset \mathbb{R}^n$ is a polytope with $o \in \mathrm{int}K$, then $\Omega(K) = \widetilde{\Omega}(K) = 0$.

One of the core properties of affine surface area and the centro-affine surface area is their inriance under $\mathrm{SL}(n)$. In order to discuss this property, we need the notion of the $\mathrm{SL}(n)$ invariant centro-affine curvature.

8.9.2 The Centro-Affine Curvature

The invariance of the centro-affine curvature under volume preserving linear transformations was already observed by Tzitzéica [555] in 1908.

Definition 8.9.2 (Centro-Affine Curvature, Tzitzéica). If $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, and ∂K is twice differentiable at $x \in \partial K$, then the centro-affine curvature at x is

$$\kappa_0(K, x) = \frac{\kappa_K(x)}{\langle x, \nu_K(x)\rangle^{n+1}} = \frac{\kappa_K(x)}{h_K(\nu_K(x))^{n+1}}.$$

Proposition 8.9.3 (Tzitzéica). *If $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, ∂K is twice differentiable at $x \in \partial K$, and $\Phi \in \mathrm{GL}(n)$, then*

$$\kappa_0(\Phi K, \Phi x) = (\det \Phi)^2 \kappa_0(K, x).$$

Remark. It follows that $\kappa_0(K, x)^{-\frac{1}{2}}$ is proportional with the volume of the osculating o-symmetric ellipsoid at x.

Proof. We may assume that $\det\Phi = 1$. It is equivalent to prove that if $\lambda \in (1,2)$, then

$$\lambda^{-1}\kappa_0(K,x) \le \kappa_0(\Phi K, \Phi x) \le \lambda\kappa_0(K,x). \tag{8.134}$$

Fix $\lambda \in (1,2)$. First we verify some formulas for a convex body $M \subset \mathbb{R}^n$ with $o \in \mathrm{int}M$ assuming that ∂M is twice differentiable at a $y \in \partial M$. For $u = -\nu_K(y)$ and $z_0 = y|u^\perp$, let $\varphi : (\mathrm{int}M)|u^\perp \to \mathbb{R}$ be the convex function such that $z + \varphi(z)u \in \partial K$ and $y = z_0 + \varphi(z_0)u$. As ∂M is twice differentiable at y, there exists a positive semi-definite quadratic form Q_y with $\kappa(y) = \det Q_y$ on $u^\perp$ such that

$$\varphi(z) - \varphi(z_0) = \frac{1}{2}Q_y(z - z_0) + o(\|z - z_0\|^2).$$

If $0 < t < \langle y, \nu_M(y)\rangle$, then we consider the cap

$$C(M, y, t) = \{w \in M : \langle w, \nu_M(y)\rangle \ge \langle y, \nu_M(y)\rangle - t\}.$$

First let $\kappa(y) > 0$. There exists $\theta(M, y) \in (0,1)$ such that if $0 < t < \theta(M, y)\langle y, \nu_M(y)\rangle$, then as subsets of $u^\perp$,

$$\lambda^{\frac{-1}{2(n-1)}} \cdot \left\{\tfrac{1}{2}Q_y \le t\right\} \le \{\varphi \le t\} \le \lambda^{\frac{1}{2(n-1)}} \cdot \left\{\tfrac{1}{2}Q_y \le t\right\},$$

which in turn yields that

$$\lambda^{\frac{-1}{2}} \cdot \frac{2^{\frac{n-1}{2}} t^{\frac{n-1}{2}}}{\kappa(y)^{\frac{1}{2}}} \le \mathcal{H}^{n-1}\Big(\{\varphi \le t\}\Big) \le \lambda^{\frac{1}{2}} \cdot \frac{2^{\frac{n-1}{2}} t^{\frac{n-1}{2}}}{\kappa(y)^{\frac{1}{2}}}.$$

Therefore if $\kappa(y) > 0$ and $0 < s < \theta(M, y)$, then

$$\begin{aligned}\lambda^{\frac{-1}{2}} \cdot \frac{2^{\frac{n+1}{2}} s^{\frac{n+1}{2}}}{n+1} \cdot \frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}} &\le |C(M, y, s\langle y, \nu_M(y)\rangle)| \\ &\le \lambda^{\frac{1}{2}} \cdot \frac{2^{\frac{n+1}{2}} s^{\frac{n+1}{2}}}{n+1} \cdot \frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}}\end{aligned} \tag{8.135}$$

where $\frac{\langle y, \nu_M(y)\rangle^{\frac{n+1}{2}}}{\kappa(y)^{\frac{1}{2}}} = \kappa_0(M, y)^{\frac{-1}{2}}$.

If $\kappa(y) = 0$, then similar argument as above yields

$$\lim_{s\to 0^+} s^{-\frac{n+1}{2}}|C(M, y, s\langle y, \nu_M(y)\rangle)| = \infty. \tag{8.136}$$

Since $\Phi(\nu_K(x)^\perp) = \nu_{\Phi K}(\Phi x)^\perp$, we deduce that

$$\Phi\, C(K, x, s\langle x, \nu_K(x)\rangle) = C(\Phi K, \Phi x, s\langle \Phi x, \nu_{\Phi K}(\Phi x)\rangle)$$

for small $s > 0$, and hence $\Big|C(K, x, s\langle x, \nu_K(x)\rangle)\Big| = \Big|C(\Phi K, \Phi x, s\langle \Phi x, \nu_{\Phi K}(\Phi x)\rangle)\Big|$. Therefore, combining (8.135) and (8.136) yields (8.134). ∎

As we have seen in Section 1.9, the polar $K^* = \{y \in \mathbb{R}^n : \langle y, x\rangle \le 1\}$ of a convex body $K \subset \mathbb{R}^n$ with $o \in \mathrm{int}K$ satisfies that $(\Phi K)^* = \Phi^{-t} K^*$ for $\Phi \in \mathrm{GL}(n)$, and if $y \in \mathbb{R}^n$ is an exterior normal $x \in \partial K$ with $\langle y, x\rangle = 1$, then x is an exterior normal at $y \in \partial K^*$. The following statement was already known at the beginning of the 20th century in the case of C^2_+ boundary, and is due to Hug [336] for general convex bodies.

Lemma 8.9.4 (Hug). *Let $K \subset \mathbb{R}^n$ be convex body with $o \in \mathrm{int}K$. If ∂K is twice differentiable at $x \in \partial K$ with $\kappa_{\partial K}(x) > 0$, then ∂K^* is twice differentiable at $x^* \in \partial K^*$ with $\kappa_{\partial K^*}(x^*) > 0$ and*

$$\kappa_0(K, x) \cdot \kappa_0(K^*, x^*) = 1 \tag{8.137}$$

where x^* is the the exterior normal at x with $\langle x^*, x\rangle = 1$.

Proof. Since this property is invariant under linear transformations by Proposition 8.9.3, we may assume that $x = \nu_K(x) = x^* = \nu_{K^*}(x^*)$. In addition, we may assume that each principal curvature at $x \in \partial K$ is one.

For any $\varepsilon > 0$, we consider the ellipsoid E_ε whose one semi axis is $\mathrm{conv}\{0, x\}$, and the other semi axes are of length $1 + \varepsilon$. It follows that $\mathrm{conv}\{0, x\}$ is a semi axis of E^*_ε, as well, and the other semi axes of E^*_ε are of length $(1 + \varepsilon)^{-1}$. In addition, for any $\varepsilon > 0$, there exists $s \in (0, 1)$ depending on ε and K such that

$$\{z \in E^*_\varepsilon : \langle z, x\rangle \ge 1 - s\} \subset \{z \in K : \langle z, x\rangle \ge 1 - s\} \subset E_\varepsilon.$$

We deduce the existence of $t \in (0, s)$ (depending on ε, K and s) such that

$$\{z \in E^*_\varepsilon : \langle z, x\rangle \ge 1 - t\} \subset \{z \in K^* : \langle z, x\rangle \ge 1 - t\} \subset E_\varepsilon,$$

which in turn yields (8.137) by the arbitraryness of $\varepsilon > 0$. ■

Similar argument like the one for Lemma 8.9.4 yields the following statement:

Lemma 8.9.5. *Let $K \subset \mathbb{R}^n$ be convex body, and let $x \in \partial K$. ∂K is twice differentiable at x with $\kappa_{\partial K}(x) > 0$ if and only if h_K is twice differentiable at $\nu_K(x)$ with $\sigma_{n-1} D^2 h_K(\nu_K(x)) > 0$, and in this case, we have*

(i) $f_K(\nu_K(x)) = \sigma_{n-1} D^2 h_K(\nu_K(x)) = \kappa_{\partial K}(x)^{-1}$;

(ii) $\kappa_0(K, x)^{-1} = f_K(\nu_K(x)) \cdot h_K(\nu_K(x))^{n+1}$.

Remark. Based on (ii), $f_K(u) \cdot h_K(u)^{n+1}$, $u \in S^{n-1}$, is frequently called the centro-affine curvature function.

Proof. We may assume that $o \in \mathrm{int}\, K$. Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ where $e_n = \nu_K(x)$, either $e_1, \ldots, e_{n-1}$ are the principal directions at $x \in \partial K$ (if ∂K is twice differentiable at x) or $e_1, \ldots, e_n$ are the principal directions for $D^2 h_K(\nu_K(x))$ (if h_K is twice differentiable at $\nu_K(x)$). In particular, if h_K is twice differentiable at $\nu_K(x)$, then any second partial derivative involving ∂_n is zero. In the argument, we use Proposition 8.9.3 and the fact that $h_{\Phi K} = h_K \circ \Phi^t$ for any convex body K and $\Phi \in \mathrm{GL}(n)$.

First applying a linear transform with $e_n \mapsto e_n + \sum_{i=1}^{n-1} \alpha_i e_i$ and $e_i \mapsto e_i$ for $i \leq n-1$, we may assume that $x = h_K(\nu_K(x)) \cdot \nu_K(x)$. Then applying a linear transform with $e_i \mapsto e_i$ for $i \leq n-1$ and $e_n \mapsto \beta\, e_n$, we may assume that $h_K(e_n) = 1$. Finally, applying a a linear transform with $e_i \mapsto \tau_i e_i$ for $i \leq n-1$ and $e_n \mapsto e_n$, we may assume that either each principal curvature at $x = e_n$, or each eigenvalue of $D^2 h_K(e_n)$ corresponding to e_i, $i \leq n-1$, is 1.

Since $h_K(u) = \|u\|_{K^*}$, we complete the argument for (i) using the ellipsoids E_ε and E_ε^* as in the proof of Lemma 8.9.4. In addition, (ii) follows from (i). ■

8.9.3 L_p Affine Surface Areas for $p > 0$ and Linear invariance

Lutwak [435] provided a rather natural generalization of the notions of affine surface area and centro-affine surface area in the case of convex bodies with C^2_+ boundaries. This new notion of L_p-affine surface area for $p \geq 0$ was extended to any convex body by Hug [335], who also verified the equivalence of the two types of definitions for any convex body (see also Schütt, Werner [529] and Werner [562] for equivalent definitions). One definition of the L_p-affine surface area in (8.139) uses the notion of auxiliary cone volume measure $\widetilde{V}_K$, introduced in Section 2.6, living on ∂K for a convex body $K \subset \mathbb{R}^n$ with $o \in \mathrm{int}K$. In particular, if $\Xi \subset \partial K$ is Borel, then $\widetilde{V}_K(\Xi) = |\cup\{\mathrm{conv}\{o, x\} : x \in \Xi\}|$, and if $\Phi \in \mathrm{GL}(n)$, then

$$\widetilde{V}_{\Phi K}(\Phi\Xi) = |\det \Phi| \cdot \widetilde{V}_K(\Xi). \tag{8.138}$$

Definition 8.9.6 (L_p-affine surface area). If $p \geq 0$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}K$, then

$$\Omega_p(K) = n \int_{\partial K} \kappa_0(K, x)^{\frac{p}{n+p}}\, d\widetilde{V}_K(x) = \int_{\partial K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, \nu_K(x)\rangle^{\frac{n(p-1)}{n+p}}}\, dx \tag{8.139}$$

$$= \int_{S^{n-1}} f_K^{\frac{n}{n+p}} \cdot h_K^{\frac{n(1-p)}{n+p}}\, d\mathcal{H}^{n-1}. \tag{8.140}$$

Remark. $\Omega_0(K) = n|K|$, $\Omega_1(K) = \Omega(K)$ and $\Omega_n(K) = \widetilde{\Omega}(K)$.

The following invariance properties of the affine surface areas directly follows from (8.138) and Proposition 8.9.3.

Theorem 8.9.7. *Let $K \subset \mathbb{R}^n$ be a convex body.*

(i) $\Omega(\Phi K + w) = \Omega(K)$ *if* $\Phi \in \mathrm{GL}(n)$, $\det \Phi = \pm 1$ *and* $w \in \mathbb{R}^n$.

(ii) $\widetilde{\Omega}(\Phi K) = \widetilde{\Omega}(K)$ *if* $o \in \mathrm{int}K$ *and* $\Phi \in \mathrm{GL}(n)$.

(iii) $\Omega_p(\Phi K) = \Omega(K)$ *for* $p \geq 0$ *if* $o \in \mathrm{int}K$ *and* $\Phi \in \mathrm{GL}(n)$, $\det \Phi = \pm 1$.

8.9.4 Upper semicontinuity, Affine Isoperimetric Inequality

We recall that $|K^*| = \frac{1}{n}\int_{S^{n-1}} \varrho_{K^*}(u)^n\,du = \frac{1}{n}\int_{S^{n-1}} h_K(u)^{-n}\,du$ if K is a convex body with $o \in \mathrm{int}K$ (see Section 1.9). We now prove a useful upper bound on the volume of the L_p affine surface area in terms of the convex body and its polar.

Proposition 8.9.8. *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \mathrm{int}K$, and let $0 < p < n$.*

(i) $\widetilde{\Omega}(K) \leq n\sqrt{|K|\cdot|K^*|}$.

(ii) $\Omega_p(K) \leq n^{\frac{n-p}{n+p}}\,\widetilde{\Omega}(K)^{\frac{2p}{n+p}}\,|K|^{\frac{n-p}{n+p}} \leq n|K^*|^{\frac{p}{n+p}}|K|^{\frac{n}{n+p}}$.

Remark. In particular, $\Omega(K) \leq n^{\frac{n-1}{n+1}}\,\widetilde{\Omega}(K)^{\frac{2}{n+1}}\,|K|^{\frac{n-1}{n+1}} \leq n|K^*|^{\frac{1}{n+1}}|K|^{\frac{n}{n+1}}$.

Proof. Applying Hölder inequality to (8.140) and to (8.139), we deduce

$$\begin{aligned}
\widetilde{\Omega}(K) &= \int_{S^{n-1}} (h_K f_K)^{\frac{1}{2}}\, h_K^{\frac{-n}{2}}\, d\mathcal{H}^{n-1} \leq \left(\int_{S^{n-1}} h_K f_K\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h_K^{-n}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}}\\
&\leq \left(\int_{S^{n-1}} h_K\, dS_K\right)^{\frac{1}{2}} \left(\int_{S^{n-1}} h_K^{-n}\, d\mathcal{H}^{n-1}\right)^{\frac{1}{2}} = (n|K|)^{\frac{1}{2}}(n|K^*|)^{\frac{1}{2}}\\
\Omega_p(K) &= n\int_{\partial K} \kappa_0(K,x)^{\frac{p}{n+p}}\, d\widetilde{V}_K(x)\\
&\leq n\left(\int_{\partial K} \kappa_0(K,x)^{\frac{1}{2}}\, d\widetilde{V}_K(x)\right)^{\frac{2p}{n+p}} \left(\int_{\partial K} 1\, d\widetilde{V}_K(x)\right)^{\frac{n-p}{n+p}} = n^{\frac{n-p}{n+p}}\,\widetilde{\Omega}(K)^{\frac{2p}{n+p}}\,|K|^{\frac{n-p}{n+p}}.
\end{aligned}$$

■

The affine surface area, that was defined at the beginning of 20th century for convex bodies with C^2_+ boundary, was long conjectured to be upper semicontinuous (see Theorem 8.9.10). Finally, Lutwak [432] provided the following representation of the affine surface area for convex bodies with C^2_+ boundary in 1991 that yields automatically the upper semicontinuity, and the representation was extended to any convex body by Dolzmann, Hug [192]. Here we prove Theorem 8.9.9 for convex bodies with C^2_+ boundary, and the general case is handled as Theorem 8.B.2 in Section 8.B.

Theorem 8.9.9 (Lutwak, Dolzmann-Hug). *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$\Omega(K) = \inf_{\substack{g:S^{n-1}\to(0,\infty)\\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1}\, dS_K\right)^{\frac{n}{n+1}}. \tag{8.141}$$

Proof of Theorem 8.9.9 if ∂K is C^2_+. Now $dS_K = f_K d\mathcal{H}^{n-1}$ where f_K is continuous and positive. On the one hand, if $g : S^{n-1} \to (0,\infty)$ is continuous, then the Hölder inequality yields

$$\left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} f_K\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} \geq \int_{S^{n-1}} f_K^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1} = \Omega(K).$$

Taking $g = f_K^{\frac{1}{n+1}}$ finishes the proof of (8.141). ■

Theorem 8.9.10 (Main properties of the Affine Surface Area). *Let $K \subset \mathbb{R}^n$ be a convex body.*

- $\Omega(\lambda K) = \lambda^{\frac{n(n-1)}{n+1}} \Omega(K)$ *for* $\lambda > 0$.
- $\Omega(z + \Phi K) = \Omega(K)$ *if* $z \in \mathbb{R}^n$ *and* $\Phi \in \mathrm{GL}(n)$, $\det \Phi = \pm 1$.
- **Affine Isoperimetric Inequality** $\Omega(K) \leq n\omega_n^{\frac{2}{n+1}} V(K)^{\frac{n-1}{n+1}}$, *with equality if and only if K is an ellipsoid.*
- **Upper semi-continuity** $\Omega(K) \geq \limsup_{m\to\infty} \Omega(K_m)$ *if* $\lim_{m\to\infty} K_m = K$.

Proof. Homogeneity of degree $\frac{n(n-1)}{n+1}$ follows from the definition, and equi-affine invariance from Theorem 8.9.7.

For the Affine Isoperimetric Inequality, we may assume that $\sigma_K = o$ by the translation invariance of $\Omega(K)$, and then use the Blaschke-Santaló inequality (6.25) and Proposition 8.9.8 (ii) with $p = 1$.

For the upper semi-continuity, one cosiders a continuous $g : S^{n-1} \to (0, \infty)$ such that

$$\left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \, dS_K\right)^{\frac{n}{n+1}}$$

is arbitrary close to $\Omega(K)$. Applying the same g in Theorem 8.9.9 for each K_m shows that $\Omega(K) \geq \limsup_{m\to\infty} \Omega(K_m)$. ■

Remark 8.9.11 (Some additional properties of the Affine Surface Area).

- The affine surface area is not decreased by Steiner symmetrization, see Section 8.C.
- The affine surface area can be characterized as an equi-affine invariant and upper semi-continuous valuation (finitely additive measure on the space of compact convex sets, see Section 8.D).
- The affine surface area is related to the floating body and plays a very important role in polytopal approximation (see the Comments to Chapter 8).

Turning to the case of L_p affine surface area for $p > 0$, we start with the analogue of Theorem 8.9.9 verified by Lutwak [435] when the boundary is C_+^2, and by Hug [335] in general (the argument for Theorem 8.B.2 in Section 8.B also proves Theorem 8.9.12 in general).

Theorem 8.9.12 (Lutwak, Hug). *If $p > 0$ and $K \subset \mathbb{R}^n$ is a convex body with $o \in \mathrm{int}\, K$, then*

$$\Omega_p(K) = \inf_{\substack{g: S^{n-1}\to(0,\infty) \\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n \, d\mathcal{H}^{n-1}\right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} g^{-p} h_K^{1-p} \, dS_K\right)^{\frac{n}{n+p}}.$$

Remark. The measure $h_K^{1-p}\, dS_K$ on S^{n-1} in Theorem 8.9.12 is the so-called L_p-surface area measure (see Section 9.3).

Proof of Theorem 8.9.9 if ∂K is C^2_+. We have $dS_K = f_K d\mathcal{H}^{n-1}$ where f_K is continuous and positive. If $g : S^{n-1} \to (0,\infty)$ is continuous, then the Hölder inequality yields

$$\begin{aligned}\Omega_p(K) &= \int_{S^{n-1}} f_K^{\frac{n}{n+1}} h_K^{\frac{n(1-p)}{n+p}} f_K\, d\mathcal{H}^{n-1}\\ &\leq \left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{p}{n+p}} \left(\int_{S^{n-1}} g^{-p} h_K^{1-p} f_K\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+p}}.\end{aligned}$$

Finally, $\Omega_p(K)$ is attained when $g = f_K^{\frac{p}{n+p}}$. ∎

It follows from (2.28) that if $M \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} M$ and $f : \partial M \to [0,\infty)$ is measurable, then

$$\int_{\partial M} f\, d\widetilde{V}_M = \frac{1}{n}\int_{S^{n-1}} f(\varrho_M(u)\cdot u)\cdot \varrho_M(u)^n\, d\mathcal{H}^{n-1}(u). \tag{8.142}$$

The following statement due to Hug [336] relates the L_p affine surface areas of a convex body and its polar.

Lemma 8.9.13 (Hug). *If $p > 0$ and $K \subset \mathbb{R}^n$ convex body with $o \in \operatorname{int} K$, then*

$$\Omega_p(K) = \Omega_{n^2/p}(K^*). \tag{8.143}$$

Proof. Let $\Xi \subset S^{n-1}$ be the measurable set of all $u \in S^{n-1}$ such that h_K is twice differentiable at u with $f_K(u) = \sigma_{n-1} D^2 h_K(u) > 0$ (see Theorem 10.6.2). According to Lemma 8.9.4 and Lemma 8.9.5, $u \in \Xi$ if and only if ∂K^* is twice differentiable at $\varrho_{K^*}(u)\cdot u \in \partial K^*$ and $\kappa_{\partial K^*}(\varrho_{K^*}(u)\cdot u) > 0$. It follows by (8.139) (applied to K^*) and (8.140) (applied to K) that $\Omega_p(K) > 0$ if and only if $\Omega_{n^2/p}(K^*) > 0$, which are in turn equivalent with $\mathcal{H}^{n-1}(\Xi) > 0$. In this case, Lemma 8.9.4 and and Lemma 8.9.5 that if $u \in \Xi$, then

$$\frac{\kappa_{\partial K^*}(\varrho_{K^*}(u)\cdot u)}{h_{K^*}(\nu_{K^*}(\varrho_{K^*}(u)\cdot u)))^{n+1}} = f_K(u) h_K(u)^{n+1}.$$

We conclude from (8.139), (8.142), (8.140) and $\frac{n^2/p}{n+(n^2/p)} = \frac{n}{n+p}$ that

$$\begin{aligned}\Omega_{n^2/p}(K^*) &= n\int_{\partial K^*} \kappa_0(K^*,x)^{\frac{n}{n+p}}\, d\widetilde{V}_{K^*}(x)\\ &= \int_\Xi \frac{\kappa_{\partial K^*}(\varrho_{K^*}(u)\cdot u)^{\frac{n}{n+p}}}{h_{K^*}(\nu_{K^*}(\varrho_{K^*}(u)\cdot u)))^{\frac{n(n+1)}{n+p}}}\varrho_{K^*}(u)^n\, d\mathcal{H}^{n-1}(u)\\ &= \int_\Xi f_K(u)^{\frac{n}{n+p}} h_K(u)^{\frac{n(n+1)}{n+p}} h_K(u)^{-n}\, d\mathcal{H}^{n-1}(u) = \Omega_p(K).\end{aligned}$$

∎

Theorem 8.9.14 (Main properties of the Centro-Affine Surface Area). *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \mathrm{int}K$.*

Linear invariance: $\widetilde{\Omega}(\Phi K) = \widetilde{\Omega}(K)$ *if* $\Phi \in \mathrm{GL}(n)$*;*

Upper semicontinuity: $\widetilde{\Omega}(K) \geq \limsup_{m\to\infty} \widetilde{\Omega}(K_m)$ *if* $\lim_{m\to\infty} K_m = K$*;*

Invariance under polarity: $\widetilde{\Omega}(K) = \widetilde{\Omega}(K^*)$*;*

Centro-Affine Isoperimetric Inequality *If $\sigma_K = o$, then*

$$\widetilde{\Omega}(K) \leq n\omega_n$$

with equality if and only if K is an o-symmetric ellipsoid.

Remark. The centro-affine surface area can be characterized as a $\mathrm{GL}(n)$ invariant and upper semicontinuous valuation (see Section 8.D), and it is also related to polytopal approximation (see the Comments to Chapter 8).

Proof. $\mathrm{GL}(n)$ invariance follows from (8.138) and Proposition 8.9.3, and upper semicontinuity follows from Theorem 8.9.12 as in the case of the affine surface area.

Next (8.143) yields $\widetilde{\Omega}(K) = \widetilde{\Omega}(K^*)$. For the Centro-Affine Isoperimetric Inequality, we use the Blaschke-Santaló inequality (6.25) and Proposition 8.9.8 (i). ■

Finally, we consider the L_p affine surface area:

Theorem 8.9.15 (Main properties of the L_p Affine Surface Area). *Let $K \subset \mathbb{R}^n$ be a convex body with $o \in \mathrm{int}K$, and let $p > 0$.*

- $\Omega_p(\Phi K) = \Omega_p(K)$ *if* $\Phi \in \mathrm{GL}(n)$ *with* $|det\Phi| = 1$.
- **Upper semicontinuity**
  $\Omega_p(K) \geq \limsup_{m\to\infty} \Omega_p(K_m)$ *if* $\lim_{m\to\infty} K_m = K$*;*
- $\Omega_p(K) = \Omega_{n^2/p}(K^*)$*;*
- L_p **Affine Isoperimetric Inequality** *If $\sigma_K = o$, then*

$$\Omega_p(K) \leq n|B^n|^{\frac{2p}{n+p}}|K|^{\frac{n-p}{n+p}}$$

with equality if and only if K is an o-symmetric ellipsoid.

Remark. The centro-affine surface area can be characterized as a $\mathrm{GL}(n)$ invariant and upper semicontinuous valuation (see Section 8.D), and it is also related to polytopal approximation (see the Comments to Chapter 8).

Proof. Theorem 8.9.7 yields the invariance under volume preserving linear maps, and upper semicontinuity follows from Theorem 8.9.12 as in the case of the affine surface area.

Next, $\Omega_p(K) = \Omega_{n^2/p}(K^*)$ is just (8.143). For the L_p-Affine Isoperimetric Inequality, if $0 < p \le n$, then we use the Blaschke-Santaló inequality (6.25) and Proposition 8.9.8. If $p > n$, then (8.143) and the case $0 < p < n$ yield

$$\Omega_p(K) = \Omega_{n^2/p}(K^*) \le n|K|^{\frac{n}{n+p}} |K^*|^{\frac{p}{n+p}} \le n|B^n|^{\frac{2p}{n+p}} |K|^{\frac{n-p}{n+p}},$$

with equality if and only if K is a centered ellipsoid. ∎

8.10 Comments to Chapter 8

The main properties of related to the second differentiability of convex bodies were worked out by Aleksandrov [2–4, 7]. Originally, Minkowski himself proposed a way to approximate any compact convex set by smooth convex bodies (see Bonnesen, Fenchel [81]), but his argument contained a gap. Here we present the probably simplest construction due to Firey [234].

Given $\varepsilon \in (0, 1)$, Schneider (see [522], Theorem 3.4.1) constructed a convex body $T_\varepsilon K$ for any compact convex set $K \in \mathcal{K}^n$ such that $h_{T_\varepsilon K}$ is C^∞ on $\mathbb{R}^n \backslash \{o\}$, $\delta_H(K, T_\varepsilon K) \le R\varepsilon$ if $K \subset R B^n$, $R > 0$, TK is a ball if K is a ball, and in general, $\Phi TK = TK$ if Φ is an isometry of $\mathbb{R}^n$ with $\Phi K = K$, $T_\varepsilon(\alpha K + \beta C) = \alpha T_\varepsilon K + \beta T_\varepsilon C$ and $\delta_H(T_\varepsilon K, T_\varepsilon C) \le (1+\varepsilon)\delta_H(K, C)$ for $\alpha, \beta > 0$ and $C \in \mathcal{K}^n$. In particular, $T_\varepsilon K + \varepsilon B^n$ is a good approximation of K with C^∞_+ boundary, and if K is of constant width $D > 0$ (*i.e.* $K - K = DB^n$), then $T_\varepsilon K + \varepsilon B^n$ is of constant width, as well.

The fact that non-negative linear combination of compact convex sets is a homogeneous polynomial in the coefficients, and the basic properties of mixed volumes, and the relation of mean curvatures to mixed volumes were established by Minkowski [464, 465]. The representation of the intrinsic volumes as mean projections is due to Kubota [388].

See Florentin, V. Milman, Schneider [235] for a characterization and related properties of the mixed discriminant of positive definite matrices, and V. Milman, Schneider [462] for some additional characterizations of mixed volumes.

Minkowski's inequality is due to Minkowski [464, 465] around 1900. Aleksandrov [3, 5, 7] already provided two proofs of the Aleksandrov-Fenchel Inequality around 1937-38 (for additional arguments still based on Aleksandrov's ideas, see also van Handel, Shenfeld [300], Schneider [522] and D. Cordero-Erausquin, B. Klartag, Q. Merigot, F. Santambrogio [176]). Fenchel only stated the inequality, never actually provided a proof. Both arguments in this monograph, the one using strongly isomorphic polytopes in Section 7.A, and the one using the theory of elliptic operators in Section 8.5.2, are based on Aleksandrov's original ideas as developed further by van Handel, Shenfeld [300]. For various problems in algebraic geometry or combinatorics, etc, related to the Aleksandrov-Fenchel inequality, see Section 7.8.

For convex bodies K and C, the first stability forms of the Brunn-Minkowski inequality were due to Minkowski himself (see Groemer [272]). If the distance of the convex bodies K and C is measured in terms of the Hausdorff distance, then Diskant [191] and Groemer [271] provided close to optimal stability versions (see Groemer [272]). However, the natural distance is in terms of the volume of the symmetric difference, and the optimal result is due to the work of Figalli, Maggi, Pratelli [224, 225] and Kolesnikov, E. Milman [381] (see Section 8.6).

The L_p version of the Brunn-Minkowski inequality for $p > 1$ was proved by Firey [231] in 1962 (see Section 7.6) as a consequence of the classical Brunn-Minkowski inequality, the $p = 1$ case. The right extension of the L_p combination for $p \in [0, 1)$ was eventually developed by Böröczky, Lutwak, Yang, Zhang [110] within Erwin Lutwak's L_p-Brunn-Minkowski theory, which started with the seminal paper Lutwak [433] in 1993. The L_p-Brunn-Minkowski conjecture for origin symmetric convex bodies in $\mathbb{R}^n$ for $p \in [0, 1)$ was stated by Böröczky, Lutwak, Yang, Zhang [110], who verified the the conjecture when $n = 2$. The conjecture has been verified if $p < 1$ is close to 1 in any dimension by Chen, Huang, Li, Liu [154] based on the local results by Kolesnikov, E. Milman [381] (see Puttermann [495] for another local-to-global approach based on results by Kolesnikov, E. Milman [381]). For $p \in [0, 1)$, the L_p Brunn-Minkowski conjecture is deeply related to many open problems in Convex Geometric Analysis, for example, it is intimately connected the uniqueness of the solution of the even L_p-Minkowski problem, a Minkowski type Monge-Ampère equation on the sphere. The L_p-Brunn-Minkowski conjecture is discussed from the point of view of the Brunn-Minkowski theory and Elliptic operators in Section 8.7 (the fundamental case $p = 0$) and Section 8.8 (the case $p \in (0, 1)$), and from the point of view of Monge-Ampère equations on the sphere in Section 9.4. A functional analogue of the L_0-addition is presented by Crasta, Fragalà [182].

The centro-affine curvature was already considered by Tzitzéica [555] in 1908. The notion of affine surface area for convex bodies with C^2_+ boundary was developed by Blaschke [74]in $\mathbb{R}^2$ and $\mathbb{R}^3$ around 1920, who established the affine invariance, the connection to the Blaschke-Santaló inequality and the affine isoperimetric inequality, and his results were extended to any dimension by Santaló [505].

It was a highly non-trivial task to extend the notion of affine surface area as it was defined by Blaschke around 1920 (cf. [74]) for convex bodies of C^2_+ boundary to any convex bodies. Leichtweiss' attempt in [398] was based on Dupin's floating body in 1986, and Schütt, Werner [528] provided a more satisfactory definition based on the convex floating body defined by Bárány, Larman [47] and Schütt, Werner [528] (see below for the definition of these two notions of floating body). The definition of affine surface area using the extremal problem (8.141) is due Lutwak [432], who showed his notion coincides with the classical definition in the case of convex bodies with

C_+^2 boundary. Lutwak's approach verified the long conjectured upper-semicontinuity of the affine surface area, had an important role in the solution of the affine Plateau problem by Trudinger and Wang [554]. Finally, Dolzmann, Hug [192] proved that all these notions of surface area (Lutwak's approach, and using centroaffince curvature or convex floating body) are equivalent for any convex body. For a survey on affine surface area, see Schütt, Werner [530].

Let us discuss some properties of the *Affine Surface Area* that are not listed in Theorem 8.9.10 and Theorem 8.D.5:

Best approximation with respect to the volume, C^2 boundary: If $K \subset \mathbb{R}^n$ convex body, ∂K is C^2, $P_m \subset K$ polytope with at most m vertices and maximal volume, and $P_{(m)} \supset K$ polytope with at most m facets and minimal volume (see Böröczky [90]), then

$$\lim_{m\to\infty} m^{\frac{2}{n-1}}|K\backslash P_m| = \frac{\mathrm{del}_{n-1}}{2}\cdot\Omega(K)^{\frac{n+1}{n-1}}$$
$$\lim_{m\to\infty} m^{\frac{2}{n-1}}|P_{(m)}\backslash K| = \frac{\mathrm{div}_{n-1}}{2}\cdot\Omega(K)^{\frac{n+1}{n-1}}$$

where $\mathrm{del}_1 = \frac{1}{6}$, $\mathrm{div}_1 = \frac{1}{12}$, $\mathrm{del}_2 = \frac{1}{2\sqrt{3}}$, and $\mathrm{div}_2 = \frac{5}{18\sqrt{3}}$; moreover, $\lim_{n\to\infty}\frac{\mathrm{del}_{n-1}}{n} = \lim_{n\to\infty}\frac{\mathrm{div}_{n-1}}{n} = (2\pi e)^{-1}$ (see Hoehner, Kur [315] for more exact estimates, which paper also discusses the history of this problem).

Random approximation: Schütt [527] proved that if $K \subset \mathbb{R}^n$ is any convex body, and Q_m is a the convex hull of m random points of K acccording to the uniform propability measure, then

$$\lim_{m\to\infty} m^{\frac{2}{n-1}}\cdot\mathbb{E}|K\backslash Q_m| = \gamma_n|K|^{\frac{2}{n+1}}\cdot\Omega(K)$$

where $\gamma_n > 0$ depends on n (see Böröczky, Fodor, Hug [100] for the value of γ_n and for a clarification concerning the argument).

Convex floating body: Let $K \subset \mathbb{R}^n$ be a convex body, let $0 < \delta < \frac{1}{e|K|}$, and for any $u \in S^{n-1}$, we consider $t_{u,\delta} \in \mathbb{R}$ satisfying that $|C_{u,\delta}| = \delta$ for the cap $C_{u,\delta} = \{x \in K : \langle x, u\rangle \geq t_{u,\delta}\}$. Now Bárány, Larman [47] and Schütt, Werner [528] defined the convex floating body K_δ by

$$K_\delta = \{x \in K : \langle x, u\rangle \leq t_{u,\delta}\ \forall\, u \in S^{n-1}\} = \mathrm{cl}\left(K\backslash \cup_{u\in S^{n-1}} C_{u,\delta}\right).$$

We note that the centroid $\sigma_K \in K_\delta$ by Lemma 1.11.4. Schütt, Werner [528] proved (see Prochno, Schütt, Werner [494] or Werner [564] for a clear formulation of the statement) that

$$\Omega(K) = 2\left(\frac{\omega_{n-1}}{n+1}\right)^{\frac{2}{n+1}}\lim_{\delta\to 0^+}\frac{|K|-|K_\delta|}{\delta^{\frac{2}{n+1}}}.$$

In 1822, Dupin [196] considered the floating body that is the envelope of the centroids of the $(n-1)$-dimensional sections $\{x \in K : \langle x, u\rangle = t_{u,\delta}\}$ for $u \in S^{n-1}$. Dupin's floating may not be convex, but it is convex when ∂K is C^2 and δ is small, and in this case, it agrees with the convex floating body K_δ). This floating body was investigated in detail for example by Blaschke [74] and Leichweiss [398].

For the Affine Isoperimetric Inequality, Blaschke's and Santaló's approach was to prove it for convex bodies with C^2_+ boundaries via Steiner symmetrization with characterization of the equality, and then deduce the Blaschke-Santaló inequality. Meyer, Pajor [451] proved the Blaschke-Santaló inequality via Steiner symmetrizatio with a characterization of the equality case for all convex bodies, and Lutwak [434] explained how the Affine Isoperimetric Inequality and the Blaschke-Santaló inequality are equivalent, even including the equality case.

We note that Hug [335] managed to characterize equality in the Affine Isoperimetric Inequality for any convex body via Steiner symmetrization. The curvature relation (8.137) is significantly generalized by Hug [338], Theorem 5.1.

The centro-affine surface area was introduced by Blaschke [74]. Due to its $\mathrm{GL}(n)$ invariance, the centro-affine surface area shows up in the asymptotic formula in the case best approximation by polytopes with respect to the Banach-Mazur distance. For o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, their Banach-Mazur distance $\delta_{\mathrm{BM}}(K, C)$ is the minimum of $\log\lambda$ where there exists $\Phi \in \mathrm{GL}(n)$ such that $K \subset \Phi C \subset \lambda K$. It is a metric on the equivalence classes of o-symmetric convex bodies with respect to linear transformations. Mow if $K \subset \mathbb{R}^n$ convex body, ∂K is C^2, and P_{2m} is an o-symmetric polytope with at most $2m$ vertices minimizing $\delta_{\mathrm{BM}}(K, P_{2m})$, and $P_{(2m)}$ is an o-symmetric polytope with at most $2m$ facets minimizing $\delta_{\mathrm{BM}}(K, P_{(2m)})$, then (see Böröczky [90])

$$\lim_{m\to\infty}(2m)^{\frac{2}{n-1}} \cdot \delta_{\mathrm{BM}}(K, P_{2m}) = \frac{1}{2}\left(\frac{\vartheta_{n-1}}{\omega_{n-1}}\right)^{\frac{2}{n-1}} \cdot \widetilde{\Omega}(K)^{\frac{2}{n-1}}$$

$$\lim_{m\to\infty}(2m)^{\frac{2}{n-1}} \cdot \delta_{\mathrm{BM}}(K, P_{(2m)}) = \frac{1}{2}\left(\frac{\vartheta_{n-1}}{\omega_{n-1}}\right)^{\frac{2}{n-1}} \cdot \widetilde{\Omega}(K)^{\frac{2}{n-1}}$$

where ϑ_{n-1} is the covering density; namely, the minimal density of a covering of $\mathbb{R}^{n-1}$ by equal balls. We note that $\lim_{n\to\infty} \vartheta_{n-1}^{\frac{2}{n-1}} = 1$.

Lutwak [432] introduced L_p-affine surface area for $p \geq 1$, which notion was extended to $p > 0$ by Hug [335], and even further to $p < 0$ by Werner, Ye [565]. If $-n < p < 0$ and $K \subset \mathbb{R}^n$ is a centered convex body ($\sigma_K = o$), then Werner, Ye [565] proved that

$$\Omega_p(K) \geq n|B^n|^{\frac{2p}{n+p}}|K|^{\frac{n-p}{n+p}}, \tag{8.144}$$

with equality if and only if K is a centered ellipsoid. In addition, Ludwig [426] verified that if $-n < p < 0$, then $\Omega_p(K)$ is lower semicontinuous on the space of convex bodies containing the origin in their interior. Ludwig [426] actually extended the L_p-affine surface area to a broad Orlicz setting, which extended affine surface area shows up in the characterization of upper or lower semicontinuous $\mathrm{SL}(n)$ invariant valuations on the space of convex bodies containing the origin in their interior by Ludwig, Reitzner [429]. Concerning the L_p-affine surface area, Steiner-type formulas are proved and investigated by Tatarko, Werner [550, 551]. Some additional extensions of the notion of L_p-affine surface area discussed by Werner [563] and Caglar, *et al* [140].

8.A Supplement: Aleksandrov's Mixed Discriminant Inequality

The sole goal of this section is to prove the following inequality for mixed discriminants (see below for their definition):

Theorem 8.A.1 (Aleksandrov's Mixed Discriminant Inequality). *If $d \geq 2$, A is any symmetric $d \times d$ matrix, and $B, M_1, \dots, M_{d-2}$ are positive-semidefinite symmetric $d \times d$ matrices, then*

$$\mathcal{D}(A, B, M_1, \dots, M_{d-2})^2 \geq \mathcal{D}(A, A, M_1, \dots, M_{d-2})\mathcal{D}(B, B, M_1, \dots, M_{d-2}) \quad (8.145)$$

where no $M_1, \dots, M_{d-2}$ occur in the case of $d = 2$.

The proof - due to Shenfeld, van Handel [300] - of the Mixed Discriminant Inequality Theorem 8.A.1 given here is rather technical, as all known arguments (see, for example, Schneider [522], Section 5.5), and is based on similar ideas like the proof of the Alexandrov-Fenchel inequality in Section 8.5.

