Valuations on Lattice Polytopes

Károly J. Böröczky and Monika Ludwig

Abstract This survey contains classification results on valuations defined on lattice polytopes that intertwine the special linear group over the integers. The basic real valued valuations, namely, the coeficients of the Ehrhart polynomials are introduced and their characterization by Betke & Kneser is discussed. More recent results include a classification of vector and convex body valued valuations on lattice polytopes.

1 From Pick's theorem to the Ehrhart polynomial

A lattice $\Lambda \subset \mathbb{R}^n$ is a discrete subgroup spanned by n independent vectors. We write $\mathscr{P}(\Lambda)$ to denote the family of lattice polytopes meaning the convex hulls of finite subsets of Λ. This section concentrates on the lattice point enumerator $G_\Lambda(X)$, which is the cardinality $\#(X \cap \Lambda)$ for a bounded set X, $G(P)$ is readily a valuation for $P \in \mathscr{P}(\Lambda)$. If $\Lambda = \mathbb{Z}^n$, then we simply write $G(P)$.

MONOGRAPHS M. Beck and S. Robins [1],

SURVEYS

The first related resultis the formula due to Georg Alexander Pick (1859-1942) who died in the Theresienstadt concentration camp. For a $P \in \mathscr{P}(\mathbb{Z}^2)$, we write $b(P)$ to denote the number of latttice points in the boundary of P if P is two-dimensional, and $b(P) = 2\#(P \cap \Lambda) - 2$ if P is a segment or a point. In particular, $b(P)$ is a valuation, as well.

Theorem 1 (Pick). *For $P \in \mathscr{P}(\mathbb{Z}^2)$, we have*

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$$G(P) = V_2(P) + \tfrac{1}{2}\, b(P) + 1.$$

The core fact behind Pick's Theorem is that if $P \in \mathscr{P}(\mathbb{Z}^2)$ is a triangle with $G(P) = 3$, then $V_2(P) = 1/2$. Thus the essential two dimensional case can be proved for examle by induction on $G(P)$, dissecting P into triangles sharing a common vertex if $G(P) \geq 4$. Due to its importance and beauty, Pick's theorem has various proofs (see e.g. B. Grünbaum, G.C. Shephard [9] and C. Blatter [5]).

In higher dimensions, there is no simple formula like Pick's theorem, as it was noted by J.E. Reeve [22, 23]. The main reason is that the volume of an n-simplex $S \in \mathscr{P}(\mathbb{Z}^n)$ with $G(S) = n+1$ can be any integer multiple of $1/n!$. However, Eugène Ehrhart, a French highschool teacher found a related fundamental formula working in all dimensions.

Theorem 2 (Ehrhart). *For $n \geq 2$ and $i = 0, \ldots, n$, there exists homogeneous rational valued valuation $G_i(P)$ of degree i of $P \in \mathscr{P}(\mathbb{Z}^n)$, such that if $k \in \mathbb{N}$, then*

$$G(kP) = \sum_{i=0}^{n} G_i(P) k^i.$$

According to the Ehrhart-Macdonald Reciprocity Law (see I.G. Macdonald [17]), if $k \in \mathbb{N}$, then

$$G(\mathrm{int}(kP)) = (-1)^n \sum_{i=0}^{n} G_i(P)(-k)^i.$$

Here, we have

$$\begin{aligned} G_n(P) &= V_n(P) \\ G_0(P) &= 1. \end{aligned}$$

In addition, let $\det_{n-1} L$ denote the determinant of a n $(n-1)$-dimensional sublattice of $\mathbb{Z}^n$, and let $\mathscr{F}(P)$ denote the family of facets of an n-dimensional polytope. If $P \in \mathscr{P}(\mathbb{Z}^n)$, then

$$G_{n-1}(P) = \begin{cases} \frac{1}{2} \sum_{F \in \mathscr{F}(P)} \frac{V_{d-1}(F)}{\det_{n-1}(\mathbb{Z}^n \cap \mathrm{aff}\, F)} & \text{if } \dim P = n, \\ \frac{V_{n-1}(P)}{\det_{n-1}(\mathbb{Z}^n \cap \mathrm{aff}\, P)} & \text{if } \dim P = n-1, \\ 0 & \text{if } \dim P \leq n-2. \end{cases}$$

In particular, we have $G_1(P) = \frac{1}{2} b(P)$ in accordance with Pick's Theorem if $n = 2$.

