Smoothness in the $L_p$ Minkowski Problem for $p < 1$

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Abstract
We discuss the smoothness and strict convexity of the solution of the $L_p$-Minkowski problem when $p < 1$ and the given measure has a positive density function.

Keywords $L_p$ Minkowski problem · Monge–Ampère equation

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1 Introduction

Given $K$ in the class $K^0_n$ of compact convex sets in $\mathbb{R}^n$ that have non-empty interior and contain the origin $o$, we write $h_K$ and $S_K$ to denote its support function and its surface area measure, respectively, and for $p \in \mathbb{R}$, $S_{K,p}$ to denote its $L_p$-area measure, where $dS_{K,p} = h^{1-p}_K dS_K$. The $L_p$-area measure defined by Lutwak [35] is a central notion in convexity, see say Barthe et al. [2], Böröczky et al. [5], Campi and Gronchi [10], Chou [15], Cianchi et al. [17], Gage and Hamilton [19], Haberl and Parapatits [23], Haberl and Schuster [24,25], Haberl et al. [26], He et al. [27], Henk and Linke

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The $L_p$ Minkowski problem asks for the existence of a convex body $K \in \mathcal{K}_0^n$ whose $L_p$ area measure is a given finite Borel measure $\nu$ on $S^{n-1}$. When $p = 1$, this is the classical Minkowski problem solved by Minkowski [40] for polytopes, and by Alexandrov [1] and Fenchel and Jessen [18] in general. The smoothness of the solution was clarified in a series of papers by Nirenberg [43], Cheng and Yau [14], Pogorelov [46] and Caffarelli [7,8]. For $p > 1$ and $p \neq n$, the $L_p$ Minkowski problem has a unique solution according to Chou and Wang [16], Guan and Lin [22] and Hug, Lutwak, Yang and Zhang [30]. The smoothness of the solution is discussed in Chou and Wang [16], Huang and Lu [29] and Lutwak and Oliker [36]. In addition, the case $p < 1$ has been intensively investigated by Böröczky et al. [4], Böröczky and Trinh [6], Chen [13], Chen et al. [11,12], Ivaki [31], Jiang [32], Lu and Wang [33], Lutwak et al. [39], Stancu [48,49] and Zhu [52–55].

The solution of the $L_p$-Minkowski problem may not be unique for $p < 1$ according to Chen et al. [12] if $0 < p < 1$, according to Stancu [49] if $p = 0$, and according to Chou and Wang [16] if $p < 0$ small.

In this paper we are interested in this problem when $p < 1$ and $\nu$ is a measure with density with respect to the Hausdorff measure $\mathcal{H}^{n-1}$ on $S^{n-1}$, i.e. in the problem

$$dS_{K,p} = f \, d\mathcal{H}^{n-1} \quad \text{on} \quad S^{n-1},$$

(1.1)

where $f$ is a non-negative Borel function in $S^{n-1}$.

According to Chou and Wang [16], if $-n < p < 1$ and the Borel function $f$ is bounded from above and below by positive constants, then (1.1) has a solution. More general existence results are provided by the recent works Chen et al. [11] if $p = 0$, Chen et al. [12] if $0 < p < 1$, and Bianchi et al. [3] if $-n < p < 0$. In particular, it is known that (1.1) has a solution if $0 \leq p < 1$ and $f$ is any non-negative function in $L_1(S^{n-1})$ with $\int_{S^{n-1}} f \, d\mathcal{H}^{n-1} > 0$, and if $-n < p < 0$ and $f$ is any non-negative function in $L_{-n+p}(S^{n-1})$ with $\int_{S^{n-1}} f \, d\mathcal{H}^{n-1} > 0$.

We observe that $h$ is a non-negative positively 1-homogeneous convex function in $\mathbb{R}^n$ which solves the Monge–Ampère equation

$$h^{1-p} \det(\nabla^2 h + hI) = f \quad \text{on} \quad S^{n-1}$$

(1.2)

in the sense of measure if and only if $h$ is the support function of a convex body $K \in \mathcal{K}_0^n$ which is the solution of (1.1) (see Sect. 2). Here $h$ is the unknown non-negative (support) function on $S^{n-1}$ to be found, $\nabla^2 h$ denotes the (covariant) Hessian matrix of $h$ with respect to an orthonormal frame on $S^{n-1}$, and $I$ is the identity matrix. The function $h$ may vanish somewhere even in the case when $f$ is positive and continuous, and when this happens and $p < 1$ the Eq. (1.2) is singular at the zero set of $h$. Naturally, if $h$ is $C^2$, then (1.2) is a proper Monge–Ampère equation.