Let us collect some some basic properties of the mixed discriminant of real $d \times d$ matrices $A_1, \dots, A_d$ for $d \geq 1$. According to Definition 8.3.2, if $A_i = [a_1^{(i)}, \dots, a_d^{(i)}]$ for $a_j^{(i)} \in \mathbb{R}^d$, then

$$\mathcal{D}(A_1, \dots, A_d) = \frac{1}{d!} \sum_{\pi:\{1,\dots,d\}\to\{1,\dots,d\} \text{ bijection}} \det[a_1^{(\pi(1))}, \dots, a_d^{(\pi(d))}]. \quad (8.146)$$

In particular, for any permutation $\pi : \{1, \dots, d\} \to \{1, \dots, d\}$, we have

$$\mathcal{D}(A_1, \dots, A_d) = \mathcal{D}(A_{\pi(1)}, \dots, A_{\pi(d)}),$$

and if $\lambda_1, \dots, \lambda_d \in \mathbb{R}$, then

$$\det\left(\sum_{i=1}^{d} \lambda_i A_i\right) = \sum_{i_1,\dots,i_d \in \{1,\dots,d\}} \mathcal{D}(A_{i_1}, \dots, A_{i_d})\lambda_{i_1}, \dots, \lambda_{i_d}. \quad (8.147)$$

We deduce from (8.147) that if $U \in \mathrm{GL}(d)$, then

$$\mathcal{D}(UA_1U^t, \dots, UA_dU^t) = \det(UU^t) \cdot \mathcal{D}(A_1, \dots, A_d). \quad (8.148)$$

We note that if $A_1, \ldots, A_d$ are symmetric positive definite, then

$$\mathcal{D}(A_1, A_2, \ldots, A_d) > 0 \tag{8.149}$$

according to Lemma 8.3.3.

For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, we use the breviation $\mathrm{diag}(x) = \mathrm{diag}[x_1, \ldots, x_d]$ to denote the corresponding diagonal matrix. For $j = 1, \ldots, d$ and $d \geq 2$, let $A_i^{(j)}$ denote the $(d-1) \times (d-1)$ matrix obtained by removing the jth column and the jth row of the $d \times d$ matrix A_i, and hence (8.146) yields that if $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, then

$$\mathcal{D}(A_1, \ldots, A_{d-1}, \mathrm{diag}(x)) = \frac{1}{d} \sum_{j=1}^{d} x_j \mathcal{D}(A_1^{(j)}, \ldots, A_{d-1}^{(j)}). \tag{8.150}$$

If $\mathcal{E}$ is a symmetric $d \times d$ matrix for $d \geq 2$, then there exist eigenvectors $x_1, \ldots, x_d$ forming an orthonormal basis of $\mathbb{R}^d$ and eigenvalues $\lambda_1 \geq \ldots \geq \lambda_d$ such that $\mathcal{E}x_i = \lambda_i x_i$ and if $j = 1, \ldots, d-1$ and $x \in \mathbb{R}^d$ satisfy that $\langle x_i, x \rangle = 0$ for $i = 1, \ldots, j$, then

$$\langle \mathcal{E}x, x \rangle \leq \lambda_{j+1} \langle x, x \rangle. \tag{8.151}$$

We also need the following special case of the Perron-Frobenius theorem (see Theorem 10.8.1 in the Appendix for a proof):

Proposition 8.A.2 (Perron-Frobenius Theorem for symmetric positive matrices). *If each entry of the symmetric $d \times d$ matrix $\mathcal{E}$ is positive, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $\mathcal{E}x_1 = \lambda_1 x_1$;*
- *any eigenvector x of $\mathcal{E}$ whose coordinates are all positive satisfy $x = r\,x_1$ for $r > 0$.*

If $\mathcal{E}$ is a symmetric matrix, its positive eigenspace is the subspace spanned by the eigeinvectors corresponding to positive eigenvalues. The following statement has been proved as Lemma 7.A.4, but the argument is essentially the same as in the case of Lemma 8.5.3:

Lemma 8.A.3 (Hyperbolic Quadratic Forms). *For a symmetric $d \times d$ matrix $\mathcal{E}$, $d \geq 2$, the following conditions are equivalent for $x, y \in \mathbb{R}^d$.*

(i) $\langle x, \mathcal{E}y \rangle^2 \geq \langle x, \mathcal{E}x \rangle \langle y, \mathcal{E}y \rangle$ if $\langle y, \mathcal{E}y \rangle \geq 0$.

(ii) There exists a $w \in \mathbb{R}^d$ such that $\langle x, \mathcal{E}x \rangle \leq 0$ if $\langle x, \mathcal{E}w \rangle = 0$.

(iii) The dimension of the positive eigenspace of $\mathcal{E}$ is at most one.

We are ready to verify the case $d = 2$ of the Mixed Discriminant Inequality (8.39):

Lemma 8.A.4. *If A is a symmetric 2×2 matrix and B is a symmetric positive definite 2×2 matrix, then*

$$\mathcal{D}(A, B)^2 \geq \mathcal{D}(A, A) \cdot \mathcal{D}(B, B). \tag{8.152}$$

Proof. We may assume that B is positive definite by repacing it with $B + \varepsilon I_2$ for small $\varepsilon > 0$. Therefore, we deduce from (8.148) that we may also assume that $B = I_2$ and $A = \mathrm{diag}(x)$ for $x = (x_1, x_2) \in \mathbb{R}^2$. In this case, (8.150) yields that $\mathcal{D}(A, B) = \frac{x_1+x_2}{2}$, $\mathcal{D}(A, A) = x_1x_2$ and $\mathcal{D}(B, B) = 1$; therefore, (8.152) is a conseuence of the AG-GM inequality. ∎

Lemma 8.A.5 is the core special case the Mixed Discriminant Inequality (8.39). The main idea of the proof of Lemma 8.A.5 is to construct a scalar product $\langle \cdot, \cdot \rangle_\mu$ on $\mathbb{R}^d$ and a symmetric $d \times d$ matrix $\mathcal{A}$ with positive coefficients such that the Mixed Discriminant Inequality (8.39) for suitable matrices is the consequence of the inequality $\langle x, \mathcal{A}y \rangle_\mu^2 \geq \langle x, \mathcal{A}x \rangle_\mu \langle y, \mathcal{A}y \rangle_\mu$ whenever $\langle y, \mathcal{A}y \rangle_\mu \geq 0$, and the vector $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ is an eigenvector of $\mathcal{A}$ with eigenvalue 1. We verify the inequality $\langle \mathcal{A}x, \mathcal{A}x \rangle_\mu^2 \geq \langle x, \mathcal{A}x \rangle_\mu$ assuming the Mixed Discriminant Inequality (8.39) for $(d-1) \times (d-1)$ matrices, which, together with the Perron-Frobenius theorem, ensures that the positive eigenspace of $\mathcal{A}$ is one dimensional. In turn, Lemma 8.A.3 completes the argument.

Lemma 8.A.5. *Let $d \geq 3$, and let $M_1, \dots, M_{d-2}$ be positive definite matrices such that $M_1 = I_d$. If $x, y \in \mathbb{R}^d$ such that each coordinate of y is positive, and the Mixed Discriminant Inequality (8.39) holds for $(d-1) \times (d-1)$ matrices, then*

$$\mathcal{D}((x), (y), M_1, \dots, M_{d-2})^2 \geq \mathcal{D}((x), (x), M_1, \dots, M_{d-2})\mathcal{D}((y), (y), M_1, \dots, M_{d-2}) \tag{8.153}$$

Proof. Let $e_1, \dots, e_d$ be an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$. We consider the scalar product $\langle \cdot, \cdot \rangle_\mu$ on $\mathbb{R}^d$ and the $d \times d$ matrix $\mathcal{A}$ such that if $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ and $y = (y_1, \dots, y_d) \in \mathbb{R}^d$

$$\langle x, y \rangle_\mu = \sum_{j=1}^{d} x_j y_j \, \mathcal{D}(M_1^{(j)}, M_1^{(j)}, \dots, M_{d-2}^{(j)}); \tag{8.154}$$

$$(\mathcal{A}x)_j = \frac{\mathcal{D}((x)^{(j)}, M_1^{(j)}, \dots, M_{d-2}^{(j)})}{\mathcal{D}(M_1^{(j)}, M_1^{(j)}, \dots, M_{d-2}^{(j)})} \tag{8.155}$$

$$= \sum_{k \neq j} \frac{x_k}{d-1} \cdot \frac{\mathcal{D}(M_1^{(jk)}, \dots, M_{d-2}^{(jk)})}{\mathcal{D}(M_1^{(j)}, M_1^{(j)}, \dots, M_{d-2}^{(j)})} \tag{8.156}$$

where $(\mathcal{A}x)_j$ is the jth coordinate of $\mathcal{A}x$, $j = 1, \dots, d$, $\mathcal{D}(M_1^{(j)}, M_1^{(j)}, \dots, M_{d-2}^{(j)}) > 0$ and $\mathcal{D}(M_1^{(jk)}, \dots, M_{d-2}^{(jk)}) > 0$ by (8.149), and we used (8.150). In particular, (8.150), (8.155) and $I_d = (\mathbf{1})$ yield

$$\langle x, \mathcal{A}y\rangle_\mu = \langle \mathcal{A}x, y\rangle_\mu = d \cdot \mathcal{D}((x),(y),M_1,\dots,M_{d-2}); \tag{8.157}$$

$$\mathcal{A}\mathbf{1} = \mathbf{1}. \tag{8.158}$$

Since there exist $t_1,\dots,t_d > 0$ such that $t_1e_1,\dots,t_de_d$ is an orthonormal basis with respect to $\langle\cdot,\cdot\rangle_\mu$ by (8.154), we deduce from (8.156) that each coefficient of $\mathcal{A}$ is positive in this basis, and from (8.158) that $\mathbf{1}$ is a positive eigenvector of $\mathcal{A}$ in this basis with eigenvalue 1. Since $\mathcal{A}$ is symmetric in this basis by (8.157), the Perron-Frobenius Theorem 8.A.2 yields that 1 is a simple eigenvalue of $\mathcal{A}$ that is the maximal eigenvalue.

Next, first we apply Mixed Discriminant Inequality (8.39) for $(d-1)\times(d-1)$ matrices, then (8.150) and $M_1 = I_d$, and finally (8.157) to obtain

$$\begin{aligned}\langle \mathcal{A}x, \mathcal{A}x\rangle_\mu &= \sum_{j=1}^{d} \frac{\mathcal{D}((x)^{(j)}, M_1^{(j)},\dots,M_{d-2}^{(j)})^2}{\mathcal{D}(M_1^{(j)}, M_1^{(j)},\dots,M_{d-2}^{(j)})}\\ &\geq \sum_{j=1}^{d} \mathcal{D}((x)^{(j)},(x)^{(j)},M_2^{(j)},\dots,M_{d-2}^{(j)})\\ &= d\cdot\mathcal{D}((x),(x),M_1,\dots,M_{d-2}) = \langle x, \mathcal{A}x\rangle_\mu \end{aligned} \tag{8.159}$$

where $\mathcal{D}((x)^{(j)},(x)^{(j)},M_2^{(j)},\dots,M_{d-2}^{(j)}) = \mathcal{D}((x)^{(j)},(x)^{(j)})$ is meant in the case of $d = 3$. It follows from (8.159) that λ is an eigenvalue of $\mathcal{A}$, then $\lambda^2 \geq \lambda$, and hence either $\lambda \geq 1$ or $\lambda \leq 0$. Since we have already seen that the maximal eigenvalue of $\mathcal{A}$ is 1, and it is a simple eigenvalue, we deduce that the the dimension of the positive eigenspace of $\mathcal{A}$ is one. In turn, Lemma 8.A.3 and (8.157) yields (8.153). ■

Proof of Theorem 8.A.1. We prove Theorem 8.A.1 by induction on $d \geq 2$ where the case $d = 2$ is just Lemma 8.A.4. Therefore, let $d \geq 3$, and we assume that the Mixed Discriminant Inequality (8.39) holds for $(d-1)\times(d-1)$ matrices. Our goal is to prove that if A is any symmetric $d\times d$ matrix, and $B, M_1,\dots,M_{d-2}$ are positive-semidefinite symmetric $d\times d$ matrices, then

$$\mathcal{D}(A,B,M_1,\dots,M_{d-2})^2 \geq \mathcal{D}(A,A,M_1,\dots,M_{d-2})\mathcal{D}(B,B,M_1,\dots,M_{d-2}). \tag{8.160}$$

Adding εI_d for small $\varepsilon > 0$, we may assume that $B, M_1,\dots,M_{d-2}$ are positive-definite. According to (8.148), we may also assume that

$$M_1 = I_d.$$

First we verify the special $B = I_d$; namely, if A is any symmetric $d \times d$ matrix, then

$$\mathcal{D}(A, I_d, M_1, \ldots, M_{d-2})^2 \geq \mathcal{D}(A, A, M_1, \ldots, M_{d-2})\mathcal{D}(I_d, I_d, M_1, \ldots, M_{d-2}). \tag{8.161}$$

Applying any orthogonal transformation U as in (8.148) keeps the condition $M_1 = I_d$, and hence we may assume that A is a diagonal matrix. However, Lemma 8.A.5 yields (8.161) in this case.

Let $\mathcal{M}_d$ denote vector space of dimension $d(d+1)/2$ of real $d \times d$ symmetric matrices. Considering the symmetric bilinear form

$$Q(A, B) = \mathcal{D}(A, B, M_1, \ldots, M_{d-2})$$

for $A, B \in \mathcal{M}_d$, we have $Q(A, I_d)^2 \geq Q(A, A)Q(I_d, I_d)$ for any $A \in \mathcal{M}_d$ according to (8.161) where $Q(I_d, I_d) > 0$ by (8.149). In particular, if $Q(A, I_d) = 0$, then $Q(A, A) \leq 0$; therefore, the equivalence of (i) and (ii) in Lemma 8.A.3 imples that $Q(A, B)^2 \geq Q(A, A)Q(B, B)$ for any positive definite $B \in \mathcal{M}_d$. In turn, we conclude Theorem 8.A.1. ∎

8.B Supplement: Upper semicontinuity of the Affine Surface Area

The key statement to prove the upper semicontinuity of the affine surface area of general convex bodies (cf. Theorem 8.9.10) is Theorem 8.B.2 below proved by Dolzmann, Hug [192]. The following observation is needed in the argument for Theorem 8.B.2.

Lemma 8.B.1. *If $\eta > 0$ and $f \geq \eta$ for $f \in L_1(S^{n-1})$, then there exists a sequence $h_m \in C^\infty(S^{n-1})$ such that $h_m \geq \frac{\eta}{2^{n-1}}$ for each m and* $\lim_{m\to\infty} \int_{S^{n-1}} |h_m - f|\, d\mathcal{H}^{n-1} = 0.$

Remark. Similar statement can be proved using spherical convolution (see Feng Dai, Yuan Xu [185], Chapter 2).

Proof. For an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^{n+1}$, let $e_{i+n} = -e_i$ for $i = 1, \ldots, n$, therefore, $e_1, \ldots, e_{2n}$ are the exterior unit normals of the facets $F_1, \ldots, F_{2n}$, respectively, of the cube $W = \sum_{i=1}^n [-1, 1]e_i$. For $i = 1, \ldots, 2n$, let $F_i' = F_i|e_i^\perp = W \cap e_i^\perp$, let Θ_i be the radial projection of F_i onto S^{n-1}, and let $\pi_i : e_i^\perp \to S^{n-1}$ be defined by $\pi_i(x) = \frac{x+e_i}{\|x+e_i\|}$, and hence $\pi_i(F_i') = \Theta_i$ and $\Theta_1, \ldots, \Theta_{2n}$ tile S^{n-1}.

For $i = 1, \ldots, 2n$, let $f_{(i)} = f \cdot \mathbf{1}_{\Theta_i}$ on S^{n-1}, let $\tilde{f}_{(i)} : e_i^\perp \to [0, \infty)$ satisfy $\tilde{f}_{(i)}(x) = f(\pi_i(x))$ if $x \in F_i'$ and $\tilde{f}_{(i)}(x) = 0$ if $x \notin F_i'$, and as in Lemma 10.2.3 in the Appendix, let $k_i(x) = \gamma\varphi(1 - \|x\|^2)$ for $x \in e_i^\perp$ be a $\mathbb{C}^\infty$ function where $\varphi(t) = 0$ if $t \leq 0$, and $\varphi(t) = e^{\frac{-1}{t^2}}$ if $t > 0$, and the constant $\gamma > 0$ is chosen in a way such that $\int_{e_i^\perp} k_i \, d\mathcal{H}^{n-1} = 1$. For $m \geq 1$, we consider the approximate identity $k_{(i),m}(z) = m^{n-1}k(mz)$ and $\tilde{h}_{(i),m} = \tilde{f}_{(i)} * k_{(i),m}$

on $e_i^\perp$, and hence $\tilde{h}_{(i),m} \geq 0$, $\lim_{m\to\infty} \int_{e_i^\perp} |\tilde{h}_{(i),m} - \tilde{f}_{(i)}|\, d\mathcal{H}^{n-1} = 0$ and if $x \in F_i'$, then

$$\tilde{h}_{(i),m}(x) = \int_{e_i^\perp} \tilde{f}_{(i)}(y) k_{(i),m}(x-y)\, d\mathcal{H}^{n-1}(y) \geq \eta \int_{F_i'} k_{(i),m}(x-y)\, d\mathcal{H}^{n-1}(y) \geq \frac{\eta}{2^{n-1}}$$

as F_i' is an $(n-1)$-cube. Now we define $h_m = \sum_{i=1}^{2n} h_{(i),m}$ where $h_{(i),m}(\pi_i(x)) = \tilde{h}_{(i),m}(x)$ for $x \in e_i^\perp$ and $h_{(i),m}(u) = 0$ if $u \in S^{n-1}$ with $\langle u, e_i\rangle \leq 0$. Since each π_i is a contraction, and $f(u) = \sum_{i=1}^{2n} f_{(i)}(u)$ for $\mathcal{H}^{n-1}$ a.e. $u \in S^{n-1}$, we conclude Lemma 8.B.1.

■

We also need the following consequence of the Mean Value Theorem: If $\eta > 0$ and $a, b \geq \eta$, then

$$|a^{\frac{n}{n+1}} - b^{\frac{n}{n+1}}| \leq \eta^{\frac{-1}{n+1}} |a - b|. \tag{8.162}$$

Theorem 8.B.2 (Dolzmann-Hug). *If $K \subset \mathbb{R}^n$ is a convex body, then*

$$\Omega(K) = \inf_{\substack{g: S^{n-1}\to(0,\infty)\\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1}\, dS_K\right)^{\frac{n}{n+1}}. \tag{8.163}$$

Proof. We recall that $dS_K = f_K d\mathcal{H}^{n-1} + dS_K^s$ on S^{n-1} where $f_K = \sigma_{n-1} D^2 h_K$ is measurable and the singular part S_K^s is a regular Borel measure (cf. Theorem 10.1.3) s.t. $\exists\, \mathcal{H}^{n-1}$-measurable $X \subset S^{n-1}$ satisfying $\mathcal{H}^{n-1}(X) = 0$ and $S_K^s(S^{n-1}\backslash X) = 0$.

The right hand side RHS of (8.163) is estimated by the fact that $f_K d\mathcal{H}^{n-1}$ is the absolutely continuous part of dS_K and by the Hölder inequality:

$$\begin{aligned} \text{RHS} &\geq \inf_{\substack{g: S^{n-1}\to(0,\infty)\\ g \text{ continuous}}} \left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1} \cdot f_K\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} \\ &\geq \int_{S^{n-1}} f_K^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1} = \Omega(K). \end{aligned}$$

Therefore all we have to prove is that for any $\varepsilon > 0$, there exists continuous $g : S^{n-1} \to (0, \infty)$ such that

$$\left(\int_{S^{n-1}} g^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g^{-1}\, dS_K\right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_K^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1} + \varepsilon. \tag{8.164}$$

For $\eta \in (0, 1)$, let $f_\eta = \max\{\eta, f_K\} \in L_1(S^{n-1})$. Our first step towards verifying (8.164) is that for any $\eta \in (0, 1)$, there exists continuous $g_\eta : S^{n-1} \to (0, \infty)$ such that

$$\left(\int_{S^{n-1}} g_\eta^n\, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g_\eta^{-1} f_\eta\, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_\eta^{\frac{n}{n+1}}\, d\mathcal{H}^{n-1} + \eta. \tag{8.165}$$

To prove (8.165), we consider a sequence of continuous $h_m : S^{n-1} \to [\frac{\eta}{2^{n-1}}, \infty)$ provided by Lemma 8.B.1 such that $\lim_{m\to\infty} \int_{S^{n-1}} |h_m - f_\eta| \, d\mathcal{H}^{n-1} = 0$. It follows from (8.162) that

$$\lim_{m\to\infty} \int_{S^{n-1}} |h_m^{\frac{n}{n+1}} - f_\eta^{\frac{n}{n+1}}| \, d\mathcal{H}^{n-1} = 0. \tag{8.166}$$

For $\psi_m = h_m^{\frac{1}{n+1}}$, $\psi_m^{-1} \le \frac{2}{\eta^{\frac{1}{n+1}}}$ yields

$$\lim_{m\to\infty} \int_{S^{n-1}} \left|\psi_m^{-1} f_\eta - h_m^{\frac{n}{n+1}}\right| \, d\mathcal{H}^{n-1} = \lim_{m\to\infty} \int_{S^{n-1}} \psi_m^{-1} \left|f_\eta - h_m\right| \, d\mathcal{H}^{n-1} = 0,$$

which estimate combined with (8.166) implies that we can take $g_\eta = \psi_m$ for large m in (8.165).

According to Lebesgue's Dominated Convergence Theorem, we can fix a a small $\eta \in (0, 1)$ in (8.165) such that

$$\left(\int_{S^{n-1}} g_\eta^n \, d\mathcal{H}^{n-1}\right)^{\frac{1}{n+1}} \left(\int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1}\right)^{\frac{n}{n+1}} < \int_{S^{n-1}} f_K^{\frac{n}{n+1}} \, d\mathcal{H}^{n-1} + \frac{\varepsilon}{2}. \tag{8.167}$$

Fix a $G > 1$ such that $G^{-1} < g_\eta < G$.

To deduce (8.164) from (8.167), we note that there exist compact $C_m \subset X$ and open $U_m \supset X$ such that

$$S_K^s(S^{n-1} \backslash C_m) < \frac{1}{m} \quad \text{and} \quad \mathcal{H}^{n-1}(U_m) < \frac{1}{m^{n+1}}$$

as S_K^s and $\mathcal{H}^{n-1}$, being finite Borel mesures on S^{n-1}, are regular (cf. Theorem 10.1.3). For $m > G$, let $\theta_m : S^{n-1} \to [0, m]$ be continuous such that $\theta_m(x) = m$ if $x \in C_m$ and $\theta_m(x) = 0$ if $x \notin U_m$, and let $\varphi_m = \max\{g_\eta, \theta_m\}$. In particular,

$$\begin{aligned} \int_{S^{n-1}} \varphi_m^n \, d\mathcal{H}^{n-1} &\le \int_{S^{n-1}\backslash U_m} g_\eta^n \, d\mathcal{H}^{n-1} + \int_{U_m} m^n \, d\mathcal{H}^{n-1} \\ &\le \int_{S^{n-1}} g_\eta^n \, d\mathcal{H}^{n-1} + \frac{1}{m}, \end{aligned}$$

and as $\varphi_m^{-1} \le g_\eta^{-1} \le G$ on S^{n-1} and $\varphi_m^{-1} = \frac{1}{m}$ on C_m, we also have

$$\begin{aligned} \int_{S^{n-1}} \varphi_m^{-1} \, dS_K &\le \int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1} + \int_{U_m\backslash C_m} G \, dS_K^s + \int_{C_m} \frac{1}{m} \, dS_K^s \\ &\le \int_{S^{n-1}} g_\eta^{-1} f_K \, d\mathcal{H}^{n-1} + \frac{G}{m} + \frac{S_K^s(S^{n-1})}{m}. \end{aligned}$$

Therefore, (8.167) yields that we can choose $g = \varphi_m$ for large m in (8.164). ∎

8.C Supplement: Affine Isoperimetric Inequality via Steiner symmetrization

Lemma 8.C.1 (Minkowski's determinantal inequality). *If A, B positive semidefinite symmetric $n \times n$ matrices and $p \geq n$, then*

$$\frac{1}{2}(\det A)^{\frac{1}{p}} + \frac{1}{2}(\det B)^{\frac{1}{p}} \leq \det\left(\frac{1}{2}(A+B)\right)^{\frac{1}{p}}.$$

Assuming that A, B positive definite, equality holds if and only if $A = B$.

Proof. It follows from the Jensen inequality (10.4) that we may assume that $p = n$. In addition, we may assume that A, B positive definite, and hence also that both A, B are diagonal with eigenvalues $a_1, \ldots, a_n > 0$ and $b_1, \ldots, b_n > 0$, respectively. In this case, the inequality is

$$\left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} b_i\right)^{\frac{1}{n}} \leq \left(\prod_{i=1}^{n}(a_i + b_i)\right)^{\frac{1}{n}},$$

which follows from the AM-GM inequality as

$$\left(\prod_{i=1}^{n} \frac{a_i}{a_i + b_i}\right)^{\frac{1}{n}} + \left(\prod_{i=1}^{n} \frac{b_i}{a_i + b_i}\right)^{\frac{1}{n}} \leq \frac{1}{n}\left(\sum_{i=1}^{n} \frac{a_i}{a_i + b_i}\right) + \frac{1}{n}\left(\sum_{i=1}^{n} \frac{b_i}{a_i + b_i}\right) = 1.$$

■

Let us recall the well-known formula for the Gaussian curvature for a graph of a convex function. For (relatively) open and convex $\Omega \subset u^{\perp}$, $u \in S^{n-1}$, and for convex $\varphi : \Omega \to \mathbb{R}$, let $Z = \{x + \varphi(x)u : x \in \Omega\} \subset \mathbb{R}^n$ be the graph of φ. If φ is twice differentiable at $x \in \Omega$, then the Gauss curvature $\kappa(z)$ at $z = x + \varphi(x)u$ is

$$\kappa(z) = \frac{\det D^2 f(x)}{(1 + \|Df(x)\|^2)^{\frac{n+1}{2}}}. \tag{8.168}$$

The following statement is due to Santaló [505].

Theorem 8.C.2 (Santaló). *If $K \subset \mathbb{R}^n$ is a convex body and $u \in S^{n-1}$, then*

$$\Omega(\Theta_{u^{\perp}} K) \geq \Omega(K). \tag{8.169}$$

Assuming in addition that ∂K is C^2_+, equality holds in (8.169) if and only if the midpoints of the secants of K parallel to u are contained in a hyperplane.

Proof. There exist continuous concave functions f and g on $K|u^\perp$ such that

$$K = \left\{x + tu : -g(x) \le t \le f(x) \text{ for } x \in K|u^\perp\right\}.$$

Let $X \subset (\mathrm{int}K)|u^\perp$ be the set of points $x \in (\mathrm{int}K)|u^\perp$ where both f and g twice differentiable, and hence $\mathcal{H}^{n-1}\left((K|u^\perp)\setminus X\right) = 0$. For $Z^+ = \{x + f(x)u : x \in X\} \subset \partial K$ and $Z^- = \{x + g(x)u : x \in X\} \subset \partial K$, we have

$$\begin{aligned}\Omega(K) &= \int_{Z^+} \kappa(z)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(z) + \int_{Z^-} \kappa(z)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(z)\\ &= \int_X (\det D^2 f(x))^{\frac{1}{n+1}} + (\det D^2 g(x))^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x).\end{aligned}$$

Since

$$\Theta_{u^\perp} K = \left\{x + tu : -\frac{f(x)+g(x)}{2} \le t \le \frac{f(x)+g(x)}{2} \text{ for } x \in K|u^\perp\right\},$$

we deduce that

$$\Omega(\Theta_{u^\perp} K) = 2\int_X \left(\det \frac{D^2 f(x) + \det D^2 g(x)}{2}\right)^{\frac{1}{n+1}} d\mathcal{H}^{n-1}(x).$$

As $D^2 f(x)$ and $D^2 g(x)$ are positive semidefinite for $x \in X$, Lemma 8.C.1 yields (8.169).

Now let us assume that ∂K is C^2_+ and equality holds in (8.169). As $D^2 f(x)$ and $\det D^2 g(x)$ are positive definite and continuous functions of x, Lemma 8.C.1 yields that $D^2 f(x) = \det D^2 g(x)$ for all $x \in (\mathrm{int}K)|u^\perp$. Therefore, there exist $v \in u^\perp$ and $\gamma \in \mathbb{R}$ such that if $x \in K|u^\perp$, then $f(x) = g(x) + \langle v, x\rangle + \gamma$, and hence

$$\frac{1}{2}\left((x + f(x)u) + (x - g(x)u)\right) = \left\langle \frac{v}{2}, x\right\rangle + \frac{\gamma}{2}.$$

■

In turn, we deduce the Affine Isoperimetric Inequality following Affine following Santaló [505].

Theorem 8.C.3 (Affine Isoperimetric Inequality, Santaló). *If $K \subset \mathbb{R}^n$ convex body, then*

$$\Omega(K) \le n\omega_n^{\frac{2}{n+1}} V(K)^{\frac{n-1}{n+1}}. \tag{8.170}$$

Assuming that ∂K is C^2_+, equality holds in (8.170) if and only if K is an ellipsoid.

Proof. Since starting from K, a sequence of Steiner symmetrizations lead to a ball of volume $|K|$ (cf. Theorem 1.10.7 or Theorem 1.A.3), we deduce (8.170) from Theorem 8.C.2.

If ∂K is C^2_+ and equality holds in (8.170), then Theorem 8.C.2 yields that the midpoints of the secants of K parallel to u are contained in a hyperplane for any $u \in S^{n-1}$. Therefore, Theorem 6.2.1 implies that K is an ellipsoid. ■

8.D Supplement: Valuations on convex bodies

Let $\mathcal{K}^n$ be the space of compact convex sets in $\mathbb{R}^n$ equipped with the Hausdorff metric. We observe that the Minikowski addition makes $\mathcal{K}^n$ a cancellative abelian semigroup. In addition, let $\mathcal{K}^n_{(o)}$ be the subspace of $\mathcal{K}^n$ consisting of convex bodies K with $o \in \operatorname{int} K$.

Definition 8.D.1 (Valuations on convex compact sets). For a cancellative abelian semigroup $\mathcal{A}$, a function $Z : \mathcal{K}^n \to \mathcal{A}$ (or $Z : \mathcal{K}^n_{(o)} \to \mathcal{A}$) is called a valuation if $K, C \in \mathcal{K}^n$ (or $K, C \in \mathcal{K}^n_{(o)}$) satisfy that $K \cup C \in \mathcal{K}^n$, then

$$Z(K \cup C) + Z(K \cap C) = Z(K) + Z(C). \tag{8.171}$$

We observe that if $K, C \in \mathcal{K}^n$ satisfy that $K \cup C$ is convex, and hence lies in $\mathcal{K}^n$, then

$$h_{K\cup C} = \max\{h_K, h_C\} \text{ and } h_{K\cap C} = \min\{h_K, h_C\}. \tag{8.172}$$

In this section, we only consider some fundamental properties of continuous or semicontimuous valuations on compact convex sets. For surveys on various related aspects of the theory of valuations, see for example the monograph Alesker [9], the survey papers McMullen, Schneider [448] and Schneider [523].

Examples of valuations on compact convex sets

(a) $Z(K) = \gamma$ for a constant $\gamma \in \mathbb{R}$ for $K \in \mathcal{K}^n$;

(b) $Z(K) = |K| = V_n(K)$ for $K \in \mathcal{K}^n$;

(c) Given $A \in \mathcal{K}^n$, $Z(K) = |K + A|$ for $K \in \mathcal{K}^n$ as $(K + A) \cup (C + A) = (K \cup C) + A$;

(d) Given $i = 1, \ldots, n-1$, $Z(K) = V_i(K)$ for $K \in \mathcal{K}^n$ (follows from (b) and the Kubota formula (7.5) representing $V_i(K)$ as mean projection);

(e) Generalizing (d), given $C_1, \ldots, C_m \in \mathcal{K}^n$ where $1 \le m < n$, $Z(K) = V(C_1, \ldots, C_m, K, \ldots, K)$ for $K \in \mathcal{K}^n$ (follows from (c) and Theorem 7.3.1 (iii));

(f) $Z(K) = K - K$ for $K \in \mathcal{K}^n$ (here $\mathcal{A} = \mathcal{K}^n$);

(g) $Z(K) = \Omega(K)$ for $K \in \mathcal{K}^n$ as $\Omega(K) = \int_{\partial' K} \kappa_K(x)^{\frac{1}{n+1}}\, d\mathcal{H}^{n-1}(x)$ if K is a convex body;

(h) $Z(K) = \int_K x\, dx$ for $K \in \mathcal{K}^n_{(o)}$ (here $\mathcal{A} = \mathbb{R}^n$);

(i) Given $p > 0$, $Z(K) = \Omega_p(K)$ for $K \in \mathcal{K}^n_{(o)}$ (in particular, $Z(K) = \widetilde{\Omega}(K)$) as $\Omega_p(K) = \int_{\partial' K} \frac{\kappa(x)^{\frac{p}{n+p}}}{\langle x, \nu_K(x)\rangle^{\frac{n(p-1)}{n+p}}}\, dx$ (use also (8.172)).

For references about properties of continuous valuations discussed in this section, see Alesker [9]. Any continuous valuation $Z : \mathcal{K}^n \to \mathbb{R}$ satifies the inclusion-exlusion principle; namely, if $K_1 \cup \ldots \cup K_m \in \mathcal{K}^n$ for $K_1, \ldots, K_m \in \mathcal{K}^n$, then

$$\begin{aligned} Z(K_1 \cup \ldots \cup K_m) &= \sum_{i=1}^{m} Z(K_i) - \sum_{1 \le i < j \le m} Z(K_i \cap K_j) + \ldots \\ &= \sum_{i=1}^{m} (-1)^{i-1} \sum_{1 \le j_1 < \ldots < j_i \le m} Z(K_{j_1} \cap \ldots \cap K_{j_i}). \end{aligned}$$

Remark. According to Groemer [270], if a valuation Z satisfies the inclusion-exlusion principle, then Z can be extended to be a finitely additive measure on the finite unions of convex compact sets. For example, the constant one valuation on $\mathcal{K}^n$ extends to the Euler characteristic of finite unions of convex compact sets (a result due to Hadwiger [295]).

Let us review some results about valuations that characterize quantities related to this book. If G is a subgroup of the group of affine transformations of $\mathbb{R}^n$ (any element of G is of the form $x \mapsto \Phi x + w$ for $\Phi\mathrm{GL}(n)$ and $w \in \mathbb{R}^n$), then we say that a valuation $Z : \mathcal{K}^n \to \mathcal{A}$, $\mathcal{A}$ cancellative abelian semigroup, is G invariant if $Z(gK) = Z(K)$ for $g \in G$ and $K \in \mathcal{K}^n$, and a valuation $Z : \mathcal{K}^n \to \mathbb{R}^n$ or a valuation $Z : \mathcal{K}^n \to \mathcal{K}^n$ is G equivariant if $Z(gK) = gZ(K)$ for $g \in G$ and $K \in \mathcal{K}^n$. The modern theory of valuations started with Hadwiger's characterization theorem in 1957 (see Klain [368] for a simpler proof).

Theorem 8.D.2 (Hadwiger [295]). *$Z : \mathcal{K}^n \to \mathbb{R}$ is a continuous valuation invariant under isometries of $\mathbb{R}^n$ if and only if there exist $\gamma_0, \ldots, \gamma_n \in \mathbb{R}$ such that*

$$Z(K) = \sum_{i=0}^{n} \gamma_i V_i(K).$$

Remark. It is sufficient that Z is $\mathrm{SO}(n)$ and translation invariant and continuity of Z can be replaced by monotoncity ($Z(K) \le Z(C)$ if $K \subset C$).

We say that a valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is homogeneous of degree $q \in \mathbb{R}$ if $Z(\lambda K) = \lambda^q Z(K)$ for $\lambda > 0$. Concerning volume, Hadwiger [295] verified that any translation invariant and n-homogeneous valuation $Z : \mathcal{K}^n \to \mathbb{R}$ is of the form

$$Z(K) = \gamma |K| \ \text{ for a} \gamma \in \mathbb{R}, \tag{8.173}$$

and Klain [368] proved the same conlusion for any continuous, translation invariant and even ($Z(-K) = Z(K)$) valuation $Z : \mathcal{K}^n \to \mathbb{R}$ that vanishes on lower dimensional compact convex sets. In addition, Schneider [521] characterizes any continuous, translation invariant valuation $Z : \mathcal{K}^n \to \mathbb{R}$ that vanishes on lower dimensional compact convex sets; namely, there exist $\gamma \in \mathbb{R}$ and an odd continuous function $g : S^{n-1} \to \mathbb{R}$ $(g(-u) = g(u))$ such that

$$Z(K) = \gamma |K| + \int_{S^{n-1}} g \, dS_K.$$

Next, Haberl and Parapatits characterized the moment vector among vector valued valuations, and Ludwig characterized the difference body among Minkowski (compact convex set valued) valuations. Let us remark that the centroid of a convex body is not a valuation.

Theorem 8.D.3 (Haberl, Parapatits [293]). *$Z : \mathcal{K}^n_{(o)} \to \mathbb{R}^n$ is a continuous and* $\mathrm{SL}(n)$ *equivariant valuation for $n \geq 3$ if and only if there exists $\gamma \in \mathbb{R}$ such that*

$$Z(K) = \gamma \int_K x \, dx.$$

Theorem 8.D.4 (Ludwig [425]). *$Z : \mathcal{K}^n \to \mathcal{K}^n$ is a continuous,* $\mathrm{SL}(n)$ *equivariant and translation invariant valuation if and only if there exists $\gamma \geq 0$ such that*

$$Z(K) = \gamma(K - K).$$

Ludwig, Reitzner [428] characterized the affine surface area among valuations as follows.

Theorem 8.D.5 (Ludwig, Reitzner [428]). *$Z : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, translation and* $\mathrm{SL}(n)$ *invariant valuation if and only if there exist $\gamma_0, \gamma_1 \in \mathbb{R}$ and $\gamma_2 \geq 0$ such that*

$$Z(K) = \gamma_0 + \gamma_1 |K| + \gamma_2 \Omega(K).$$

Remark. $Z : \mathcal{K}^n \to \mathbb{R}$ is an upper semicontinuous, translation and $\mathrm{SL}(n)$ invariant valuation that vanishes on polytopes if and only if $Z(K) = \gamma\Omega(K)$ for $\gamma \geq 0$.

Ludwig, Reitzner [429] characterized all upper semicontinuous and $\mathrm{SL}(n)$ invariant valuations on $\mathcal{K}^n_{(o)}$. Their result yields the following characterization of the centro-affine surface area.

Theorem 8.D.6 (Ludwig, Reitzner [429]). *$Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ is an upper semicontinuous, and* $\mathrm{GL}(n)$ *invariant valuation if and only if there exist $\gamma_0 \in \mathbb{R}$ and $\gamma_1 \geq 0$ such that*

$$Z(K) = \gamma_0 + \gamma_1 \widetilde{\Omega}(K).$$

Remark. In particular, $Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ is an upper semicontinuous and $\mathrm{GL}(n)$ invariant valuation that vanishes on polytopes if and only if $Z(K) = \gamma\widetilde{\Omega}(K)$ for $\gamma \geq 0$.

For $0 < p < n$, Ludwig, Reitzner [429] even characterized the L_p-surface area $\Omega_p(K)$ as upper semicontinuous and $\mathrm{SL}(n)$ invariant valuation

$Z : \mathcal{K}^n_{(o)} \to \mathbb{R}$ that is homogeneous of degree $q = n(n-p)/(n+p)$ (recall that $\Omega(K) = \Omega_1(K)$).

Let us provide some hints about the flourishing theory of the structure of the space of continuous and translation invariant valuation that is main topic of Alesker [9]. According to McMullen's decomposition theorem in 1977, the existence of mixed volumes depends on the properties that the volume of a compact convex set is a continuous translation invariant valuation that is homogeneous of degree n.

Theorem 8.D.7 (McMullen's Decomposition [467]). *Let $Z : \mathcal{K}^n \to \mathbb{R}$ be a continuous and translation invariant valuation.*

(i) $Z = Z_0 + Z_1 + \ldots + Z_n$ where Z_i is a continuous and translation invariant valuation that is homogeneous of degree i.

(ii) Given $K_1, \ldots, K_m \in \mathcal{K}^n$, $Z(\lambda_1 K_1 + \ldots + \lambda_m K_m)$ is a polynomial in $\lambda_1, \ldots, \lambda_m \geq 0$ of degree at most n.

Observe that McMullen's decomposition is in line with Hadwiger's Theorem 8.D.2.

The next crucial step towards understanding continuous and translation invariant valuations is Alesker's theory introducing representation theory as the main tool around 2000. We write $\mathrm{Val}(\mathbb{R}^n)$ to denote the real (or complex) topological vector space of continuous and translation invariant valuations on $\mathcal{K}^n$ that is actually a Banach space. There is a natural $\mathrm{GL}(n)$ action on $\mathrm{Val}(\mathbb{R}^n)$ defined by $gZ(K) = Z(g^{-1}K)$ for $g \in \mathrm{GL}(n)$ and $K \in \mathcal{K}^n$. In addition, any valuation $Z : \mathcal{K}^n \to \mathbb{R}$ can be written (uniquely) as $Z = Z^+ + Z^-$ where Z^+ is even ($Z^+(-K) = Z^+(-K)$) and Z^- is odd ($Z^-(-K) = -Z^-(-K)$); namely; $Z^+(K) = \frac{1}{2}(Z(K) + Z(-K))$ and $Z^-(K) = \frac{1}{2}(Z(K) - Z(-K))$. According to McMullen's decomposition Theorem 8.D.7,

$$\mathrm{Val}(\mathbb{R}^n) = \bigoplus_{i=0}^{n} \left(\mathrm{Val}_i^+(\mathbb{R}^n) \oplus \mathrm{Val}_i^-(\mathbb{R}^n)\right)$$

where $\mathrm{Val}_i^+(\mathbb{R}^n)$ ($\mathrm{Val}_i^-(\mathbb{R}^n)$) is the subspace of even (odd) continuous and translation invariant valuations of $\mathrm{Val}_i(\mathbb{R}^n)$ of valuations homogeneous of degree i. We note that $\mathrm{Val}_0(\mathbb{R}^n) = \mathbb{R}\, V_0$ and $\mathrm{Val}_n(\mathbb{R}^n) = \mathbb{R}\, V_n$ by Hadwiger's theorem (8.173). According to Alesker's irreducibility theorem (see [9]), the irreducible closed subspaces with respect to the natural action of $\mathrm{GL}(n)$ on $\mathrm{Val}(\mathbb{R}^n)$ are exactly $\mathrm{Val}_i^+(\mathbb{R}^n)$ and $\mathrm{Val}_i^-(\mathbb{R}^n)$, $i = 0, \ldots, n$. As a consequence, Alesker proved McMullen's conjecture that the mixed volumes (see Example (e)) are dense among continuous and translation invariant valuations.