Unfortunately, there seems to be no "geometric formula" for $G_i(P)$ if $n \geq 3$ and $1 \leq i \leq n-2$, and actually $G_i(P)$ might be negative in this case. Still, if $P \in \mathscr{P}(\mathbb{Z}^n)$ is n-dimensional and $i = 1, \ldots, n-1$ good bounds of the form

$$a(n,i)V(P) + b(n,i) \leq G_i(P) \leq c(n,i)V(P) + d(n,i)$$

involving the so called Stirling numbers are known. Here the optimal upper bound on $G(P)$ for $i = 1, \ldots, n-1$ is due to U. Betke and P. McMullen [4]. The lower bound is due to M. Henk and M. Tagami [13] and A. Tsuchiya [25], and it is known to be optimal if $i = 1,2,3,n-3,n-2$, and if $n-i$ is even.

There is a natural representation of the Ehrhart polynomial via the projective toric variety X_P assosiated to an n-dimensional $P \in \mathscr{P}(\mathbb{Z}^n)$. More precisely, the Ehrhart polynomial of P coincides with Hilbert polynomial given by the Todd Class of the very ample divisor corresponding to P on X_P (see W. Fulton and Cox, D.A., Little, J.B., Schenck, H.K. [6]).

J. Pommersheim [21] gives a formula for $G_1(P)$ in terms of Dedeking sums if $P \in \mathscr{P}(\mathbb{Z}^3)$ is a tetrahedon using the associated toric variety.

J.-M. Kantor and A. Khovanskii [14] provide formulas for the coefficients of the Ehrhart polynomial if $n = 3,4$ using combinatorial analogues of the algebraic geometric approach.

R. Diaz, S. Robins [7] provided formulas using valuation theory and Fourier analysis in any dimension.

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2 Exclusion-inclusion principle and polynomial valuations

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$$a \times b = c\ , \tag{1}$$

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$$
\begin{aligned}
a \times b &= c \\
\mathbf{a} \cdot \mathbf{b} &= \mathbf{c} \qquad (2)
\end{aligned}
$$

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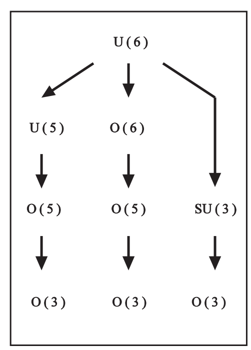

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1. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   a. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
   b. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.
2. Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development.

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- Livelihood and survival mobility are oftentimes coutcomes of uneven socioeconomic development, cf. Table 1.
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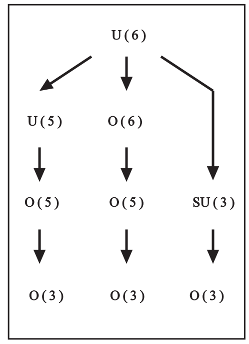

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Table 1 Please write your table caption here

Classes	Subclass	Length	Action Mechanism
Translation	mRNA[a]	22 (19–25)	Translation repression, mRNA cleavage
Translation	mRNA cleavage	21	mRNA cleavage
Translation	mRNA	21–22	mRNA cleavage
Translation	mRNA	24–26	Histone and DNA Modification

[a] Table foot note (with superscript)

5 Section Heading

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5.1 Subsection Heading

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Theorem 3. *Theorem text goes here.*

Definition 1. Definition text goes here.

Proof. Proof text goes here. □

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Theorem 4. *Theorem text goes here.*

Definition 2. Definition text goes here.

Proof. Proof text goes here. □

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$$a \times b = c \tag{3}$$

References

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