In this paper we study the smoothness and strict convexity of a solution $K \in \mathcal{K}_0^n$ of (1.1) assuming $\tau_2 > f > \tau_1$ for some constants $\tau_2 > \tau_1 > 0$. Concerning these aspects for $p < 1$, we summarise the known results in Theorem 1.1, and the new results in Theorem 1.2.
We say that $x \in \partial K$ is a $C^1$-smooth point if there is a unique tangent hyperplane to $K$ at $x$, and observe that $\partial K$ is $C^1$ if and only if each $x \in \partial K$ is $C^1$-smooth (see Sect. 2 for all definitions). In addition, we note that $h_K$ is $C^1$ on $S^{n-1}$ if and only if $\partial K$ is $C^1$ (see Theorem 3.6), and Theorem 1.1(iii) is due to Chou and Wang [16]. If the function $f$ in (1.1) is $C^\alpha$ for $\alpha > 0$, then Caffarelli [8] proves (iv).

**Theorem 1.1** (Caffarelli, Chou, Wang) If $K \in K^n_0$ is a solution of (1.1) for $n \geq 2$ and $p < 1$, and $f$ is bounded from above and below by positive constants, then the following assertions hold:

(i) The set $X_0$ of the points $x \in \partial K$ with $N(K, x) \subset N(K, o)$ is closed, each point of $X = \partial K \backslash X_0$ is $C^1$-smooth and $X$ contains no segment.
(ii) If $o \in \partial K$ is a $C^1$-smooth point, then $\partial K$ is $C^1$.
(iii) If $p \leq 2 - n$, then $o \in \text{int} K$, and hence $K$ is strictly convex and $\partial K$ is $C^1$.
(iv) If $o \in \text{int} K$ and the function $f$ in (1.1) is positive and $C^\alpha$, for some $\alpha > 0$, then $\partial K$ is $C^{2,\alpha}$.

Concerning strict convexity, assertion (iii) here is optimal because Example 4.2 shows that if $2 - n < p < 1$, then it is possible that $o$ belongs to the relative interior of an $(n - 1)$-dimensional face of a solution $K$ of (1.1) where $f$ is a positive continuous function. Therefore, the only question left open is the $C^1$ smoothness of the boundary of the solution if $2 - n < p < 1$.

We note that if $p < 1$ and $K$ is a solution of (1.2) with $f$ positive and $o \in \partial K$, then

$$\dim N(K, o) \leq n - 1.$$  \hspace{1cm} (1.3)

Therefore, Theorem 1.1(ii) yields that $\partial K$ is $C^1$ for the solution $K$ if $n = 2$. In general, we have the following partial results.

**Theorem 1.2** If $K \in K^n_0$ is a solution of (1.1) for $n \geq 2$ and $p < 1$, and $f$ is bounded from above and below by positive constants, then the following assertions hold:

(i) If $n = 2$, $n = 3$ or $n > 3$ and $p < 4 - n$, then $\partial K$ is $C^1$.
(ii) If $\mathcal{H}^{n-1}(X_0) = 0$ for the $X_0$ in Theorem 1.1(i), then $\partial K$ is $C^1$.

Our results differ in some cases from the ones in Chou and Wang [16], possibly because [16] considers the equation

$$\det(\nabla^2 h + h I) = fh^{p-1} \text{ on } S^{n-1}$$  \hspace{1cm} (1.4)

instead of (1.2). In the context of non-negative convex functions, being a solution of this last equation is a priori more restrictive than being a solution of (1.2), even if obviously the two notions coincide when $h$ is positive (see Sect. 2 for more on this point). Chou and Wang [16] proves, under our same assumptions on $f$, the strict convexity of the solution $h$ of (1.4) on hyperplanes avoiding the origin, and uses this to prove that $\partial K$ is $C^1$ for the convex body $K$. We note that if $K \in K^n_0$ is a solution of
(1.4) for $p < 1$ and $f$ is bounded from below and above by positive constants, then combining Theorem 1.2(ii) with the simple observation (2.11) in Sect. 2 shows that $\partial K$ is $C^1$, as it was verified by Chou and Wang [16]. In our opinion (1.2) is the right equation to consider and using it we obtain weaker results.

To give an example of how the two equations differ, the support function $h$ of the body $K$ in Example 4.2 (where $o$ belongs to the relative interior of an $(n - 1)$-dimensional face) is a solution of (1.2) but not a solution of (1.4).

According to Chou and Wang [16] (see also Lemma 3.1 below), the Monge-Ampère equation (1.2) can be transferred to a Monge-Ampère equation

$$v^{1-p} \det(D^2v) = g$$

(1.5)

for a convex function $v$ on $\mathbb{R}^{n-1}$ where $g$ is a given non-negative function and $D^2$ stands for the Hessian in $\mathbb{R}^{n-1}$.