The dense subspace $\mathrm{Val}^\infty(\mathbb{R}^n)$ of the so-called smooth valuations where $Z \in \mathrm{Val}^\infty(\mathbb{R}^n)$ if the map $\mathrm{GL}(n) \to \mathrm{Val}^\infty(\mathbb{R}^n)$, $g \mapsto gZ$ is C^∞ is equipped with two natural products through the work by Alesker, Bernig and Fu (see [9]). We note the valuation $K \mapsto |K + A|$ in Example (c) and the mixed volume in Example (e) are smooth if $A, C_1, \ldots, C_m$

have C^∞_+ boundary. For the Alesker product $\mathrm{Val}^\infty(\mathbb{R}^n)\times\mathrm{Val}^\infty(\mathbb{R}^n)\to\mathrm{Val}^\infty(\mathbb{R}^n)$, if $i+j\leq n$, then $Z\cdot\widetilde{Z}\in\mathrm{Val}^\infty_{i+j}(\mathbb{R}^n)$ for $Z\in\mathrm{Val}^\infty_i(\mathbb{R}^n)$ and $\widetilde{Z}\in\mathrm{Val}^\infty_j(\mathbb{R}^n)$, and $\mathrm{Val}^\infty_i(\mathbb{R}^n)\times\mathrm{Val}^\infty_{n-i}(\mathbb{R}^n)\to\mathrm{Val}^\infty_n(\mathbb{R}^n)=\mathbb{R}\cdot V_n$ is a perfect pairing. In addition, it satisfies the Hard Lefschetz theorem stating that $Z\mapsto Z\cdot V_1^{n-2i}$ is an isomorphism $\mathrm{Val}^\infty_i(\mathbb{R}^n)\to\mathrm{Val}^\infty_{n-i}(\mathbb{R}^n)$ for $0\leq i<n/2$. The convolution $Z*\widetilde{Z}$ goes the other way, and the two products are connected by the Fourier transform $\mathbb{F}:\mathrm{Val}^\infty(\mathbb{R}^n)\to\mathrm{Val}^\infty(\mathbb{R}^n)$ mapping $\mathrm{Val}^\infty_i(\mathbb{R}^n)$ into $\mathrm{Val}^\infty_{n-i}(\mathbb{R}^n)$ and satisfying $\mathbb{F}\circ\mathbb{F}(Z)(K)=Z(-K)$ and $\mathbb{F}(Z\cdot\widetilde{Z})=\mathbb{F}(Z)*\mathbb{F}(\widetilde{Z})$.

Concerning some applications of the theory of valuations on compact convex sets, Hadwiger's Theorem 8.D.2 directly yields the principal kinematic formula, and Bernig, Hug [64] prove very general kinematic formulas based results on valuation. Haberl, Schuster [294] applies results on Minkowski valuations to obtain new Sobolev type inequalities. In addition, Aleksandrov-Fenchel-type inequalities are verified by Alesker [10] and Kotrbatý, Wannerer [384, 385] based on Kotrbatý's Hodge-Riemann relations on valuations in [383].

We note that other important families of valuations that are defined on polytopes (see Alesker [9] and Böröczky, Ludwig [109] for classical results, and for example Haberl, Parapatits [293], Jochemko, Sanyal [357, 358] for recent advances), on polyhedra (see e.g. Barvinok [57]), on lattice polytopes (see Böröczky, Ludwig [109]), on fans of polyhedra (see Backman, Manecke, Sanyal [32]), on hyperplanes (see Gates, Hug, Schneider [262]), on function spaces (see Ludwig [427] or Colesanti, Ludwig, Mussnig [170]), and even on manifolds (see Alesker [8] and Alesker, Bernig [11]).

Chapter 9

The Minkowski Problem, the L_p-Minkowski problem, and the L_p-Brunn-Minkowski inequality/conjecture

9.1 Monge-Ampère equations in $\mathbb{R}^d$

In this section, we collect some properties of convex functions (see Section 10.6) that are related to Monge-Ampère equations. Our main references are Figalli [222] and Trudinger, Wang [553]. For the whole section, let $\Omega \subset \mathbb{R}^d$, $d \geq 1$, be a convex, bounded, open set, and let $\varphi : \Omega \to \mathbb{R}$ be a convex function. We consider three key notions:

Subdifferential: $\partial\varphi(x) = \{z \in \mathbb{R}^d : \varphi(y) \geq \varphi(x) + \langle z, y - x\rangle \text{ for } y \in \Omega\}$ for $x \in \Omega$, which is a compact convex set.

Monge-Ampeȑe measure: If $\omega \subset \Omega$ is a Borel set, then

$$\mu_\varphi(\omega) = \mathcal{H}^d\left(\bigcup_{x\in\omega} \partial\varphi(x)\right). \tag{9.1}$$

If φ is C^2, then

$$\mu_\varphi(\omega) = \int_\omega \det D^2\varphi \, d\mathcal{H}^d.$$

Monge-Ampère equation: For given finite Borel measure μ on Ω, find convex φ on Ω such that

$$\mu_\varphi = \mu \quad (\mu_\varphi \text{ is the solution Aleksandrov's sense, or in the sense of measure})$$

$$\det D^2\varphi = f \quad \text{if } d\mu = f\, d\mathcal{H}^n. \tag{9.2}$$

The regularity of the solution of the Minkowski problem (9.2) was intensively investigated by Nirenberg [476], Cheng, Yau [160] and Pogorelov [490] in the middle of the 20th century, and finally, Caffarelli [135, 136] settled this issue around 1990. The first step is in Caffarelli [135] where for a convex, compact $K \subset \mathbb{R}^n$, an $x \in K$ is called an extreme point if $x = (1-\lambda)y + \lambda z$ for $y, z \in K$ and $\lambda \in (0,1)$ implies $y = z = x$ (cf. Definition 1.6.4 and Lemma 1.6.5).

Theorem 9.1.1 (Caffarelli). *For $\lambda_2 > \lambda_1 > 0$ and convex open bounded $\Omega \subset \mathbb{R}^d$, $d \geq 1$, let convex $\varphi : \Omega \to \mathbb{R}$ satisfy*

$$\lambda_1 \leq \det D^2\varphi \leq \lambda_2 \tag{9.3}$$

in the sense of measure; namely, $\lambda_1\mathcal{H}^d(\omega) \leq \mu_\varphi(\omega) \leq \lambda_2\mathcal{H}^d(\omega)$ for Borel for $\omega \subset \Omega$.

(i) *If $\varphi(y) \geq 0$ for all $y \in \Omega$ and $S = \{y \in \Omega : \varphi(y) = 0\}$ is not a point, then S (more precisely,* cl S*) has no extreme point in Ω.*

(ii) *If φ is strictly convex, then φ is C^1.*

Remark. We note that (i) does not rule out the possibility that S is an open segment whose endpoints are in $\partial\Omega$.

Caffarelli [136] deals with the case when the Monge-Ampére measure has a Hölder continuous density functions (see Theorem 9.1.3).

Definition 9.1.2 (Hölder continuity). For $\alpha \in (0, 1]$ and $Z \subset \mathbb{R}^d$, a function $f : Z \to \mathbb{R}$ is $C^{0,\alpha}$, if there exists $c > 0$ such that $|f(x) - f(y)| \leq c \cdot \|x - y\|^\alpha$ for $x, y \in Z$, and f is locally $C^{0,\alpha}$, if each $z \in Z$ has a neighborhood where f is $C^{0,\alpha}$ (and the implied constant depends on the neighborhood).

In addition, for open $U \subset \mathbb{R}^d$ and integer $m \geq 1$, a function $h : U \to \mathbb{R}$ is $C^{m,\alpha}$ if h is C^m, and each partial derivative of h of order m is $C^{0,\alpha}$.

Remark. The $C^{0,1}$ functions are the Lipschitz functions.

Theorem 9.1.3 (Caffarelli). *For an open bounded convex $\Omega \subset \mathbb{R}^d$, $d \geq 1$, let the functions φ, f on Ω satisfy that φ is strictly convex, $\lambda^{-1} \leq f \leq \lambda$ for a constant $\lambda > 1$, and*

$$\det D^2\varphi = f$$

in the sense of measure.

(i) *If f is continuous, then φ is locally $C^{1,\alpha}$ for any $\alpha \in (0, 1)$; and in general, if f is C^k for an integer $k \geq 0$, then φ is locally $C^{k+1,\alpha}$ for any $\alpha \in (0, 1)$.*

(ii) *If f is locally $C^{k,\alpha}$ in Ω for some $\alpha \in (0, 1)$ and integer $k \geq 0$, then φ is locally $C^{k+2,\alpha}$.*

Remark. Instead of (i), Caffarelli [136] actually proves that if f is continuous, then for any open ball B whose closure is in Ω, φ is in the Sobolev space $W^{2,l}(B)$ for any $l > n$. Since $W^{2,l}(B) \subset C^{1,\alpha}(B)$ if $\frac{n}{l} = 1 - \alpha$ according the Sobolev Embedding Theorem (cf. Demengel, Demengel [188]), φ is locally $C^{1,\alpha}$ for any $\alpha \in (0, 1)$.

9.2 The Minkowski Problem

This section discusses the classical Minkowski problem dating back to around 1900. We use several notions introduced in Section 8.1 and Section 8.2, like the curvature function f_K and the surface area measure S_K on S^{n-1}, etc.

Remark 9.2.1 (Minkowski Problem). Characterize a finite non-trivial Borel measure μ on S^{n-1} such that

$$\mu = S_K \tag{9.4}$$

for a convex body $K \subset \mathbb{R}^n$, and characterize the case when $S_K = S_C$ for another convex body $C \subset \mathbb{R}^n$. If $dS_K = f_K \, d\mathcal{H}^{n-1}$, then determine how smooth K has to be under some smoothness assumptions on f_K.

Let us discusses the results in Section 8.1 and Section 8.2 from the point of view of the Minkowski problem. For any Borel set $\omega \subset S^{n-1}$,

$$S_K(\omega) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} F(K,u)\big) = \mathcal{H}^{n-1}\big(\cup_{u\in\omega} \partial h_K(u)\big);$$

therefore, S_K is the "Monge-Ampère measure" for the restriction $h = h_K|_{S^{n-1}}$. On the other hand, for any C^2 function $h: S^{n-1} \to \mathbb{R}$, there exists a convex body $K \subset \mathbb{R}^n$ (or a point) such that $h = h_K|_{S^{n-1}}$ if and only if $\nabla^2 h + hI_{n-1}$ is positive semidefinite where ∇ (or ∇^2) is the covariant differentiation (Hessian) on S^{n-1}. Here $o \in K$ ($o \in \mathrm{int}K$) if and only if $h \geq 0$ ($h > 0$) on S^{n-1}.

If ∂K is C^2_+ for a convex body $K \subset \mathbb{R}^n$, $x \in \partial K$, $u = \nu_K(x)$, and $\tilde{h} = h_K|_{S^{n-1}}$, then

- $x = Dh_K(u) = \nabla h(u) + h(u)\,u$;
- $dS_K = \det\left(\nabla^2 h + hI_{n-1}\right)\, d\mathcal{H}^{n-1}$;
- the eigenvalues of $\nabla^2 h(u) + h(u)I_{n-1}$ are $\frac{1}{\kappa_1(x)}, \ldots, \frac{1}{\kappa_{n-1}(x)}$ where $\kappa_1(x), \ldots, \kappa_{n-1}(x)$ are the principal curvatures at $x \in \partial K$, and $\det(\nabla^2 h(u) + h(u)I_{n-1}) = f_K(u) = \frac{1}{\kappa(x)}$ for the Gaussian curvature $\kappa(x) = \prod_{i=1}^{n-1} \kappa_i(x)$ and curvature function f_K.

Minkowski Problem as a Monge-Ampère equation on S^{n-1}: Given a measurable function $f: S^{n-1} \to [0,\infty)$, find $h: S^{n-1} \to \mathbb{R}$ such that

$$\det(\nabla^2 h + hI_{n-1}) = f \tag{9.5}$$

Remark. We search for an h that is the restriction of a convex 1-homogeneous function on $\mathbb{R}^n$. The Minkowski Problem Remark 9.2.1 is the version of (9.5) in sense of measure (or Aleksandrov solution).

Our proof of Theorem 9.2.3 is based on the variational method, and an essential tool is the Aleksandrov lemma Theorem 7.5.2 that we now recall:

Lemma 9.2.2 (Aleksandrov Lemma for Wulff shapes). *Let $\Omega \subset S^{n-1}$ be closed set not contained in a closed hemisphere, let $\varphi: \Omega \times (-t_0, t_0) \to (0,\infty)$, $t_0 > 0$, be continuous such that*

$$\lim_{t\to 0} \frac{\varphi(u,t) - \varphi(u,0)}{t} = \partial_t \varphi(u,0)$$

exists uniformly in $u \in \Omega$ *where* $\partial_t \varphi(u,0)$ *is continuous on* Ω. *For* $t \in (-t_0, t_0)$, *the Wulff shape* $K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq \varphi(u,t)$ *for* $u \in S^{n-1}\}$ *satisfies that*

$$\lim_{t\to 0} \frac{|K_t| - |K_0|}{t} = \int_{S^{n-1}} \partial_t \varphi(u,0)\, dS_{K_0}(u).$$

We will use the notion of a spherical cap. For $u \in S^{n-1}$ and $\delta \in [0,1)$, an open spherical cap centered at u is

$$\Omega(u,\delta) = \{v \in S^{n-1} : \langle v, u\rangle > \delta\} \tag{9.6}$$

where $\Omega(u,0)$ is an open hemisphere. The solution of the Minkowski problem about characterization of the surface area measure is due to Minkowski [464, 465] around 1900 if the measure is discrete (the case of polytopes) or absolutely continuous with positive C^∞ density function, and to Aleksandrov [4, 7] in 1938 in general.

Theorem 9.2.3 (Minkowski problem). *For a finite Borel measure* μ *on* S^{n-1}, *there exists a convex body* K *with* $\mu = S_K$ *if and only if*

(a) supp μ *intersects each open hemisphere of* S^{n-1};

(b) $\displaystyle\int_{S^{n-1}} u\, d\mu(u) = o \in \mathbb{R}^n$.

In addition, $S_K = S_C$ *if and only if* K *and* C *are translates.*

Proof. The necessity of the conditions (a) and (b) have been proved in Lemma 2.5.6 and Lemma 2.5.7. The uniqueness of the surface area measure has been characterized in Theorem 2.5.11.

For the most involved part of the proof, for the sufficiency of the conditions in Theorem 9.2.3, let μ be a finite Borel measure on S^{n-1} satisfying (a) and (b). The main idea comes from the fact that if $S_M = \mu$ for a convex body M, then the form (2.24) or (7.28) of the Minkowski inequality says that $\int_{S^{n-1}} h_C\, d\mu \geq \int_{S^{n-1}} h_M\, d\mu$ for any convex body C with $|C| = |M|$; or in other words, $C = M$ minimizes the integral $\int_{S^{n-1}} h_C\, d\mu$ over all convex bodies C with $|C| = |M|$.

Let $\mathcal{C}$ be the set of convex bodies C with $o \in C$ and $|C| = 1$, and we consider $\inf \int_{S^{n-1}} h_C\, d\mu$ for $C \in \mathcal{C}$. Let $C_m \in \mathcal{C}$ such that

$$\lim_{m\to\infty} \int_{S^{n-1}} h_{C_m}\, d\mu = \inf\left\{\int_{S^{n-1}} h_C\, d\mu : C \in \mathcal{C}\right\}.$$

We claim that $\{C_m\}$ is bounded. Otherwise (proving the claim indirectly), there exist $R_m > 0$ with $\lim_{m\to\infty} R_m = \infty$ and $u_m \in S^{n-1}$ such that $R_m u_m \in C_m$. We may assume that $u_m \to u \in S^{n-1}$. Now (a) yields the existence of $\delta \in (0,1)$ such that $q = \mu(\Omega(u,\delta)) > 0$. In turn, there exists threshold $m_0 > 0$ such that if $m > m_0$, then $\Omega(u,\delta) \subset \Omega(u_m, \delta/2)$, and hence $\mu(\Omega(u_m, \delta/2)) \geq q$. It follows that

$$\begin{aligned}\lim_{m\to\infty}\int_{S^{n-1}} h_{C_m}\,d\mu &\geq \lim_{m\to\infty}\int_{\Omega(u_m,\delta/2)} \langle R_m u_m, v\rangle\, d\mu(v)\\ &\geq \lim_{m\to\infty}\int_{\Omega(u_m,\delta/2)} \frac{R_m\delta}{2}\,d\mu(v) \geq \lim_{m\to\infty}\frac{R_m\delta}{2}\cdot q = \infty.\end{aligned}$$

This contradicts the definition of $\{C_m\}$; therefore, $\{C_m\}$ is bounded.

Since $\{C_m\}$ is bounded, we may assume that C_m tends to convex compact set K by the Blaschke Selection Theorem 1.7.3. As each $|C_m| = 1$, we have $|K| = 1$ by the continuity of volume (cf. Lemma 1.7.4), thus $K \in \mathcal{C}$; therefore,

$$\int_{S^{n-1}} h_K\,d\mu = \min\left\{\int_{S^{n-1}} h_C\,d\mu : C \in \mathcal{C}\right\}. \tag{9.7}$$

We claim that

$$\mu = \lambda \cdot S_K \quad \text{for } \lambda = \frac{1}{n}\int_{S^{n-1}} h_K\,d\mu, \tag{9.8}$$

where (9.8) is equivalent with saying that

$$\int_{S^{n-1}} g\,d\mu = \lambda\cdot\int_{S^{n-1}} g\,dS_K \tag{9.9}$$

for any continuous function $g:\ S^{n-1}\to\mathbb{R}$. For $t \geq 0$, we consider the Wulff-shape

$$K_t = \{x\in\mathbb{R}^n : \langle x,u\rangle \leq h_K(u) + tg(u)\ \forall u\in S^{n-1}\},$$

and hence $K_0 = K$, and the Aleksandrov Lemma 9.2.2 that

$$\left.\frac{d}{dt}|K_t|\right|_{t=0} = \int_{S^{n-1}} g\,dS_K. \tag{9.10}$$

If $|t|$ is small, then $|K_t|^{\frac{-1}{n}}\cdot K_t \in \mathcal{C}$, and we deduce from $h_{K_t} \leq h_K(u) + tg(u)$ and from (9.7) that

$$f(t) = \log\int_{S^{n-1}}(h_K + tg)\,d\mu - \frac{1}{n}\log|K_t| \geq \log\frac{\int_{S^{n-1}} h_{K_t}\,d\mu}{|K_t|^{\frac{1}{n}}} \geq f(0).$$

In particular, the differentiable function f has a minimum at $t = 0$, and hence (9.10) implies that

$$0 = f'(0) = \lambda^{-1}\int_{S^{n-1}} g\,d\mu - \frac{1}{n}\int_{S^{n-1}} g\,dS_K,$$

proving (9.9), and in turn (9.8). Therefore, $\mu = S_M$ for $M = \lambda^{\frac{1}{n-1}}K$. ∎

Actually, uniqueness in the Minkowski inequality $\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_K\, dS_K$ for convex bodies $K, C \subset \mathbb{R}^n$ up to translation is equivalent with the uniqueness of the solution of the Minkowski Problem (9.4) up to translation:

Remark 9.2.4 (The Minkowski inequality and the uniqueness of the solution of the Minkowski Problem). The uniqueness of the surface area measure has been characterized in Theorem 2.5.11 based on uniqueness in the Minkowski inequality.

On the other hand, let us asssume that we know the uniqueness solution of the solution of the Minkowski Problem (9.4) for $\mu = S_K$ for a convex body $K \subset \mathbb{R}^n$ up to translation. The proof of Theorem 9.2.3 taking $\mu = S_K$ for a convex body $K \subset \mathbb{R}^n$ shows that $\int_{S^{n-1}} h_C\, dS_K$ among convex bodies C with $|C| = |K|$ attains its minimum at a $C = \widetilde{C}$ and $S_{\widetilde{C}} = S_K$ (the latter follows by the variational argument). Now the uniqueness of the solution of the Minkowski Problem (9.4) implies that $\widetilde{C} = K + z$ for a $z \in \mathbb{R}^n$; therefore,

$$\int_{S^{n-1}} h_C\, dS_K \geq \int_{S^{n-1}} h_{\widetilde{C}}\, dS_K = \int_{S^{n-1}} h_K\, dS_K$$

for any convex body $C \subset \mathbb{R}^n$ with $|C| = |K|$.

In order to apply the results discussed in Section 9.1 to the regularity of the solution of the Minkowski problem on S^{n-1}, we need to transfer the local equation into an equation on a convex set in $\mathbb{R}^{n-1}$.

Remark 9.2.5 (Transferring Monge-Ampére equation on S^{n-1} to Monge-Ampére in $\mathbb{R}^{n-1}$). Let $e \in S^{n-1}$, and let $h = h_K|_{S^{n-1}}$ be a solution of the Monge-Ampère equation (9.5) for a convex body $K \subset \mathbb{R}^n$. Setting $\varphi(y) = h_K(y + e)$ for $y \in e^\perp$, (8.8) implies that

$$\det D^2\varphi(y) = g(y) \;\text{ got } y \in e^\perp \tag{9.11}$$

in the sense of measure where for $y \in e^\perp$, we have

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+1}{2}} \cdot f\left(\frac{e + y}{\sqrt{1 + \|y\|^2}}\right).$$

Combining (9.11) with the results in Section 9.1 mostly due to Caffarelli [135,136] imply the differentiability of the solution of the Minkowski problem in the case of Hölder continuous density function (cf. Definition 9.1.2).

Theorem 9.2.6 (Caffarelli). *If $\int_{S^{n-1}} u f(u)\, du = o$ for a $C^{0,\alpha}$ function $f : S^{n-1} \to (0, \infty)$ for $\alpha \in (0, 1)$, then any convex body K with $dS_K(u) = f(u)\, du$ has $C^{2,\alpha}_+$ boundary, h_K is $C^{2,\alpha}$ on $\mathbb{R}^n \backslash \{o\}$, and $f = f_K$ is the curvature function.*

If in addition, if f is $C^{m,\alpha}$ for an integer $m \geq 1$, then ∂K and h_K are $C^{m+2,\alpha}$.

Proof. We may assume that $o \in \operatorname{int} K$.

First we show that ∂K is C^1. The argument is indirect, we suppose that there exists a $z \in \partial K$ such that $\dim N_K(z) \geq 2$ for the normal cone $N_K(z)$ at z, and seek a contradiction (cf. Section 1.5). For a $w \in \mathbb{R}^n$ such that $z - w \in \operatorname{int} K$, $(w + w^\perp) \cap N_K(z)$ is a compact convex set of dimension at least 1, and let $\tilde{e}$ be an extreme point of this set (cf. Lemma 1.6.5). It follows that $e = \tilde{e}/\|\tilde{e}\| \in S^{n-1}$ is an extreme point of the compact, convex set $S' = N_K(z) \cap (e + e^\perp) \cap (e + B^n)$ of dimension at least 1, and hence o is an extreme point of $S = S' - e$.

We deduce from Remark 9.2.5 that $\tilde{\varphi}(y) = h_K(y + e)$ for $y \in e^\perp$ satisfies the Monge-Ampère equation

$$\det D^2 \tilde{\varphi}(y) = g(y)$$

for a positive continuous function g on $e^\perp$. It follows from the definition of a support function that $\tilde{\varphi}(y) = h_K(y + e) \geq \langle y + e, z\rangle$ for $y \in e^\perp$, and $\tilde{\varphi}(y) = \langle y + e, z\rangle$ if and only if $y + e \in N_K(z)$. We deduce that $\varphi(y) = tilde\varphi(y) - \langle y + e, z\rangle \geq 0$ is convex on $\Omega = e^\perp \cap \operatorname{int} B^n$,

$$\det D^2 \varphi(y) = g(y) \tag{9.12}$$

for the function g on Ω such that $\lambda^{-1} < g(y) < \lambda$ for a constant $\lambda > 1$, and

$$\operatorname{cl}\{y \in \Omega : \varphi(y) = 0\} = S.$$

This contradicts Theorem 9.1.1 as o is an extreme point of S; therefore, ∂K is C^1.

Next we show that h_K is $C^{0,\alpha}$ near any $e \in S^{n-1}$ (here we redefine the notions from the previous paragraph). As ∂K is C^1, we deduce from Lemma 1.9.6 that the function $\varphi(y) = h_K(y + e)$ for $y \in e^\perp$ is strictly convex, and it satisfies (9.12) for a function g on $\Omega = e^\perp \cap \operatorname{int} B^n$ that extends to a positive $C^{0,\alpha}$ function on $e^\perp \cap B^n$ according to Remark 9.2.5. Therefore, Theorem 9.1.3 yields that φ on Ω, and in turn h_K on $e + \operatorname{int} B^n$ is locally $C^{2,\alpha}$. ∎

Given a measure $d\mu = \varphi\, d\mathcal{H}^n$ for a continuous density function φ on $\mathbb{R}^n$, Livshyts [418] defined the *weighted surface area measure* $S_{\mu,K}$ of a convex body $K \subset \mathbb{R}^n$ in a way such that

$$S_{\mu,K}(\omega) = \int_{\nu_K^{-1}(\omega)} \varphi\, \mathcal{H}^{n-1}$$

for a measurable $\omega \subset S^{n-1}$ (note that this notion is unrelated to the L_p surface area measure $S_{K,p}$, $p \in \mathbb{R}$, discussed in Section 9.3). In addition, Livshyts [418] proved the variational formula

$$\lim_{\varrho \to 0^+} \frac{\mu(K + \varrho C) - \mu(K)}{\varrho} = \int_{S^{n-1}} h_C\, dS_{\mu,K}$$

for any convex body $C \subset \mathbb{R}^n$ with $o \in \operatorname{int} C$. The Minkowski problem for the weighted surface area measure is considered in various settings by Livshyts [418], Kryvonos, Langharst [387] and Fradelizi, Langharst, Madiman, Zvavitch [246].

9.3 The L_p-Minkowski problem

In this section, we summarize some major results and conjectures concerning the L_p-Minkowski problem initiated by Erwin Lutwak in the 1990's (starting with the case $p > 1$ in Lutwak [433]) that extends the classical Minkowski problem ($p = 1$, cf. Section 9.2), and Firey's [233] Logarithmic Minkowski problem about the cone volume measure when $p = 0$. In the case $p \geq 0$, more detailed discussions about these topics are provided in Sections 9.A, 9.B and 9.C because the uniqueness of the solution of a Monge-Ampère equation is connected to Brunn-Minkowski type inequalities in this case. Since convex bodies in the rest of the chapter contain the origin, let $\mathcal{K}_o^n$ and $\mathcal{K}_{(o)}^n$ denote the family of convex bodies $K \subset \mathbb{R}^n$ such that $o \in K$ or $o \in \operatorname{int} K$, respectively.

Definition 9.3.1 (L_p-surface area measure $S_{K,p}$). For $p \in \mathbb{R}$, and a convex body $K \in \mathcal{K}_o^n$, the L_p-surface area measure $S_{K,p}$ is the Borel measure

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

on S^{n-1} where in the case when $p > 1$ and $o \in \partial K$, we assume that $S_K(\{u \in S^{n-1} : h_K(u) = 0\}) = 0$; or equivalently, $S_K(N_K(o) \cap S^{n-1}) = 0$. This last condition we typically write as $S_K(\{h_K = 0\}) = 0$.

Let us list some basic properties that directly follow from the definition where $p \in \mathbb{R}$ and $K \in \mathcal{K}_o$, and we also assume $S_K(\{h_K = 0\}) = 0$ if $p > 1$ and $o \in \partial K$:

- $\Phi_* S_{K,p} = S_{\Phi K,p}$ for $\Phi \in O(n)$; namely, $S_{K,p}$ is equivariant under orthogonal transformations.
- If $g : S^{n-1} \to \mathbb{R}$ Borel, then (cf. (2.15))
$$\int_{S^{n-1}} g\, dS_{K,p} = \int_{\partial' K} g(\nu_K(x)) \cdot \langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x) \tag{9.13}$$
where the integral makes sense even if $p > 1$ and $o \in \partial K$ as in this case, the conditions $S_K(\{h_K = 0\}) = 0$ and $\mathcal{H}^{n-1}(\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\}) = 0$ are equivalent.
- $S_{K,p}$ is a finite measure if $o \in \operatorname{int} K$ or $p \leq 1$. In the case $p > 1$ and $o \in \partial K$, $S_{K,p}(S^{n-1})$ might be infinite even if $S_K(\{h_K = 0\}) = 0$.
- If $\lambda > 0$, then
$$S_{\lambda K,p} = \lambda^{n-p} \cdot S_{K,p}. \tag{9.14}$$
- $S_{K,p}$ is weakly continuous for $K \in \mathcal{K}_{(o)}$, and even for $K \in \mathcal{K}_o$ if $p \leq 1$.

For the last property, if K_m tends to K for $K_m, K \in \mathcal{K}_o$, then S_{K_m} tends weakly to S_K according to Proposition 2.6.12 (see also Proposition 8.4.1). On the other hand, $h_{K_m}^{1-p}$ tends uniformly to h_K^{1-p} either if $p \leq 1$, or if $p > 1$ and $K_m, K \in \mathcal{K}_{(o)}$ (see Lemma 9.A.1 for a statement in the case $p > 1$, $K_m \in \mathcal{K}_{(o)}$ and $o \in \partial K$).

Example 9.3.2 (Some fundamental L_p surface area measures).

$p = 1$: $S_{K,1} = S_K$ (surface area measure, cf. Sections 2.5 and 8.2):

$$S_{K+z} = S_K \text{ for } z \in \mathbb{R}^n.$$

$p = 0$: $S_{K,0} = nV_K$ (V_K cone volume measure, cf. Section 2.6):
If $\omega \subset S^{n-1}$ Borel set and $\Phi \in \mathrm{GL}(n)$ with $\det \Phi = \pm 1$, then

$$S_{K,0}(\omega) = S_{\Phi K,0}\left(\left\{\frac{\Phi^{-t}u}{\|\Phi^{-t}u\|} : u \in \omega\right\}\right).$$

$p = -n$: If ∂K is C^2_+, and $\nu_K(x(u)) = u$ for $u \in S^{n-1}$ and $x(u) \in \partial K$, then

$$dS_{K,-n}(u) = \kappa_0(K, x(u))^{-1}\, d\mathcal{H}^{n-1}(u)$$

where $\kappa_0(K,x) = \frac{\kappa(x)}{\langle x, \nu_K(x)\rangle^{n+1}} = \kappa_0(\Phi K, \Phi x)$ if $\Phi \in \mathrm{GL}(n)$ with $\det \Phi = \pm 1$ (cf. Section 8.9.2). In particular, characterizing $S_{K,-n}$ is equivalent with characterizing the centro-affine curvature $\kappa_0(K, x(u))$.

The main question addressed by this section is the following version of the Minkowski problem proposed by Lutwak [433] in 1993.

Remark 9.3.3 (L_p-Minkowski Problem). Let $p \in \mathbb{R}$ and $n \geq 2$.

Monge-Ampère equation on S^{n-1}*:* Given measurable $f : S^{n-1} \to [0, \infty)$, find suitably differentiable $h : S^{n-1} \to [0, \infty)$ such that

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \qquad \text{if } p \geq 1; \tag{9.15}$$

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = f \qquad \text{if } p \leq 1. \tag{9.16}$$

In the sense of measure:

- $h = h_K|_{S^{n-1}}$ for $K \in \mathcal{K}^n_o$ is an *Aleksandrov solution* of (9.15) or (9.16) if

$$dS_{K,p} = f\, d\mathcal{H}^{n-1} \tag{9.17}$$

  where $S_K(\{h_K = 0\}) = 0$ if $p > 1$.
- For $p \in \mathbb{R}$, characterize L_p-surface area measure $S_{K,p}$ of a $K \in \mathcal{K}^n_o$ as a Borel measure on S^{n-1} where $S_K(\{h_K = 0\}) = 0$ if $p > 1$.

In particular, the L_p-Minkowski Problem is just the classical Minkowski problem if $p = 1$ (cf. Section 9.2), and the Logarithmic Minkowski problem about the cone volume measure posed by Firey [233] in 1974 if $p = 0$.

Let us list the known not too technical existence results about the solution of the L_p-Minkowski Problem without symmetry assumption. If $p = n$, then no scaling is possible ($S_{K,p} = S_{\lambda K,p}$ for $\lambda > 0$, cf. (9.14)); therefore, the solution is only known up to homothety (cf. Hug, Lutwak, Yang, Zhang [339]). Out of the three fundamental cases; namely, $p = 1, 0, -n$, we exclude the classical $p = 1$ case because it is discussed in Section 9.2:

Remark 9.3.4 (Some known results about the L_p-Minkowski Problem).

$p > 0$, $p \neq 1, n$: If μ is a non-trivial finite Borel measure on S^{n-1} not concentrated on any closed hemi-sphere, then $\mu = S_{K,p}$ for convex body $K \in \mathcal{K}_o^n$ where $S_K(\{h_K = 0\}) = 0$ if $p > 1$ (cf. Theorem 9.C.1).
This result is due to Chou, Wang [162] and Hug, Lutwak, Yang, Zhang [339] if $p > 1$, and to Chen, Li, Zhu [156] if $0 < p < 1$ (cf. Section 9.C).

$p = 0$ (Logarithmic Minkowski Problem): If μ is a non-trivial finite Borel measure on S^{n-1} such that

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \quad \text{for any proper linear subspace } L \subset \mathbb{R}^n, \tag{9.18}$$

then $\mu = S_{K,0} = nV_K$ for a convex body $K \in \mathcal{K}_o^n$ (e.g. any absolutely continuous measure is a cone volume measure). This result, proved as Theorem 9.C.1, is due to Chen, Li, Zhu [157]. Actually, Chen, Li, Zhu [157] also prove that the conditions (i) and (ii) in Theorem 9.3.6 are sufficient (the argument is similar to the one for Theorem 9.B.5 in the even case).
On the other hand, there are some non-trivial obstructions for a finite Borel measure on S^{n-1} to be a cone volume measure. For example, for any convex body $K \in \mathcal{K}_o^n$ and $u \in S^{n-1}$, Böröczky, Hegedűs [102] (cf. (2.35)) prove that

$$V_K(u) + V_K(-u) + 2(n-1)\sqrt{V_K(u)V_K(-u)} \leq V_K(S^{n-1})(= |K|). \tag{9.19}$$

$-n < p < 0$: If $d\mu = f\, d\mathcal{H}^{n-1}$ is a non-trivial measure for a non-negative function $f \in L^{\frac{n}{n+p}}(S^{n-1}, \mathcal{H}^{n-1})$, then $\mu = S_{K,p}$ for a convex body $K \in \mathcal{K}_o^n$, see Bianchi, Böröczky, Colesanti, Yang [69].

$p = -n$: The $p = -n$ case of the L_p-Minkowski problem is the critical case because its link with the $\mathrm{SL}(n)$ invariant century old notion of centro-affine curvature. If $K \in \mathcal{K}_{(o)}^n$ has C_+^2 boundary, then $dS_{K,-n}(u) = \kappa_0(x(u))^{-1}\, du$ where $\nu_K(x(u)) = u$ for $u \in S^{n-1}$ and $x(u) \in \partial K$, and $\kappa_0(K, x(u)) = \kappa_K(x(u))/h_K(u)^{n+1}$ is the $\mathrm{SL}(n)$ invariant centro-affine curvature (cf. Proposition 8.9.3). This case is far the most mysterious, hardly anything is known.
If the f in (9.16) is unconditional and satisfies certain additional technical conditions, then Jian, Lu, Zhu [356] verify the existence of a solution. Moreover, the paper Guang, Li, Wang [289] solves a variant of the centro-affine Minkowki problem. Stancu [539] links the centro-affine Minkowki problem to the logarithmic Minkowski problem (when $p = 0$).
On the other hand, Du [194] constructs the first explicit example of a positive $C^{0,\alpha}$, $\alpha \in (0, 1)$, function f on S^{n-1} such that (9.16) has no solution when $p = -n$, and

Chou, Wang [162] proved an implicit condition on possible functions f in (9.16) (see also [69]).

$p < -n$: In this super-critical case, if μ is an absolutely continuous measure on S^{n-1} with positive continuous density, then $\mu = S_{K,p}$ for a $K \in \mathcal{K}^n_{(o)}$ according to the groundbreaking paper Li, Guang, Wang [288] where $h_K|_{S^{n-1}}$ is $C^{1,\alpha}$.
On the other hand, for $p < -n$, Du [193] constructs a non-negative $C^{0,\alpha}$, $\alpha \in (0,1)$, function f on S^1 that is positive everywhere but a fixed pair of antipodal points and the L_p Minkowski problem (9.16) has no solution, not even in the Aleksandrov sense (see Du [193] for the case $n = 2$).

Both the variational and the flow methods - the two main methods leading to the results above - are reviewed in Remark 9.3.7.

Concerning the super-critical case $p < -n$ open for many decades in spite of intense research, it is not surprising that the flow method is the one succeeding, as E. Milman [459] points out the limitations of the variational argument in this case. The groundbreaking paper Li, Guang, Wang [288] solves the problem by introducing a whole new approach, mixing the flow method with homology theory to find the right initial condition. When $n = 2$, the case $p \le -2 = -n$ is further investigated by Ivaki [346] and Yang, Liu, Fang [576].

Let μ be a finite Borel measure on S^{n-1} not concentrated on any closed hemisphere. If $K_{\mu,p} \in \mathcal{K}^n_o$ satisfies $\mu = S_{K_p,p}$ for $p > n$, then Zou [583] proves that

$$\lim_{p\to\infty} K_{\mu,p} = K_\mu = \{x \in \mathbb{R}^n : \langle x, u\rangle \le 1,\ \forall u \in \operatorname{supp}\mu\};$$

for example, $h_{K_p}|_{S^{n-1}}$ tends uniformly to the constant 1 function as p tends to infinity if μ has a positive density function f in (9.15).

A general sufficiency condition for all $p \le 0$ (see Theorem 9.C.2) is due to Zhu [581]: if μ is a discrete measure on S^{n-1} such that $\operatorname{supp}\mu$ is not contained in a closed hemisphere and any n elements of $\operatorname{supp}\mu$ are independent, then there exists a polytope $Q \in \mathcal{K}^n_{(o)}$ such that $\mu = S_{Q,p}$. However, this result does seem to be useful to construct solutions for more general measures if $p < 0$ because no method is known how to control the diameter of the solution in the case of weak approximation by such discrete measures.

Remark 9.3.5 (Open problems).

$p \le 0$: For $p \le 0$, not even a conjecture is known for any $p \le 0$. Probably, the most challenging case is when $p = -n$.

$p \in (0,1)$ *and* $\operatorname{supp} S_{K,p}$ *is lower dimensional:* Let $p \in (0,1)$, and for a non-trivial finite Borel measure μ on S^{n-1}, let $L = \operatorname{lin}\operatorname{supp}\mu$. If $n = 2$, then Böröczky, Trinh [119] and Chen, Li, Zhu [156] prove that $\mu = S_{K,p}$ for $K \in \mathcal{K}^2_o$ if and only if $\operatorname{supp}\mu$ is not a pair of antipodal points; or in other words, $\dim L \ne 1$. However, a full characterization of the L_p surface area measure under the condition $\dim L \le n-1$ is still not known. So let $\dim L \le n-1$. If $\operatorname{supp}\mu \subset L$ is contained in a closed hemisphere centered at a point of $L \cap S^{n-1}$, then μ is an L_p surface area measure according to Bianchi, Böröczky, Colesanti, Yang [69]. On the other hand, Saroglou [511] proved that if $\mu(\omega)$ is the $\dim L$-dimensional Lebesgue measure of $\omega \cap L$ for any Borel $\omega \subset S^{n-1}$, then μ is not an L_p surface area measure.

We shortly discuss the L_0 (Logarithmic) Minkowski Problem in more details because this has been very intensely investigated the last decade. The even case has been completely characterized by Böröczky, Lutwak, Yang, Zhang [111] (cf. Theorem 9.B.5).

Theorem 9.3.6. *For a non-trivial finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = S_{K,0} = nV_K$ if and only if*

(i) $\mu(L \cap S^{n-1}) \le \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any proper linear subspace $L \subset \mathbb{R}^n$;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with* $\operatorname{supp}\mu \subset L \cup L'$*, and in this case, $K = C + C'$ for o-symmetric compact convex sets $C \subset L^\perp$ and $C' \subset L'^\perp$.*

Remarks.

- For any non-trivial finite Borel measure μ on S^{n-1} satisfying (i) and (ii), Chen, Li, Zhu [157] prove that $\mu = S_{K,0} = nV_K$ for a $K \in \mathcal{K}_o^n$.
- The cone volume measure $V_K = \mu$ of any centered convex body $K \subset \mathbb{R}^n$ (the centroid $\sigma_K = o$) satisfies the properties (i) and (ii) according to Böröczky, Henk [104]. In addition, Böröczky, Henk [105] prove a stability version of (ii), and Freyer, Henk, Kipp [248] prove an analoguos property of centered convex polytopes with respect to affine subspaces.
- A Borel probability measure μ on S^{n-1} satisfies the conditions (i) and (ii) if and only if there exists an isotropic linear image $\Phi_*\mu$ for a $\Phi \in \mathrm{GL}(n)$ according to Böröczky, Lutwak, Yang, Zhang [112] where μ is isotropic if $n\int_{S^{n-1}} u \otimes u \, d\mu(u) = \mathrm{Id}_n$, and for any Borel set $\omega \subset S^{n-1}$, we have
$$\Phi_*\mu(\omega) = \mu\left(\left\{\frac{\Phi^{-1}(u)}{\|\Phi^{-1}(u)\|} : u \in \omega\right\}\right).$$

Let us review the main approaches to find a solution of the L_p-Minkowski problem. One approach based on weak approximation by discrete measures if $p \ge 0$ is discussed in more detail in Section 9.C because that approach is the least technical.

Remark 9.3.7 (How to find a solution of the L_p-Minkowski problem).
Overall strategy: For $p \in \mathbb{R}$, a fundamental tool is the entropy of a $K \in \mathcal{K}_o^n$ with respect to a Borel probability measure μ on S^{n-1}. If $p > 1$, then the entropy is

$$\mathcal{E}_{\mu,p}(K) = \frac{1}{p}\int_{S^{n-1}} h_K^p \, d\mu,$$

and if $p < 1$ and $\xi \in \operatorname{int} K$, then the corresponding entropy is

$$\mathcal{E}_{\mu,p}(K,\xi) = \begin{cases} \frac{1}{p} \int_{S^{n-1}} h_{K-\xi}^p \, d\mu & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{K-\xi} \, d\mu & \text{if} \quad p = 0. \end{cases}$$

First one considers special measures, either discrete ones, or ones that are absolutely continuous with a positive continuous density function, and finds a solution with suitably bounded entropy using either the flow method or the variational method in this case. Then the more general measures in Remark 9.3.4 are weakly approximated by the special type of measures, and the entropy bound ensures that the solutions for the approximating special type of measures are bounded for $p > -n$ (cf. Proposition 9.A.2), and hence a subsequence converges to a solution of the original problem.