The proofs of Claims (i) and (ii) in Theorem 1.1 use as an essential tool a result proved by Caffarelli in [7] regarding smoothness and strict convexity of convex solutions of certain Monge–Ampère equation of type (1.5) (see Theorem 3.6). Proving that $\partial K$ is $C^1$ is equivalent to prove that $h_K$ is strictly convex, and [7] is the key to prove this property in $\{y \in S^{n-1} : h_K(y) > 0\}$.

The proof of Claim (i) in Theorem 1.2 is based on the following result for the singular inequality $v^{1-p} \det D^2v \geq g$.

**Proposition 1.3** Let $\Omega \subset \mathbb{R}^n$ be an open convex set, and let $v$ be a non-negative convex function in $\Omega$ with $S = \{x \in \Omega : v(x) = 0\}$. If for $p < 1$ and $\tau > 0$, $v$ is the solution of

$$v^{1-p} \det D^2v \geq \tau \quad \text{in} \ \Omega \setminus S$$

(1.6)

in the sense of measure, and $S$ is $r$-dimensional, for $r \geq 1$, then $p \geq -n + 1 + 2r$.

We mention that in Caffarelli [9] a corresponding result for $p = 1$ is established.

The underlying idea behind the proof of this result is the following: On the one hand, the graph of $v$ near $S$ is close to being ruled. Hence, the total variation of the derivative is “small”. On the other hand, the total variation of the derivative is “large” because of the Monge-Ampère inequality (1.6).

The inequality $p \geq -n + 1 + 2r$ in this result is close to being optimal, at least when $r = 1$. Indeed, Example 3.2 shows that, for any $p > -n + 3$, there exists a non-negative convex solution of (1.6) in $\Omega$ which vanishes on the intersection of $\Omega$ with a line. For the version $p = 1$ of Proposition 1.3, Caffarelli [9] proves that $\dim S < n/2$ and that this inequality is optimal.

Proposition 1.3 yields actually somewhat more than Claim (i) in Theorem 1.2; namely, if $r \geq 2$ is an integer, $p < \min\{1, 2r - n\}$ and $K \in K^p_{\Omega}$ is a solution of (1.1) with $o \in \partial K$, then $\dim N(K, o) < r$. As a consequence, we have the following technical statements about $K$, where we also use Theorem 1.2 (ii) for Claim (ii).

**Corollary 1.4** If $p < 1$ and $K \in K^p_{\Omega}$, $n \geq 4$, is a solution of (1.1) with $o \in \partial K$, then

(i) $\dim N(K, o) < \frac{n+1}{2}$;
(ii) if in addition \( n = 4, 5 \) and \( \partial K \) is not \( C^1 \), then \( \dim N(K, o) = 2 \) and \( \dim F(K, u) = n - 1 \) for some \( u \in N(K, o) \).

In Section 2 we review the notation used in this paper. Section 3 contains results and examples regarding Monge-Ampère equations in \( \mathbb{R}^n \), namely Proposition 1.3, Example 3.2 and Proposition 3.4. This last result is the key to prove Theorem 1.2 (ii). In Section 4 we show, for the sake of completeness, how to prove Theorem 1.1 using ideas due to Caffarelli [7,8] and Chou and Wang [16]. Theorem 1.2 and Corollary 1.4 are proved in Section 5.

2 Notation and Preliminaries

As usual, \( S^{n-1} \) denotes the unit sphere and \( o \) the origin in the Euclidean \( n \)-space \( \mathbb{R}^n \). The symbol \( B^n \) denotes the unit ball in \( \mathbb{R}^n \) centred at \( o \) and \( \omega_n \) denotes its volume. If \( x, y \in \mathbb{R}^n \), then \( \langle x, y \rangle \) is the scalar product of \( x \) and \( y \), while \( \| x \| \) is the euclidean norm of \( x \). By \([x, y]\) we denote the segment with endpoint \( x \) and \( y \).

We write \( \mathcal{H}^k \) for \( k \)-dimensional Hausdorff measure in \( \mathbb{R}^n \).

We denote by \( \partial E, \text{int} E, \text{cl} E, \) and \( 1_E \) the boundary, interior, closure, and characteristic function of a set \( E \) in \( \mathbb{R}^n \), respectively. The symbols aff \( E \) and lin \( E \) denote, respectively, the affine hull and the linear hull of \( E \). The dimension \( \dim E \) is the dimension of aff \( E \). With the symbol \( E | L \) we denote the orthogonal projection of \( E \) on the linear space \( L \).