The flow method for $p < 1$: Here we mostly follow Böröczky, Guan [101] (see Bryan, Ivaki, Scheuer [132] in the even case). Given $p \in (-n, 1)$, and a positive C^∞ probability density function f on S^{n-1}, let $\alpha = \frac{1}{1-p} > \frac{1}{n+1}$, and our aim is to solve the Monge-Ampère equation

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = h^{1/\alpha} \det(\nabla^2 h + hI_{n-1}) = f. \tag{9.20}$$

For a convex body $M_0 \subset \mathbb{R}^n$ with C^∞_+ boundary and $|M_0| = 1$, the theory of anisotropic flows (cf. e.g. Böröczky, Guan [101] for references) provides us a family of convex bodies M_t of C^∞_+ boundary for $t \in [0,T)$ that satisfy the evolution equation

$$\frac{\partial}{\partial t} X(x,t) = -f(\nu)^\alpha \kappa(x,t)^\alpha \, \nu(x,t) \tag{9.21}$$

where for $t \in [0,T)$ and $x \in \partial M_t$, we have $X(x,t) = x$, $\nu = \nu(x,t) = \nu_{M_t}(x)$, and $\kappa(x,t)$ is the Gaussian curvature at $x \in \partial M_t$, and in addition, M_t tends to a point $p \in \operatorname{int} M_0$ as t tends to T. Assuming $p = o$, the dilate $\widetilde{M}_t$ of M_t with $|M_t| = 1$ corresponds to the normalized flow satisfying the evolution equation

$$\frac{\partial}{\partial t} \widetilde{X}(x,t) = -\frac{f(\tilde{\nu})^\alpha \tilde{\kappa}(x,t)^\alpha}{\int_{S^{n-1}} f(u)^\alpha \bar{\kappa}(u,t)^{\alpha-1} \, du} \cdot \tilde{\nu}(x,t) + \widetilde{X}(x,t) \tag{9.22}$$

where for $t \in [0,T)$ and $x \in \partial \widetilde{M}_t$, we have $\widetilde{X}(x,t) = x$, $\tilde{\nu} = \tilde{\nu}(x,t) = \nu_{\widetilde{M}_t}(x)$, and $\tilde{\kappa}(x,t)$ is the Gaussian curvature at $x \in \partial \widetilde{M}_t$; moreover, $\bar{\kappa}(\tilde{\nu}(x,t),t) = \tilde{\kappa}(x,t)$. Now the supremum $\sup_{\xi \in \operatorname{int} \widetilde{M}_t} \mathcal{E}_{\mu,p}(\widetilde{M}_t, \xi)$ of the entropy for the probability measure $d\mu = f \, d\mathcal{H}^{n-1}$ is non-increasing in t. Therefore, as Guan, Ni [285] prove, diameter bounds in terms of the entropy for $p > -n$ (cf. Proposition 9.A.2) yield that $\widetilde{M}_t$ converges to a convex body $K \in \mathcal{K}_o$ as $t \to T$, and $h_K|_{S^{n-1}}$ satisfies (9.20).

When $p = -n$, the argument breaks down because the entropy does not bound the diameter anymore - consider for example centered ellipsoids, whose L_{-n}-surface area is a constant multiple of the Lebesgue measure - and $\widetilde{M}_t$ can actually be unbounded (see Andrews [21]). When $p < -n$, Li, Guang, Wang [288] use homology theory to find a suitable initial condition M_0 to tackle this issue.

Historically, Firey [233] indicated in 1974 that the surface of the worn stone can be modeled by the differential equation (9.21) when f is a suitable constant function and $\alpha = 1$ (and hence $p = 0$), and the support function of the limit shape of the normalized equation (9.22) satisfies (9.20).

The variational method for $p > -n$: When $p \geq 0$, one can find a solution of the L_p-Minkowski problem for a rather general measure μ on S^{n-1} by first using the variational method to find a polytopal solution in the case when μ is a discrete measure (see Section 9.C). The discrete case is due to Hug, Lutwak, Yang, Zhang [339] if $p > 1$, to Zhu [580] if $p \in (0, 1)$, and to Böröczky, Hegedűs, Zhu [103] if $p = 0$.

If $p \in (-n, 1)$ and f is a positive continuous function on S^{n-1}, then the variational method can be also used to solve the Monge-Ampére equation

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = f, \tag{9.23}$$

as it is done by Chou, Wang [162] and Bianchi, Böröczky, Colesanti, Yang [69]. We sketch the main steps. Let $d\mu = f\, d\mathcal{H}^{n-1}$, and let C be the family of convex bodies $C \in \mathcal{K}_o^n$ with $|C| = 1$. For any $C \in C$, there exists a unique $\xi_C \in C$ (depending also on p and μ) such that the entropy $\mathcal{E}_{\mu,p}(K, \xi)$ for $\xi \in C$ is maximized at $\xi = \xi_C$. Using the diameter bounds in Proposition 9.A.2 in terms of entropy, we obtain a $\widetilde{C} \in C$ such that

$$\mathcal{E}_{\mu,p}(\widetilde{C}, \xi) = \min_{C \in C} \mathcal{E}_{\mu,p}(C, \xi), \tag{9.24}$$

such that $h = h_{\widetilde{C} - \xi_{\widetilde{C}}}$ satisfies

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \tag{9.25}$$

in the sense of measure. The path from (9.24) to (9.25) is a variational argument (similar to the one in Theorem 9.2.3 in the case of the classical Minkowski-problem) if $\xi_{\widetilde{C}} \in \operatorname{int} \widetilde{C}$. However, $\xi_{\widetilde{C}}$ may lie in $\partial \widetilde{C}$ if $p > 2 - n$ (see Example 9.3.9); therefore, we need a twist in the argument (see Chou, Wang [162] or Bianchi, Böröczky, Colesanti, Yang [69]). For small $\varepsilon > 0$, we replace the function $t \mapsto \frac{1}{p} t^p$ (or $t \mapsto \log t$ if $p = 0$) in the definition of the entropy by a C^1 strictly concave function $\varphi_\varepsilon(t)$ that coincides with the original function if $t \geq 3\varepsilon$ and $\varphi_\varepsilon(t) = \frac{-1}{|p|} t^{-n}$ (or $\varphi_\varepsilon(t) = -t^{-n}$ if $p = 0$) if $t \in (0, \varepsilon]$. Now the $\widetilde{C}_\varepsilon$ minimizing this modified entropy satisfies that the corresponding "center" (the analogue of $\xi_{\widetilde{C}}$) lies in $\operatorname{int} \widetilde{C}_\varepsilon$, and hence assuming the the corresponding "center" is the origin, a variational argument yields that

$$\det(\nabla^2 h + hI_{n-1}) = \varphi'_\varepsilon(h) f$$

holds for the support function. Finally, letting ε tending to zero, we obtain a common solution of (9.24) and (9.25).

Starting with Haberl, Lutwak, Yang, Zhang [292], Orlicz versions of the L_p-Minkowski problem have been intensively investigated; namely, the function $t \mapsto t^{1-p}$ in the Monge-Ampére equations (9.15) and (9.16) corresponding to the L_p-Minkowski problem is replaced by certain $\varphi : (0,\infty) \to (0,\infty)$, and hence the new Monge-Ampére equations is of the form

$$\varphi(h)\det(\nabla^2 h + h\, I_{n-1}) = f \tag{9.26}$$

where f is a given non-negative function on S^{n-1}. Typically, the solution is only up to a constant factor; namely, there exists some $c > 0$ such that $\varphi(h)\det(\nabla^2 h + h\,\mathrm{Id}) = c \cdot f$. The known existence results about the L_p-Minkowski problem for $p > -n$ have been generalized to the Orlicz L_p-Minkowski problem where $\varphi(t)$ replaces t^{1-p} by Huang, He [329] if $p > 1$ (see also Xie [572], and uniqueness if f is constant in (9.26) is clarified by Ivaki [349]), by Jian, Lu [355] if $p \in (0,1)$, and by Bianchi, Böröczky, Colesanti [67] if $p \in (-n, 0)$.

9.3.1 Positiveness and smoothness in the L_p-Minkowski problem

According to Böröczky, Fodor [99] (correcting a formula in Chou, Wang [162]), the L_p-Minkowski problem - that is a Monge-Ampère equation on S^{n-1} (cf. (9.30) and (9.31)) - can be locally written as a Monge-Ampère equation on $\mathbb{R}^{n-1}$ in a rather natural way using the homogeneity of the support function (cf. (8.8)).

Remark 9.3.8 (Transfering local equation on S^{n-1} to Monge-Ampère in $\mathbb{R}^{n-1}$). Let $p \in \mathbb{R}$ and $e \in S^{n-1}$. Setting $\varphi(y) = h_K(y+e)$ for $y \in e^\perp$, a solution $h = h_K|_{S^{n-1}}$, $K \subset \mathbb{R}^n$ convex body, of the Monge-Ampère equations (9.30) and (9.31), (8.8) implies that

$$\det(D^2\varphi) = \varphi^{p-1} g \qquad \text{if } p \geq 1; \tag{9.27}$$

$$\varphi^{1-p}\det(D^2\varphi) = g \qquad \text{if } p \leq 1. \tag{9.28}$$

in the sense of measure where for $y \in e^\perp$, we have

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+p}{2}} \cdot f\left(\frac{e+y}{\sqrt{1+\|y\|^2}}\right).$$

As it was observed by Hug, Lutwak, Yang, Zhang [339] if $1 < p < n$ and by Bianchi, Böröczky, Colesanti [68] if $2-n < p < 1$, the solution of the L_p-Minkowski problem may not be positive even if the f in (9.15) and (9.16) is positive and $C^{0,\alpha}$.

Example 9.3.9. For $p \in (2-n, n)$, we show that there exists a convex body $K \in \mathcal{K}_o^n$ such that ∂K and $h = h_K|_{S^{n-1}}$ are $C^{1,\alpha}$ for some $\alpha > 0$, $o \in \partial K$ (and hence $h(e) = 0$ for the exterior unit normal e at o), and

$$dS_{K,p} = f\, d\mathcal{H}^{n-1}$$

on S^{n-1} for a positive and $C^{0,\alpha}$ function f.

We fix $e \in S^{n-1}$, and write $x + te = (x, t)$ for $x \in e^\perp$ and $t \in \mathbb{R}$. For

$$q = \frac{2(n-1)}{n+p-2} > 1 \text{ with } \frac{q}{q-1} = \frac{2n-2}{n-p} > 1,$$

we choose the convex body $K \subset \mathbb{R}^n$ in a way such that $(x, -\|x\|^q) \in \partial K$ if $x \in e^\perp \cap B^n$ and $\partial K \backslash \{o\}$ is C_+^2. For $\psi(x) = \|x\|^q$, $x \in e^\perp \cap \operatorname{int} B^n$, we observe that $D\psi(x) = q\|x\|^{q-2}x$ if $x \neq o$ and $D\psi(o) = o$, and one exterior normal to K at $(x, -\|x\|^q) \in \partial K$ is $\tilde{v}(x) = (D\psi(x), 1)$ and

$$h_K(\tilde{v}(x)) = (q-1)\|x\|^q.$$

In particular, writing θ_1, θ_2 to denote positive constants that depend on p and n, if $y = D\psi(x) = q\|x\|^{q-2}x$, then

$$\begin{aligned} \varphi(y) = h_K(y, 1) &= \theta_1 \cdot \|y\|^{\frac{q}{q-1}} = \theta_1 \cdot \|y\|^{\frac{2(n-1)}{n-p}} \\ D^2\varphi(y) &= \theta_2 \cdot \|y\|^{(n-1)(\frac{q}{q-1}-2)} = \theta_2 \cdot \|y\|^{(n-1)\frac{2(p-1)}{n-p}}. \end{aligned}$$

We deduce that φ solves the Monge-Ampère equation

$$\varphi^{1-p} D^2\varphi(y) = \theta_1^{1-p}\theta_2$$

on $e^\perp \cap B^n$, and hence the statement in Example 9.3.9 follows from Remark 9.3.8.

Next we show, following Chou, Wang [162], that Example 9.3.9 is optimal in the sense that if $p \geq n$ or $p \leq 2-n$ and the f in the L_p Minkowski problem (9.17) is bounded and bounded away from zero, then the solution $h = h_K|_{S^{n-1}}$ for $K \in \mathcal{K}_o^n$ is positive; namely, $o \in \operatorname{int} K$.

Lemma 9.3.10. *For $f : S^{n-1} \to (\lambda^{-1}, \lambda)$, $\lambda > 1$, if $K \in \mathcal{K}_o^n$ is a solution of the L_p Minkowski problem (9.17) where*

- *either $p \geq n$ and $S_K(\{h_K = 0\}) = 0$;*
- *or $p \leq 2-n$;*

then $o \in \operatorname{int} K$, and hence $h > 0$ for $h = h_K|_{S^{n-1}}$.

Proof. We suppose that $o \in \partial K$, and seek a contradiction. According to Lemma 1.2.9, there exist $t, r \in (0, 1)$ and an exterior unit normal $e \in S^{n-1}$ to K at o such that

$$-te + rB^n \subset \operatorname{int} K. \tag{9.29}$$

First let $p \geq n$ and $S_K(\{h_K = 0\}) = \mathcal{H}^{n-1}(\{x \in \partial' K : \langle \nu_K(x), x\rangle = 0\}) = 0$. As $h_K(\nu_K(x)) = \langle \nu_K(x), x\rangle \leq \|x\|$ for $x \in \partial' K$, (9.13) and (9.29) yield

$$\begin{aligned}\lambda \cdot \mathcal{H}^{n-1}(S^{n-1}) &\geq \int_{S^{n-1}} f = \int_{S^{n-1}} h_K^{1-p}\, dS_K = \int_{\partial' K} \langle \nu_K(x), x\rangle^{1-p}\, d\mathcal{H}^{n-1}(x) \\ &\geq \int_{\partial' K} \|x\|^{1-p}\, d\mathcal{H}^{n-1}(x) \geq \int_{e^\perp \cap r B^n} \|y\|^{1-p}\, d\mathcal{H}^{n-1}(y) \\ &= (n-1)\omega_{n-1} \int_{s=0}^{r} s^{n-1-p}\, ds = \infty,\end{aligned}$$

which is absurd, verifying Lemma 9.3.10 if $p \geq n$.

Assuming $p \leq 2-n$, we consider the spherical cap $\Omega(e,\varrho) = \{u \in S^{n-1} : \langle u, e\rangle > \cos\varrho\}$ for $\varrho \in (0, \frac{\pi}{2})$, which satisfies $\mathcal{H}^{n-1}(\Omega(e,\varrho)) > \omega_{n-1}(\sin\varrho)^{n-1}$. If $D = \operatorname{diam} K$, $x \in K$ and $u \in \Omega(e,\varrho)$, then writing x in the form $y - se$ for $s \geq 0$ and $y \in e^\perp$, we have $\|y\| \leq \|x\| \leq D$, and there exists an orthonormal basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ with $e = v_1$ and $\|y\| v_2 = y$, and hence

$$\langle x, u\rangle \leq \langle y, u\rangle \leq D\langle v_2, u\rangle \leq D\sqrt{1 - \langle v_1, u\rangle^2} \leq D \sin\varrho.$$

Therefore $h_K(u) \leq D \sin\varrho$ for any $u \in \Omega(e,\varrho)$ and $\varrho \in (0,, \frac{\pi}{2})$, thus

$$\begin{aligned}\lambda^{-1} &\leq \frac{1}{\mathcal{H}^{n-1}(\Omega(e,\varrho)\backslash\{e\})} \int_{\Omega(e,\varrho)\backslash\{e\}} f = \frac{1}{\mathcal{H}^{n-1}(\Omega(e,\varrho)\backslash\{e\})} \int_{\Omega(e,\varrho)\backslash\{e\}} h_K^{1-p}\, dS_K \\ &\leq \frac{D^{1-p}}{\omega_{n-1}} \cdot (\sin\varrho)^{2-n-p} \cdot S_K(\Omega(e,\varrho)\backslash\{e\}) \leq \frac{D^{1-p}}{\omega_{n-1}} \cdot S_K(\Omega(e,\varrho)\backslash\{e\}).\end{aligned}$$

Since $\lim_{\varrho\to 0^+} S_K(\Omega(e,\varrho)\backslash\{e\}) = 0$, we arrive at a contradiction, which in turn proves Lemma 9.3.10. ■

Concerning the smoothness of the solution of the L_p-Minkowski problem (9.15) and (9.16), Chou, Wang [162] explained that the same argument based on Theorem 9.1.1 and Theorem 9.1.3 due to Cafarelli [135, 136] that lead to Theorem 9.2.6 yields the following.

Theorem 9.3.11 (Caffarelli). *If $p \in \mathbb{R}$ and the f in (9.15) and (9.16) is positive and $C^{0,\alpha}$, and in addition, the solution h is positive (or equivalently, $o \in \operatorname{int} K$ for the corresponding convex body K), then h is $C^{2,\alpha}$ (and ∂K is $C^{2,\alpha}_+$).*

Remark. We note that, under the condition that f is positive and $C^{0,\alpha}$, the solution h in (9.15) or (9.16) is positive

- if $p \geq n$ or $p \leq 2-n$ (cf. Lemma 9.3.10),
- or if f and h are even.

Additional results about the smoothness of the solution are provided by Bianchi, Böröczky, Colesanti [68] in the case $2-n < p < 1$. For example, if $p < \min\{1, 4-n\}$, and the f in (9.16) satisfies $\lambda^{-1} < f < \lambda$ for some $\lambda > 1$, then h is strictly convex even if $h(u) = 0$ for some $u \in S^{n-1}$ (or equivalently, ∂K is C^1 for the corresponding convex body K even if $o \in \partial K$).

9.4 Uniqueness in the L_p-Minkowski problem

Uniqueness of the solution of a differential equations has always been a central problem. In the case of the L_p-Minkowski problem for $p \geq 0$, the role of uniqueness is even more significant as it is intimately connected to Brunn-Minkowski type inequalities (see Section 7.6 for $p > 1$, Section 9.2 for the classical case $p = 1$, Section 8.8 for $p \in (0, 1)$ and Section 8.7 for $p = 0$).

We recall (cf. (9.15) and (9.16)) that for $p \in \mathbb{R}$ and measurable $f : S^{n-1} \to [0, \infty)$, the L_p-Minkowski Problem is the Monge-Ampère equation

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \qquad \text{if } p \geq 1; \tag{9.30}$$

$$h^{1-p} \det(\nabla^2 h + hI_{n-1}) = f \qquad \text{if } p \leq 1. \tag{9.31}$$

Remark 9.4.1 (Uniqueness of the solution of the L_p-Minkowski problem).

Unique solution for $p > 1$: See Proposition 7.6.8 due to Hug, Lutwak, Yang, Zhang [339] in the case (9.30) where we have uniqueness even in the Aleksandrov sense.

No uniqueness if $p < 1$, not even for even measures if $p < 0$: See Chen, Li, Zhu[156] if $p \in (0, 1)$, and Li, Liu, Lu [402] if $p < 0$ (in the letter case, we may assume that the f and h in (9.31) are even and have axial rotational symmetry according to [402]). In addition, E. Milman [459] shows that for any o-symmetric convex body $C \subset \mathbb{R}^n$ with C^2_+ boundary, one finds a $q \in [-n, 0)$ where $q = -n$ if and only if C is a centered ellipsoid with the property that for any $p < q$, there exist multiple even solutions of the L_p-Minkowski problem (9.31); or in other words, there exists o-symmetric convex body $K \neq C$ with C^2_+ boundary with $S_{K,p} = S_{C,p}$.
One of the useful tools for non-uniqueness results is that if $p \in (-n, 1)$ and f is a positive continuous function on S^{n-1}, then there always exists a solution of (9.31) minimizing the entropy (cf. (9.24) and (9.25)); therefore, one just needs to produce a solution with higher entropy.

According Remark 9.4.1, the uniqueness question concerning (9.31) is still an intriguing one if $p \in [0, 1)$, but before discussing this problem, we survey results about Firey's classical question whether the solution of (9.31) is unique if f is a constant function.

Remark 9.4.2 (Firey's isotropic L_p Minkowski problem). In his seminal work about the "shapes of worn stones" in 1974, Firey [233] proved that if $p = 0$, then there is a unique even solution of the Monge-Ampére equation - the now called "isotropic L_p-Minkowski problem" -

$$h^{1-p}\det(\nabla^2 h + hI_{n-1}) = 1 \text{ on } S^{n-1}; \tag{9.32}$$

namely, the constant 1 function corresponding to the unit ball B^n. After a long sequence of partial results, finally Brendle, Choi, Daskalopoulos [126] verified that the constant 1 function is the unique solution of (9.32) without the evenness condition if $-n < p < 1$. On the other hand, the $\mathrm{SL}(n)$ invariance Proposition 8.9.3 of the centro-affine curvature yields that any centered ellipsoid of volume $|B^n|$ is a solution of (9.32), and they are actually the only solutions (see the "classical papers" Calabi [142], Pogolerov [489], Cheng, Yau [161], and the novel approaches Crasta, Fragalá [182], Ivaki, E. Milman [350] and Saroglou [510]). Concerning the uniqueness of the solution of (9.32) assuming $p < -n$, if $n = 2$, then it was clarified by Andrews [22]; namely, (9.32) has the unique constant 1 solution if $-7 \le p < -2$, and the solution is not unique if $p < -7$. If $n > 2$, then Du [194] proves that (9.32) has multiple solutions if $p < -2n - 3$.

As a variation of the isotropic L_p-Minkowski problem, uniqueness of the solution of (9.31) if $p \in [0, 1)$ and f is $C^{0,\alpha}$ close to 1 (without the evenness assumption) is proved by Böröczky, Saroglou [117] (the case $p = 0$ and $n = 3$ was handled earlier by Chen, Feng, Liu [152]). In addition, for any $p > -n$ (with evenness condition for certain range of p), Ivaki [348] proved that if the f in (9.31) is close to be a constant, then any solution has to be close to be a ball.

Lutwak [433] proposed the following fundamental conjecture for the uniqueness of the solution in the even L_p-Minkowski problem if $p = 0$ in 1993, and the case $p \in (0, 1)$ was proposed by Böröczky, Lutwak, Yang, Zhang [110].

Conjecture 9.4.3 (Even L_p-Minkowski conjecture for $p \in [0, 1)$ and smooth data). *If $p \in [0, 1)$ and f is a positive even C^∞ function in (9.31), then (9.31) has a unique even solution.*

Remark. Kolesnikov, E. Milman [381] prove that knowing the L_p-Minkowski conjecture for some $p \in [0, 1)$ yields the L_q-Minkowski inequality when $q \in (p, 1)$.

Conjecture 9.4.3 has been verified in the following cases:

- *$n = 2$:* Proved by Böröczky, Lutwak, Yang, Zhang [110].
- *f and h are unconditional in $\mathbb{R}^n$:* Proved by Saroglou [508].
- *$p \in (p_n, 1)$ where $0 < p_n < 1 - \frac{c}{n \log n}$ for an absolute constant $c > 0$:* Combination of the local result by Kolesnikov, E. Milman [381] and the local to global approach based on Schrauder estimates in PDE by Chen, Huang, Li, Liu [154] (see Puttermann [495] for an Aleksandrov-type argument for the local to global approach).

More generally, the following two conjectures are due to Böröczky, Lutwak, Yang, Zhang [110] in the o-symmetric case.

Conjecture 9.4.4 (Even L_p-Minkowski Conjecture for $p \in (0, 1)$). *If $p \in (0, 1)$ and $K, C \subset \mathbb{R}^n$ are o-symmetric convex bodies with $S_{K,p} = S_{C,p}$, then $K = C$.*

We say that the o-symmetric convex bodies $K, C \subset \mathbb{R}^n$ have dilated summands if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for $m \geq 1$ and o-symmetric compact convex sets $C_i, K_i \subset \mathbb{R}^n$ of dimension at least 1 such that $\sum_{i=1}^m \dim K_i = n$ and $C_i = \lambda_i K_i$ for $\lambda_i > 0$, $i = 1, \ldots, m$. If in addition we have $|K| = |C|$ in this case, then $V_K = V_C$ according to Corollary 9.B.4.

Conjecture 9.4.5 (Even log-(L_0-)Minkowski Conjecture). *For o-symmetric convex bodies $K, C \subset \mathbb{R}^n$, $S_{K,0} = S_{C,0}$ (or equivalently, $V_K = V_C$) if and only if $|K| = |C|$ and K and C have dilated summands.*

Again, Conjecture 9.4.4 and Conjecture 9.4.5 have been verified in the planar $n = 2$ case by Böröczky, Lutwak, Yang, Zhang [110], and if both K and C are unconditional in any dimension by Saroglou [508].

Given a non-trivial finite even Borel measure μ on S^{n-1}, while solving the even L_p-Minkowski problem for $p \in (0, 1)$ by Haberl, Lutwak, Yang, Zhang [292], and for $p = 0$ by Böröczky, Lutwak, Yang, Zhang [110], the authors considered the entropy function

$$\mathcal{E}_{\mu,p}(C) = \begin{cases} \frac{1}{p} \int_{S^{n-1}} h_C^p \, d\mu & \text{if} \quad p \in (0, 1) \\ \int_{S^{n-1}} \log h_C \, d\mu & \text{if} \quad p = 0 \end{cases}$$

of an o-symmetric convex body $C \subset \mathbb{R}^n$, and proved the following properties (cf. Proposition 9.B.1).

Proposition 9.4.6. *Let $p \in [0, 1)$, and let μ be a finite even Borel measure μ on S^{n-1} such that any open hemisphere has positive measure, and if $p = 0$, then even*

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \ \text{ for any non-trivial linear subspace } L \subset \mathbb{R}^n.$$

(i) $\mathcal{E}_{\mu,p}(C)$ attains its minimum for o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$.

(ii) If an o-symmetric convex body $\widetilde{C} \subset \mathbb{R}^n$ with $|\widetilde{C}| = 1$ minimizes $\mathcal{E}_{\mu,p}(C)$ among o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$, then there exists $\lambda > 0$ with $\mu = S_{\lambda \widetilde{C},p}$.

For $p \in [0, 1)$, let us discuss the connection between uniqueness of the solution of the even L_p-Minkowski problem and the conjectured L_p-Minkowski inequality for o-symmetric convex bodies (cf. Conjecture 8.8.2 and Conjecture 8.7.3). Let $K \subset \mathbb{R}^n$ be an o-symmetric convex body such that if $p = 0$, then K is not of the form $K = K_0 + K_1$ for the at least one dimensional compact convex sets $K_0, K_1 \subset \mathbb{R}^n$ with

$\dim K_0 + \dim K_1 = n$, and hence $S_{K,0}(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot S_{K,0}(S^{n-1})$ for any non-trivial linear subspace $L \subset \mathbb{R}^n$ according to Theorem 9.3.6. In particular, for any $p \in [0,1)$, Proposition 9.4.6 yields that $\mathcal{E}_{S_{K,p},p}(C)$ attains its minimum for o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = |K|$, and any minimizer $\widetilde{C}$ satisfies $S_{K,p} = S_{\lambda \widetilde{C},p}$ for a $\lambda > 0$. Now if Conjecture 9.4.4 and Conjecture 9.4.5 about the uniqueness of the solution of the even L_p-Minkowski problem hold, then $\widetilde{C} = K$ for any minimizer; therefore, any o-symmetric convex body $C \subset \mathbb{R}^n$ with $|C| = |K|$ satisfies

$$\begin{array}{rclll} \int_{S^{n-1}} h_C^p \, dS_{K,p} & \geq & \int_{S^{n-1}} h_K^p \, dS_{K,p} & \text{if} & p \in (0,1) \\ \int_{S^{n-1}} \log h_C \, dS_{K,0} & \geq & \int_{S^{n-1}} \log h_K \, dS_{K,0} & \text{if} & p = 0. \end{array}$$

As $dS_{K,p} = n h_K^{-p} \, dV_K$ for the cone volume measure $dV_K = \frac{1}{n} h_K \, dS_K$ satisfying $V_K(S^{n-1}) = |K|$ (see Section 2.6), we deduce Remark 9.4.7.

Remark 9.4.7 (Uniqueness for L_p-Minkowski problem versus L_p-Minkowski inequality). Conjecture 9.4.4 and Conjecture 9.4.5 about the uniqueness of the solution of the even L_p-Minkowski problem for $p \in [0,1)$ are equivalent to saying that if $K, C \subset \mathbb{R}^n$ are o-symmetric convex bodies, then

$$\int_{S^{n-1}} \frac{h_C^p}{h_K^p} \, dV_K \geq |K|^{\frac{n-p}{n}} |C|^{\frac{p}{n}} \text{ provided } p \in (0,1), \tag{9.33}$$

with equality if and only if K and C are dilates, and

$$\int_{S^{n-1}} \log \frac{h_C}{h_K} \, dV_K \geq \frac{|K|}{n} \log \frac{|C|}{|K|} \tag{9.34}$$

with equality if and only if $K = K_1 + \ldots + K_m$ and $C = C_1 + \ldots + C_m$ for centered compact convex sets $K_1, \ldots, K_m, C_1, \ldots, C_m$ of dimension at least one where $\sum_{i=1}^m \dim K_i = n$ and K_i and C_i are dilates, $i = 1, \ldots, m$.

According to Lemma 8.8.3, (9.33) is equivalent with the L_p-Brunn-Minkowski Conjecture for o-symmetric convex bodies and $p \in (0,1)$, and according to Lemma 8.7.4, (9.34) is equivalent to the L_0 or Logarithmic Brunn-Minkowski Conjecture for o-symmetric convex bodies.

Conjecture 9.4.3 is equivalent to the cases of (9.33) and (9.34) when ∂K and ∂C are C_+^∞. On the other hand, assuming that Conjecture 9.4.3 holds for $p = q$ for some $q \in [0,1)$, then the inequalities in (9.33) and (9.34) hold for $p \in [q,1)$, and equality holds in (9.33) provided $p \in (q,1)$ and ∂K is C^1 if and only if $K = C$ according to Remark 8.8.4.

9.5 Comments to Chapter 9

In this section, we collect some recently well-investigated analogues of the L_p-Minkowski problem. For Minkowski-type problems for log-concave functions, see Section 10.9.2. Recall that $\mathcal{K}^n_{(o)}$ ($\mathcal{K}^n_o$) is the set of convex bodies $K \subset \mathbb{R}^n$ with $o \in \operatorname{int} K$ ($o \in K$).

9.5.1 The Christoffel-Minkowski problem about the radii of curvatures

Given a positive continuous function f on S^{n-1}, the Christoffel problem - posed by Christoffel [163] in 1865, and hence preceeding the Minkowski problem - considers the existence of a convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary such that the sum of the principal radii of curvature at $x \in \partial K$ is $f(\nu_K(x))$.

From the point of view of measures (cf. Schneider [522]), Theorem 8.3.5 about mixed volumes yields that for any convex body $K \subset \mathbb{R}^n$ and $i = 1, \ldots, n-1$, there exists the so-called *ith area measure* $S_i(K, \cdot)$ - that is a finite Borel measure on S^{n-1} introduced by Aleksandrov [2] and Fenchel, Jessen [220] - satisfying that if $C \subset \mathbb{R}^n$ is a compact convex set, then

$$n\binom{n-1}{i} V(\overbrace{K, \ldots, K}^{i}, \overbrace{B^n, \ldots, B^n}^{n-1-i}, C) = \int_{S^{n-1}} h_C(u)\, dS_i(K, u). \tag{9.35}$$

In particular, the $(n-1)$th area measure $S_{n-1}(K, \cdot)$, is just the classical surface area measure S_K. The normalization is chosen in a way such that if $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary, then for the C^2 function $h = h_K|_{S^{n-1}}$ and $u \in S^{n-1}$ (cf. Definition 8.1.6 and (8.12)) and the differential operators

$$\nabla^2 h(u) + h(u) I_{n-1} = \widetilde{D}^2 h_K(u) = D^2 h_K(u)|_{u^\perp},$$

(8.26) and (8.27) in Theorem 8.3.4 yields that

$$dS_i(K, \cdot) = \sigma_i\left(\widetilde{D}^2 h_K\right)\, d\mathcal{H}^{n-1} = \sigma_i\left(\nabla^2 h + h I_{n-1}\right)\, d\mathcal{H}^{n-1}$$

where $\sigma_i(A)$ is the ith symmetric function of the eigenvalues of an $(n-1) \times (n-1)$ symmetric matrix A. In other words, if $K \subset \mathbb{R}^n$ is a convex body with C^2_+ boundary, then the density function of $S_i(K, \cdot)$ at $\nu_K(x)$ for $x \in \partial K$ is the ith symmetric function of the principal radii of curvatures at x. The corresponding differential equation on S^{n-1} is the *Christoffel-Minkowski problem*

$$\sigma_i\left(\nabla^2 h + h I_{n-1}\right) = f \tag{9.36}$$

where a C^2 function h on S^{n-1} is thought such that $\nabla^2 h + h I_{n-1}$ is positive definite everywhere. The last condition says that $h = h_K|_{S^{n-1}}$ for convex body $K \subset \mathbb{R}^n$ with C^2_+ boundary (cf. Lemma 8.1.8).

The ith area measure $S_i(K,\cdot)$ is readily translation invariant and $O(n)$ equivariant. Applying (9.35) to the case when C is a point $p \in \mathbb{R}^n$, and hence $h_C(u) = \langle p, u\rangle$, shows that for any convex body $K \subset \mathbb{R}^n$, we have

$$\int_{S^{n-1}} u \, dS_i(K,u) = o,$$

and hence the function f in the Christoffel-Minkowski problem (9.36) should satisfy

$$\int_{S^{n-1}} u \, f(u)\, du = o. \tag{9.37}$$

The original *Christoffel problem* posed by by Christoffel [163] in 1865 is the case $i = 1$ of (9.36); namely, it is about the sum of the radii of curvatures. After various attempts to characterize the possible f in (9.36) for $i = 1$, among others by Hilbert and Hurwitz around 1900, it was Firey [232] in 1967 first providing a full but rather complicated classification (similar results are due to Berg [61] and Goodey, Yaskin, Yaskina [266]). A more accessible classification is provided by Li, Wan, Wang [405], who link the Christoffel problem on S^{n-1} to the Laplace equation $\Delta h_K = \mathring{f}$ on $\mathbb{R}^n \backslash \{o\}$ where $\mathring{f}(\lambda u) = \lambda^{-1} f(u)$ for $\lambda > 0$ and $u \in S^{n-1}$. Schneider [517] solved the discrete Christoffel problem on S^{n-1}; namely, characterized $S_1(P,\cdot)$ for n-polytopes $P \subset \mathbb{R}^n$.

The Christoffel-Minkowski problem (9.36) when $1 < i < n-1$ is even more mysterious, see Guan, Ma [282], Guan, Lin, Ma [283], Guan, Ma, Zhou [284] up to 2006, and the recent paper Bryan, Ivaki, Scheuer [133] (see Guan [279] for a survey of the methods of the earlier papers).

For L_p versions of the Christoffel-Minkowski problem, see, for example, Guan, Xia [286], Ivaki [347], Chen [151], Li, Ju, Liu [400] and Hu, Ivaki [326].

9.5.2 Aleksandrov's problem on integral curvature and some of its versions (prescribed curvature measures, L_p Aleksandrov problem)

For $K \in \mathcal{K}^n_{(o)}$, we parametrize the boundary of K using the radial image as

$$r_K(u) = \varrho_K(u)\, u \in \partial K$$

for $u \in S^{n-1}$ and the radial function $\varrho_K(u) > 0$. Then Aleksandrov [6] defined the *integral curvature measure* of a Borel set $\omega \subset S^{n-1}$ as

$$J_K(\omega) = \mathcal{H}^{n-1}\left(\left\{v \in S^{n-1} : \exists u \in \omega \text{ such that } v \in N_K(r_K(u))\right\}\right), \tag{9.38}$$

where $v \in N_K(r_K(u))$ says that v is an exterior normal at $r_K(u) \in \partial K$. It follows that

$$J_K(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1}) = n\omega_n.$$

If ∂K is C^2 (and hence ϱ_K is C^2, as well), then (cf. Lemma 2.2.3)

$$J_K(\omega) = \int_{r_K(\omega)} \kappa\left(r_K(u)\right)\, d\mathcal{H}^{n-1} \tag{9.39}$$

$$= \int_{\omega} \kappa\left(r_K(u)\right) \varrho_K(u)^{n-2} \sqrt{\varrho_K(u)^2 + \|\nabla \varrho_K(u)\|^2}\, d\mathcal{H}^{n-1}(u). \tag{9.40}$$

The formula (9.39) explains the name integral curvature measure.

Aleksandrov [6] characterized integral curvature measures, and hence solving what later was called Aleksandrov problem.

Theorem 9.5.1 (Aleksandrov). *For a Borel measure μ on S^{n-1} with $\mu(S^{n-1}) = \mathcal{H}^{n-1}(S^{n-1})$, $\mu = J_K$ for a $K \in \mathcal{K}^n_{(o)}$ if and only if*

$$\mu(\omega) < \mathcal{H}^{n-1}\left(\{u \in S^{n-1} : \exists v \in \omega,\ \langle u, v\rangle > 0\}\right) \tag{9.41}$$

for any Borel set ω contained in a closed hemisphere, and K is unique up to dilation.

Remark. The condition (9.41) can be expressed in terms of the polar

$$C^* = \{u \in S^{n-1} : \langle u, v\rangle \leq 0\ \forall v \in \omega\}$$

of a spherically convex compact set $C \subset S^{n-1}$. Here C^* is a spherically convex compact set, as well, which satisfies $C^{**} = C$. Now (9.41) is equivalent to saying that for any spherically convex compact set $C \subset S^{n-1}$, we have

$$\mu(C) + \mathcal{H}^{n-1}(C^*) < \mathcal{H}^{n-1}(S^{n-1}).$$

Regularity of the solution of the Aleksandrov problem is investigated by Guan, Li [280].

Remark 9.5.2 (Known proofs of Theorem 9.5.1).

- Aleksandrov [6,7]: First proving for polytopes (discrete measures), and then using topology to handle the case of general convex bodies.
- Oliker [478] and Bertrand [66]: Using optimal transport.
- Böröczky, Lutwak, Yang, Zhang, Zhao [114]: Variational argument.

As (9.40) indicates (cf. Guan, Li, Li [281]), given a positive continuous function f on S^{n-1}, finding a convex body $K \in \mathcal{K}^n_{(o)}$ with $dJ_K = f\, d\mathcal{H}^{n-1}$ leads to a fully non-linear elliptic partial differential equation S^{n-1}. On the other hand, Lemma 9.5.3 below shows that the Aleksandrov problem for the polar (dual) body is a Monge-Ampère equation, a familiar setting in this book.

First, in line with (2.5), the *reverse radial Gauss image* of a Borel set $\omega \subset S^{n-1}$ is the set of $u \in S^{n-1}$ such that some $v \in \omega$ is an exterior normal at $\varrho_K(u)u$; namely,

$$\alpha^*_K(\omega) = \left\{u \in S^{n-1} : N_K(r_K(u)) \cap \omega \neq \emptyset\right\}, \tag{9.42}$$

and hence (9.38) and Proposition 1.9.3 yield that

$$J_{K^*}(\omega) = \mathcal{H}^{n-1}\left(\alpha_K^*(\omega)\right). \tag{9.43}$$

In order to provide an integral formula based on (9.43), let $\pi(x) = x/\|x\|$ be the radial projection $\mathbb{R}^n \backslash \{o\} \to S^{n-1}$, and for $K \in \mathcal{K}^n_{(o)}$, let $\aleph_K = (\pi\partial' K) \cap (\pi\partial' K^*)$ be the set of points $u \in S^{n-1}$ such that

$$\alpha_K(u) = \nu_K(r_K(u)) = \nu_K(\varrho_K(u)) \tag{9.44}$$

is well defined for $u \in \aleph_K$, and $\alpha_K(u)$ is an exterior normal only at $r_K(u) \in \partial K$ and

$$\alpha_K^*(\alpha_K(u)) = u. \tag{9.45}$$

Here we used that for $z = r_{K^*}(\alpha_K(u)) \in \partial' K^*$, we have $\pi(z) = u$ and $\alpha_K^*(\pi(z)) = \nu_{K^*}(z)$. Since $\mathcal{H}^{n-1}$ a.e. boundary point of a convex body in $\mathbb{R}^n$ is regular, and π is locally Lipschitz, we deduce that

$$\mathcal{H}^{n-1}\left(S^{n-1} \backslash \aleph_K\right) = 0. \tag{9.46}$$

Using the function $f(v) = \varrho_K\left(\alpha_K^*(v)\right)^{-n}$ for $v \in \aleph_K$ in Proposition 2.6.8, and the formulas (9.43), (9.45) and (9.46), we deduce from Lemma 2.2.4 that

$$J_{K^*}(\omega) = \mathcal{H}^{n-1}\left(\alpha_K^*(\omega)\right) = n\int_{\omega\cap\aleph_K} \varrho_K^{-n}\, dV_K = \int_\omega h\left(\|\nabla h\| + h^2\right)^{\frac{-n}{2}} dS_K \tag{9.47}$$

for the Lipschitz function $h = h_K|_{S^{n-1}}$ and any Borel set $\omega \subset S^{n-1}$. Thus (8.14) yields the Monge-Ampére equation for h if J_{K^*} is absolutely continuous.

Lemma 9.5.3. *If $K \in \mathcal{K}^n_{(o)}$ is a convex body with C^2_+ boundary, then $dJ_{K^*} = f\, d\mathcal{H}^{n-1}$ where for the C^2 function $h = h_K|_{S^{n-1}}$, we have*

$$h\left(\|\nabla h\| + h^2\right)^{\frac{-n}{2}} \det\left(\nabla^2 h + hI_{n-1}\right) = f. \tag{9.48}$$

Let us discuss briefly two generalizations of the Aleksandrov problem. The first is the characterization of curvature measures introduced by Federer [212] originally for sets of positive reach, and discussed in depth by Schneider [522] in the case of convex bodies. Here we only consider the case of a convex body $K \in \mathcal{K}^n_{(o)}$ with C^2_+ boundary. For $i = 0, \ldots, n-1$ and Borel $\eta \subset \partial K$, the ith curvature measure $C_i(K, \eta)$ satisfies (cf. Lemma 2.2.3)

$$\begin{aligned}(n-i)\omega_{n-i} \cdot C_i(K,\eta) &= \int_\eta \sigma_{n-1-i}(\kappa_1(x), \ldots, \kappa_{n-1}(x))\, dx \\ &= \int_{r_K^{-1}(\eta)} \sigma_{n-1-i}(\kappa_1, \ldots, \kappa_{n-1}) \circ r_K \cdot \\ &\quad \cdot \varrho_K^{n-2}\sqrt{\varrho_K^2 + \|\nabla \varrho_K\|^2}\, d\mathcal{H}^{n-1}\end{aligned}$$

where $\sigma_j(\kappa_1,\ldots,\kappa_{n-1})$ at $x\in\partial K$ stands for the jth symmetric function of the principal curvatures $\kappa_1(x),\ldots,\kappa_{n-1}$ at x. Here the normalization is chosen in a way such that $C_i(K,S^{n-1}) = V_i(K)$ (cf. (8.27)). According to Guan, Li, Li [281], characterizing the measure $C_i(K, r_K(\omega))$ of Borel sets $\omega\subset S^{n-1}$ leads to a fully non-linear elliptic partial differential equation S^{n-1}, which coincides with the Aleksandrov problem if $i=0$ by (9.40).

For $p\in\mathbb{R}$, the L_p version of the Aleksandrov problem posed by Huang, Lutwak, Yang, Zhang [332], asks for characterization of the L_p *Aleksandrov integral curvature measure*

$$dJ_{K,p} = \varrho_K^p\, dJ_K \tag{9.49}$$

for $K\in\mathcal{K}^n_{(o)}$ where the classical Aleksandrov problem is the case $p=0$. It follows from Lemma 9.5.3 that the Monge-Ampère equation for the L_p Aleksandrov integral curvature measure $J_{K^*,p}$ of the polar body is

$$\det(\nabla^2 h + h\, I_{n-1}) = h^{p-1}(\|\nabla h\|^2 + h^2)^{\frac{n}{2}}\cdot f.$$

For related results, see, for example, Zhao [579], Li, Sheng, Ye, Yi [406] Mui [468] and Wu, Wu, Xiang [567].

9.5.3 L_p Dual curvature measures

In order to define the dual curvature measures for $q\in\mathbb{R}$, first let $K\in\mathcal{K}^n_{(o)}$. For a Borel set $\omega\subset S^{n-1}$, using the reverse radial Gauss image $\alpha^*(\omega)$ in (9.42), Huang, Lutwak, Yang, Zhang [331]) defines the *qth dual curvature measure* by the formula

$$\widetilde{C}_{K,q}(\omega) = \int_{\alpha^*(\omega)} \varrho_K^q\, d\mathcal{H}^{n-1}.$$

We deduce from (9.47) and Lemma 2.2.4 that

$$\widetilde{C}_{K,q}(\omega) = n\int_\omega \varrho_K(\alpha^*(u))^{q-n}\, dV_K(u) \tag{9.50}$$

$$= \int_\omega h\left(\|\nabla h\| + h^2\right)^{\frac{q-n}{2}} dS_K \tag{9.51}$$

for $h = h_K|_{S^{n-1}}$. For example,

- $\widetilde{C}_{K,n} = nV_K$ (cf. Section 2.6);
- $\widetilde{C}_{K,0} = J_{K^*}$ (cf. Section 9.5.2, especially (9.47)).

For $q\in\mathbb{R}$, Lutwak [433] defined the *qth dual volume* of a $K\in\mathcal{K}_{(o)}$ as

$$\widetilde{V}_q(K) = \frac{1}{n}\int_{S^{n-1}} \varrho^q\, d\mathcal{H}^{n-1},$$

and the same formula works if $q > 0$ and $K \in \mathcal{K}_o^n$ (so possibly $o \in \partial K$ in this case). Readily, $\widetilde{V}_q(K) = |K|$.

Remark 9.5.4 (Continuity of $\widetilde{V}_q$ and weak continuity of $\widetilde{C}_{K,q}$). If $q \in \mathbb{R}$ and $K_m \in \mathcal{K}_{(o)}$ tends to $K \in \mathcal{K}_{(o)}$, then readily $\widetilde{V}_q(K_m)$ tends to $\widetilde{V}_q(K)$, and (9.50) and the weak continuity of V_K yields that $\widetilde{C}_{K_m,q}$ tends weakly to $\widetilde{C}_{K,q}$.

If $q > 0$, then the same holds even if $K \in \mathcal{K}_o$ according to Böröczky, Fodor [99].

Huang, Lutwak, Yang, Zhang [331] established the characteristic property of the dual curvature measure that it encodes the first variation of the dual volume.

Theorem 9.5.5. *For $K \in \mathcal{K}_{(o)}^n$, $q \in \mathbb{R}\backslash\{0\}$ and $g : S^{n-1} \to \mathbb{R}$ continuous, the Wulff shape $K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \le h_K(u)(1 + tg(u)),\ \forall u \in S^{n-1}\}$ satisfies*

$$\lim_{t\to 0} \frac{\widetilde{V}_q(K_t) - \widetilde{V}_q(K)}{t} = \frac{q}{n} \int_{S^{n-1}} g \, d\widetilde{C}_{K,q}.$$

We observe that the cone volume measure V_K (cf. Section 2.6, Section 9.3 and Section 9.4) lies at the cross road: nV_K is the L_0 surface area measure (cf. Section 9.3) on the one hand, and the nth dual curvature measure on the other hand.

Concerning the dual Minkowski Problem for $q \in \mathbb{R}$, it follows from (9.51) that the corresponding Monge-Ampère equation is

$$(\|\nabla h\|^2 + h^2)^{\frac{q-n}{2}} \cdot h \det(\nabla^2 h + h\,\mathrm{Id}) = f. \tag{9.52}$$

The case $q = n$ about the cone volume measure is discussed in Sections 9.3 and 9.4, and the case $q = 0$ about the Aleksandrov problem is covered in Section 9.5.2. If $q \neq 0, n$, then the following results are known:

- If $q < 0$, then any Borel measure on S^{n-1} not concentrated on a closed hemisphere is a qth dual Minkowski curvature measure according to Zhao [578] and Li, Sheng, Wang [404].
- If $q > 0$ and $n = 2$, then (9.52) has a solution for any measurable f provided $\frac{1}{c} < f < c$ for a $c > 1$ according to Chen, Li [155].
- If $0 < q < n$, then a finite even Borel measure μ on S^{n-1} is a qth dual Minkowski curvature measure if and only if

$$\mu(L \cap S^{n-1}) < \tfrac{\dim L}{q} \cdot \mu(S^{n-1})$$

  for any proper linear subspace $L \subset \mathbb{R}^n$ according to Böröczky, Lutwak, Yang, Zhang, Zhao [113] where one needs to add that μ is not concentrated onto a great subsphere if $q < 1$.
- If $q \geq n + 1$ and $K \in \mathcal{K}_{(o)}^n$ is origin symmetric, then Henk, Pollehn [307] prove the necessary condition

$$\widetilde{C}_{K,q}(L \cap S^{n-1}) < \tfrac{q-n+\dim L}{q} \cdot \widetilde{C}_{K,q}(S^{n-1})$$

for any proper linear subspace $L \subset \mathbb{R}^n$.

Remark. In particular, it is an intriguing open problem to characterize a qth dual Minkowski curvature measure on S^{n-1} if $q > 0$, or an even qth dual Minkowski curvature measure on S^{n-1} if $q > n$.

No uniqueness of the solution of the dual Minkowski problem (9.52) in general, for example, if $q > 2n$, even assuming that $f \equiv 1$ and h is even according to Chen, Chen, Li [149].

Next we discuss the L_p dual Minkowski problem introduced by Lutwak, Yang, Zhang [437] that is a common generalization of the L_p-Minkowski problem and the L_p-Aleksandrov problem. For $p, q \in \mathbb{R}$, Lutwak, Yang, Zhang [437] defines the *qth L_p dual curvature measure* on S^{n-1} by

$$d\widetilde{C}_{K,p,q} = h_K^{-p}\, d\widetilde{C}_{K,q},$$

and hence

$$\begin{aligned}
\widetilde{C}_{K,p,n} &= S_{K,p};\\
\widetilde{C}_{K,0,q} &= \widetilde{C}_{K,q};\\
\widetilde{C}_{K,p,0} &= J_{K^*,p}.
\end{aligned}$$

Given a Borel measure μ on S^{n-1}, the simplest version of the qth L_p dual Minkowski problem asks for a $K \in \mathcal{K}^n_{(o)}$ with $\mu = \widetilde{C}_{K,p,q}$, and the correspondig Monge-Ampére equation is (cf. (9.52))

$$h^{1-p} \det(\nabla^2 h + h\,\mathrm{Id}) = (\|\nabla h\|^2 + h^2)^{\frac{n-q}{2}} \cdot f. \tag{9.53}$$

Improving on Böröczky, Fodor [99] and Huang, Zhao [334], Chen, Li [150] and Lu, Pu [424] prove that if $p > 0$ and $q \neq p, 0$, then any Borel measure not concentrated on a closed hemisphere is a qth L_p dual Minkowski curvature measure (more precisely, if $p \leq q$, then some modification of the Monge-Ampére equation might be needed). Huang, Zhao [334] proved the same for $p, q < 0$ and $p \neq q$ within the category of even measures. See also Guang, Li, Wang [287] for a flow approach when $p < 0$ and $q > n$ under regularity assumptions.

Uniqueness of the solution of the qth L_p dual Minkowski problem (9.53) is thoroughly investigated by Chen, Chen, Li [149] and Li, Liu, Lu [402]. The case when $n = 2$ and f is a constant function has been completely clarified by Li, Wan [401]. Uniqueness in the the isotropic case, when the f in (9.53) is a constant, was verified for certain ranges of values of p and q by Ivaki, E. Milman [350] in the even case, and by Hu, Ivaki [325] in the general case.

Orlicz versions of these Monge-Ampère equations have been considered by Li, Sheng, Ye, Yi [406], Feng, Hu, Liu [221] and Hu, Liu, Ma [322] in the case of the

Aleksandrov problem, by Xing, Ye [573] in the case of the dual Minkowski problem, and Gardner, Hug, Weil, Xing, Ye [255, 256], Xing, Ye, Zhu [574] and Liu, Lu [416] in the case of the L_p dual Minkowski problem in general.

Another important related variant of the dual Minkowski problem is the so-called "Chord Minkowski Problem" (*cf.* Lutwak, Xi, Yang, Zhang [436]) and its L_p version by Xi, Yang, Zhang, Zhao [571] for $p > 0$, and by Li [407, 408] for $p < 0$, see also Guo, Xi, Zhao [290] and Xi, Yang, Zhang, Zhao [571]. In addition, the "Affine dual Minkowski problem" is proposed by Cai, Leng, Wu, Xi [141].

9.5.4 Gaussian Minkowski problem

For the Gaussian density $d\gamma_n(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}} dx$ in $\mathbb{R}^n$ and a convex body $K \in \mathcal{K}^n_{(0)}$, in line with the more general set up in Livshyts [418], Huang, Xi, Zhao [333] defined the *Gaussian surface area measure* on S^{n-1} by the formula

$$\lim_{t\to 0} \frac{\gamma_n(K + tC) - \gamma_n(K)}{t} = \int_{S^{n-1}} h_C \, dS_{\gamma_n,K}$$

for any convex body $C \subset \mathbb{R}^n$. In other words, for any Borel $\omega \subset S^{n-1}$, we have

$$S_{\gamma_n,K}(\omega) = \int_{\nu_K^{-1}(\omega)} \frac{1}{(2\pi)^{\frac{n}{2}}} e^{-\frac{\|x\|^2}{2}} \, d\mathcal{H}^{n-1}(x).$$

In particular, if K is strictly convex (no segment on the boundary), and hence the α^*_K (cf. (9.42)) is a continuous function $S^n \to S^n$, then

$$dS_{\gamma_n,K}(u) = \frac{1}{(2\pi)^{\frac{n}{2}}} \exp\left(-\frac{\varrho_K(\alpha^*(u))^2}{2}\right) dS_K.$$

We deduce from Lemma 2.2.4 that the *Gaussian Minkowski problem* leads to the Monge-Ampère equation

$$\frac{1}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{\|\nabla h\|^2 + h^2}{2}} \cdot \det(\nabla^2 h + h\, I_{n-1}) = f.$$

These notions are due to Huang, Xi, Zhao [333], who obtain significant results about the even Gaussian Minkowski problem. Their results are extended to the not necessarily even case by Feng, Liu, Xu [219] and Chen, Hu, Liu, Zhao [153]. Uniqueness of the solution is discussed in the works above and in Ivaki, E. Milman [350]. Various properties of the Gaussian surface area measure is discussed in depth by Fradelizi, Langharst, Madiman, Zvavitch [246].

Remark 9.5.6. Uniqueness can't be expected in the Gaussian Minkowski Problem in general as, for example, a small ball and suitable large ball have the same Gaussian surface area measure. For $K, C \in \mathcal{K}^n_{(0)}$ with $\gamma_n(K), \gamma_n(C) \geq \frac{1}{2}$, Huang, Xi, Zhao [333] prove that $S_{\gamma_n,K} = S_{\gamma_n,C}$ implies $K = C$. Therefore, the uniqueness question is really interesting under the assumptions $\gamma_n(K), \gamma_n(C) < \frac{1}{2}$.

Concerning sufficient conditions for the Gaussian Minkowski Problem, let us quote the following result from Feng, Liu, Xu [219].

Theorem 9.5.7. *If μ is a Borel measure on S^{n-1} with $\mu(S^{n-1}) < \frac{1}{\sqrt{2\pi}}$, then $\mu = S_{\gamma_n,K}$ for a $K \in \mathcal{K}^n_{(o)}$ with $\gamma_n(K) > \frac{1}{2}$.*

The L_p-Gaussian Minkowski problem is considered for example by Liu [415] and Feng, Hu, Xu [218]. Orlicz versions of the Gaussian Minkowski problem are discussed by Li, Sheng, Ye, Yi [406].

9.5.5 Capacity, Torsion rigidity and the first eigenvalue of the Laplacian

This section discusses theories analogues to the Brunn-Minkowski theory that have been built for the classical notion of electrostatic (Newtonian) capacity and the torsion rigidity.

The *electrostatic capacity* $C_2(X)$ (see Colesanti, *et al* [172] for background) of a compact set $X \subset \mathbb{R}^n$, $n \geq 3$, is defined by

$$C_2(X) = \inf\left\{\int_{\mathbb{R}^n} \|D\varphi\|^2 : \ \varphi \in C_c^\infty(\mathbb{R}^n),\ \varphi(x) \geq 1 \text{ for } x \in X\right\},$$

and if $\Omega \subset \mathbb{R}^n$ is a bounded open convex set, then

$$C_2(\Omega) = \sup\left\{C_2(X) : \ X \subset \Omega \text{ compact}\right\},$$

which satisfies $C_2(s\,\Omega) = s^{n-2}C_2(\Omega)$ for $s > 0$. For $n \geq 3$, $s \in (0,1)$ and bounded open convex sets $\Omega_0, \Omega_1 \subset \mathbb{R}^n$, Borell [87] proved the Brunn-Minkowski type inequality

$$C_2((1-s)\Omega_0 + s\,\Omega_1)^{\frac{1}{n-2}} \geq (1-s)C_2(\Omega_0)^{\frac{1}{n-2}} + s\,C_2(\Omega_1)^{\frac{1}{n-2}}, \tag{9.54}$$

where equality holds if and only if Ω_0 and Ω_1 are homothetic according to Caffarelli, Jerison, Lieb [139].

For a bounded open convex set $\Omega \subset \mathbb{R}^n$, the equilibrium potential $U = U_\Omega$ associated to Ω is the unique solution of the boundary value problem

$$\begin{array}{rcll} \Delta U & = & 0 & \text{on } \mathbb{R}\backslash(\operatorname{cl}\Omega) \\ U(x) & = & 1 & \text{for } x \in \partial\Omega \text{ and } \lim_{x\to\infty} U(x) = 0 \end{array} \tag{9.55}$$

where Δ is the Laplace operator. Now $\Omega = \operatorname{int} K$ for the convex convex body $K = \operatorname{cl}\Omega$, and the corresponding finite Borel measure $\mu_{2,\Omega}$ on S^{n-1} - the so-called electrostatic capacitary measure - is defined by the formula

$$\mu_{2,\Omega}(\omega) = \int_{\nu_K^{-1}(\omega)} \|DU_\Omega\|^2 \, d\mathcal{H}^{n-1}$$

for Borel $\omega \subset S^{n-1}$ where the integral on $\partial K = \partial\Omega$ makes sense and finite according to Dahlberg [184]. According to the Poincaré formula, we have

$$C_2(\Omega) = \frac{1}{n-2} \int_{S^{n-1}} h_K \, d\mu_{2,\Omega}. \tag{9.56}$$

For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x, u\rangle < h_K(u) + tg(u), \ \forall u \in S^{n-1}\}$, the electrostatic capacitary measure satisfies the variational formula (cf. Jerison [354])

$$\frac{d}{dt} C_2(\Omega_t)\bigg|_{t=0} = \int_{S^{n-1}} g \, d\mu_{2,\Omega} \tag{9.57}$$

which actually yields (9.56) by taking $g = h_K$.

The Minkowski problem for capacity and a finite Borel measure μ on S^{n-1} has been solved by Jerison [353], proving that $\mu = \mu_{2,\Omega}$ for a bounded open convex set $\Omega \subset \mathbb{R}^n$ if and only if μ satisfies Minkowki's conditions (a) and (b) in Theorem 9.2.3; namely, the centroid of μ is the origin of $\mathbb{R}^n$, and μ is not concentrated on any closed hemisphere. Uniqueness of the solution up to translation was clarified by Caffarelli, Jerison, Lieb [139] using their result on (9.54).

If $n = 2$, then the notion of capacity has been replaced by the notion of so-called *transfinite diameter*, and the corresponding Minkowski problem is solved again by Jerison [354].

For $n \geq 3$, $p \in (1, n)$ and a bounded open convex set $\Omega \subset \mathbb{R}^n$, the p-capacity of Ω is $C_p(\Omega) = \sup \{C_p(X) : X \subset \Omega \text{ compact}\}$, where

$$C_p(X) = \inf \left\{ \int_{\mathbb{R}^n} \|D\varphi\|^p : \varphi \in C_c^\infty(\mathbb{R}^n), \ \varphi(x) \geq 1 \text{ for } x \in X \right\}$$

for any compact $X \subset \mathbb{R}^n$. We note that $C_p(s\,\Omega) = s^{n-p} C_p(\Omega)$ for $s > 0$. Colesanti, *et al* [172] associates a so-called p-capacitary measure - a finite Borel measure - $\mu_{p,\Omega}$ on S^{n-1} to Ω, and proves the variational formula

$$\frac{d}{dt} C_p(\Omega_t)\bigg|_{t=0} = \int_{S^{n-1}} g \, d\mu_{p,\Omega}$$

using the notion of (9.57). It follows that

$$C_p(\Omega) = \frac{1}{n-p} \int_{S^{n-1}} h_K \, d\mu_{p,\Omega}$$

where $K = \mathrm{cl}\,\Omega$. Here the case $p = 2$ is the classical case.

The analogue of the Brunn-Minkowski-type inequality (9.54) was proved by Colesanti, Salani [168] for $p \in (1, n)$ where the exponent $\frac{1}{n-2}$ is replaced by $\frac{1}{n-p}$, and Colesanti, *et al* [172]) propose the Minkowski problem for p-capacity for $p \in (1, n)$, characterize uniqueness for $p \in (1, n)$, and characterize the solution for $p \in (1, 2)$. For some additional generalizations of the notion of p-capacity, see, for example, Hong, Ye [316], Hong, Ye, Zhang [317] and Liu, Sheng [417].

Next we turn to the torsional rigidity $\tau(\Omega)$ of a bounded open convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, (cf. Colesanti, Fimiani [167] or Langharst, Ulivelli [391]), namely,

$$\tau(\Omega) = \int_\Omega \|DU\|^2$$

where U is the unique solution of the boundary value problem

$$\begin{array}{rcll} -\Delta U & = & 2 & \text{on } \Omega, \\ U(x) & = & 0 & \text{for } x \in \partial\Omega, \end{array} \tag{9.58}$$

and hence $\tau(s\,\Omega) = s^{n+2}\tau(\Omega)$ for $s > 0$. Using the U in (9.58) and the convex body $K = \mathrm{cl}\,\Omega$, the corresponding finite Borel measure $\mu_{\tau,\Omega}$ on S^{n-1} is defined by

$$\mu_{\tau,\Omega}(\omega) = \int_{\nu_K^{-1}(\omega)} \|DU\|^2 \, d\mathcal{H}^{n-1}$$

for Borel $\omega \subset S^{n-1}$ where the integral on $\partial K = \partial\Omega$ makes sense and finite according to Dahlberg [184].

Colesanti, Fimiani [167] handled the first variation of torsional rigidity at a bounded open convex set $\Omega \subset \mathbb{R}^n$ with $K = \mathrm{cl}\,\Omega$. For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x, u\rangle < h_K(u) + tg(u), \ \forall u \in S^{n-1}\}$, [167] proves that

$$\frac{d}{dt}\tau(\Omega_t)\bigg|_{t=0} = \int_{S^{n-1}} g \, d\mu_{\tau,\Omega},$$

and hence

$$\tau(\Omega) = \frac{1}{n+2}\int_{S^{n-1}} h_K \, d\mu_{\tau,\Omega}.$$

Concerning the Minkowski problem for torsion rigity, still no characteristic necessary condition is known, while Colesanti, Fimiani [167] proved that given a finite Borel measure μ on S^{n-1}, $\mu = \mu_{\tau,\Omega}$ for a bounded open convex set $\Omega \subset \mathbb{R}^n$ if μ satisfies Minkowki's conditions (a) and (b) in Theorem 9.2.3, and the solution is unique up to translation in this case.

The L_p torsional rigidity Minkowski problem was posed by Chen, Dai [159] to characterize the Borel measure $h_K^{1-p}\, d\mu_{\tau,\Omega}$ on S^{n-1} where $K = \mathrm{cl}\,\Omega$ for the bounded open convex set $\Omega \subset \mathbb{R}^n$, and [159] proves existence and uniqueness results when $p > 1$. In addition, existence results have been provided by Hu, Liu [320] if $p \in (0, 1)$, and by Hu [319] if $p = 0$, where in the latter case, "subspace concentration conditions" like in (9.18) for the cone volume measure play a crucial role. Additional versions of the Minkowski problem for torsional rigidity have been considered by Hu, Liu, Ma [321], Hu, Zhang [323] and Hu, Li [328].

Another related problem is about the first eigenvalue of the Laplacian. For any open bounded convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, let $\lambda(\Omega)$ be the smallest positive (the principal) eigenvalue of $-\Delta$ on Ω with Dirichlet condition; namely, the smallest positive number such that there exists a continuous function $V = V_\Omega$ on $\mathrm{cl}\,\Omega$ such that $\int_\Omega V^2 = 1$, V is C^∞ on Ω, and

$$\begin{array}{rcll} -\Delta V & = & \lambda(\Omega)V & \text{on } \ \Omega, \\ V(x) & = & 0 & \text{for } \ x \in \partial\Omega \end{array}$$

(cf. Jerison [354]) where - using that $\lambda(\Omega)$ is the principal eigenvalue - we may also assume that V_Ω is positive on Ω in order to make it unique. Here $\lambda(s\,\Omega) = s^{-2}\lambda(\Omega)$ for $s > 0$. For a continuous $g : S^{n-1} \to \mathbb{R}$ and the open Wulff shape $\Omega_t = \{x \in \mathbb{R}^n : \langle x, u\rangle < h_K(u) + tg(u),\ \forall u \in S^{n-1}\}$, the first eigenvalue of the Laplacian satisfies the variational formula (cf. Jerison [354])

$$\frac{d}{dt}\lambda(\Omega_t)\bigg|_{t=0} = \int_{\partial K} g(\nu_K(x))\|V_\Omega(x)\|^2 \, d\mathcal{H}^{n-1} \tag{9.59}$$

for the convex body $K = \mathrm{cl}\,\Omega$ where the integral on $\partial K = \partial\Omega$ makes sense and finite according to Dahlberg [184]. In turn, one deduces from (9.59) taking $g = h_K$ that

$$\lambda(\Omega) = \frac{1}{2}\int_{\partial K} h_K(\nu_K(x))\|V_\Omega(x)\|^2 \, d\mathcal{H}^{n-1}.$$

Concerning the even Minkowski problem for the first eigenvalue of the Laplacian, Langharst, Ulivelli [391] handles the sufficiency part under the "subspace concentration conditions" (9.18).

Jerison [354] placed the notions of capacity, torsion rigidity and the first eigenvalue of the Laplacian into a more general framework where the homogeneity and the formula for the first variation play crucial role, see Colesanti [166] for the corresponding Brunn-Minkowski type inequalities, and Crasta, Fragalà [183] for the corresponding Firey-type evolution equations, and Langharst, Ulivelli [391] for the sufficiency concerning the even Minkowski problem under the "subspace concentration conditions" (9.18).

9.A Supplement: Weak continuity of the L_p surface area measures and diameter bounds in terms of the entropy

In this section, we discuss two fundamental technical properties related to the L_p surface area measure for $p \in \mathbb{R}$. The first is the weak continuity of the L_p surface area measure, and the second property is a bound of the diameter of a convex body in terms of the "L_p entropy".

Lemma 9.A.1. *Let $p \in \mathbb{R}$, and let K_m tend to K for $K_m, K \in \mathcal{K}_o$.*

(i) $S_{K_m,p}$ tends weakly to $S_{K,p}$ if either $p \leq 1$, or $p > 1$ and $K_m, K \in \mathcal{K}_{(o)}$.

(ii) If $p > 1$, $K_m \in \mathcal{K}_{(o)}$, and $\{S_{K_m,p}(S^{n-1})\}$ is bounded, then $S_K(\{h_K = 0\}) = 0$, and hence $S_{K,p}$ is well-defined, and $S_{K_m,p}$ tends weakly to $S_{K,p}$.

Proof. We deduce (i) from the facts that S_{K_m} tends weakly to S_K according to Proposition 2.6.12 (see also Proposition 8.4.1), and $g \cdot h_{K_m}^{1-p}$ tends uniformly to $g \cdot h_K^{1-p}$ for any continuous $g : S^{n-1} \to \mathbb{R}$ as h_{K_m} tends uniformly to h_K (cf. Definition 1.7.1).

For (ii), we may assume that $o \in \partial K$ by (i), and let $R > 0$ such that $S_{K_m,p}(S^{n-1}) < R$ for every m. First we show that $S_{K,p}$ is well-defined; namely, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$S_K(\Xi_\delta) \leq \varepsilon \ \text{ for } \ \Xi_\delta = \left\{u \in S^{n-1} : \ h_K(u) < \delta\right\}. \tag{9.60}$$

We choose $\delta > 0$ in a way such that $(2\delta)^{1-p} \cdot \frac{\varepsilon}{2} > R$. If m is large, then $h_{K_m}(u) < 2\delta$ for $u \in \Xi_\delta$, and as Ξ_δ is a non-empty open subset of ∂K, we have $S_{K_m}(\Xi_\delta) > \frac{1}{2} \cdot S_K(\Xi_\delta)$ by the weak convergence of S_{K_m} (cf. Remark 10.1.1). We deduce that

$$\begin{aligned}(2\delta)^{1-p} \cdot \frac{S_K(\Xi_\delta)}{2} &\leq (2\delta)^{1-p} \cdot S_{K_m}\left(\left\{u \in S^{n-1} : \ h_{K_m}(u) \leq 2\delta\right\}\right)\\ &\leq \int_{\left\{u \in S^{n-1} : \, h_{K_m}(u) \leq 2\delta\right\}} h_{K_m}^{1-p} \, dS_{K_m} \leq R,\end{aligned}$$

yielding (9.60), and in turn the well-definedness of $S_{K,p}$.

Given a continuous $g : S^{n-1} \to \mathbb{R}$, $g \cdot h_{K_m}^{1-p}$ tends uniformly to $g \cdot h_K^{1-p}$ on $S^{n-1} \backslash \Xi_\delta$ for any small $\delta > 0$. Therefore, $\lim_{m\to\infty} \int_{S^{n-1}} g \, dS_{K_m,p} = \int_{S^{n-1}} g \, dS_{K,p}$ follows from (9.60). ■

One of the key tools in handling the L_p-Minkowski problem are the diameter bounds in terms of the entropy. Let μ be a Borel probability measure on S^{n-1}, and let $C \in \mathcal{K}_o$. If $p > 1$, then we use the entropy function

$$\mathcal{E}_{\mu,p}(C) = \frac{1}{p} \int_{S^{n-1}} h_C^p \, d\mu \tag{9.61}$$

of a C, which is continuous in C. For $p < 1$ and $\xi \in \operatorname{int} C$, we consider the entropy function

$$\mathcal{E}_{\mu,p}(C,\xi) = \begin{cases} \frac{1}{p}\int_{S^{n-1}} h_{C-\xi}^p \, d\mu & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{C-\xi} \, d\mu & \text{if} \quad p = 0, \end{cases} \tag{9.62}$$

which is continuous in C and $\xi \in \operatorname{int} C$.

We introduce some additional notions which help to describe how much a measure on S^{n-1} is concentrated to great sub-spheres. Let $\delta \in [0,1)$. For a linear i-subspace L of $\mathbb{R}^n$ with $1 \leq \dim L \leq n-1$, we consider the collar

$$\Psi(L \cap S^{n-1}, \delta) = \{x \in S^{n-1} : \langle x, y \rangle \leq \delta \text{ for } y \in L^\perp \cap S^{n-1}\}.$$

In addition, for a $u \in S^{n-1}$, an open spherical cap centered at u is

$$\Omega(u,\delta) = \{v \in S^{n-1} : \langle v, u \rangle > \delta\}$$

where $\Omega(u,0)$ is an open hemisphere (cf. (9.6)). We recall that σ_K is the centroid of a convex body $K \subset \mathbb{R}^n$ (cf. Definition 1.11.1).

Proposition 9.A.2. *Let μ be a Borel probability measure on S^{n-1}, let $K \in \mathcal{K}_o^n$ with* $\operatorname{diam} K = D$, *and let $\delta, \tau \in (0,1)$.*

(i) *If $\mu(\Omega(u,\delta)) > \tau$ for any $u \in S^{n-1}$, then*

$$\frac{\tau\,\delta^p}{2^p} \cdot D^p \leq \begin{cases} p\,\mathcal{E}_{\mu,p}(K) & \textit{provided} \quad p > 1 \\ p\,\mathcal{E}_{\mu,p}(K,\sigma_K) & \textit{provided} \quad p \in (0,1). \end{cases}$$

(ii) *If $p = 0$, $|K| = 1$, and*

$$\mu\left(\Psi(L \cap S^{n-1}, \delta)\right) < \frac{(1-\tau)i}{n}$$

for any linear i-subspace $L \subset \mathbb{R}^n$, $i = 1, \ldots, n-1$, then

$$\mathcal{E}_{\mu,0}(K,\sigma_K) \geq \tau \log D + \log\delta - 6\log n.$$

(iii) *If $-n < p < 0$, and $d\mu = f\,\mathcal{H}^{n-1}$ for $f \in L^{\frac{n}{n+p}}(S^{n-1}, \mathcal{H}^{n-1})$ where*

$$\int_{\Psi(u^\perp \cap S^{n-1},\delta)} f^{\frac{n}{n+p}} \leq (n\omega_n^2)^{\frac{p}{n+p}} \cdot \tau^{\frac{n}{n+p}}$$

for any $u \in S^{n-1}$, then any K with $\tau \leq \frac{p}{2}\,\mathcal{E}_{\mu,p}(K,\sigma_K)$ satisfies

$$\textit{either } D \leq 4n^2/\delta^2, \quad \textit{or } \ D \leq \left(\frac{p}{2}\,\mathcal{E}_{\mu,p}(K,\sigma_K)\right)^{\frac{2}{p}}.$$

Remark. In the applications, the K in (iii) is obtained via minimizing the entropy, e.g. it satisfies $\mathcal{E}_{\mu,p}(K,\sigma_K) \leq \sup_{\xi \in \operatorname{int} \widetilde{B}} \mathcal{E}_{\mu,p}(\widetilde{B},\xi)$ for the centered ball $\widetilde{B}$ of volume 1, and hence the τ in (iii) can be chosen independently of K.

Proof. Let $R = \max_{x\in K} \|x\|$, and hence $D \le 2R$, and let $x_0 \in K$ and $u_0 \in S^{n-1}$ with $u_0 \in S^{n-1}$ and $x_0 = R\,u_0$.

If $p < 1$, then we may assume that $\sigma_K = o$, and hence the KLS ellipsoid E (cf. Lemma 1.11.5) satisfies that

$$E \subset K \subset n\,E, \tag{9.63}$$

which in turn also yields that

$$-x/n \in K \quad \text{for } x \in K. \tag{9.64}$$

For (i), we observe that $h_K(u) \ge \delta R$ for $u \in \Omega(u_0, \delta)$ and $\mu\,(\Omega(u_0, \delta)) \ge \tau$; therefore, $\int_{S^{n-1}} h_K^p \,d\mu \ge \tau\,(R\delta)^p \ge \tau\,(D\delta/2)^p$.

For (ii), let $e_1, \dots, e_n \in S^{n-1}$ be orthonormal basis of $\mathbb{R}^n$ forming the principal directions associated to the ellipsoid E in (9.63), and let $r_1, \dots, r_n > 0$ be the half axes of E with $r_i e_i \in \partial E$ where we may assume that $r_1 \le \dots \le r_n$. In particular, (9.63) yields that

$$\prod_{i=1}^{n} r_i = \frac{|(nE|}{n^n \omega_n} \ge \frac{|K|}{n^n \omega_n} = \aleph_n \quad \text{for } \aleph_n = n^{-n}\omega_n^{-1} < 1. \tag{9.65}$$

We observe that for any $v \in S^{n-1}$, there exists e_i such that $|\langle v, e_i\rangle| \ge \frac{1}{\sqrt{n}} > \frac{\delta}{n}$. For $i = 1, \dots, n$, we define

$$B_i = \left\{ v \in S^{n-1} : |\langle v, e_i\rangle| \ge \frac{\delta}{n} \text{ and } |\langle v, e_j\rangle| < \frac{\delta}{n} \text{ for } j > i \right\}.$$

In particular, $B_i \subset \Psi(L_i \cap S^{n-1}, \delta)$ for $i = 1, \dots, n$ and $L_i = \text{lin}\{e_1, \dots, e_i\}$.

It follows that S^{n-1} is partitioned into the Borel sets $B_1, \dots, B_n$, and as $B_i \subset \Psi(L_i \cap S^{n-1}, \delta)$ for $i = 1, \dots, n-1$, we have

$$\mu(B_1) + \dots + \mu(B_i) \le \frac{i(1-\tau)}{n} \quad \text{for } i = 1, \dots, n-1 \tag{9.66}$$

$$\mu(B_1) + \dots + \mu(B_n) = 1. \tag{9.67}$$

For $\zeta = \frac{1-\tau}{n}$, we have $0 < \zeta < \frac{1}{n}$, and we define

$$\beta_i = \mu(B_i) - \zeta \quad \text{for } i = 1, \dots, n-1 \tag{9.68}$$

$$\beta_n = \mu(B_n) - \zeta - \tau \tag{9.69}$$

where (9.66) and (9.67) yield

$$\beta_1 + \dots + \beta_i \le 0 \quad \text{for } i = 1, \dots, n-1 \tag{9.70}$$

$$\beta_1 + \dots + \beta_n = 0. \tag{9.71}$$

As $r_ie_i \in K$, it follows from the definition of B_i that $h_K(u) \geq \langle u, r_ie_i\rangle \geq r_i \cdot \frac{\delta}{n}$ for $u \in B_i$, $i = 1, \ldots, n$. We deduce from applying (9.65), (9.67), (9.68), (9.69), (9.70), (9.71), $r_1 \leq \ldots \leq r_{n+1}$ and $\zeta < \frac{1}{n}$ that

$$\begin{aligned}
\int_{S^{n-1}} \log h_K\, d\mu &= \sum_{i=1}^n \int_{B_i} \log h_K\, d\mu \\
&\geq \sum_{i=1}^n \mu(B_i) \log r_i + \sum_{i=1}^n \mu(B_i) \log \frac{\delta}{n} = \sum_{i=1}^n \mu(B_i) \log r_i + \log \frac{\delta}{n} \\
&= \sum_{i=1}^n \beta_i \log r_i + \sum_{i=1}^n \zeta \log r_i + \tau \log r_n + \log \frac{\delta}{n} \\
&\geq \sum_{i=1}^n \beta_i \log r_i + \zeta \log \aleph_n + \tau \log r_n + \log \frac{\delta}{n} \\
&= (\beta_1 + \ldots + \beta_n) \log r_n + \sum_{i=1}^{n-1} (\beta_1 + \ldots + \beta_i)(\log r_i - \log r_{i+1}) \\
&\quad + \zeta \log \aleph_n + \tau \log r_n + \log \frac{\delta}{n} \\
&\geq \tau \log r_n + \log \delta + \log \aleph_n^{1/n} - \log n.
\end{aligned}$$

Now $D \leq n \operatorname{diam} E = 2nr_n \leq n^2 r_n$ and $\tau < 1$, and hence $\tau \log r_n \geq \tau \log D - 2 \log n$. In addition, $\aleph_n^{1/n} = \frac{\Gamma(\frac{n}{2}+1)^{\frac{2}{n}}}{n\sqrt{\pi}} > (2e\pi n)^{\frac{-1}{2}} > n^{-3}$; therefore,

$$\tau \log r_n + \log \delta + \log \aleph_n^{1/n} - \log n \geq \tau \log D + \log \delta - 6 \log n.$$

For (iii), when $-n < p < 0$, we may assume that

$$D \geq 4n^2/\delta^2,$$

and we consider

$$\begin{aligned}
\Phi_0 &= \left\{ u \in S^{n-1} : h_K(u) > \sqrt{2R} \right\}; \\
\Phi_1 &= \left\{ u \in S^{n-1} : h_K(u) \leq \sqrt{2R} \right\}.
\end{aligned}$$

Concerning Φ_0, we have

$$\int_{\Phi_0} f \cdot h_K^p \leq (2R)^{p/2} \int_{\Phi_0} f \leq D^{\frac{p}{2}}. \tag{9.72}$$

On the other hand, we have $\pm \frac{R}{n} u_0 \in K$ by (9.64). Thus any $u \in \Phi_1$ satisfies

$$\sqrt{2R} \geq h_K(u) \geq \left| \left\langle u, \frac{R}{n} u_0 \right\rangle \right|,$$

and hence $|\langle u, u_0\rangle| \le n\sqrt{\frac{2}{R}} \le \frac{2n}{\sqrt{D}} \le \delta$; or in other words,

$$\Phi_1 \subset \Psi(u_0^\perp \cap S^{n-1}, \delta).$$

It follows from $|K| = 1$ and the from (6.26) of the Blaschke-Santaló inequality that

$$\int_{S^{n-1}} h_K^{-n} \le n\omega_n^2.$$

For $p \in (-n, 0)$, Hölder's inequality and $\int_{\Phi_1} f^{\frac{n}{n+p}} < (n\omega_n^2)^{\frac{p}{n+p}} \cdot \tau^{\frac{n}{n+p}}$ yield

$$\int_{\Phi_1} f \cdot h_K^p \le \left(\int_{\Phi_1} f^{\frac{n}{n+p}}\right)^{\frac{n+p}{n}} \left(\int_{\Phi_1} h_K^{-n}\right)^{\frac{|p|}{n}} \le \tau.$$

Finally, adding the last estimate to (9.72) yields

$$p\,\mathcal{E}_{\mu,p}(K, \sigma_K) = \int_{S^{n-1}} f \cdot h_K^p \le D^{\frac{p}{2}} + \tau,$$

and hence the condition $\tau \le \frac{p}{2}\,\mathcal{E}_{\mu,p}(K)$ on τ implies (iii). ■

9.B Supplement: The even L_p-Minkowski problem for $p \in [0, 1)$

This section about the even L_p-Minkowski surface area measures discusses work by Haberl, Lutwak, Yang, Zhang [292] if $p \in (0, 1)$ (cf. Corollary 9.B.2), and by Böröczky, Lutwak, Yang, Zhang [111] if $p = 0$, and proves Proposition 9.4.6 (cf. Proposition 9.B.1), and Theorem 9.3.6 (cf. Theorem 9.B.5). The key tool is the notion of entropy defined in (9.62); namely, if $C \subset \mathbb{R}^n$ is an o-symmetric convex body, then

$$\mathcal{E}_{\mu,p}(C, o) = \begin{cases} \frac{1}{p}\int_{S^{n-1}} h_C^p\,d\mu & \text{if} \quad p \in (0, 1), \\ \int_{S^{n-1}} \log h_C\,d\mu & \text{if} \quad p = 0, \end{cases} \tag{9.73}$$

which is continuous in C. For the reader's convenience, we restate Proposition 9.4.6.

Proposition 9.B.1. *For $p \in [0, 1)$, and an even probability Borel measure μ on S^{n-1} such that any open hemisphere has positive measure, and, in addition*

$$\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1}) \tag{9.74}$$

holds for any non-trivial linear subspace $L \subset \mathbb{R}^n$ provided $p = 0$, then

(a) *$\mathcal{E}_{\mu,p}(C, o)$ attains its minimum among o-symmetric convex bodies $C \subset \mathbb{R}^n$ satisfying $|C| = 1$.*

(b) Moreover, if an o-symmetric convex body $K \subset \mathbb{R}^n$ with $|K| = 1$ minimizes $\mathcal{E}_{\mu,p}(C, o)$ among o-symmetric convex bodies $C \subset \mathbb{R}^n$ with $|C| = 1$, then $\mu = S_{\theta K,p}$ for $\theta = \left(\frac{1}{n}\int_{S^{n-1}} h_K^p \, d\mu\right)^{\frac{1}{n-p}}$.