Given a function \( v \) defined on a subset of \( \mathbb{R}^n \), \( Dv \) and \( D^2v \) denote its gradient and its Hessian, respectively.

Our next goal is to recall a standard notion of generalised solution of Monge-Ampère equations, usually referred to as solution in the sense of measure. Our general reference for notions and facts about Monge-Ampère equations is the survey by Trudinger and Wang [51]. Let \( v \) be a convex function defined in an open convex set \( \Omega \); the subgradient \( \partial v(x) \) of \( v \) at \( x \in \Omega \) is defined as

\[
\partial v(x) = \{ z \in \mathbb{R}^n : v(y) \geq v(x) + \langle z, y - x \rangle \text{ for each } y \in \Omega \},
\]

which is a non-empty compact convex set. Note that \( v \) is differentiable at \( x \in \Omega \) if and only if \( \partial v(x) \) consists of exactly one vector, which is the gradient of \( v \) at \( x \). If \( \omega \subset \Omega \) is a Borel set, then we denote by \( N_v(\omega) \) the image of \( \omega \) through the gradient map of \( v \), i.e.

\[
N_v(\omega) = \bigcup_{x \in \omega} \partial v(x).
\]

Note that as \( \omega \) is a Borel set, then \( N_v(\omega) \) is measurable. Hence, we may define the Monge–Ampère measure associated to \( v \) as follows

\[
\mu_v(\omega) = \mathcal{H}^n \left( N_v(\omega) \right).
\]
For \( p < 1 \) and non-negative \( g \) on \( \mathbb{R}^n \), we say that the non-negative convex function \( v \) satisfies the Monge-Ampère equation
\[
v^{1-p} \det(D^2v) = g
\]
in the sense of measure (or in the Alexandrov sense) if
\[
v^{1-p} \, d\mu_v = g \, d\mathcal{H}^n.
\]
Equivalently
\[
\int_\omega v^{1-p}(x) \, d\mu_v(x) = \int_\omega g(x) \, dx
\]
for every Borel subset \( \omega \) of \( \Omega \).

A convex body in \( \mathbb{R}^n \) is a compact convex set with non-empty interior. The treatises Gardner [20], Gruber [21] and Schneider [47] are excellent general references for convex geometry. The function
\[
h_K(u) = \max\{\langle u, y \rangle : y \in K\},
\]
for \( u \in \mathbb{R}^n \), is the support function of \( K \). When it is clear the convex body to which we refer we will drop the subscript \( K \) from \( h_K \) and write simply \( h \). Any convex body \( K \) is uniquely determined by its support function. A set \( C \subset \mathbb{R}^n \) is a convex cone if
\[
\alpha_1 u_1 + \alpha_2 u_2 \in C \quad \text{for any} \quad u_1, u_2 \in C \quad \text{and} \quad \alpha_1, \alpha_2 \geq 0.
\]
If \( S \) is a convex set in \( \mathbb{R}^n \), then \( z \in S \) is an extremal point if
\[
z = \alpha x_1 + (1 - \alpha) x_2
\]
for \( x_1, x_2 \in S \) and \( \alpha \in (0, 1) \) imply \( x_1 = x_2 = z \). We note that if \( S \) is compact and convex, then \( S \) is the convex hull of its extremal points. If \( C \) is a convex cone and \( u \in C \setminus \{0\} \), we say that \( \sigma = \{\lambda u : \lambda \geq 0\} \) is an extremal ray if \( \alpha_1 x_1 + \alpha_2 x_2 \in \sigma \) for \( x_1, x_2 \in C \) and \( \alpha_1, \alpha_2 > 0 \) imply \( x_1, x_2 \in \sigma \). Now if \( C \not= \{0\} \) is a closed convex cone such that the origin is an extremal point of \( C \), then \( C \) is the convex hull of its extremal rays.

The normal cone of a convex body \( K \) at \( z \in K \) is defined as
\[
N(K, z) = \{u \in \mathbb{R}^n : \langle u, y \rangle \leq \langle u, z \rangle \text{ for all } y \in K\},
\]
where \( N(K, z) = \{0\} \) if \( z \in \text{int}K \) and \( \dim N(K, z) \geq 1 \) if \( z \in \partial K \). This definition can be written also as
\[
N(K, z) = \{u \in \mathbb{R}^n : h_K(u) = \langle z, u \rangle\}. \quad (2.2)
\]
In particular, \( N(K, z) \) is a closed convex cone such that the origin is an extremal point, and
\[
h_K(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 h_K(u_1) + \alpha_2 h_K(u_2) \quad \text{for} \quad u_1, u_2 \in N(K, z) \quad \text{and} \quad \alpha_1, \alpha_2 > 0.
\]
(2.3)
A convex body $K$ is $C^1$-smooth at $p \in \partial K$ if $N(K, p)$ is a ray, and $\partial K$ is $C^1$ if each $p \in \partial K$ is a $C^1$-smooth point. Therefore, $\partial K$ is $C^1$ if and only if the restriction of $h_K$ to any hyperplane not containing $o$ is strictly convex, by (2.3).