Proof. Let $\mathcal{C}$ be the set of o-symmetric convex bodies C with $|C| = 1$. As for any $i = 1, \dots, n-1$, the Grassmannian space of all i-dimensional linear subspaces is compact, the conditions on μ imply that there exist $\delta, \tau \in (0, 1)$ such that

$$\begin{array}{rcll} \mu\left(\Omega(u, \delta)\right) & > & \tau & \text{if } p \in (0,1) \text{ and } u \in S^{n-1}; \\ \mu\left(\Psi(L \cap S^{n-1}, \delta)\right) & < & \frac{(1-\tau)i}{n} & \text{if } p = 0 \text{ and } L \subset \mathbb{R}^n \text{ linear } i\text{-subspace}, \\ & & & i = 1, \dots, n-1; \end{array} \tag{9.75}$$

namely, μ satisfies the conditions in Proposition 9.A.2.

The argument is easier presented using another notion of entropy that is equivalent to the one defined (9.73) for our purposes: if $C \subset \mathbb{R}^n$ is an o-symmetric convex body, then let

$$\overline{\mathcal{E}}_{\mu,p}(C, o) = \begin{cases} \frac{1}{p}\log \int_{S^{n-1}} h_C^p \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p \in (0, 1), \\ \int_{S^{n-1}} \log h_C \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p = 0, \end{cases}$$

which notion is invariant under rescaling; namely, it satisfies

$$\overline{\mathcal{E}}_{\mu,p}(\lambda C, o) = \overline{\mathcal{E}}_{\mu,p}(C, o) \ \text{ for } \lambda > 0. \tag{9.76}$$

Let $\widetilde{B}^n$ be the o-symmetric Euclidean ball with $|\widetilde{B}^n| = 1$, and let $C_m \in \mathcal{C}$ satisfy that $\overline{\mathcal{E}}_{\mu,p}(C_m, o) \leq \overline{\mathcal{E}}_{\mu,p}(\widetilde{B}^n, o)$ and $\overline{\mathcal{E}}_{\mu,p}(C_m, o)$ tends to $\inf\left\{\overline{\mathcal{E}}_{\mu,p}(C, o) : C \in \mathcal{C}\right\}$. It follows from Proposition 9.A.2 that the sequence $\{C_m\}$ is bounded; therefore, we may assume that C_m tends to an o-symmetric convex compact set K by the Blaschke Selection Theorem 1.7.3. As each $|C_m| = 1$, we have $|K| = 1$ by the continuity of volume (cf. Lemma 1.7.4), thus $K \in \mathcal{C}$; therefore,

$$\overline{\mathcal{E}}_{\mu,p}(K, o) = \min\left\{\overline{\mathcal{E}}_{\mu,p}(C, o) : C \in \mathcal{C}\right\}. \tag{9.77}$$

We claim that

$$\mu = \lambda \cdot S_{K,p} \quad \text{for } \lambda = \frac{1}{n}\int_{S^{n-1}} h_K^p \, d\mu, \tag{9.78}$$

where (9.78) is equivalent with saying that

$$\int_{S^{n-1}} g \cdot h_K^{p-1} \, d\mu = \lambda \cdot \int_{S^{n-1}} g \, dS_K \tag{9.79}$$

for any continuous function $g : S^{n-1} \to \mathbb{R}$. For $t \geq 0$, we consider the Wulff-shape

$$K_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_K(u) + tg(u) \ \forall u \in S^{n-1}\},$$

and hence $K_0 = K$, and the Aleksandrov Lemma 9.2.2 that

$$\frac{d}{dt}|K_t|\bigg|_{t=0} = \int_{S^{n-1}} g\, dS_K. \tag{9.80}$$

If $|t|$ is small, then we consider the differentiable function

$$f(t) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}}(h_K + tg)^p\, d\mu - \frac{1}{n}\log|K_t| & \text{if} \quad p \in (0,1), \\ \int_{S^{n-1}}\log(h_K + tg)\, d\mu - \frac{1}{n}\log|K_t| & \text{if} \quad p = 0, \end{cases}$$

which, according to $|K_t|^{\frac{-1}{n}} \cdot K_t \in C$, $h_{K_t} \leq h_K(u) + tg(u)$, (9.76) and (9.77), satisfies that

$$f(t) \geq \overline{\mathcal{E}}_{\mu,p}(K_t, o) = \overline{\mathcal{E}}_{\mu,p}\left(|K_t|^{\frac{-1}{n}} \cdot K_t, o\right) \geq f(0).$$

In particular, f has a minimum at $t = 0$, and hence (9.80) implies that

$$0 = f'(0) = \frac{1}{n\lambda}\int_{S^{n-1}} g \cdot h_K^{p-1}\, d\mu - \frac{1}{n}\int_{S^{n-1}} g\, dS_K,$$

proving (9.79), and in turn (9.78). Therefore, $\mu = S_{M,p}$ for $M = \lambda^{\frac{1}{n-p}} K$. ■

Proposition 9.B.1 directly implies the characterization of even L_p-surface area measures for $p \in (0,1)$ due to Haberl, Lutwak, Yang, Zhang [292].

Corollary 9.B.2. *For $p \in (0,1)$, and a finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = S_{K,p}$ if and only if the measure of any open hemi-sphere is positive.*

The characterization of even L_0-surface area measure (cone volume measure) due to Böröczky, Lutwak, Yang, Zhang [111] is more involved. We recall that according to (2.27) and (2.30), if $\omega \subset \mathbb{R}^n$ is a Borel set and $K \in \mathcal{K}_o^n$, then

$$V_K(\omega) = \left|\bigcup\{\mathrm{conv}\{o,x\} : x \in \partial K \text{ and } N_K(x) \cap \omega \neq \emptyset\}\right| \tag{9.81}$$

$$= \frac{1}{n}\int_{\nu_K^{-1}(\omega)} \langle \nu_K(x), x\rangle\, d\mathcal{H}^{n-1}(x). \tag{9.82}$$

We say that the linear subspaces $L_1, \ldots, L_m \subset \mathbb{R}^n$ of dimension at least 1 are complementary if $\sum_{i=1}^m L_i = \mathbb{R}^n$ and $L_i \cap L_j = \{o\}$ for $i \neq j$. The following statement is a direct consequence of (9.82).

Lemma 9.B.3. *Let $C_1, \ldots, C_m \subset \mathbb{R}^n$ be compact convex sets of dimension $d_i \geq 1$ and containing o such that $L_i = \mathrm{lin}\, C_i$, $i = 1, \ldots, m$, are complementary and pair-wise orthogonal linear subspaces. In this case, writing V_{C_i} to denote the cone volume measure of C_i on $L_i \cap S^{n-1}$,*

- $\mathrm{supp}\, V_K \subset \bigcup_{i=1}^m L_i$ *for $K = \sum_{i=1}^m C_i$,*

- *if $\omega \subset L_i \cap S^{n-1}$ is Borel for $i \in \{1, \ldots, m\}$, then*

$$V_K(\omega) = \frac{d_i}{n} \left(\prod_{j \neq i} \mathcal{H}^{d_j}(C_j) \right) \cdot V_{C_i}(\omega).$$

We deduce via the $\mathrm{SL}(n)$ equivariance of the cone volume measure (cf. Proposition 2.6.15) the following ambiguity. According to the Logarithmic Minkowski Conjecture 9.4.5, this would be the only ambiguity concerning recovering the o-symmetric convex body its even cone volume measure.

Corollary 9.B.4. *If $\sum_{i=1}^m C_i$ is a convex body in $\mathbb{R}^n$ for compact convex sets $C_1, \ldots, C_m \subset \mathbb{R}^n$ of dimension $d_i \geq 1$ and containing o such that $\sum_{i=1}^m \dim C_i = n$, and $\lambda_1, \ldots, \lambda_m > 0$ satisfy that $\prod_{i=1}^m \lambda_i^{d_i} = 1$, or equivalently, $|\sum_{i=1}^m C_i| = |\sum_{i=1}^m \lambda_i C_i|$, then*

$$V_{\sum_{i=1}^m C_i} = V_{\sum_{i=1}^m \lambda_i C_i}.$$

We arrived at the main result of this section.

Theorem 9.B.5. *For a non-trivial finite even Borel measure μ on S^{n-1}, there exists an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = V_K = \frac{1}{n} S_{K,0}$ if and only if*

(i) $\mu(L \cap S^{n-1}) \leq \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *for any proper linear subspace $L \subset \mathbb{R}^n$;*

(ii) $\mu(L \cap S^{n-1}) = \frac{\dim L}{n} \cdot \mu(S^{n-1})$ *in (i) is equivalent with the existence of a complementary linear subspace $L' \subset \mathbb{R}^n$ with $\operatorname{supp} \mu \subset L \cup L'$, and in this case, $K = C + C'$ for o-symmetric compact convex sets $C \subset L^\perp$ and $C' \subset L'^\perp$.*

Proof. Step 1. If a non-trivial even Borel measure μ on S^{n-1}, $n \geq 1$, satisfies (i) and (ii), then $\mu = V_K$ for an o-symmetric convex body $K \subset \mathbb{R}^n$.

We prove this by induction on $n \geq 1$. When $n = 1$, then the only condition on μ on S^0 is its evenness, and $\mu = V_K$ for the o-symmetric segment $K \subset \mathbb{R}$ with $|K| = \mu(S^0)$. Therefore, let $n \geq 2$.

If $\mu(L \cap S^{n-1}) < \frac{\dim L}{n} \cdot \mu(S^{n-1})$ for any non-trivial linear subspace $L \subset \mathbb{R}^n$, then Proposition 9.B.1 yields the existence of an o-symmetric convex body $K \subset \mathbb{R}^n$ with $\mu = V_K$.

Next, we assume that, in line with condition (ii), there exist complementary linear subspaces $L_1, L_2 \subset \mathbb{R}^n$ of dimensions $d_1, d_2 \geq 1$ such that $\operatorname{supp} \mu \subset L_1 \cup L_2$, and hence (i) yields that $\mu(L_i \cap S^{n-1}) = \frac{d_i}{n} \cdot \mu(S^{n-1})$, $i = 1, 2$. For $i = 1, 2$, the restriction μ_i of μ onto $L_i \cap S^{n-1}$ also satisfies the conditions (i) and (ii). Let $\Phi \in SL(n)$ such that ΦL_1 and ΦL_2 are orthogonal. It follows from the linear equivariance of the cone volume measure (cf. Proposition 2.6.15) and Lemma 9.B.3 that it is sufficient to find o-symmetric full dimensional compact convex sets $C_i \subset \Phi L_i$ for $i = 1, 2$ such that

$$\text{the restriction of } V_{C_1 + C_2} \text{ to } S^{n-1} \cap \Phi L_i \text{ is } \Phi_* \mu_i \text{ for } i = 1, 2. \tag{9.83}$$

For $i = 1, 2$, $\Phi_*\mu_i$ also satisfies the conditions (i) and (ii), and hence the induction hypothesis yields the existence of full dimensional o-symmetric compact convex sets $M_i \subset \Phi L_i$ for $i = 1, 2$, whose cone volume measures are $\Phi_*\mu_1$ and $\Phi_*\mu_2$. In particular,

$$\mathcal{H}^{d_i}(M_i) = \frac{d_i}{n} \cdot \mu(S^{n-1}) \quad \text{for } i = 1, 2.$$

Therefore, to construct the C_1 and C_2 in (9.83), all we need is to adjust the volume of $M_1 + M_2$ according to Lemma 9.B.3. Thus, we choose $\lambda_1, \lambda_2 > 0$ such that $\mu(S^{n-1}) = \prod_{i=1}^{2} \lambda_i^{d_i} \mathcal{H}^{d_i}(M_i)$, and the $C_i = \lambda_i M_i$, $i = 1, 2$, satisfy (9.83). In turn, $\mu = \Phi_*^{-1} V_{C_1+C_2} = V_{\Phi^t(C_1+C_2)}$.

Step 2. If $K \subset \mathbb{R}^n$ is an o-symmetric convex body, and $L \subset \mathbb{R}^n$ is an i-dimensional linear subspace for $i \in \{1, \dots, n-1\}$, then (i) and (ii) hold.

We consider the orthogonal projection $K' = \Pi_L K$. For any $x \in \operatorname{relbd} K'$, let $M(x) = K \cap (x + L^\perp)$, and hence $-x \in \operatorname{relbd} K'$ and $M(-x) = -M(x)$. If $z = tx$ for $t \in [0, 1]$, then $z = (1-\lambda)x + \lambda(-x)$ for $\lambda = \frac{1-t}{2} \in [0, \frac{1}{2}]$, and the convexity of K, $M(-x) = -M(x)$ and the Brunn-Minkowski inequality Theorem 1.12.3 imply

$$\mathcal{H}^{n-i}\left(K \cap (z + L^\perp)\right) \geq \mathcal{H}^{n-i}\left((1-\lambda)M(x) + \lambda\, M(-x)\right) \geq \mathcal{H}^{n-i}\left(M(x)\right). \tag{9.84}$$

We deduce from the Fubini theorem and using polar coordinates in L (cf. (1.26)) that

$$\begin{aligned}
|K| &= \int_{K'} \mathcal{H}^{n-i}\left(K \cap (z + L^\perp)\right) d\mathcal{H}^i(z) \\
&= \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \mathcal{H}^{n-i}\left(K \cap (ru + L^\perp)\right) r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\
&\geq \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right) r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\
&= \frac{1}{i} \int_{S^{n-1}\cap L} \varrho_{K'}(u)^i \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right) d\mathcal{H}^{i-1}(u)
\end{aligned} \tag{9.85}$$

where $\varrho_{K'}(u)u \in \operatorname{relbd} K'$ for $u \in S^{n-1} \cap L$ and the radial function $\varrho_{K'}(u) > 0$. On the other hand, (9.81), the Fubini theorem and using polar coordinates in L (cf. (1.26)) yield that

$$\begin{aligned}
V_K(L \cap S^{n-1}) &= \left|\left\{t\, y :\ t \in [0, 1] \text{ and } y \in (L^\perp + \operatorname{relbd} K')\right\}\right| \\
&= \int_{S^{n-1}\cap L} \int_0^{\varrho_{K'}(u)} \frac{r^{n-i} \cdot \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right)}{\varrho_{K'}(u)^{n-i}} \cdot r^{i-1}\, dr\, d\mathcal{H}^{i-1}(u) \\
&= \frac{1}{n} \int_{S^{n-1}\cap L} \varrho_{K'}(u)^i \mathcal{H}^{n-i}\left(M\left(\varrho_{K'}(u)u\right)\right) d\mathcal{H}^{i-1}(u).
\end{aligned} \tag{9.86}$$

We conclude (i) from comparing (9.85) and (9.86).

Let us assume that that equality holds in (i), and hence equality holds in (9.84) for any $x \in \operatorname{relbd} K'$, $\lambda \in [0, \frac{1}{2}]$ and $z = (1-\lambda)x + \lambda(-x)$. We deduce from the equality conditions in the Brunn-Minkowski inequality Theorem 1.12.3 that $M(x)$ and $M(-x) = -M(x)$ are translates. In particular, equality in (9.84) yields that $K \cap (z + L^\perp)$ is a translate of C for any $z \in K'$ and the $(n-i)$-dimensional o-symmetric compact convex set $C = K \cap L^\perp$. In other words, there exists an even function $\varphi : K' \to L^\perp$ such that $K \cap (z + L^\perp) = C + \varphi(z)$ for any $z \in K'$. Now if $z_1, z_2 \in K'$ and $\alpha \in [0,1]$, then $\varphi((1-\alpha)z_1 + \alpha z_2) = (1-\alpha)\varphi(z_1) + \alpha\,\varphi(z_2)$ follows from the convexity of K, and hence φ is a linear function. We deduce that $K = C + C'$ for the i-dimensional o-symmetric compact convex set $C' = \{z + \varphi(z) : z \in K'\}$, and in turn $\operatorname{supp} V_K \subset L \cup L'$ where $L' = (\operatorname{lin} C')^\perp$. ∎

9.C Supplement: The L_p-Minkowski problem in general for $p \geq 0$

For $p \geq 0$ with $p \neq 1, n$, and for non-negative $f \in L_1(S^{n-1})$ with $\mathcal{H}^{n-1}(\{f = 0\}) = 0$, the main goal of this section is to show the existence of the solution of the Monge-Ampére equation

$$\det(\nabla^2 h + hI_{n-1}) = h^{p-1} f \qquad \text{if } p > 1; \tag{9.87}$$

$$h^{1-p}\det(\nabla^2 h + hI_{n-1}) = f \qquad \text{if } p < 1 \tag{9.88}$$

in the sense of measure. As defined in Section 9.3, for a convex body $K \in \mathcal{K}_o^n$ and $p \in \mathbb{R}$, the L_p-surface area measure $S_{K,p}$ is the Borel measure

$$dS_{K,p} = h_K^{1-p}\, dS_K$$

on S^{n-1} where in the case when $p > 1$ and $o \in \partial K$, we assume that $S_K(\{h_K = 0\}) = 0$. In particular, if $p \in \mathbb{R}$, $K \in \mathcal{K}_o^n$ and $\lambda > 0$, then

$$S_{\lambda K,p} = \lambda^{n-p} \cdot S_{K,p}. \tag{9.89}$$

For a $p \in \mathbb{R}$ and finite non-trivial Borel μ on S^{n-1}, the L_p-Minkowski problem (cf. Section 9.3) asks whether there exists a convex body $K \in \mathcal{K}_o^n$ such that

$$\mu = S_{K,p}. \tag{9.90}$$

In this section, we prove the following sufficiency conditions concerning the L_p-Minkowski problem (9.90) for $p \geq 0$ using the variational method where the case $p > 1$ is due to Hug, Lutwak, Yang, Zhang [339] and Chou, Wang [162], and the case $p \in [0, 1)$ is due to Chen, Li, Zhu [156, 157].

Theorem 9.C.1. *Let μ be a finite Borel measure on S^{n-1}.*

$p > 0$, $p \neq 1, n$: *If μ is not concentrated on any closed hemi-sphere, then $\mu = S_{K,p}$ for convex body $K \in \mathcal{K}_o^n$ (where $S_K(\{h_K = 0\}) = 0$ if $p > 1$ and $o \in \partial K$).*

$p = 0$: *If $\mu(L \cap S^{n-1}) < \frac{i}{n} \cdot \mu(S^{n-1})$ for any linear i-space $L \subset \mathbb{R}^n$, $i = 1, \ldots, n-1$, then $\mu = S_{K,0} = nV_K$ for a convex body $K \in \mathcal{K}_o^n$.*

Remarks.

- In particular, any non-trivial absolutely continuous measure on S^{n-1} is a cone volume measure.
- If $p = n$, what known is (cf. Hug, Lutwak, Yang, Zhang [339]) - and what our method yields, as well - that if μ is a finite Borel measure on S^{n-1} not concentrated on any closed hemi-sphere, then $\mu = \lambda\, S_{K,p}$ for a convex body $K \in \mathcal{K}_o^n$ and $\lambda > 0$. In this case, rescaling does not help (cf. (9.89)).
- Here we do not handle the case $p \in (-n, 0)$ even if a variational argument does exist (cf. Bianchi, Böröczky, Colesanti, Yang [69]) because the diameter bound in terms of entropy in Proposition 9.A.2 works only for certain absolutely continuous measures, and here we use discrete measures for weak approximation, not absolutely continuous measures.
- Let μ be a finite Borel measure on S^{n-1} not concentrated on any closed hemi-sphere. If $K_p \in \mathcal{K}_o^n$ satisfies $\mu = S_{K_p,p}$ for $p > n$, then Zou [583] proves that

  $$\lim_{p\to\infty} K_p = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq 1,\ \forall u \in \operatorname{supp}\mu\}.$$

  In particular, if $f \in L_1(S^{n-1})$ is non-negative and $\mathcal{H}^{n-1}(\{f = 0\}) = 0$, then the solution h_p of the L_p Monge-Ampère equation (9.87) on S^{n-1} tends uniformly to the constant 1 function as p tends to infinity.

The idea to prove Theorem 9.C.1 is to prove it first for discrete measures whose support is in general position (cf. Theorem 9.C.2), and for the more general measures in Theorem 9.C.1, to use weak approximation by discrete measures. The conditions in Theorem 9.C.1 ensure that the convex bodies that we use in the case of weak approximation are of bounded diameter (cf. Proposition 9.A.2).

Theorem 9.C.2 (The L_p-Minkowski problem for "general" discrete measures). *If $p \in \mathbb{R}$, $p \neq 1, n$, and μ is a discrete measure on S^{n-1} such that* suppμ *is not contained in a closed hemi-sphere, and any n vectors in* suppμ *are independent, then there exists a $Q \in \mathcal{K}_{(o)}^n$ such that $\mu = S_{Q,p}$.*

Remark. The measures in Theorem 9.C.2 are very special, but any finite measure can be weakly approximated by them, and we have a result for essentially all $p \in \mathbb{R}$. Theorem 9.C.2 is due to Hug, Lutwak, Yang, Zhang [339] if $p > 1$, and to Zhu [580, 581] if $p < 1$.

In order to handle discrete measures whose support is in general position for $p \in \mathbb{R}^n$, let $\mathcal{U}^n$ be the family of all finite subsets $U \subset S^{n-1}$ that are not contained in a closed hemisphere and any n elements of U are independent. For a $U \in \mathcal{U}^n$, let $\mathcal{P}(U) \subset \mathcal{K}_o^n$ be the family of polytopes whose facets' exterior unit normals are from U (possibly a proper subset). This idea of considering polytopes whose exterior unit normals are in general position is due to Zhu [581]. To prove Theorem 9.C.2, we borrow ideas from Hug, Lutwak, Yang, Zhang [339] if $p > 1$, from Zhu [580] if $0 < p < 1$, and from Zhu [581] if $p \leq 0$.

Lemma 9.C.3. *Let $U \in \mathcal{U}^n$.*

(i) There exists $D_U > 0$ such that $\operatorname{diam} Q \leq D_U |Q|^{\frac{1}{n}}$ *for $Q \in \mathcal{P}(U)$.*

(ii) Any sequence $Q_m \in \mathcal{P}(U)$ with $|Q_m| = 1$ has a convergent subsequence whose limit lies in $\mathcal{P}(U)$.

Proof. Let $\widetilde{D}_U$ be the maximum of the diameters of any simplices of the form $\{x \in \mathbb{R}^n : \langle x, u_i \rangle \leq 1,\ i = 1, \ldots, n+1\}$ for $u_1, \ldots, u_{n+1} \in U$ not contained in any closed hemisphere.

For a $Q \in \mathcal{P}(U)$, let $z + rB^n \subset Q$, $r > 0$ be a ball of maximal radius contained in Q. As any n elements of U are independent, there exist $u_1, \ldots, u_{n+1} \in U$ not contained in any closed hemisphere such that $\langle x - z, u_i \rangle \leq r$ holds for $x \in Q$ and $i = 1, \ldots, n+1$, and hence $\operatorname{diam} Q \leq \widetilde{D}_U r$. We conclude (i), and in turn (ii). ■

After translating a polytope $Q \in \mathcal{P}(U)$ in a way such that the translate contains the origin in its interior, the next auxiliary statement follows from the Aleksandrov Lemma Theorem 7.5.2 for the Wulff shapes.

Lemma 9.C.4. *For $U \in \mathcal{U}^n$, $Q \in \mathcal{P}(U)$, and a function $g : U \to \mathbb{R}$, the polytope $Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_Q(u) + g(u) \cdot t,\ \forall u \in U\}$ satisfies*

$$\lim_{t \to 0} \frac{|Q_t| - |Q|}{t} = \int_{S^{n-1}} g \, dS_Q. \tag{9.91}$$

Let μ be a Borel probability measure on S^{n-1}, and let $C \in \mathcal{K}_o$. The main tool in proving Theorem 9.C.2 is to consider an entropy function that is similar to the one also used in Section 9.B. If $p > 1$, then the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(C) = \frac{1}{p} \log \int_{S^{n-1}} h_C^p \, d\mu - \frac{1}{n} \log |C| \tag{9.92}$$

of a convex body $C \in \mathcal{K}_o^n$, which is continuous in C and zero homogeneous; namely, if $\lambda > 0$, then

$$\overline{\mathcal{E}}_{\mu,p}(\lambda C) = \overline{\mathcal{E}}_{\mu,p}(C). \tag{9.93}$$

For $p < 1$ and $\xi \in \operatorname{int} C$, the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(C,\xi) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}} h_{C-\xi}^p \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{C-\xi} \, d\mu - \frac{1}{n}\log|C| & \text{if} \quad p = 0, \end{cases} \tag{9.94}$$

which is again zero homogeneous; namely, if $\lambda > 0$, then

$$\overline{\mathcal{E}}_{\mu,p}(\lambda C, \lambda\xi) = \overline{\mathcal{E}}_{\mu,p}(C,\xi) \tag{9.95}$$

(here we use that $\mu(S^{n-1}) = 1$ if $p = 0$).

We note that according to Lemma 9.C.6, if $p < 1$, and μ is a discrete probability measure on S^{n-1} such that $\operatorname{supp}\mu \in \mathcal{U}^n$, then for $Q \in \mathcal{P}(\operatorname{supp}\mu)$, there exists a unique $\xi_Q \in \operatorname{int} Q$ (that depends on p and μ, as well) such that $\overline{\mathcal{E}}_{\mu,p}(Q) = \overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)$. When proving Theorem 9.C.2, we will actually show that there exists a solution that has bounded entropy.

Proposition 9.C.5. *If $p \in \mathbb{R}$, $p \neq 1, n$, and μ is a discrete probability measure on S^{n-1} such that* $\operatorname{supp}\mu$ *is not contained in a closed hemi-sphere, and any n vectors in* $\operatorname{supp}\mu$ *are independent, and $o \in \operatorname{int}\check{Q}$ holds for a $\check{Q} \in \mathcal{P}(\operatorname{supp}\mu)$, then there exists $Q \in \mathcal{K}^n_{(o)}$ such that $\mu = S_{\theta Q,p}$,*

$$\log\operatorname{diam}\check{Q} - \frac{\log|\check{Q}|}{n} \geq \begin{cases} \overline{\mathcal{E}}_{\mu,p}(Q) & \textit{if} \quad p > 1, \ p \neq n \\ \overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) & \textit{if} \quad p < 1 \end{cases} \tag{9.96}$$

where $\xi_Q = o$ if $p < 1$, and

$$n|Q|^{\frac{n-p}{n}} = \begin{cases} \exp\left(p \cdot \overline{\mathcal{E}}_{\mu,p}(Q)\right) & \textit{if} \quad p > 1, \ p \neq n \\ \exp\left(p \cdot \overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)\right) & \textit{if} \quad p < 1, \ p \neq 0, \\ \overline{\mathcal{E}}_{\mu,0}(Q,\xi_Q) & \textit{if} \quad p = 0. \end{cases} \tag{9.97}$$

For any probability measure μ as in Theorem 9.C.1, the idea, due to Hug, Lutwak, Yang, Zhang [339] if $p > 1$, and Chen, Li, Zhu [156, 157] if $p \in (0,1)$ and $p = 0$, is using weak convergence. We approximate the measure μ in Theorem 9.C.1 by a sequence $\{\mu_m\}$ of discrete probability measures that also satisfy the same conditions as μ and whose support is in general position. For each μ_m, let $Q_m \in \mathcal{K}^n_{(0)}$ be a solution of the L_p-Minkowski problem provided by Proposition 9.C.5 such that the entropy $\overline{\mathcal{E}}_{\mu_m,p}(Q_m, \sigma_{Q_m})$ stays bounded. We deduce via Proposition 9.A.2 that the sequence $\{Q_m\}$ is bounded and $|Q_m|$ stays bounded away from zero, and hence a subsequence of $\{Q_m\}$ tends to a convex body K with $S_{K,p} = \mu$.

9.C.1 Proof of Theorem 9.C.2 and Proposition 9.C.5 if $p > 1$

First we prove Theorem 9.C.2 when $p > 1$ and $p \neq n$ because the argument is much easier in that case.

Proof of Theorem 9.C.2 when $p > 1$. Let $p > 1$, and let μ be a discrete measure on S^{n-1} with $\operatorname{supp}\mu \in \mathcal{U}^n$. We deduce from Lemma 9.C.3 that there exists a $\widetilde{Q} \in \mathcal{P}(\operatorname{supp}\mu)$ such that $|\widetilde{Q}| = 1$ and (cf. (9.93))

$$\begin{aligned}\overline{\mathcal{E}}_{\mu,p}(\widetilde{Q}) &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q) : Q \in \mathcal{P}(\operatorname{supp}\mu) \text{ and } |Q| = 1\right\}\\ &= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q) : Q \in \mathcal{P}(\operatorname{supp}\mu)\right\}. \qquad (9.98)\end{aligned}$$

Step 1. We claim that

$$o \in \operatorname{int}\widetilde{Q}. \qquad (9.99)$$

Otherwise, there exists $u_0 \in \operatorname{supp} S_{\widetilde{Q}}$ such that $h_{\widetilde{Q}}(u_0) = 0$, and we seek a contradiction. Let $U_0 \subset U$ be the set of all $u \in \operatorname{supp}\mu$ such that $h_{\widetilde{Q}}(u) = 0$, and let

$$\widetilde{Q}_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_{\widetilde{Q}}(u) + \tilde{g}(u)\cdot t,\ \forall u \in \operatorname{supp}\mu\}$$

where

$$\tilde{g}(u) = \begin{cases} 1 & \text{if} \quad u \in U_0 \\ 0 & \text{if} \quad u \in (\operatorname{supp}\mu)\backslash U_0 \end{cases}$$

As $u_0 \in U_0 \cap \operatorname{supp} S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t}\left|\widetilde{Q}_t\right|\Big|_{t=0} > 0. \qquad (9.100)$$

On the other hand, there exists $\tilde{\theta} > 0$ depending on μ and $\widetilde{Q}$ such that if $t \geq 0$, then

$$\int_{S^{n-1}} h_{\widetilde{Q}}^p\, d\mu \leq \int_{S^{n-1}} h_{\widetilde{Q}_t}^p\, d\mu \leq \int_{S^{n-1}} h_{\widetilde{Q}}^p\, d\mu + \tilde{\theta}\cdot t^p,$$

and hence combining these estimates with $p > 1$, and then using (9.100) yields

$$\frac{\partial}{\partial t}\mathcal{E}_{\mu,p}(\widetilde{Q}_t)\Big|_{t=0^+} = -\frac{1}{n}\cdot\frac{\partial}{\partial t}\left|\widetilde{Q}_t\right|\Big|_{t=0^+} < 0.$$

Since $\widetilde{Q}_t \in \mathcal{P}(\operatorname{supp}\mu)$ for small $t \geq 0$, and $\widetilde{Q}_0 = \widetilde{Q}$, the last estimate contradicts the minimality property (9.98), and in turn proves (9.99).

Step 2. We claim that

$$\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp}\mu. \qquad (9.101)$$

Otherwise, there exists $\bar{u} \in (\operatorname{supp}\mu)\backslash\operatorname{supp} S_{\widetilde{Q}}$, and let

$$\overline{Q}_t = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq h_{\widetilde{Q}}(u) + \bar{g}(u)\cdot t,\ \forall u \in \operatorname{supp}\mu\}$$

where if $u \in \operatorname{supp} \mu$

$$\bar{g}(u) = \begin{cases} -1 & \text{if} \quad u = \bar{u} \\ 0 & \text{if} \quad u \neq \bar{u}. \end{cases}$$

As $\bar{u} \notin S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t} \left| \overline{Q}_t \right| \Big|_{t=0} = 0. \tag{9.102}$$

On the other hand, $h_{\overline{Q}_t}(\bar{u}) = h_{\widetilde{Q}}(\bar{u}) - t$ and $h_{\overline{Q}_t}(u) \leq h_{\widetilde{Q}}(u)$ for $u \neq \bar{u}$ for small $t > 0$, and hence

$$\frac{1}{p} \log \int_{S^{n-1}} h_{\overline{Q}_t}^p \, d\mu < \frac{1}{p} \log \int_{S^{n-1}} h_{\widetilde{Q}}^p \, d\mu - \bar{\theta} \cdot t$$

for a $\bar{\theta} > 0$ depending on $\widetilde{Q}$, $\bar{u}$ and μ. Therefore, combining this estimate with (9.102) yields

$$\overline{\mathcal{E}}_{\mu,p}(\overline{Q}_t) < \overline{\mathcal{E}}_{\mu,p}(\widetilde{Q})$$

for small $t > 0$. Since $\overline{Q}_t \in \mathcal{P}(\operatorname{supp} \mu)$ for small $t > 0$, the last estimate contradicts the minimality property (9.98), and in turn proves (9.101).

Step 3. We claim that

$$\mu = \frac{\lambda}{n} \cdot S_{\widetilde{Q},p} \quad \text{for} \quad \lambda = \int_{S^{n-1}} h_{\widetilde{Q}}^p \, d\mu. \tag{9.103}$$

Since $h_{\widetilde{Q}}(u) > 0$ for $u \in \operatorname{supp} \mu$ by (9.99) in Step 1., it is equivalent to saying that for any $g : \operatorname{supp} \mu \longrightarrow \mathbb{R}$, we have

$$\int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1} \, d\mu = \frac{\lambda}{n} \cdot \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.104}$$

Let us consider

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_{\widetilde{Q}}(u) + g(u) \cdot t, \ \forall u \in \operatorname{supp} \mu\}.$$

We deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t} |Q_t| \Big|_{t=0} = \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.105}$$

If $|t|$ is small, then $o \in \operatorname{int} Q_t$ by (9.99) in Step 1., and $h_{Q_t}(u) = h_{\widetilde{Q}}(u) + g(u) \cdot t$ for $u \in \operatorname{supp} \mu$ as $\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp} \mu$ by (9.101). In turn, we conclude that

$$\frac{\partial}{\partial t} \frac{1}{p} \log \int_{S^{n-1}} h_{Q_t}^p \, d\mu \Big|_{t=0} = \lambda^{-1} \int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1} \, d\mu.$$

Since $Q_t \in \mathcal{P}(\operatorname{supp}\mu)$ if t is small, and $Q_0 = \widetilde{Q}$, combining the last formula with (9.105) and the minimality property (9.98) yields that

$$0 = \frac{\partial}{\partial t} \overline{\mathcal{E}}_{\mu,p}(Q_t) \Big|_{t=0} = \lambda^{-1} \int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1}\, d\mu - \frac{1}{n} \int_{S^{n-1}} g\, dS_{\widetilde{Q}}.$$

We conclude (9.104), and in turn (9.103).

Finally, it follows from (9.103) and (9.89) that $\mu = S_{Q,p}$ for $Q = (\lambda/n)^{\frac{1}{n-p}} \widetilde{Q}$. ■

Proof of Proposition 9.C.5 *if* $p > 1$. The proof of Theorem 9.C.2 if $p > 1$ above provides a $Q \in \mathcal{P}(\operatorname{supp}\mu)$ such that $\mu = S_{Q,p}$ and $\overline{\mathcal{E}}_{\mu,p}(Q) \leq \overline{\mathcal{E}}_{\mu,p}(\check{Q})$. Since $h_{\check{Q}}(u) \leq \operatorname{diam} \check{Q}$ for any $u \in \operatorname{supp}\mu$, and μ is a probability measure, we have $\overline{\mathcal{E}}_{\mu,p}(Q) \leq \log \operatorname{diam} \check{Q} - \frac{1}{n} \log |\check{Q}|$, verifying (9.96).

To prove (9.97), $|\widetilde{Q}| = 1$ yields that $|Q| = (\lambda/n)^{\frac{n}{n-p}}$. ■

9.C.2 Proof of Theorem 9.C.2 and Proposition 9.C.5 if $p < 1$

Let $p < 1$, and let μ be a discrete probability measure on S^{n-1} such that $\operatorname{supp}\mu \in \mathcal{U}^n$. We recall that if $Q \in \mathcal{P}(\operatorname{supp}\mu)$ and $\xi \in \operatorname{int} Q$, then the entropy function is

$$\overline{\mathcal{E}}_{\mu,p}(Q,\xi) = \begin{cases} \frac{1}{p} \log \int_{S^{n-1}} h_{Q-\xi}^p \, d\mu - \frac{1}{n} \log |Q| & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{Q-\xi}\, d\mu - \frac{1}{n} \log |Q| & \text{if} \quad p = 0. \end{cases}$$

We note that if $u \in S^{n-1}$ and $\xi \in Q$, then

$$h_{Q-\xi}(u) = h_Q(u) - \langle \xi, u \rangle. \tag{9.106}$$

We deduce from (9.106) that $\overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ is a continuous function of $Q \in \mathcal{P}(\operatorname{supp}\mu)$ and $\xi \in \operatorname{int} Q$, and for fixed $Q \in \mathcal{P}(\operatorname{supp}\mu)$, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ is C^1 on $\operatorname{int} Q$. Let us verify some useful properties of our entropy function.

Lemma 9.C.6. *Let $p < 1$, and let μ be a discrete probability measure on S^{n-1} such that* $\operatorname{supp}\mu \in \mathcal{U}^n$.

(i) For $Q \in \mathcal{P}(\operatorname{supp}\mu)$, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q,\xi)$ over all $\xi \in \operatorname{int} Q$ attains its maximum at a unique $\xi_Q \in \operatorname{int} Q$ (ξ_Q naturally also depends on p and μ), and

$$\int_{S^{n-1}} u \cdot h_{Q-\xi_Q}(u)^{p-1}\, d\mu(u) = o. \tag{9.107}$$

(ii) The functions ξ_Q and $\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q)$ of $Q \in \mathcal{P}(\operatorname{supp}\mu)$ are continuous.

(iii) If $g : \operatorname{supp}\mu \to \mathbb{R}$, $Q \in \mathcal{P}(\operatorname{supp}\mu)$ satisfies that $\operatorname{supp} S_Q = \operatorname{supp}\mu$, *and* $\operatorname{supp} S_{Q_t} = \operatorname{supp}\mu$ *and $o \in Q_t$ hold for $t \in (-t_0, t_0)$, $t_0 > 0$, and*

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_Q(u) + t\, g(u),\ \forall u \in \operatorname{supp}\mu\},$$

then ξ_{Q_t} is a differentiable function of t.

Proof. If $p \le 0$, then the existence of $\xi_Q \in \operatorname{int} Q$ in (i) follows from the observations that the function $\xi \mapsto \log(h_Q(u) - \langle \xi, u\rangle)$ is strictly concave, and the function $(h_Q(u) - \langle \xi, u\rangle)^p$ is strictly convex for $\xi \in \operatorname{int} Q$ where $u \in \operatorname{supp} \mu$; moreover, if $\xi_m \in \operatorname{int} Q$ tends to a $\tilde{\xi} \in \partial Q$, then there exists a $u \in \operatorname{supp} \mu$ such that $\lim_{m\to\infty} h_Q(u) - \langle \xi, u\rangle = 0$, and hence $\lim_{m\to\infty} \overline{\mathcal{E}}_{\mu,p}(Q, \xi_m) = -\infty$.

To consider (i) if $p \in (0,1)$, we note that the function $\xi \mapsto (h_Q(u) - \langle \xi, u\rangle)^p$ is strictly concave even on Q for any $u \in \operatorname{supp} \mu$, therefore, the function $\xi \mapsto \overline{\mathcal{E}}_{\mu,p}(Q, \xi)$ attains its maximum at a unique $\xi_Q \in Q$.

We suppose that $\xi_Q \in \partial Q$, and seek a contradiction. We choose a $w \in S^{n-1}$ and $t_0 > 0$ such that $\xi(t) = \xi_Q - tw \in \operatorname{int} Q$ if $t \in (0, t_0)$, and $\varrho > 0$ such that $h_{Q-\xi_Q}(u) > \varrho$ if $u \in \operatorname{supp} \mu$ and $h_{Q-\xi_Q}(u) > 0$. For the non-empty subset $U_0 \subset \operatorname{supp} \mu$ such that $h_{Q-\xi_Q}(u) = 0$ for $u \in U_0$ and $\theta = \min_{u\in U_0} |\langle u, w\rangle| > 0$, we have

$$\begin{array}{rcll} h_{Q-\xi(t)}(u)^p & \ge & \theta^p t^p = h_{Q-\xi_Q}(u)^p + \theta^p t^p & \text{if} \quad u \in U_0, \\ h_{Q-\xi(t)}(u)^p & \le & h_{Q-\xi_Q}(u)^p + p\varrho^{p-1} \cdot t & \text{if} \quad u \in (\operatorname{supp} \mu) \backslash U_0. \end{array}$$

We deduce from $p \in (0,1)$ that $\overline{\mathcal{E}}_{\mu,p}(Q, \xi(t)) < \overline{\mathcal{E}}_{\mu,p}(Q, \xi_Q)$ for small $t > 0$, which is a contradiction proving that $\xi_Q \in \operatorname{int} Q$ also if $p \in (0,1)$.

Now (9.107) follows from the maximality property of $\xi_Q \in \operatorname{int} Q$, and (ii) follows from the uniqueness of ξ_Q.

For (iii), (9.107) and (ii) yield that for $\xi(t) = \xi_{Q_t}$ and

$$F(t, \xi) = \int_{S^{n-1}} u \cdot \big(h_{Q_t}(u) - \langle \xi, u\rangle\big)^{p-1}\, d\mu(u),$$

there exists $\varrho > 0$ such $F(t, \xi)$ is C^1 on the set $|t| < \varrho$ and $\|\xi - \xi_Q\| < \varrho$, $\xi(t)$ is a continuous function on $(-\varrho, \varrho)$, and $\xi(t)$ is the solution of the functional equation $F(t, \xi(t)) = o$. Since the derivative $D_\xi F$ of F with respect to ξ at $t = 0$ and $\xi = \xi_Q \in \operatorname{int} Q$ is

$$D_\xi F(0, \xi_Q) = (p-1) \int_{S^{n-1}} (u \otimes u) \cdot \big(h_Q(u) - \langle \xi_Q, u\rangle\big)^{p-2}\, d\mu(u)$$

(here $u \otimes u = u\, u^t$), which is negative definite as $u \otimes u$ is positive semi-definite for any $u \in \operatorname{supp} \mu$, and $\operatorname{supp} \mu$ contains n independent vectors. Therefore, the implicit function theorem yields (iii). ∎

Proof of Theorem 9.C.2 if $p < 1$. Let $p < 1$, and let μ be a discrete measure on S^{n-1} with $\operatorname{supp} \mu \in \mathcal{U}^n$.

We deduce from the diameter bound Lemma 9.C.3 and the continuity of $\mathcal{E}_{\mu,p}(Q, \xi_Q)$ and ξ_Q (cf. Lemma 9.C.6) that there exists a $\widetilde{Q} \in \mathcal{P}(\operatorname{supp} \mu)$ such that $|\widetilde{Q}| = 1$ and (cf.