We say that a convex body $K$ is strictly convex if $\partial K$ contains no segment. The face of $K$ with outer normal $u \in \mathbb{R}^n$ is defined as

$$F(K, u) = \{z \in K : h_K(u) = \langle z, u \rangle\},$$

which lies in $\partial K$ if $u \neq o$. Schneider [47, Theorem 1.7.4] proves that

$$\partial h_K(u) = F(K, u).$$

Therefore, $K$ is strictly convex if and only if $h_K$ is $C^1$ on $\mathbb{R}^n \setminus \{o\}$.

A crucial notion for this paper is the one of surface area measure $S_K$ of a convex body $K$, which is a Borel measure on $S^{n-1}$, defined as follows. For any Borel set $\omega \subset S^{n-1}$:

$$S_K(\omega) = \mathcal{H}^{n-1}(\bigcup_{u \in \omega} F(K, u)) = \mathcal{H}^{n-1}(\bigcup_{u \in \omega} \partial h_K(u)).$$

Hence, $S_K$ is the analogue of the Monge–Ampère measure for the restriction of $h_K$ to $S^{n-1}$.

Given a convex body $K$ containing $o$ and $p < 1$, let $S_{K,p}$ denote the $L_p$ area measure of $K$; namely,

$$dS_{K,p} = h_K^{1-p} dS_K.$$  \hfill (2.5)

Let $f$ be a positive and measurable function on $S^{n-1}$; we say that $h_K$ is a solution of (1.2) in the sense of measure if

$$\int_\omega h_K(y)^{1-p} dS_K(y) = \int_\omega f(y) d\mathcal{H}^{n-1}(y)$$  \hfill (2.6)

for every Borel subset $\omega$ of $S^{n-1}$.

In what follows we will always assume that $f$ is bounded between two positive constants. Our first remark is that the previous definition is equivalent to the following conditions (a) and (b):

(a) $\dim N(K, o) < n$; or equivalently,

$$\mathcal{H}^{n-1}(\{y \in S^{n-1} : h_K(y) = 0\}) = \mathcal{H}^{n-1}(N(K, o) \cap S^{n-1}) = 0,$$  \hfill (2.7)

(b) for each Borel set $\omega \subset \{y \in S^{n-1} : h_K(y) > 0\}$, we have

$$\int_\omega h_K^{1-p} (y) dS_K(y) = \int_\omega f(y) d\mathcal{H}^{n-1}(y).$$  \hfill (2.8)

Moreover, condition (b) is in turn equivalent to
for each Borel set $\omega \subset \{ y \in S^{n-1} : h_K(y) > 0 \}$, we have

$$S_K(\omega) = \int_{\omega} f(y)h_K(y)^{p-1} \, d\mathcal{H}^{n-1}(y). \tag{2.9}$$

To prove that (b) and \((b')\) are equivalent is a simple exercise (in which one has to take into account the fact that $h_K$ is continuous). Indeed, both claims are in turn equivalent to the following fact: the measure $S_K$ is absolutely continuous with respect to $\mathcal{H}^{n-1}$ on $S^{n-1} \setminus \{ y \in S^{n-1} : h_K(y) = 0 \}$, and the Radon–Nikodym derivative of $S_K$ with respect to $\mathcal{H}^{n-1}$ is $f h_K^{p-1}$.

Let us prove the equivalence between (2.6) and (a)–(b). To this end, it will be useful the following observation: the set

$$\{ x \in \mathbb{R}^n : h_K(x) = 0 \}$$

is a closed convex cone. Indeed, it is the set where the non-negative, convex and 1-homogeneous function $h_k$ attains its minimum. For convenience, we set $\omega_0 = \{ y \in S^{n-1} : h_K(y) = 0 \}$. Assume that (2.6) holds; then (b) follows immediately. If, by contradiction, (a) is false, then $\omega_0$ has non-empty interior so that

$$0 = \int_{\omega_0} h_K(y)^{1-p} \, dS_K(y) = \int_{\omega_0} f(y) \, d\mathcal{H}^{n-1}(y) > 0,$$

i.e. a contradiction (in the last inequality we have used the fact that $f$ is bounded from below by a positive constant). Vice versa, assume that (a) and (b) hold. Given a Borel subset $\omega$ of $S^{n-1}$ we may write it as the disjoint union of $\omega' = \omega \cap \omega_0$ and $\omega'' = \omega \setminus \omega'$. By (a), $\mathcal{H}^{n-1}(\omega') = 0$, moreover $h_K = 0$ on $\omega'$; hence,

$$\int_{\omega} h_K(y)^{1-p} \, dS_K(y) = \int_{\omega''} h_K(y)^{1-p} \, dS_K(y) = \int_{\omega''} f(y) \, d\mathcal{H}^{n-1}(y) = \int_{\omega} f(y) \, d\mathcal{H}^{n-1}(y),$$

i.e. (2.6).

Our next step is to compare the solutions considered by Chou and Wang [16] with the ones introduced here. In particular, we will show that if $h_K$ is a solution of (1.4), then it verifies conditions (a) and (b) as well [and consequently (2.6)]. Note that being a solution of (1.4) in the sense of measures means that

$$S_K(\omega) = \int_{\omega} f(y)h_K(y)^{p-1} \, d\mathcal{H}^{n-1}(y). \tag{2.10}$$
has to hold for every Borel subset of $S^{n-1}$. In particular (2.9) follows (and then (b)). Moreover, as $S_K$ is finite, $h_K \geq 0$ and $f$ is bounded between two positive constants, the previous relation implies that

$$\int_{S^{n-1}} h_K(y)^{p-1} \, d\mathcal{H}^{n-1}(y) < +\infty.$$ 

As $p - 1 < 0$, this yields that the set $\omega_0$ where $h_K$ vanishes on $S^{n-1}$ has zero $(n - 1)$-dimensional measure. On the other hand this is the intersection of $S^{n-1}$ with a convex cone. Hence we get condition (a).

In addition, if we now apply (2.10) to $\omega_0$, we get that when $h_K$ is a solution of (1.4) then

$$S_K\left(N(K, o) \cap S^{n-1}\right) = 0.$$ 

(2.11)

Note that (2.11) implies that $\mathcal{H}^{n-1}(X_0) = 0$, in the notation of Theorem 1.2, because $X_0 \subset \bigcup \{F(K, u) : u \in N(K, o) \cap S^{n-1}\}$ and (2.11) means, by definition,

$$\mathcal{H}^{n-1}\left(\bigcup_{u \in N(K, o) \cap S^{n-1}} F(K, u)\right) = 0.$$ 

Hence, applying Theorem 1.2(ii) we deduce that if $K \in \mathcal{K}_0^n$ is a solution of (1.4) for $p < 1$ and $f$ is bounded from below and above by positive constants, then $\partial K$ is $C^1$, as it was verified by Chou and Wang [16].

### 3 Some Results on Monge–Ampère Equations in Euclidean Space

Lemma 3.1 is the tool to transfer the Monge–Ampère equation (1.2) on $S^{n-1}$ to a Euclidean Monge–Ampère equation on $\mathbb{R}^{n-1}$. For $e \in S^{n-1}$, we consider the restriction of a solution $h$ of (1.2) to the hyperplane tangent to $S^{n-1}$ at $e$.

**Lemma 3.1** If $e \in S^{n-1}$, $h$ is a convex positively $1$-homogeneous non-negative function on $\mathbb{R}^n$ that is a solution of (1.2) for $p < 1$ and positive $f$, and $v(y) = h(y + e)$ holds for $v : e^\perp \to \mathbb{R}$, then $v$ satisfies

$$v^{1-p} \det(D^2v) = g \quad \text{on } e^\perp,$$ 

(3.1)

where, for $y \in e^\perp$, we have

$$g(y) = \left(1 + \|y\|^2\right)^{-\frac{n+p}{2}} f\left(\frac{e + y}{\sqrt{1 + \|y\|^2}}\right).$$

**Proof** Let $h = h_K$ for $K \in \mathcal{K}_0^n$, and let

$$\tilde{S} = \{u \in S^{n-1} : h_K(u) = 0\}.$$
which is a possibly empty spherically convex compact set whose spherical dimension is at most \( n - 2 \), by (2.7). According to (2.9), the Monge–Ampère equation for \( h_K \) can be written in the form

\[
dS_K = h_K^{p-1} f \ d\mathcal{H}^{n-1} \quad \text{on} \quad S^{n-1} \setminus \tilde{S}.
\]

We consider \( \pi : e^\perp \to S^{n-1} \) defined by

\[
\pi(x) = (1 + \|x\|^2)^{-\frac{1}{2}} (x + e),
\]

which is induced by the radial projection from the tangent hyperplane \( e + e^\perp \) to \( S^{n-1} \). Since \( \langle \pi(x), e \rangle = (1 + \|x\|^2)^{-\frac{1}{2}} \), the Jacobian of \( \pi \) is

\[
\det D\pi(x) = (1 + \|x\|^2)^{-\frac{n}{2}}.
\]