(9.95) for (9.108))

$$\overline{\mathcal{E}}_{\mu,p}(\widetilde{Q},\xi_{\widetilde{Q}}) = \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) : Q \in \mathcal{P}(\operatorname{supp}\mu) \text{ and } |Q| = 1\right\}$$
$$= \min\left\{\overline{\mathcal{E}}_{\mu,p}(Q,\xi_Q) : Q \in \mathcal{P}(\operatorname{supp}\mu)\right\}. \tag{9.108}$$

In the estimates below, we also use the expression

$$\widetilde{\mathcal{E}}_{\mu,p}(Q,\xi) = \begin{cases} \frac{1}{p}\log\int_{S^{n-1}} h_{Q-\xi}^p\,d\mu & \text{if} \quad p \neq 0 \\ \int_{S^{n-1}} \log h_{Q-\xi}\,d\mu & \text{if} \quad p = 0 \end{cases}$$

for $Q \in \mathcal{P}(\operatorname{supp}\mu)$ and $\xi \in \operatorname{int} Q$, which then satisfies that if $\xi \in \operatorname{int} Q$, then

$$\widetilde{\mathcal{E}}_{\mu,p}(Q,\xi) \leq \widetilde{\mathcal{E}}_{\mu,p}(Q,\xi_Q). \tag{9.109}$$

Step 1. We claim that

$$\operatorname{supp} S_{\widetilde{Q}} = \operatorname{supp}\mu. \tag{9.110}$$

Otherwise, there exists $\bar{u} \in (\operatorname{supp}\mu)\backslash\operatorname{supp} S_{\widetilde{Q}}$, and let

$$\overline{Q}_t = \{x \in \mathbb{R}^n : \langle x,u\rangle \leq h_{\widetilde{Q}}(u) + \bar{g}(u)\cdot t,\ \forall u \in \operatorname{supp}\mu\}$$

where if $u \in \operatorname{supp}\mu$

$$\bar{g}(u) = \begin{cases} -1 & \text{if} \quad u = \bar{u} \\ 0 & \text{if} \quad u \neq \bar{u}. \end{cases}$$

As $\bar{u} \notin S_{\widetilde{Q}}$, we deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t}\left|\overline{Q}_t\right|\Big|_{t=0} = 0. \tag{9.111}$$

On the other hand, let $\xi(t) = \xi_{\overline{Q}_t}$. We deduce from the continuity of $\xi(t)$ (cf. Lemma 9.C.6) that there exists $\varrho, t_0 > 0$ such that if $t \in (0,t_0)$, then $\xi(t) \in \xi_{\widetilde{Q}} + \varrho\,B^n$, $\xi_{\widetilde{Q}} + 2\varrho\,B^n \subset \overline{Q}_t$ and $\overline{Q}_t \in \mathcal{P}(\operatorname{supp}\mu)$. Since $h_{\overline{Q}_t-\xi(t)}(\bar{u}) = h_{\widetilde{Q}-\xi(t)}(\bar{u}) - t$ and $h_{\overline{Q}_t-\xi(t)}(u) \leq h_{\widetilde{Q}-\xi(t)}(u)$ for $u \neq \bar{u}$ for small $t > 0$, we deduce using (9.109) that

$$\widetilde{\mathcal{E}}_{\mu,p}(\overline{Q}_t,\xi(t)) < \widetilde{\mathcal{E}}_{\mu,p}(\widetilde{Q},\xi(t)) - \bar{\theta}\cdot t \leq \widetilde{\mathcal{E}}_{\mu,p}(\widetilde{Q},\xi_{\widetilde{Q}}) - \bar{\theta}\cdot t$$

for $\bar{\theta} > 0$ depending on μ, $\widetilde{Q}$, $\bar{u}$ and ϱ. Therefore, combining this estimate with (9.111) yields

$$\overline{\mathcal{E}}_{\mu,p}(\overline{Q}_t,\xi(t)) < \overline{\mathcal{E}}_{\mu,p}(\widetilde{Q},\xi_{\widetilde{Q}})$$

for small $t > 0$. This last estimate contradicts the minimality property (9.108), and in turn proves (9.110).

Step 2. Assuming that $\xi_{\widetilde{Q}} = o$, we claim that

$$\mu = \frac{\lambda}{n} \cdot S_{\widetilde{Q},p} \quad \text{for} \quad \lambda = \int_{S^{n-1}} h_{\widetilde{Q}}^p \, d\mu. \tag{9.112}$$

Since $h_{\widetilde{Q}}(u) > 0$ for $u \in \operatorname{supp} \mu$ by $o = \xi_{\widetilde{Q}} \in \operatorname{int} \widetilde{Q}$, it is equivalent to saying that for any $g : \operatorname{supp} \mu \to \mathbb{R}$, we have

$$\int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1} \, d\mu = \frac{\lambda}{n} \cdot \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.113}$$

Let us consider

$$Q_t = \{x \in \mathbb{R}^n : \langle x, u \rangle \le h_{\widetilde{Q}}(u) + g(u) \cdot t, \ \forall u \in \operatorname{supp} \mu\},$$

and let $\xi(t) = \xi_{\widetilde{Q}_t}$. We deduce from Lemma 9.C.4 that

$$\frac{\partial}{\partial t} \, |Q_t| \bigg|_{t=0} = \int_{S^{n-1}} g \, dS_{\widetilde{Q}}. \tag{9.114}$$

We note that if $|t|$ is small, then $\operatorname{supp} \mu = \operatorname{supp} S_{\widetilde{Q}}$ (cf. (9.110)) yields that $\operatorname{supp} \mu = \operatorname{supp} S_{Q_t}$, and hence

$$\widetilde{\mathcal{E}}_{\mu,p}(Q_t, \xi(t)) = \frac{1}{p} \cdot \log \int_{S^{n-1}} \Big(h_{\widetilde{Q}}(u) + g(u) \cdot t - \langle \xi(t), u \rangle \Big)^p \, d\mu(u).$$

It follows from the differentiability of $\xi(t)$ (cf. Lemma 9.C.6) and $\xi(o) = o$ that

$$\frac{\partial}{\partial t} \, \widetilde{\mathcal{E}}_{\mu,p}(Q_t, \xi(t)) \bigg|_{t=0} = \lambda^{-1} \cdot \int_{S^{n-1}} h_{\widetilde{Q}}^{p-1}(u)(g(u) - \langle \xi'(o), u \rangle) \, d\mu(u).$$

Here $\int_{S^{n-1}} h_{\widetilde{Q}}^{p-1}(u) \cdot \langle \xi'(o), u \rangle \, d\mu(u) = 0$ by (9.107); therefore,

$$\frac{\partial}{\partial t} \, \widetilde{\mathcal{E}}_{\mu,p}(Q_t, \xi(t)) \bigg|_{t=0} = \lambda^{-1} \cdot \int_{S^{n-1}} h_{\widetilde{Q}}^{p-1} \cdot g \, d\mu.$$

Combining the last formula with (9.114) and the minimality property (9.108) yields that

$$0 = \frac{\partial}{\partial t} \, \overline{\mathcal{E}}_{\mu,p}(Q_t) \bigg|_{t=0} = \lambda^{-1} \int_{S^{n-1}} g h_{\widetilde{Q}}^{p-1} \, d\mu - \frac{1}{n} \int_{S^{n-1}} g \, dS_{\widetilde{Q}}.$$

We conclude (9.113), and in turn (9.112).

Finally, to prove Theorem 9.C.2 if $p < 1$, we may assume that $\mu(S^{n-1}) = 1$ by the homogeneity (9.89) of $S_{K,p}$, and we also assume that $\xi_{\widetilde{Q}} = o$ as in Step 2.. It follows from (9.112) and (9.89) that $\mu = S_{Q,p}$ for $Q = (\lambda/n)^{\frac{1}{n-p}} \widetilde{Q}$. ∎

Proof of Proposition 9.C.5 if $p < 1$. The proof of Theorem 9.C.2 if $p < 1$ above provides a $Q \in \mathcal{P}(\operatorname{supp}\mu)$ such that $\mu = S_{Q,p}$, $\xi_Q = o$ and $\overline{\mathcal{E}}_{\mu,p}(Q, o) \le \overline{\mathcal{E}}_{\mu,p}(\check{Q}, \xi_{\check{Q}})$. Since $h_{\check{Q}-\xi_{\check{Q}}}(u) \le \operatorname{diam}\check{Q}$ for any $u \in \operatorname{supp}\mu$, and μ is a probability measure, we have

$$\overline{\mathcal{E}}_{\mu,p}(Q, \xi_Q) \le \overline{\mathcal{E}}_{\mu,p}(\check{Q}, \xi_{\check{Q}}) \le \log \operatorname{diam}\check{Q} - \frac{1}{n}\log|\check{Q}|,$$

verifying (9.96). To prove (9.97), $|\widetilde{Q}| = 1$ yields that $|Q| = (\lambda/n)^{\frac{n}{n-p}}$. ■

9.C.3 Proof of Theorem 9.C.1 for $p \ge 0$, $p \ne 1, n$

For $p \ge 0$, $p \ne 1, n$, we may assume that the finite Borel measure μ on S^{n-1} satisfying the conditions in Theorem 9.C.1 is a probability measure (cf. (9.89)). In particular, the measure of any open hemi-sphere is positive, and if $p = 0$, then μ also satisfies the condition $\mu(L \cap S^{n-1}) < \frac{i}{n}$ for any linear i-space $L \subset \mathbb{R}^n$, $i = 1, \dots, n-1$. Using the compactness of the space of the i-dimensional linear subspaces of $\mathbb{R}^n$, we deduce the existence of $\delta, \tau \in (0, \frac{1}{2})$ such that if $p > 0$, then

$$\mu(\Omega(u, 2\delta)) > 2\tau \tag{9.115}$$

for any $u \in S^{n-1}$ where $\Omega(u, \delta) = \{v \in S^{n-1} : \langle v, u\rangle > \delta\}$, and if $p = 0$ and $L \subset \mathbb{R}^n$ is any non-trivial linear subspace, then

$$\mu\left(\Psi(L \cap S^{n-1}, 2\delta)\right) < \frac{(1-2\tau)i}{n} \tag{9.116}$$

where $\Psi(L \cap S^{n-1}, \delta) = \{x \in S^{n-1} : \langle x, y\rangle \le \delta \text{ for } y \in L^\perp \cap S^{n-1}\}$.

We approximate weakly the measure μ by a sequence $\{\mu_m\}$ of discrete probability measures. For large integer m, let Ξ_m be a finite $\frac{1}{m}$-net on S^{n-1} such that any n points of Ξ_m are independent as vectors of $\mathbb{R}^n$, and hence for any $u \in S^{n-1}$, there exists $v \in \Xi_m$ such that $\|u - v\| \le \frac{1}{m}$. We write $N(m)$ to denote the cardinality of Ξ_m, and for any $v \in \Xi_m$, let

$$\widetilde{\mathcal{D}}_{v,m} = \left\{u \in S^{n-1} : \|u - v\| \le \|u - w\| \text{ for any } w \in \Xi_m\right\}$$

be the Dirichlet-Voronoi cell of v that is a closed spherically convex set of diameter at most $\frac{2}{m}$. Enumerating the elements of Ξ_m as v_i, $i = 1, \dots, N(m)$, let $\mathcal{D}_{v_1,m} = \widetilde{\mathcal{D}}_{v_1,m}$, and for $i \ge 2$, let

$$\mathcal{D}_{v_i,m} = \widetilde{\mathcal{D}}_{v_i,m} \Big\backslash \bigcup_{j=1}^{i-1} \widetilde{\mathcal{D}}_{v_j,m},$$

and hence $\mathcal{D}_{v,m}$, $v \in \Xi_m$, are Borel measurable pairwise disjoint subsets of S^{n-1} of diameter at most $\frac{2}{m}$, form a partition of S^{n-1}, and $v \in \mathcal{D}_{v,m}$ for $v \in \Xi_m$.

Now for large m, let μ_m be the probability discrete measure on S^{n-1} such that $\operatorname{supp}\mu_m = \Xi_m$, and for $v \in \Xi_m$,

$$\mu_m(v) = \left(1 - \frac{1}{m}\right)\mu\left(\mathcal{D}_{v,m}\right) + \frac{1}{m \cdot N(m)}.$$

It follows that $\{\mu_m\}$ tends weakly to μ, and as the diameter of $\mathcal{D}_{v,m}$ for $v \in \Xi_m$ is at most $\frac{2}{m}$, we deduce that if $p > 0$, then

$$\mu_m\left(\Omega(u,\delta)\right) > \tau \tag{9.117}$$

for any $u \in S^{n-1}$, and if $p = 0$ and $L \subset \mathbb{R}^n$ is any non-trivial linear subspace, then

$$\mu_m\left(\Psi(L \cap S^{n-1}, \delta)\right) < \frac{(1-\tau)i}{n}. \tag{9.118}$$

In order to apply Proposition 9.C.5 to μ_m, we take $\check{Q}_m$ to be the polytope whose facets touch B^n in the points of Ξ_m; therefore,

$$\operatorname{diam}\check{Q}_m < 3 \ \text{ and } \ |\check{Q}_m| > \omega_n \tag{9.119}$$

hold for large m. Applying Proposition 9.C.5 to μ_m, we have $\mu_m = S_{\theta_m Q_m, p}$ where $\theta_m > 0$ and $Q_m \in \mathcal{K}^n_{(o)}$ is a polytope with $|Q_m| = 1$, and $\theta_m Q_m$ satisfies (9.96) and (9.97). We claim that there exists $\eta_1 > 1$ depending on p, n, δ, τ such that

$$\begin{array}{lllllll} \eta_1^{-1} & \leq & \overline{\mathcal{E}}_{\mu,p}(Q_m) & \leq & \eta_1 & & \text{if } p > 1, \\ \eta_1^{-1} & \leq & \overline{\mathcal{E}}_{\mu,p}(Q_m, \sigma_{Q_m}) & \leq & \overline{\mathcal{E}}_{\mu,p}(Q_m, \xi_{Q_m}) & \leq \ \eta_1 & \text{if } p \in [0,1). \end{array} \tag{9.120}$$

The upper bound in (9.120) directly follows from (9.119) and from (9.96) in Proposition 9.C.5. For the lower bound in (9.120), we note that the two notions of entropy in Proposition 9.A.2 and in Proposition 9.C.5 are equivalent for our purposes; for example, $\overline{\mathcal{E}}_{\mu,p}(Q_m) = \frac{1}{p}\log p\mathcal{E}_{\mu,p}(Q_m)$ if $p > 1$. Since $\operatorname{diam} Q_m$ is at least the diameter of a ball in $\mathbb{R}^n$ of volume one according to the Isodiametric Inequality Theorem 1.10.5, the lower bound in (9.120) follows from Proposition 9.A.2. In turn, (9.120), $|\theta_m Q_m| = \theta_m^n$ and (9.97) in Proposition 9.C.5 yield that there exists $\eta_2 > 1$ depending on p, n, δ, τ such that

$$\eta_2^{-1} \leq \theta_m \leq \eta_2. \tag{9.121}$$

We also deduce from (9.120) and Proposition 9.A.2 that there exists $\eta_3 > 1$ depending on p, n, δ, τ such that

$$\operatorname{diam} Q_m \leq \eta_3. \tag{9.122}$$

Since $|\theta_m Q_m| \geq \eta_2^{-n}$ and $\operatorname{diam}(\theta_m Q_m) \leq \eta_2\eta_3$, a subsequence of $\{\theta_m Q_m\}$ tends to a convex body $K \in \mathcal{K}^n_o$, and the weak continuity of the L_p surface area measure (cf. Lemma 9.A.1) yields that $S_{K,p} = \mu$.

Chapter 10

Appendix: Background from Analysis and Algebra

10.1 Weak convergence, Regular measures and the Lebesgue measure

In this section, we summarize the knowledge needed about general measures, and then we focus on the Lebesgue measure. The properties of the notion of weak convergence of measures we need are the following:

Remark 10.1.1 (Weak convergence of measures). For a compact metric space X, let μ_m and μ be finite Borel measures on X. Then the following properties are equivalent, and characterize the weak convergence of $\{\mu_m\}$ to μ.

- $\lim_{m\to\infty} \int_X g\, d\mu_m = \int_X g\, d\mu$ for any continuous function $g : S^{n-1} \to \mathbb{R}$.
- $\liminf_{m\to\infty} \mu_m(U) \geq \mu(U)$ for any open $U \subset X$.
- $\limsup_{m\to\infty} \mu_m(C) \leq \mu(C)$ for any closed $C \subset X$.

Next we discuss the notion of a regular or a Radon measure.

Definition 10.1.2 (Regularity of Borel measures). A Borel measure μ on a topological space X is regular or a Radon measure if for any measurable $A \subset X$,

- $\mu(A) = \inf\{\mu(U) : U \supset A \text{ open}\}$;
- $\mu(A) = \sup\{\mu(C) : C \subset A \text{ compact}\}$;
- μ is finite on compact subsets.

For such a measures, the term *regular Borel measure* appears for example in Rudin [504], and the term *Radon measure* occurs in Ambrosio, Fusco, Pallara [19] and Maggi [439]. For properties of regular measures, see for example Rudin [504], especially Theorem 2.18.

Theorem 10.1.3 (Regularity of some Borel measures). *If X is a locally compact Hausdorff space where every open set is the union of countable many compact subsets, then any Borel measure μ on X that is finite on compact subsets is regular.*

Remark. In particular, the Lebesgue measure on $\mathbb{R}^n$, or any finite Borel measure on S^n are regular.

A popular version of Theorem 10.1.3 is that any finite Borel measure on a separable complete metric space (Polish space) is regular (see Aliprantis, Border [13], Theorem 12.7).

One of the cornerstones of analysis are the versions of the Riesz Representation Theorem; namely, various types of well behaved linear functionals can be represented by integrals with respect to certain measures. For a locally compact Hausdorff space X, we write $C_c(X)$ to denote the family of continuous functions with compact support, and we call a linear operator $L : C_c(X) \to \mathbb{R}$ positive if $L(f) \geq 0$ whenever $f \geq 0$.

The first relevant version of Riesz' theorem is in Cohn [165].

Theorem 10.1.4 (Riesz Representation Theorem for Positive functionals). *If X is a locally compact Hausdorff space, and L is a positive linear functional on $C_c(X)$, then there exists a unique regular Borel measure (i.e. Radon measure) on X such that*

$$L(f) = \int_X f \, d\mu$$

holds for every $f \in C_c(X)$.

The second version of Riesz' theorem in Ambrosio, Fusco, Pallara [19] and Maggi [439] is about linear functionals on $C_c(\mathbb{R}^n; \mathbb{R}^n)$; namely, the space of vector valued continuous functions with compact support. For an ($\mathbb{R}$ valued) linear functional L on $C_c(\mathbb{R}^n; \mathbb{R}^n)$, the total variation $|L|$ is an outer measure on $\mathbb{R}^n$ such that if $A \subset \mathbb{R}^n$ is open, then

$$|L|(A) = \sup \left\{ L(\varphi) : \varphi \in C_c(A; \mathbb{R}^n) \text{ and } \|\varphi\|_\infty \leq 1 \right\},$$

and for any $E \subset \mathbb{R}^n$,

$$|L|(E) = \inf \left\{ |L|(A) : A \subset \mathbb{R}^n \text{ open and } E \subset A \right\}.$$

Theorem 10.1.5 (Riesz Representation Theorem for Vector fields). *If L is a bounded linear functional on $C_c(\mathbb{R}^n; \mathbb{R}^n)$, then its total variation $|L|$ is a Radon measure on $\mathbb{R}^n$, and there exist vector valued measure μ on $\mathbb{R}^n$ and an $|L|$-measurable function $\nu : \mathbb{R}^n \to \mathbb{R}^n$ such that $\|\nu\| = 1$ $|L|$ a.e. on $\mathbb{R}^n$, $d\mu = \nu \, d|L|$ and*

$$L(\varphi) = \langle \varphi, d\mu \rangle = \int_{\mathbb{R}^n} \langle \nu, \varphi \rangle \, d|L|$$

holds for every $\varphi \in C_c(\mathbb{R}^n; \mathbb{R}^n)$.

Remark. We frequently use the notation $|\mu| = |L|$ for the total variation measure.

Next, we turn to properties of the Lebesgue measure. In this book, an $X \subset \mathbb{R}^n$ is called measurable if it is Lebesgue measurable. Let us list some related notions:

- If $X \subset \mathbb{R}^n$ measurable, then $|X|$ is the Lebesgue measure of X where

$$|X| = \sup\{|Y| : Y \subset X \text{ compact}\} = \inf\{|Z| : Z \supset X \text{ open}\}.$$

- If $A \subset \mathbb{R}^n$, then $|A|^* = \text{outer measure} = \min\{|X| : X \supset A \text{ measurable}\}$.
- If $A \subset \mathbb{R}^n$, then $|A|_* = \text{inner measure} = \max\{|X| : X \subset A \text{ measurable}\}$.

Remark 10.1.6 (Volume of the Unit ball). For the Euclidean unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| \le 1\}$ centered at the origin, $|B^n| = \omega_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$ where Γ is Euler's Gamma function satisfying $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$. For $n \ge 2$,

$$\sqrt{\frac{n}{2\pi}} < \frac{\omega_{n-1}}{\omega_n} < \sqrt{\frac{n+1}{2\pi}} \tag{10.1}$$

as $\frac{\omega_m}{\omega_{m-1}} = \int_{-1}^{1}(1-t^2)^{\frac{m-1}{2}}\,dt > \frac{\omega_{m+1}}{\omega_m}$ and $\frac{\omega_m}{\omega_{m-2}} = \frac{2\pi}{m}$ for $m \ge 2$.

We set $\omega_\alpha = \frac{\pi^{\frac{\alpha}{2}}}{\Gamma(\frac{\alpha}{2}+1)}$ for any $\alpha \ge 0$.

Remark 10.1.7 (Inner Density points). For $A \subset \mathbb{R}^n$, we define
$A_* = \left\{x \in \mathbb{R}^n : \lim_{r\to 0^+} \frac{|A\cap(x+rB^n)|_*}{|rB^n|} = 1\right\}$.

- A_* is Borel (as for fixed $r > 0$, $x \mapsto |A \cap (x + rB^n)|_*$ is continuous);
- $|A_*| = |A|_*$;
- if A is measurable, then $|A_* \cap A| = |A|$.

10.2 Lebesgue Integral, Convolution, Fourier transform

In this section, we review the basic properties of integrals that we use in the book.

- For (Lebesgue) measurable $f : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x)\,dx$ is the Lebesgue integral
- For $f : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, the Outer Lebesgue integral is

$$\int_{\mathbb{R}^n}^{*} f = \min\left\{\int_{\mathbb{R}^n} g : g \ge f \text{ and } g \text{ measurable}\right\}.$$

- For $f : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$, the Inner Lebesgue integral is

$$\int_{*,\mathbb{R}^n} f = \max\left\{\int_{\mathbb{R}^n} g : g \le f \text{ and } g \text{ measurable}\right\}.$$

  Actuallly there exists a measurable $\tilde{f}$ ("a witness") such that $\tilde{f} \le f$ and $\{f > \tilde{f}$ contains no subset of positive Lebesgue measure; therefore, $\int_{*,\mathbb{R}^n} f = \int_{\mathbb{R}^n} \tilde{f}$.

- $f \in L_1(\mathbb{R}^n)$ if and only if f is measurable and $\int_{\mathbb{R}^n} |f| < \infty$.
- $\lim_{k\to\infty} f_k = \int_{\mathbb{R}^n} f$ if $f_k, f, g \in L_1(\mathbb{R}^n)$, $|f_k| \le |g|$ and $\{f_k\}$ tends to f pointwise.

Concerning convolutions, we also sketch the proof of the statement about the differetiability of convolutions.

Definition 10.2.1 (Convolution). For $f, g \in L_1(\mathbb{R}^n)$, their convolution is $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$.

Lemma 10.2.2. *Let $f, g \in L_1(\mathbb{R}^n)$,*

(i) $f * g \in L_1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} f * g = \left(\int_{\mathbb{R}^n} f\right)\left(\int_{\mathbb{R}^n} g\right)$;

(ii) If either g is C^k for $k \geq 1$ and $\operatorname{supp} g$ compact, or g is C^{k+1} and all partial derivatives of g of order at most $k+1$ are bounded, then $f * g \in C^k(\mathbb{R}^n)$

Proof. (i) is just consequence of the Fubini Theorem, and (ii) follows from the fact that if g is C^1 and $\|Dg(y) - Dg(x)\|$ is bounded assuming $\|y - x\| \leq 1$, then Lebesgue Dominated Convergence theorem implies $f * g \in C^1(\mathbb{R}^n)$. ■

Lemma 10.2.3 (Approximate Identity). *Let $k_\varepsilon(x) = \varepsilon^{-n} k(\frac{x}{\varepsilon})$ for $\varepsilon \in (0,1)$ where $k : \mathbb{R}^n \to [0,\infty)$ is C^∞, $\int_{\mathbb{R}^n} k = 1$, and k and all of its partial derivatives are bounded.*

If $f \in L_1(\mathbb{R}^n)$, then $\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n} |f - k_\varepsilon * f| = 0$ and $\lim_{\varepsilon \to 0^+} k_\varepsilon * f(x) = f(x)$ for any density point x of f for the C^∞ function $k_\varepsilon * f$.

Remark. If $\operatorname{supp} k \subset B^n$ and $U_0 \subset U \subset U_1$ open with $\mathrm{cl}U_0 \subset U$ and $\mathrm{cl}U \subset U_1$, then $\mathbf{1}_{U_0} \leq k_\varepsilon * \mathbf{1}_U \leq \mathbf{1}_{U_1}$ for small $\varepsilon > 0$.

Example 10.2.4 (Approximate Identity).

- $k(x) = e^{-\pi\|x\|}$;
- k is C^∞ with $\operatorname{supp} k \subset B^n$, for example $k(x) = \gamma\varphi(1 - \|x\|^2)$ where $\gamma > 0$ constant, $\varphi(t) = 0$ if $t \leq 0$, and $\varphi(t) = e^{\frac{-1}{t^2}}$ if $t > 0$.

Definition 10.2.5 (Fourier transform). If $h \in L_1(\mathbb{R}^n)$, then $\hat{h}(z) = \int_{\mathbb{R}^n} h(x) e^{-2\pi i\langle x,z\rangle}\,dx$ is the Fourier transform.

Remark. If $\psi(x) = e^{-\pi\|x\|^2}$, then $\hat{\psi} = \psi$.

Lemma 10.2.6. *Let $f, g \in L_1(\mathbb{R}^n)$.*

- $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
- *If $\hat{f} = \hat{g}$, then $f(x) = g(x)$ for a.e. $x \in \mathbb{R}^n$.*

10.3 Hölder's, Jensen's and Minkowski's inequalities

For a topological space X, and non-trivial Borel measure μ on X, we write $L_1(X,\mu)$ the space of μ measurable functions f such that $\int_X |f|\,d\mu < \infty$. We discuss the probably three most basic inequalities in analysis (see, for example, Rudin [504]), Hölder's, Jensen's and Minkowski's inequalities. The most fundamental is Hölder's inequality, which we state in both forms how we use it.

Theorem 10.3.1 (Hölder inequality I). *Let X be a topological space, and μ be a non-trivial Borel measure on X. If $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ (where $p = 1$ if and only if $q = \infty$), then*

$$\int_X |fg|\, d\mu \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q\, d\mu\right)^{\frac{1}{q}}. \tag{10.2}$$

Theorem 10.3.2 (Hölder inequality II). *Let X be a topological space, and μ be a non-trivial Borel measure on X. For non-negative $f_1, \ldots, f_k \in L_1(X, \mu)$ and $\lambda_1, \ldots, \lambda_k > 0$ with $\sum_{i=1}^k \lambda_i = 1$, $k \geq 2$, we have*

$$\int_X \left(\prod_{i=1}^k |f_i|^{\lambda_i}\right) d\mu \leq \prod_{i=1}^k \left(\int_X |f_i|\, d\mu\right)^{\lambda_i}. \tag{10.3}$$

Assuming that every $\int |f_i| > 0$, equality holds if and only if there exist $a_2, \ldots, a_k > 0$ such that $|f_i(x)| = a_i |f_1(x)|$ for μ a.e. $x \in X$, $i = 2, \ldots, k$.

The Jensen inequality we need is about p means for $p \in \mathbb{R}$. Let X be a topological space, and let μ be a probabilityl Borel measure on X; namely, $\mu(X) = 1$. If $p > 0$ and $f \geq 0$ is μ measurable on X with $\int_X f^p\, d\mu < \infty$, then its p-mean is

$$M_\mu^p(f) = \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}}.$$

In addition, let $p \in \mathbb{R}$, and let $f > 0$ be μ measurable on X with $\int_X f^p\, d\mu < \infty$ if $p \neq 0$, and $\int_X |\log f|\, d\mu < \infty$ if $p = 0$. In this case,

$$M_\mu^p(f) = \begin{cases} \left(\int_X f^p\, d\mu\right)^{\frac{1}{p}} & \text{if} \quad p \neq 0; \\ \exp\left(\int_X \log f\, d\mu\right) & \text{if} \quad p = 0. \end{cases}$$

We provide the simple proof of the Jensen inequality based on the Hölder inequality because the Jensen inequality exists in many forms in the literature.

Theorem 10.3.3 (Jensen inequality). *For a topological space X, and a Borel probability measure μ on X, if $q > p$, and $f \geq 0$ is a μ measurable function on X such that*

(i) either $q > p > 0$ and $\int_X f^q\, d\mu < \infty$,

(ii) or $q > p$ and there exists $R > 3$ such that $R^{-1} < f < R$, then

$$M_\mu^p(f) \leq M_\mu^q(f), \tag{10.4}$$

with equality if and only if there exists $\lambda \geq 0$ such that $f(x) = \lambda$ for μ a.e. $x \in X$. Moreover, if there exists $R > 3$ such that $R^{-1} < f < R$, then

$$\lim_{p \to 0} M_\mu^p(f) = M_\mu^0(f). \tag{10.5}$$

Proof. In the case of (i), we apply the Hölder inequality (10.3) with $f_1 = f^q$, $f_2 \equiv 1$ and $\lambda_1 = \frac{p}{q}$. In the case of (ii) and assuming $0 > q > p$, we apply the Hölder inequality (10.3) with $f_1 = f^p$, $f_2 \equiv 1$ and $\lambda_1 = \frac{q}{p}$.

Therefore, all we need is to prove (10.5). For this, we note that if $|t| \le \frac{1}{2}$, then $e^t = 1 + t + O(t^2)$ and $\log(1+t) = t + O(t^2)$, and hence

$$\begin{aligned}\lim_{p\to 0} \log \left(\int_X f^p \, d\mu\right)^{\frac{1}{p}} &= \lim_{p\to 0} \frac{1}{p} \log \int_X e^{p\log f} \, d\mu \\ &= \lim_{p\to 0} \frac{1}{p} \log \int_X 1 + p\log f + O\left(p^2 (\log R)^2\right) \, d\mu = \int_X \log f \, d\mu,\end{aligned}$$

verifying (10.5). ■

Finally, we prove the Minkowski inequality for integrals:

Theorem 10.3.4 (Minkowski inequality). *Let X be a topological space, and μ be a non-trivial Borel measure on X. If $p > 1$ and $|f|^p, |g|^p \in L_1(X,\mu)$, then*

$$\left(\int_X |f+g|^p \, d\mu\right)^{\frac{1}{p}} \le \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}. \tag{10.6}$$

Proof. For the $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we deduce from the Hölder inequality (10.2) that

$$\begin{aligned}\int_X |f+g|^p \, d\mu &= \int_X |f+g| \cdot |f+g|^{p-1} \, d\mu \\ &\le \int_X |f| \cdot |f+g|^{p-1} \, d\mu + \int_X |g| \cdot |f+g|^{p-1} \, d\mu \\ &\le \left[\left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p \, d\mu\right)^{\frac{1}{p}}\right] \cdot \left(\int_X |f+g|^{q(p-1)} \, d\mu\right)^{\frac{1}{q}}.\end{aligned}$$

We conclude (10.6) by $q(p-1) = p$ and $1 - \frac{1}{q} = \frac{1}{p}$. ■

Definition 10.3.5 (L_p spaces). For $1 \le p \le \infty$, a topological space X, non-trivial Borel measure μ on X and μ measurable $f : X \to \mathbb{R}$, we have

$$f \in L_p(X) \text{ if and only if } \begin{cases} f \text{ is bounded} & \text{provided } p = \infty, \\ \int_X |f|^p \, d\mu < \infty & \text{provided } 1 \le p < \infty. \end{cases}$$

Then $L_p(X)$ is a Banach space with the corresponding L_p-norm (cf. (10.6))

$$\|f\|_p = \begin{cases} \sup |f| & \text{if } p = \infty, \\ \left(\int_X |f|^p \, d\mu\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty. \end{cases}$$

10.4 Hausdorff measure and Lipschitz functions

In this section, we discuss the Hausdorff measure on any metric space and some fundamental properties of it based on Falconer [208] and Federer [212]. We note that some basic properties have been established in Section 1.B.

Definition 10.4.1 (Hausdorff measure $\mathcal{H}^s$). For a metric space (Ξ, d), $s \geq 0$, $\delta > 0$ and $X \subset \Xi$, let

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} \omega_s \left(\frac{\operatorname{diam} Z_i}{2}\right)^s : X \subset \cup_{i=1}^{\infty} Z_i \text{ and } \forall \operatorname{diam} Z_i < \delta\right\}.$$

where $\omega_s = \pi^{\frac{s}{2}}/\Gamma(\frac{s}{2}+1)$ and $\operatorname{diam} Z = \sup\{d(x,y) : x, y \in Z\}$. The Hausdorff outer measure is $\mathcal{H}^{*,s}(X) = \lim_{\delta \to 0^+} \mathcal{H}^s_\delta(X)$, and let $\mathcal{H}^s_\Xi$ be the corresponding Borel measure (which naturally depends on the metric d on Ξ, as well).

Example 10.4.2. Let (Ξ, d) be a metric space.

(i) $\mathcal{H}^0_\Xi$ is the counting measure; namely, $\mathcal{H}^0_\Xi = \#X$ if X is finite, and $\mathcal{H}^0_\Xi = \infty$ if X is infinite.

(ii) The Hausdorff measure is normalized in a way such that if (Ξ, d) is either the Euclidean space $\mathbb{R}^n$, the spherical space S^n, or the hyperbolic space H^n, then $\mathcal{H}^n_\Xi(X) = |X|$ for a Borel set $X \subset \Xi$ where $|X|$ is the Haar measure correponding to the transitive isometry group of Ξ (or in other words, the Lebesgue measure). This is proved in Theorem 1.B.5 if $\Xi = \mathbb{R}^n$, and the argument in the other two cases is similar, and the only essential difference is that concerning the volume of the ball $B(z,r) = \{x \in \Xi : d(x,z) \leq r\}$, $r > 0$, we only have $|B(z,r)| = \omega_n r^n + O(r^{n+2})$ as $r \to 0^+$ if $\Xi = S^n, H^n$ where the implied constant in $O(\cdot)$ depends only on n.

Remark 10.4.3 (Hausdorff dimension). For any (Ξ, d) metric space and $X \subset \Xi$, there exists a Hausdorff dimension $\alpha \geq 0$ of X such that $\mathcal{H}^s_\Xi(X) = \infty$ if $0 \leq s < \alpha$ and $\mathcal{H}^s_\Xi(X) = 0$ if $s > \alpha$. In particular, if $0 < \mathcal{H}^\alpha_\Xi(X) < \infty$, then α is the Hausdorff dimension.

Example 10.4.4. (i) The Hausdorff dimension of a finite set is 0.

(ii) The Hausdorff dimension of any mesurable subset $X \subset \mathbb{R}^n$ with $0 < |X| \leq \infty$ is n according to Example 10.4.2 (ii).

Remark 10.4.5 (Lipschitz function). If (Ξ, d), $(\widetilde{\Xi}, \tilde{d})$ are metric spaces, then $f : \Xi \to \widetilde{\Xi}$ is Lipschitz if there exists $L > 0$ such that $\tilde{d}(f(x), f(y)) \leq L \cdot d(x,y)$ for $x, y \in \Xi$. For an $\mathcal{H}^s$ measurable $X \subset \Xi$, $s \geq 0$ and $Z = f(X)$, we have $\mathcal{H}^s_\Xi(Z) \leq L^s \cdot \mathcal{H}^s(X)$.

In particular, $\mathcal{H}^s(Z) = 0$ if $\mathcal{H}^s(X) = 0$.

Lemma 10.4.6 (Rademacher's theorem). *If $\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}^m$ locally Lipschitz (Lipschitz on any compact $K \subset \Omega$), then f is differentiable at $\mathcal{H}^n$ a.e. $z \in \Omega$; namely,* *there exists $m \times n$ matrix $Df(z)$ at with*

$$f(x) = f(z) + Df(z)(x - z) + o(\|x - z\|);$$

or in other words, $\lim_{x\to z} \frac{f(x)-f(z)-Df(z)(x-z)}{\|x-z\|} = 0$.

For properties of Lipschitz manifolds, see Federer [212].

Remark 10.4.7 (locally Lipschitz map from a Lipschitz manifold). Let $\Xi \subset \mathbb{R}^q$ be an embedded Lipschitz n-manifold; namely, any point $x \in X$ has a neighbourhood bi-Lipschitz equivalent to $\mathbb{R}^n$, and the distance of $x, y \in \Xi$ is the infimum of $\text{length}(\gamma) = \int_0^1 \|D\gamma(t)\|\, dt$ for any Lipschitz curve $\gamma : [0, 1] \to \Xi$ with $\gamma(0) = x$ and $\gamma(1) = y$. In this case the Hausdorff measure $\mathcal{H}^s_\Xi$ on Ξ coincides with the Hausdorff measure $\mathcal{H}^s$ with respect to $\mathbb{R}^n$.

(i) For $\mathcal{H}^n_\Xi$ a.e. $x \in X$, the tangent space $T_{\Xi,x}$ exists (and isomorphic to $\mathbb{R}^n$).

(ii) If $F : \Xi \to \mathbb{R}^m$ is locally Lipschitz, then there exists the differential (a linear map) $DF(x) : T_{\Xi,x} \to \mathbb{R}^m$ at $\mathcal{H}^n_\Xi$ a.e. $x \in X$ by Rademacher's theorem.

(iii) In the setting of (ii), if $m \le n$ and there exists $T_{\Xi,x}$ and $DF(x)$ at $x \in \Xi$, then $J(F, x) = \sqrt{\det DF(x)\, DF(x)^t}$ is the Jacobian.
In particular, if $m = 1$, then $J(F, x) = \|DF(x)\|$.

We note that the spherical space $S^n \subset \mathbb{R}^{n+1}$ is an embedded manifold.

Theorem 10.4.8 (Coarea formula (Federer [212])). *For $m \le k < q$ and Lipschitz embedded k-manifold $\Xi \subset \mathbb{R}^q$, if $F : \Xi \to \mathbb{R}^m$ is locally Lipschitz and $\varphi : \Xi \to [0, \infty)$ measurable, then*

$$\int_\Xi \varphi(x) \cdot J(F, x)\, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \int_{F^{-1}(y)} \varphi(x)\, d\mathcal{H}^{k-m}(x)\, d\mathcal{H}^m(y).$$

Remark. In both Theorem 10.4.8 and Corollary 10.4.9, the integral on the left is finite if and only if the integral on the right hand side is finite.

Corollary 10.4.9 follows from Theorem 10.4.8 by taking $\varphi = \psi \circ F$.

Corollary 10.4.9 (Coarea formula #2). *For $m \le k < q$ and embedded Lipschitz k-manifold $X \subset \mathbb{R}^q$, if $F : X \to \mathbb{R}^m$ is locally Lipschitz and $\psi : \mathbb{R}^m \to [0, \infty)$ measurable, then*

$$\int_X \psi(F(x)) \cdot J(F, x)\, d\mathcal{H}^k(x) = \int_{\mathbb{R}^m} \psi(y)\mathcal{H}^{k-m}\left(F^{-1}(y)\right)\, d\mathcal{H}^m(y).$$

We need the basic properties of compact sets with rectifiable boundary in the Euclidean, Spherical and Hyperbolic space that are n-dimensional Riemannian manifolds (see Federer [212] for properties of compact sets with rectifiable boundary in an n-dimensional Riemannian manifolds).

Remark 10.4.10 (Rectifiable boundary on Riemannian manifolds). Let (Ξ, d) be an n-dimensional Riemannian manifold. A compact $X \subset \Xi$ has rectifiable boundary if $\mathrm{int}X \neq \emptyset$ and ∂X is the union of finitely many sets that are Lipschitz images of compact subsets of $\mathbb{R}^{n-1}$. In this case, $0 < \mathcal{H}_\Xi^{n-1}(\partial X) < \infty$, and the parallel domain $X^{(\varrho)} = \{z \in \Xi : \exists x \in X \text{ with } d(x,z) \leq \varrho\}$ for $\varrho > 0$ satisfies

$$\mathcal{H}_\Xi^{n-1}(\partial X) = \lim_{\varrho \to 0^+} \frac{\mathcal{H}_\Xi^n(X^{(\varrho)}) - \mathcal{H}_\Xi^n(X)}{\varrho}.$$

10.5 The Stone-Weierstrass theorem

We need the Stone-Weierstrass theorem for compact Hausdorff spaces (see Rudin [504]):

Theorem 10.5.1 (Stone-Weierstrass). *Let X be a compact Hausdorff space, and let $\mathcal{A} \subset C(X, \mathbb{R})$ contain all constant functions, and satisfy that $f \cdot g \in \mathcal{A}$ and $f + g \in \mathcal{A}$ for $f, g \in \mathcal{A}$. If for any different $x, y \in X$, there exists an $f \in \mathcal{A}$ with $f(x) \neq f(y)$, then $\mathcal{A}$ is dense in $C(X, \mathbb{R})$ with respect to the $\|\cdot\|_\infty$ metric.*

Corollary 10.5.2 (Stone-Weierstrass on S^{n-1}). *If $\mathcal{A}$ is the family of restrictions of polynomials in $\mathbb{R}^n$ to S^{n-1}, then $\mathcal{A}$ is dense in $C(S^{n-1}, \mathbb{R})$ with respect to the $\|\cdot\|_\infty$ metric.*

10.6 Convex functions

In this section, we collect some basic properties of convex functions (see Rockafellar [498] for in depth study).

Definition 10.6.1 (Convex functions). If $\Omega \subset \mathbb{R}^n$ is convex, then $\varphi : \Omega \to \mathbb{R}$ is convex, if $\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y)$ for any $x, y \in \Omega$ and $t \in (0,1)$.