For \( x \in e^\perp \), (2.4) and writing \( h_K \) in terms of an orthonormal basis of \( \mathbb{R}^n \) containing \( e \) yield that \( v \) satisfies

\[
\partial v(x) = \partial h_K(x + e)|e^\perp = F(K, x + e)|e^\perp = F(K, \pi(x))|e^\perp.
\]

Let \( S = \pi^{-1}(\tilde{S}) \). For a Borel set \( \omega \subset e^\perp \setminus S \), we have

\[
\mathcal{H}^{n-1}(N v(\omega)) = \mathcal{H}^{n-1} \left( \bigcup_{x \in \omega} \partial v(x) \right)
= \mathcal{H}^{n-1} \left( \bigcup_{u \in \pi(\omega)} \left( F(K, u)|e^\perp \right) \right) = \int_{\pi(\omega)} \langle u, e \rangle dS_K(u)
= \int_{\pi(\omega)} \langle u, e \rangle h_K^{p-1}(u) f(u) d\mathcal{H}^{n-1}(u)
= \int_{\omega} (1 + \|x\|^2)^{-\frac{n-p}{2}} f(\pi(x)) v(x)^p d\mathcal{H}^{n-1}(x),
\]

where we used at the last step that

\[
v(x) = h_K(x + e) = (1 + \|x\|^2)^{\frac{1}{2}} h_K(\pi(x)).
\]

In particular, \( v \) satisfies the Monge–Ampère type differential equation

\[
\det D^2 v(x) = (1 + \|x\|^2)^{-\frac{n-p}{2}} f(\pi(x)) v(x)^p \quad \text{on} \quad e^\perp \setminus S.
\]

Since \( \text{dim} \ S \leq n - 2 \) by (1.3), \( v \) satisfies (3.1) on \( e^\perp \).

Having Lemma 3.1 at hand showing the need to understand related Monge-Ampère equations in Euclidean spaces, we prove Propositions 1.3 and 3.4, and quote Caffarelli’s Theorem 3.6.

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Proof of Proposition 1.3

Up to changing coordinate system, we may assume, without loss of generality, that \( S \subset \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : x_2 = 0\} \) and the origin is contained in the relative interior of \( S \). Therefore, up to restricting \( \Omega \), we may also assume that \( v \) is continuous on \( \text{cl} \Omega \), that \( \Omega = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| < s_1, \|x_2\| < s_2\} \) for some constants \( s_1, s_2 > 0 \) and that \( S = \{(x_1, x_2) \in \Omega : x_2 = 0\} \).

Let \( \alpha = \max_{\text{cl} \Omega} v \) and let us consider the convex body

\[
M = \{(x_1, x_2, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \times \mathbb{R} : \|x_1\| \leq s_1, \|x_2\| \leq s_2, v(x_1, x_2) \leq y \leq \alpha\}.
\]

For \( t \in (0, s_2/2] \), let

\[
\Omega_t = \{(x_1, x_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} : \|x_1\| \leq s_1/2, \|x_2\| \leq t\}.
\]

We estimate \( \mathcal{H}^n(N_v(\Omega_t \setminus S)) \). Let \((x_1, x_2) \in \Omega_t \setminus S \) and let \((z_1, z_2) \in \mathbb{R}^r \times \mathbb{R}^{n-r} \) belong to \( \partial v(x_1, x_2) \). We prove that

\[
\|z_2\| \leq \frac{2\alpha}{s_2} \quad \text{and} \quad \|z_1\| \leq \frac{4\alpha}{s_1s_2} t.
\] (3.4)

If \( z_2 = 0 \) the first inequality in (3.4) holds true. Assume \( z_2 \neq 0 \). The vector \((z_1, z_2, -1)\) is an exterior normal to \( M \) at \( p = (x_1, x_2, v(x_1, x_2)) \). Since

\[
q_1 = \left(x_1, x_2 + \frac{s_2z_2}{2\|z_2\|}, \alpha\right) \in M
\]

(because \( \|x_2 + s_2z_2/(2\|z_2\|)\| \leq \|x_2\| + s_2/2 \leq s_2 \)) then \( \langle q_1 - p, (z_1, z_2, -1) \rangle \leq 0 \). This implies

\[
\|z_2\| \leq \frac{2}{s_2}(\alpha - v(x_1, x_2))
\]

and the first inequality in (3.4). Again, if \( z_1 = 0 \), then the second inequality (3.4) holds true. Assume \( z_1 \neq 0 \). We have

\[
q_2 = \left(x_1 + \frac{s_1z_1}{2\|z_1\|}, 0, v(x_1, x_2)\right) \in M,
\]

because \( \|x_1 + s_1z_1/(2\|z_1\|)\| \leq s_1, (x_1 + s_1z_1/(2\|z_1\|), 0) \in S \) and therefore \( v(x_1, x_2) \geq 0 = v(x_1 + s_1z_1/(2\|z_1\|), 0) \). The inequality \( \langle q_2 - p, (z_1, z_2, -1) \rangle \leq 0 \) implies the second inequality (3.4).