Similarly, $\psi : \Omega \to \mathbb{R}$ is concave, if $-\psi$ is convex; or in other words, if $\psi((1-t)x + ty) \geq (1-t)\psi(x) + t\psi(y)$ for $x, y \in \Omega$ and $t \in (0,1)$.

Remark. For fixed $t \in (0,1)$ fixed, if $\Omega \subset \mathbb{R}^n$ is open convex, $\varphi : \Omega \to \mathbb{R}$ is measurable and $\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y)$ for any $x, y \in \Omega$, then φ is convex (cf. Theorem 10.9.11).

Theorem 10.6.2 (The differentiability of a convex function). *Let φ be a convex function on a convex open $\Omega \subset \mathbb{R}^n$.*

(i) Rademacher's theorem: *φ locally Lipschitz, and differentiable a.e in* Ω.
φ is C^1 if and only if it is differentiable at each $x \in \Omega$.

(ii) Aleksandrov's theorem: *φ is twice differentiable almost everywhere in Ω in the following sense: For a.e. $z \in \Omega$, there exist $D\varphi(z)$ and a positive semidefinite quadratic form Q_z such that*

$$\varphi(x) = \varphi(z) + \langle D\varphi(z), x - z\rangle + \frac{1}{2} \cdot Q_z(x - z) + o(\|x - z\|^2);$$

namely, $\lim_{x\to z} \frac{\varphi(x)-\varphi(z)-\langle D\varphi(z),x-z\rangle-\frac{1}{2}\cdot Q_z(x-z)}{\|x-z\|^2} = 0.$
We write $D^2\varphi(z)$ to denote the $n \times n$ symmetric positive semidefinite matrix associated to Q_z ($D^2\varphi(z)$ is the derivative of $z \mapsto D\varphi(z)$ for a.e $z \in \Omega$). Then $\det D^2\varphi$ *is a measurable function on Ω.*

Definition 10.6.3 (Subdifferential of a convex function). If $\Omega \subset \mathbb{R}^n$ is open and convex, and $\varphi : \Omega \to \mathbb{R}$ is convex, then for $z \in \Omega$, the subdifferential is

$$\partial\varphi(z) = \{u \in \mathbb{R}^n : \varphi(x) - \varphi(z) \geq \langle u, x - z\rangle \ \forall x \in \Omega\}.$$

Note that $\partial\varphi(z)$ is nonempty, convex and compact.

Remark. If $\Omega \subset \mathbb{R}^n$ is open and convex, and $\varphi : \Omega \to \mathbb{R}$ is convex, then φ is differentiable at $z \in \Omega$ with derivative $D\varphi(z)$ if and only if $\partial\varphi(z) = \{D\varphi(z)\}$ ($\partial\varphi(z)$ has one element).

10.7 Self-adjoint Elliptic Linear Operators on the sphere S^{n-1}

For this section, see Evans [206], Chapter 6, Gilbarg, Trudinger [263], Chapter 8, Caffarelli, Cabré [138].

For a C^2 function $h : S^{n-1} \to \mathbb{R}$, let $\tilde{h}(tu) = t \cdot h(u)$ and $\bar{h}(tu) = h(u)$ for $t \geq 0$ and $u \in S^{n-1}$, and hence $\tilde{h}, \bar{h} : \mathbb{R}^n \to \mathbb{R}$ are C^2 on $\mathbb{R}^n \backslash \{o\}$. Following Definition 8.1.6, we write

$$\nabla h(u) = D\tilde{h}(u)|_{u^\perp} = D\bar{h}(u)|_{u^\perp} \tag{10.7}$$

where ∇h is the spherical gradient (see Schneider [522], Section 2.5). In addition, $\mathbb{R}u$ is an eigenspace (with eigenvalue zero) of $D^2\tilde{h}(u)$ and $D^2\bar{h}(u)$, and we define

$$\nabla^2 h(u) = D^2\bar{h}(u)|_{u^\perp} \tag{10.8}$$

$$\widetilde{D}^2 h(u) = \widetilde{D}^2\tilde{h}(u) = D^2\tilde{h}(u)|_{u^\perp}. \tag{10.9}$$

According to Schneider [522], Section 2.5, $\nabla^2 h$ is the spherical Hessian of h with respect to a moving orthogonal frame in the sense of Riemannian geometry, and $\tilde{h}(x) = \|x\| \cdot \bar{h}(x)$ implies that if $u \in S^{n-1}$, then

$$\widetilde{D}^2 h(u) = D^2\tilde{h}(u)|_{u^\perp} = \nabla^2 h(u) + h(u) I_{n-1} \ \text{ on the tangent space } u^\perp. \tag{10.10}$$

For an absolutely continuous measure μ on S^{n-1} with positive C^∞ density function, we consider the corresponding scalar product

$$(\varphi, \psi)_\mu = \int_{S^{n-1}} \varphi\psi \, d\mu$$

for $\varphi, \psi \in L_2(S^{n-1}, \mu)$.

First we provide the traditional definition of a self-adjoint (symmetric) elliptic operators in terms of partial derivatives: Let ∂_i denote the ith partial derivative with respect to a moving frame on S^{n-1}, $i = 1, \ldots, n-1$, and hence $\nabla\varphi = (\partial_1\varphi, \ldots, \partial_{n-1}\varphi)$ for a C^∞ function φ on S^{n-1}. If e^w is the C^∞ density function of μ, then $\mathcal{E}$ is a self adjoint (symmetric) elliptic linear operator on $C^\infty(S^{n-1})$ (extendable to $L_2(S^{n-1}, \mu)$) if there exists C^∞ functions a_{ij}, $i, j = 1, \ldots, n$, and c on S^{n-1} such that $a_{ij} = a_{ji}$ and

$$\mathcal{E}\varphi = e^{-w} \sum_i \partial_i \left(\sum_j e^w a_{ij} \partial_j \varphi \right) + c \cdot \varphi; \tag{10.11}$$

$$\sum_{ij} a_{ij}(x)\xi_i\xi_j > 0 \ \text{ for } \xi = (\xi_1, \ldots, \xi_{n-1}) \in x^\perp \backslash \{o\}; \tag{10.12}$$

where, as $\psi \sum_{i=1}^{n-1} \partial_i T_i = -\sum_{i=1}^{n-1} \partial_i\psi \cdot T_i$ for any C^1 tangent vector field $(T_1(x), \ldots, T_{n-1}(x)) \in x^\perp$ on S^{n-1} by the divergence theorem on S^{n-1}, we have

$$(\psi, \mathcal{E}\varphi)_\mu = -\int_{S^{n-1}} \sum_{ij} a_{ij} \partial_i\psi \partial_j\varphi \, e^w \, d\mathcal{H}^{n-1} = (\mathcal{E}\psi, \varphi)_\mu.$$

Let us describe the way how we meet self adjoint elliptic linear operators in this book. We write $\mathcal{M}_d$ to denote the space of $d \times d$ symmetric matrices. For a real C^∞ function $F_2(M, x)$ of $M \in \mathcal{M}_{n-1}$ and $x \in S^{n-1}$, and C^∞ function $F_1(v, t, x)$ of $x \in S^{n-1}$, $v \in x^\perp$ and $t \in \mathbb{R}$,

$$\begin{array}{rcl} \mathcal{E}\varphi & = & F_2(\nabla^2\varphi, \cdot) + F_1(\nabla\varphi, \varphi, \cdot); \\ & & F_2(\cdot, x) \text{ and } F_1(\cdot, \cdot, x) \text{ are linear for fixed } x \in S^{n-1}; \\ & & F_2(M, x) > 0 \ \text{ if } M \in \mathcal{M}_{n-1} \text{ is positive semi-definite with } M \neq 0, \end{array} \tag{10.13}$$

is a uniformly elliptic (sometimes called strictly elliptic) operator of $\varphi \in C^\infty(S^{n-1})$.

Then the uniformly elliptic operator $\mathcal{E}$ defined on $C^\infty(S^{n-1})$ is symmetric with respect to $L_2(S^{n-1}, \mu)$; or in other words, has a self-adjoint extension to $L_2(S^{n-1}, \mu)$ if

$$(\mathcal{E}\varphi, \psi)_\mu = (\varphi, \mathcal{E}\psi)_\mu$$

for $\varphi, \psi \in C^\infty(S^{n-1})$.

Concerning spectral properties of the self-adjoint elliptic linear operator $\mathcal{E}$, the theory of compact operators on Hilbert spaces and some classical results in PDE yield the existence of C^∞ eigenfunctions $\varphi_1, \varphi_2, \ldots$ on S^{n-1} that form an orthogonal basis of $L_2(S^{n-1}, \mu)$, and corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \ldots$ with $\lim_{k\to\infty} \lambda_k = -\infty$ and $\mathcal{E}\varphi_i = \lambda_i \varphi_i$. It follows from the strong maximal principle that

$$\text{if } \varphi \text{ is an eigenfunction with } \mathcal{E}\varphi = \lambda_1\varphi, \text{ then } \varphi(x) \neq 0 \text{ for } x \in S^{n-1}. \tag{10.14}$$

According to the variational characterization of eigenvalues, if $j \geq 1$, and $(\varphi_i, \psi)_\mu = 0$ for $\psi \in C^2(S^{n-1})$ and $i = 1, \ldots, j$, then

$$(\mathcal{E}\psi, \psi)_\mu \leq \lambda_{j+1}(\psi, \psi)_\mu. \tag{10.15}$$

The following well-known properties, that explain why λ_1 is called the principal eigenvalue, are not discussed in many textbooks; therefore, we provide the simple arguments.

Proposition 10.7.1. *Using the notation as above, we have*

(i) $\lambda_i < \lambda_1$ for $i \geq 2$ (and hence λ_1 is a simple eigenvalue),

(ii) assuming that $\varphi_1(x) > 0$ for every $x \in S^{n-1}$ (cf. (10.14)),

$$\textit{if } \varphi \geq 0 \textit{ for an eigenfunction } \varphi \textit{ of } \mathcal{E}, \textit{ then } \varphi = r\varphi_1 \textit{ for } r > 0. \tag{10.16}$$

Proof. First, let ψ be an eigenfucntion with $\mathcal{E}\psi = \lambda_1\psi$, and hence we may assume that $\psi > 0$ by (10.14). If $x_0 \in S^{n-1}$ and $r = \psi(x_0)/\varphi_1(x_0)$, then $\mathcal{E}(\psi - r\varphi_1) = \lambda_1(\psi - r\varphi_1)$ and $\psi - r\varphi_1$ has a zero; therefore, (10.14) yields that $\psi - r\varphi_1 \equiv 0$. In particular, λ_1 is a simple eigenvalue.

Finally let $\varphi \geq 0$ for an eigenfunction φ. As $(\varphi, \varphi_1)_\mu > 0$, and eigenfunctions corresponding to different eigenvalues are orthogonal, the only possible eigenvalue for φ is λ_1. ∎

10.8 Matrices: The Hilbert-Schmidt norm, Hadamard's inequality and the Perron Frobenius Theorem

Let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$. For an $n \times n$ real matrix A; or equivalently, for a linear transform $A : \mathbb{R}^n \to \mathbb{R}^n$, its Hilbert-Schmidt norm is $\|A\| = \sqrt{\sum_{i=1}^n \|Ae_i\|^2}$; that is, the Euclidean norm of the n^2-dimensional vector constructed from the entries of A. This second definition shows that $\|A^t\| = \|A\|$. The Hilbert-Schmidt norm of a linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ is independent of the orthonomal basis of $\mathbb{R}^n$ because if $f_1, \ldots, f_n$ is another orthonormal basis of $\mathbb{R}^n$, then

$$\|A\|^2 = \sum_{i=1}^n \|Ae_i\|^2 = \sum_{i=1}^n \sum_{j=1}^n \langle f_j, Ae_i\rangle^2 = \sum_{j=1}^n \sum_{i=1}^n \langle A^t f_j, e_i\rangle^2 = \sum_{j=1}^n \|A^t f_j\|^2 = \sum_{j=1}^n \|Af_j\|^2.$$

For any $n \times n$ matrix $A = [a_1, \dots, a_n]$ with columns $a_1, \dots, a_n \in \mathbb{R}^n$, Hadamard inequality states that

$$|\det A| \leq \prod_{i=1}^{n} \|a_i\|. \tag{10.17}$$

Our last topic is the Perron-Frobenius theorem. We provide the argument due to Pál Hegedűs to handle the case of symmetric matrices, see for example Bapat, Raghavan [45] for the general case.

Theorem 10.8.1 (Perron-Frobenius Theorem for positive symmetric matrices). *If A is a symmetric $d \times d$ matrix such that every entry is positive, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue, and $|\lambda| < \lambda_1$ for any other eigenvalue λ;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $Ax_1 = \lambda_1 x_1$;*
- *any eigenvector x of A whose coordinates are all non-negative satisfy $x = r\,x_1$ for $r > 0$.*

Proof. For $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^n$ and $y = (y^{(1)}, \dots, y^{(n)}) \in \mathbb{R}^n$, we write $x \succeq y$ if $x^{(i)} \geq y^{(i)}$ for $i = 1, \dots, n$, and $x \succ y$ if $x^{(i)} > y^{(i)}$ for $i = 1, \dots, n$. In particular, $x \succeq o$ is equivalent to saying that the coordinates of x are non-negative. Since every entry of A is positive, we deduce that $x \succeq y$ and $x \neq y$ yield

$$Ax \succ Ay \text{ and if in addition, } y \succeq o, \text{ then } \|y\| > \|x\|. \tag{10.18}$$

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A in a way such that $|\lambda_i| \leq |\lambda_1|$ for $i = 1, \dots, n$. Considering A in an orthonormal basis of eigenvectors shows that

$$\|Ax\| \leq |\lambda_1| \cdot \|x\| \text{ holds for any } x \in \mathbb{R}^n. \tag{10.19}$$

We claim that if $x = (x^{(1)}, \dots, x^{(n)}) \in \mathbb{R}^n$ is an eigenvector with $Ax = \lambda_1 x$, then

$$\lambda_1 > 0 \text{ and either } x \succ o \text{ or } x \prec o. \tag{10.20}$$

We may assume that $x^{(j)} > 0$ for a $j \in \{1, \dots, n\}$. First we verify that $x \succeq o$ using an indirect argument. We suppose that there exists an $x^{(i)} < 0$, and seek a contradiction. Let $\tilde{x} = (|x^{(1)}|, \dots, |x^{(n)}|)$ Since every entry of A is positive and x has both negative and positive coordinate, the absolute value of each coordinate of $A\tilde{x}$ is larger than the corresponding coordinate of Ax, thus $\|A\tilde{x}\| > \|Ax\| = \|\lambda_1 x\| = |\lambda_1| \cdot \|\tilde{x}\|$. This contradicts (10.19), and hence yields $x \succeq o$. In turn, we deduce from (10.18) that $Ax \succ o$; therefore, $Ax = \lambda_1 x$ implies (10.20).

Let us fix an eigenvector $x_1 = (x_1^{(1)}, \dots, x_1^{(n)})$ for λ_1, and hence $x_1 \succ o$ by (10.20). If $y = (y^{(1)}, \dots, y^{(n)})$ is any other eigenvector for λ_1, then we may assume that $y \succ 0$

by (10.20). For $\alpha = y^{(1)}/x_1^{(1)} > 0$, $y - \alpha x_1$ has a zero coordinate. As $A(y - \alpha x_1) = \lambda_1(y - \alpha x_1)$, we deduce from (10.20) that $y = \alpha x_1$; therefore, λ_1 is a simple eigenvalue.

Finally, if $z \geq 0$ is an eigenvector for A, then $\langle z, x_1 \rangle > 0$ and the fact that the eigenvectors corresponding to different eigenvalues are orthogonal yield that $Az = \lambda_1 z$, and hence $z = r\, x_1$ for some $r > 0$. ■

Let $A = [a_{ij}]$ be a $d \times d$ matrix. We say that A is non-negative, if $a_{ij} \geq 0$ for any $i, j = 1, \ldots, d$, and that the off-diagonal entries of A are non-negative, if $e_{ij} \geq 0$ when $i \neq j$. Assuming that the off-diagonal entries of A are non-negative, we say that A is an irreducible matrix with period $p \geq 1$, if for any $i \neq j$, there exist $1 \leq k \leq p$ and pairwise different $i_0, \ldots, i_k \in \{1, \ldots, d\}$ such that $i_0 = i$, $i_k = j$, and $a_{i_{m-1}, i_m} > 0$ for $m = 1, \ldots, k$. Obviously, $p \leq n - 1$, and $p = 1$ if and only if each off-diagonal entry is positive. If the actual period is irrelevant, then Now the version of the Perron-Frobenius theorem we need states the following (see Bapat, Raghavan [45] for the general version):

Theorem 10.8.2 (Perron-Frobenius Theorem for irreducible symmetric matrices). *If A is symmetric non-negative irreducible $d \times d$ matrix with positive entries on the diagonal, and λ_1 is the largest eigenvalue, then*

- *$\lambda_1 > 0$ and λ_1 is a simple eigenvalue, and $|\lambda| < \lambda_1$ for any other eigenvalue λ;*
- *there exists an eigenvector x_1 whose coordinates are all positive and $Ax_1 = \lambda_1 x_1$;*
- *any eigenvector x of A whose coordinates are all non-negative satisfy $x = r\, x_1$ for $r > 0$.*

Proof. As A is non-negative and symmetric, the same holds for A^2. Let $A = [a_{ij}]$. As the entries if A on the diagonal are positive, if $a_{ij} > 0$, then the entry of A^2 in the ith row and jth column is positive. In addition, if $a_{ij} > 0$ and $a_{j,m} > 0$, then $a_{mj} > 0$; therefore, the entry of A^2 in the ith row and mth column is positive. It follows that if A has period $p \geq 2$, then A^2 has period at most $\lceil p/2 \rceil$ (the smallest integer not smaller than $p/2$). Since $p < n$ and $2^{\lceil \log_2 n \rceil} < 2n$, we deduce that each entry of the symmetrix matrix A^{2n+1} is positive.

Let $x_1, \ldots, x_n$ be an orthonormal basis of eigenvectors of A with $Ax_i = \lambda_i x_i$, $\lambda_i \in \mathbb{R}$, where we assume that $|\lambda_i| \leq |\lambda_1|$ for $i = 1, \ldots, n$. Since, $x_1, \ldots, x_n$ are eigenvectors of A^{2n+1}, and $A^{2n+1} x_i = \lambda_i^{2n+1} x_i$, we conclude Theorem 10.8.2 from applying Theorem 10.8.1 to A^{2n+1}, which has λ_1^{2n+1} as the eigenvalue with maximal absolute value. ■

10.9 Log-concave functions

Log-concave functions on $\mathbb{R}^n$ can be considered as functional analogues of convex bodies in $\mathbb{R}^n$. One of the core properties of log-concave functions is that they are essentially the extremizers in the Prékopa-Leindler inequality (cf. Section 3.4): Given $\lambda \in (0,1)$, measurable $f, g, h : \mathbb{R}^n \to [0,\infty)$, $n \geq 1$, with $h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^\lambda$ for $x, y \in \mathbb{R}^n$, we have

$$\int_{\mathbb{R}^n} h \geq \left(\int_{\mathbb{R}^n} f \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g \right)^{\lambda}. \tag{10.21}$$

In this section, we do not discuss properties directly related to the Prékopa-Leindler inequality in detail because that is done in Chapter 3 (see also Chapter 6 and Section 8.7 for other geometrically inspired inequalities and conjectures for log-concave functions). We also do not discuss here properties related to isoperimetric type inequalities for log-concave measures (Cheeger constant, Kannan-Lovász-Simonovits conjecture) because that is done in Section 4.7.

Definition 10.9.1 (Log-concave functions). For a convex set $C \subset \mathbb{R}^n$, a function $f : C \to [0,\infty)$ is log-concave if $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^\lambda$ holds for $x, y \in C$ and $\lambda \in (0,1)$.

Remarks.

- f is log-concave if and only if $f = e^{-\varphi}$ for a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$.
- For a log-concave function f, the level sets $\{f > t\}$ are convex for $t \in \mathbb{R}$, and hence f is measurable. Actually f may not be Borel, because its value can be rather arbitrary on the boundary of the (convex) support of f.
- Typical examples are $e^{-\pi\|x\|^2}$ (Gaussian), or $f(x) = e^{-\|x\|_K}$, or $f(x) = e^{-\|x\|_K^2}$ for $\|x\|_K = \min\{t \geq 0 : x \in tK\}$ for a convex body K with $o \in \operatorname{int} K$.
- $f = \mathbf{1}_X$ log-concave for $X \subset \mathbb{R}^n$ if and only if X is convex.

Theorem 10.9.2 (Operations preserving log-concavity, Prékopa [493]).

Affine invariance: If f log-concave function on $\mathbb{R}^n$, $\Phi \in \mathrm{GL}(n)$, $w \in \mathbb{R}^n$ and $a > 0$, then $x \mapsto a \cdot f(\Phi x + w)$ is log-concave.

Product: If f, g are log-concave functions on $\mathbb{R}^n$, then $f \cdot g$ is log-concave.

Marginals: If $h(x,y)$ log-concave on $\mathbb{R}^{n+m}$ for $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, then $\varphi(x) = \int_{\mathbb{R}^m} h(x,y)\,dy$ is log-concave on $\mathbb{R}^n$.

Convolution: If f, g are log-concave functions on $\mathbb{R}^n$, then $f * g$ is log-concave.

Proof. Affine invariance and the log-concavity of a Product directly follow from Definition 10.9.1.

For the log-concavity of Marginals, let $x_1, x_2 \in \mathbb{R}^n$ and $\lambda \in (0,1)$, and hence

$$h((1-\lambda)x_1 + \lambda x_2, (1-\lambda)y + \lambda z) \geq h(x_1, y)^{1-\lambda} h(x_2, z)^\lambda$$

for the three function $h(x_1, y)$, $h(x_2, y)$ and $h((1-\lambda)x_1 + \lambda x_2, y)$ of $y \in \mathbb{R}^m$, thus the Prékopa-Leindler inequality (10.21) yields $\varphi((1-\lambda)x_1 + \lambda x_2) \geq \varphi(x_1)^{1-\lambda} \varphi(x_2)^\lambda$.

For Convolution, $h(x,y) = f(y)g(x-y)$ is a log-concave function of $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$ as it is the product of two log-concave functions; therefore, $f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy$ log-concave. ■

Definition 10.9.3 (Log-concave measures). A regular Borel (called also Radon) measure μ on $\mathbb{R}^n$ is log-concave if

$$\mu((1-\lambda)X + \lambda Y) \geq \mu(X)^{1-\lambda}\mu(Y)^{\lambda} \text{ for Borel } X, Y \subset \mathbb{R}^n \text{ and } \lambda \in (0,1).$$

According to the Brunn-Minkowski inequality (cf. Lemma 1.12.2), the Lebesgue measure is a log-concave measure. Any absolutely continuous finite Borel measure on $\mathbb{R}^n$ with log-concave density is a log-concave measure according to the Prékopa-Leindler inequality (10.21). The converse statement is due to Borell [86], Theorem 3.2.

Theorem 10.9.4 (Borell). *μ is a log-concave Radon measure on $\mathbb{R}^n$ with* $\operatorname{supp}\mu$ *not contained in a hyperplane if and only if $d\mu = f\,d\mathcal{H}^n$ for a log-concave f with $\int_{\mathbb{R}^n} f > 0$.*

The weak limit of log-concave measures is log-concave, see, for example, Dharmadhikari, Joag-Dev [190], Th. 2.10.

Theorem 10.9.5. *If μ_k, μ are Radon measures on $\mathbb{R}^n$, each μ_k is log-concave and μ_k tends weakly to μ, then μ log-concave.*

For a log-concave function $f : \mathbb{R}^n \to [0, \infty)$, the support of the corresponding log-concave measure is the closed convex set

$$K_f = \operatorname{supp} f = \operatorname{cl}\{f > 0\}.$$

A useful property of a log-concave function $f : \mathbb{R}^n \to [0, \infty)$ is that it has exponentially small tail. For example, if $f(o) > 0$, and there exists $a, R > 0$ such that $f(x) \leq f(o)e^{-aR}$ whenever $\|x\| = R$, then

$$f(x) \leq f(o)e^{-a\|x\|} \text{ whenever } \|x\| \geq R. \tag{10.22}$$

It follows from (10.22) that boundedness of the integral of a log-concave function can be nicely characterized (see, for example, Cordero-Erausquin, Klartag [175]).

Lemma 10.9.6. *For a log-concave function $f : \mathbb{R}^n \to [0, \infty)$, we have $0 < \int_{\mathbb{R}^n} f\,d\mathcal{H}^n < \infty$ if and only if* $\operatorname{supp} f$ *is not contained in a hyperplane and* $\lim_{x\to\infty} f(x) = 0$.

Recently, many notions associated to convex bodies have been generalized to log-concave functions. For example, even the John and Löwner ellipsoid (cf. Chapter 6) has been generalized to them by Ivanov, Naszódi [352], and V. Milman, Rotem [455] considered "mixed volumes" of log-concave functions and see Section 10.9.2 for associated measures, like surface area measures. We note that the "sum" and "dilation" operators for log-concave functions defined in V. Milman, Rotem [455] are different from the one commonly used (see Section 10.9.1 the latter).

For additional similar properties of log-concave functions, see for example Saumard, Wellner [513] and Klartag, V. Milman [374].

10.9.1 Summation, dilation, polar of upper semi-continuous log-concave functions

For a convex function $\varphi : \mathbb{R}^n \to (-\infty, \infty]$, its Legendre transform (or convex conjugate, sometimes denoted as φ^*) is

$$\mathcal{L}(\varphi)(x) = \sup_{y \in \mathbb{R}^n} \{\langle x, y \rangle - \varphi(y)\}. \tag{10.23}$$

A log-concave function $f : \mathbb{R}^n \to [0, \infty)$ can be chosen rather arbitrarily on the boundary ∂K_f of the support; therefore, it is natural to consider the family LC_n of upper semi-continuous log-concave functions. This family has the property that if $f = e^{-\varphi} \in \mathrm{LC}_n$ for a convex function φ, then its polar is

$$f^\circ(x) = \inf_{y \in \mathbb{R}^n} \left\{ \frac{e^{-\langle x, y \rangle}}{f(y)} \right\} = e^{-\mathcal{L}(\varphi)}$$

is also an upper semi-continuous log-concave function, and satisfies $(f^\circ)^\circ = f$ (see Section 6.7 for Santaló-type inequalities involving the polar).

For $\lambda > 0$, the "dilate" of an $f \in \mathrm{LC}_n$ is

$$\lambda \cdot f(x) = f\left(\frac{x}{\lambda}\right)^\lambda,$$

which is also an upper semi-continuous log-concave function, and the "sum" of $f, g \in \mathrm{LC}_n$ is the upper semi-continuous log-concave function (sup-convolution)

$$f \star g(x) = \sup_{y \in \mathbb{R}^n} g(y) f(x - y).$$

We observe that if $K, C \subset \mathbb{R}^n$ are convex bodies and $\lambda > 0$, then $\mathbf{1}_K, \mathbf{1}_C \in \mathrm{LC}_n$ satisfy $\lambda \cdot \mathbf{1}_K = \mathbf{1}_{\lambda K}$ and $\mathbf{1}_K \star \mathbf{1}_C = \mathbf{1}_{K+C}$. Using this notation as above, the Prékopa-Leindler inequality (10.21) for $f, g \in \mathrm{LC}_n$ and $\lambda \in (0, 1)$ reads as

$$\int_{\mathbb{R}^n} ((1 - \lambda) \cdot f \star \lambda \cdot g) \geq \left(\int_{\mathbb{R}^n} f\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g\right)^\lambda. \tag{10.24}$$

Riesz-type representation theorems using the notion above of linearity have been verified by Rotem [501]. Roysdon, Xing [503] defined the L_p linear combination of log-concave functions, and proved analogues of the Borell-Brascamp-Lieb inequalities (cf. (3.23)).

10.9.2 Surface area measures and relatives associated to log-concave functions

For an upper semi-continuous log-concave functions $f \in \mathrm{LC}_n$, its support function (see (10.23) for the Legendre transform) is the lower semi-continuous convex function

$$h_f = \mathcal{L}(-\log f)$$

on $\mathbb{R}^n$. The name is explained by the fact that if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$, then $h_f = h_K$ (the support function of K).

Extending the work of Cordero-Erausquin, Klartag [175] and Colesanti, Fragalá [170], Rotem [502] found that it is natural to consider two measures μ_f and ν_f as *surface area measures of an* $f \in \mathrm{LC}_n$ where μ_f is a Borel measure on $\mathbb{R}^n$ and ν_f is a Borel measure on S^{n-1}, which satisfy the following.

Theorem 10.9.7 (Rotem). *If $f, g \in \mathrm{LC}_n$ and $0 < \int_{\mathbb{R}^n} f < \infty$, then*

$$\lim_{t \to 0^+} \frac{\int_{\mathbb{R}^n} (f \star (t \cdot g)) - \int_{\mathbb{R}^n} f}{t} = \int_{\mathbb{R}^n} h_g \, d\mu_f + \int_{S^{n-1}} h_{K_g} \, d\nu_f. \tag{10.25}$$

In addition, $\mu_f = \mu_{\tilde{f}}$ and $\nu_f = \nu_{\tilde{f}}$ for $\tilde{f} \in \mathrm{LC}_n$ with $0 < \int_{\mathbb{R}^n} \tilde{f} < \infty$ if and only if there exists $x_0 \in \mathbb{R}^n$ such that $\tilde{f}(x) = f(x - x_0)$.

Remark. For the possibly unbounded closed convex set K_g, $h_{K_g}(u) = \sup_{z \in K_g} \langle u, z \rangle$ might be infinity in (10.25) for some $u \in S^{n-1}$, and hence both sides of (10.25) are infinity, for example, if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$ and $g(x) > 0$ for $x \in \mathbb{R}^n$.

If $f \in \mathrm{LC}_n$ is of the form $f = e^{-\varphi}$ for a convex function φ, then Rotem [502] proves that μ_f and ν_f are bounded measures with bounded first moments ($\int_{\mathbb{R}^n} |x| \, d\mu_f(x) < \infty$), and for any non-negative measurable functions α on $\mathbb{R}^n$ and β on S^{n-1}, we have

$$\int_{\mathbb{R}^n} \alpha \, d\mu_f = \int_{\mathbb{R}^n} \alpha(D\varphi(x)) f(x) \, d\mathcal{H}^n(x)$$
$$\int_{S^{n-1}} \beta \, d\nu_f = \int_{\partial K_f} \beta(\nu_{\partial K_f}(x)) f(x) \, d\mathcal{H}^{n-1}(x)$$

where $\nu_{\partial K_f}$ is continuous at the ($\mathcal{H}^{n-1}$ a.e.) regular points of ∂K_f (cf. Section 1.5). In particular, if $f = \mathbf{1}_K$ for a convex body $K \subset \mathbb{R}^n$, then $\mu_f \equiv 0$ and $\nu_f = S_K$ - that is the surface area measure of K - and if $K_f = \mathbb{R}^n$ or $f(x) = 0$ for $\mathcal{H}^{n-1}$ a.e. $x \in \partial K_f$, then $\nu_f \equiv 0$ (the latter case was considered earlier by Cordero-Erausquin, Klartag [175] using variational argument, and by Santambrogio [507] using optimal transport).

Using $g(x) = \langle x, p \rangle$ for any fixed $p \in \mathbb{R}^n$ in (10.25), we deduce that

$$\int_{\mathbb{R}^n} x \, d\mu_f(x) + \int_{S^{n-1}} u \, d\nu_f(u) = o$$

Cordero-Erausquin, Klartag [175] (see also the paper Santambrogio [507] using optimal transport) managed to characterize μ_f when $\nu_f \equiv 0$.

Theorem 10.9.8 (Cordero-Erausquin, Klartag). *For a Borel measure μ on $\mathbb{R}^n$, $\mu = \mu_f$ for an $f \in \mathrm{LC}_n$ such that $0 < \int_{\mathbb{R}^n} f < \infty$ and either $K_f = \mathbb{R}^n$, or $f(x) = 0$ for $\mathcal{H}^{n-1}$ a.e. $x \in \partial K_f$ if and only if μ is finite, is not concentrated to any hyperplane, and*

$$\int_{\mathbb{R}^n} x \, d\mu_f(x) = o.$$

Recently, Fang, Xing, Ye [209] considered the L_p-Minkowski problem for log-concave functions, and Huang, Liu, Xi, Zhao [330] and Fang, Ye, Zhang, Zhao [210] managed to extend the dual Minkowski problem to log-concave functions on $\mathbb{R}^n$ for $q > 0$.

10.9.3 Midpoint log-concave functions

What happens if we fix a $\lambda \in (0, 1)$, and we know the estimate $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ in Definition 10.9.1 for all $x, y \in \mathbb{R}^n$ only for this fixed λ? When the fixed λ is $\frac{1}{2}$, a function with this property is called *midpoint log-concave*. If the function is continuous, then midpoint log-concavity readily implies log-concavity.

Lemma 10.9.9 (Continuous "midpoint log-concave" functions). *Let $\lambda \in (0, 1)$ and $X \subset \mathbb{R}^n$ convex. If continuous $f : X \to \mathbb{R}_{\geq 0}$ satisfies $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ for $x, y \in X$, then f is log-concave.*

Example 10.9.10 (A measurable "midpoint log-concave" function may not be log-concave). Define $\Xi_k \subset [0, 1]$ by induction on $k = 0, 1, \ldots$ where $\{0, 1\} = \Xi_0 \subset \Xi_1 \subset \ldots$, and $\Xi_{k+1} = \{\frac{1}{2} t + \frac{1}{2} s : t, s \in \Xi_k\}$. Let $\Xi = \cup_{k=0}^{\infty} \Xi_k$.

For $X = \Xi \times \{0\} \cup [0, 1] \times (0, 1] \subset \mathbb{R}^2$, $\mathbf{1}_X$ is "Midpoint log-concave" on $\mathbb{R}^2$ with $\int_{\mathbb{R}^2} \mathbf{1}_X > 0$, but $\mathbf{1}_X$ is not log-concave.

Remark. A classical result of Blumberg [76] and Sierpiński [534] that if measurable $\varphi : \mathbb{R}^n \to \mathbb{R}$ satisfies $\varphi(\frac{1}{2} x + \frac{1}{2} y) \geq \frac{1}{2} \varphi(x) + \frac{1}{2} \varphi(y)$ for $x, y \in \mathbb{R}^n$, then φ is convex. Note that in this case the log-concave $f = e^{-\varphi}$ is positive; namely, it does not take zero.

In this section we prove that a measurable midpoint log-concave function is "essentially" log-concave.

Theorem 10.9.11 (Measurable "Midpoint log-concave" functions). *Let $\lambda \in (0, 1)$, and let measurable $f : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ satisfy $\int_{\mathbb{R}^n} f > 0$ and $f((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} f(y)^{\lambda}$ for $x, y \in \mathbb{R}^n$. Then there exists a log-concave function φ on $\mathbb{R}^n$ and an open convex $\Omega \subset \mathbb{R}^n$ with $f(x) = \varphi(x) > 0$ if $x \in \Omega$ and $f(x) = \varphi(x) = 0$ if $x \notin \mathrm{cl}\, \Omega$, and hence $f(x) = \varphi(x)$ for a.e. $x \in \mathbb{R}^n$.*

Proof. Let D be the set of density points of the set $\{f > 0\}$.
Step 1. D is convex and $\Omega = \operatorname{int} D \neq \emptyset$

As $\int_D f = \int_{\mathbb{R}^n} f > 0$, D is not contained in a hyperplane. We define finite $\Xi_k \subset [0,1]$ by induction on $k = 0, 1, \ldots$ with the properties that $\{0,1\} = \Xi_0 \subset \Xi_1 \subset \ldots$, and if $t, s \in \Xi_k$, then $(1-\lambda)t + \lambda s \in \Xi_{k+1}$.

For $x_1, x_2 \in D$, we prove that

$$(1-s)x_1 + sx_2 \in D \quad \text{for any } s \in (0,1). \tag{10.26}$$

Let $\varepsilon \in (0,1)$. As x_1, x_2 are density points of D, there exists $\varrho > 0$ such that if $r \in (0, \varrho)$, then $|D \cap (x_i + rB^n)| > (1 - \frac{\varepsilon}{3})|r\,B^n|$ for $i = 1, 2$, thus induction on k, the "λ-log-concavity" of f and the Brunn-Minkowski inequality Theorem 3.2.1 for inner measure yield that if $t \in \Xi_k$, then $|D \cap ((1-t)x_1 + tx_2 + rB^n)| > (1 - \frac{\varepsilon}{2})|r\,B^n|$. Taking a sequence $t_k \in \Xi_k$ with $t_k \to s$ yields that $|D \cap ((1-s)x_1 + sx_2 + rB^n)| > (1-\varepsilon)|r\,B^n|$; therefore, $(1-s)x_1 + sx_2$ is a density point of D, which in turn yields (10.26) and the convexity of D.

As $|D| = |\{f > 0\}| > 0$, we have $\operatorname{int} D \neq \emptyset$.

Step 2. $\{f > 0\} \subset \operatorname{cl} D$.

Indirectly, we suppose that there exists an $x_0 \notin \operatorname{cl} D$ with $f(x_0) > 0$, and seek a contradiction. Let $y \in \partial D$ be closest to x_0, and hence there exists $z \in \operatorname{int} D$ with $y \in \operatorname{conv}\{z, x_0\}$ and $x = (1-\lambda)x_0 + \lambda z \notin \operatorname{cl} D$. Since the "$\lambda$ log-concavity" of f yield that x is a density point of $\{f > 0\}$, we have arrived at a contradiction.

Step 3. f is continuous on $\operatorname{int} D$

We may assume that $\lambda \leq \frac{1}{2}$. We observe that the "λ log-concavity" of f yields that if $x, y, z \in \operatorname{int} D$ with $x = (1-\lambda)y + \lambda z$, then

$$f(y) \leq f(x) \cdot \left(\frac{f(x)}{f(z)}\right)^{\frac{\lambda}{1-\lambda}}. \tag{10.27}$$

The proof of Step 3 is also indirect, we suppose that f is not continuous at $z_0 \in \operatorname{int} D$. In this case, first we verify that

$$\text{there exists } x_m \to z_0,\ x_m \neq z_0 \text{ with } f(x_m) \to 0. \tag{10.28}$$

To prove (10.28), the indirect hypothesis yields the existence of a sequence $w_m \to z_0$, $w_m \neq z_0$ with $f(w_m) \to t \neq f(z_0)$. According to (10.27), we may assume that $t < f(z_0)$. For each w_m, we consider the sequence $\{u_k^{(m)}\}$ with $u_0^{(m)} = z_0$, $u_1^{(m)} = w_m$ and $u_k^{(m)} = (1-\lambda)u_{k+1}^{(m)} + \lambda u_{k-1}^{(m)}$ for $k \geq 1$. Now if $k \geq 2$, then

$$f(u_k^{(m)}) \leq f(z_0) \cdot \left(\frac{f(w_m)}{f(z_0)}\right)^{\frac{(k-1)\cdot\lambda}{1-\lambda}}$$

provided $u_k^{(m)} \in \operatorname{int} D$ by (10.27) where $\frac{f(w_m)}{f(z_0)}$ tends to $\frac{t}{f(z_0)}$. Choosing $k(m) \to \infty$ in a way such that $x_m = u_{k(m)}^{(m)} \in \operatorname{int} D$ satisfies $x_m \to z_0$, we conclude (10.28).

We may assume that

$$z_0 = o.$$

Choose $r > 0$ such that $\frac{1-\lambda}{\lambda} rB^n \subset \operatorname{int} D$ (remember, $\lambda \leq \frac{1}{2}$).

Next we claim that for any $y \in rB^n$,

$$\text{there exists a sequence } y_m \to y,\ y_m \neq y \text{ with } f(y_m) \to 0. \tag{10.29}$$

To prove (10.29), let $b = -\frac{1-\lambda}{\lambda} y$, and hence $o = (1-\lambda)y + \lambda b$. It follows from (10.27) and from using the sequence $\{x_m\}$ in (10.28) that $y_m = y + \frac{\lambda}{1-\lambda} x_m$ satisfies $y_m \to y$, $x_m = (1-\lambda)y_m + \lambda b$, $y_m \neq y$ and

$$f(y_m) \leq f(x_m) \cdot \left(\frac{f(x_m)}{f(b)} \right)^{\frac{\lambda}{1-\lambda}} \to 0.$$

In turn, we conclude (10.29).

Finally, we claim that if $\ell \geq 2$ and $y + 2\varrho B^n \subset rB^n$ for $y \in rB^n$, $\varrho > 0$, then

$$\left| \left\{ x \in y + \varrho B^n : f(x) < \frac{1}{\ell} \right\} \right| > \frac{\lambda^n}{\lambda^n + (1-\lambda)^n} \cdot |\varrho B^n|. \tag{10.30}$$

To prove (10.30), let $X = \left\{ x \in y + \varrho B^n : f(x) \geq \frac{1}{\ell} \right\}$, and for the sequence $\{y_m\}$ in (10.29), let $\widetilde{X}_m = \{z \in \mathbb{R}^n : y_m = (1-\lambda)z + \lambda x \text{ for } x \in X\}$. For any $\varepsilon \in (0,1)$, we may assume by (10.29) and $\lambda \leq \frac{1}{2}$ that $f(y_m) < \frac{1}{\ell}$ and $\widetilde{X}_m \subset y + (\varrho + \varepsilon)B^n$. It follows from the "$\lambda$ log-concavity" of f that $X \cap \widetilde{X}_m = \emptyset$, and hence

$$\left(1 + \frac{\lambda^n}{(1-\lambda)^n} \right) \cdot |X| = |X| + \left| \widetilde{X}_m \right| \leq (\varrho + \varepsilon)^n |B^n|.$$

As the last estimates holds for any $\varepsilon \in (0,1)$, we conclude (10.30).

For $\ell \geq 2$ and $\Xi_\ell = \left\{ x \in rB^n : f(x) < \frac{1}{\ell} \right\}$, it follows from (10.30) that

$$|\Xi_\ell| \geq \frac{\lambda^n}{\lambda^n + (1-\lambda)^n} \cdot |rB^n|. \tag{10.31}$$

However, $\bigcap_{\ell=2}^{\infty} \Xi_\ell = \emptyset$ because $rB^n \subset D = \{f > 0\}$. Since f is measurable, we have $\lim_{\ell \to \infty} |\Xi_\ell| = 0$, which contradiction with (10.31) verifies Step 3.

Finally, combining Lemma 10.9.9 with Steps 1-3 yields the existence of a suitable log-concave function φ. ∎

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