The inequalities in (3.4) imply

\[
\mathcal{H}^n(N_v(\Omega_t \setminus S)) \leq c t^r,
\] (3.5)

for a suitable constant \( c \) independent of \( t \).
Now we estimate \( \int_{\Omega \setminus S} v(x)^{p-1} \, dx \). The inclusion of the convex hull of \( S \times \{0\} \) and \( \{ \|x_1\| \leq s_1, \|x_2\| \leq s_2, y = \alpha \} \) in \( \mathcal{M} \) implies that \( v(x_1, x_2) \leq \frac{r{\alpha}}{\Omega} \|x_2\| \) for each \((x_1, x_2) \in \Omega\) by the convexity of \( v \). Using this estimate it is straightforward to compute that

\[
\int_{\Omega \setminus S} v(x)^{p-1} \, dx \geq d \, t^{n+p-r-1},
\]

for a suitable constant \( d \) independent on \( t \). The inequalities (3.5) and (3.6) and the differential inequality satisfied by \( v \) imply, as \( t \to 0^+ \),

\[
ct^r \geq \mathcal{H}^n(N_v(\Omega_t \setminus S)) \geq \int_{\Omega \setminus S} \tau v(x)^{p-1} \, dx \geq \tau d \, t^{n+p-r-1}.
\]

This inequality implies \( p \geq -n + 1 + 2r \). \( \square \)

**Example 3.2** Let us show that for any \( p > -n + 3 \) there exists a non-negative convex solution of (1.6) in \( \Omega = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_1 \in [-1, 1], \|x_2\| \leq 1\} \) which vanishes on the 1-dimensional space \( S = \{(x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{n-1} : x_2 = 0\} \).

To prove this let

\[
v(x_1, x_2) = \|x_2\| + f(\|x_2\|) g(x_1),
\]

where \( f(r) = r^\alpha \), with \( \alpha = (p - n - 1)/2 \), and \( g(x_1) = (1 + \beta x_1^2) \), with \( \beta > 0 \) sufficiently small. Note that \( \alpha > 1 \) exactly when \( p > -n + 3 \).

The function \( v \) is invariant with respect to rotations around the line containing \( S \). To compute \( \det D^2 v \) at an arbitrary point, it suffices to compute it at \((x_1, 0, \ldots, 0, r)\), \( r \geq 0 \). We get

\[
\begin{align*}
v_{x_1 x_1} &= f(r) g''(x_1), \\
v_{x_1 x_i} &= 0 \quad \text{when } 1 < i < n, \\
v_{x_1 x_n} &= f'(r) g'(x_1), \\
v_{x_i x_j} &= \frac{1}{r} + \frac{f'(r)}{r} g(x_1) \quad \text{when } 1 < i < n, \\
v_{x_i x_j} &= 0 \quad \text{when } i \neq j, (i, j) \neq (1, n), (i, j) \neq (n, 1), \\
v_{x_n x_n} &= f''(r) g(x_1).
\end{align*}
\]

The function \( v \) is convex if \( \beta \) is sufficiently small. Indeed, the eigenvalues of \( D^2 v \) are \( \frac{1}{r} + \frac{f'(r)}{r} g(x_1) \), with multiplicity \( n - 2 \), and those of the matrix

\[
\begin{pmatrix}
f g'' & f' g' \\
f' g' & f'' g
\end{pmatrix}.
\]

The determinant of the latter matrix is

\[
2\alpha \beta r^{2(\alpha - 1)} \left( \alpha - 1 - (1 + \alpha) \beta x_1^2 \right),
\]
which is positive if $\beta > 0$ is sufficiently small. Thus, all eigenvalues of $D^2 v$ are positive.

We get

$$\det D^2 v = \left( f'' g f'' - (f' g')^2 \right) \left( \frac{1}{r} + \frac{f'}{r g} \right)^{n-2},$$

which has the same order as $r^{2n-2}$ as $r \to 0^+$. Clearly $v$ has order $r$, and $v^{1-p} \det D^2 v$ has order $r^{2n-2p+1-p}$, which is uniformly bounded from above and below for our choice of $\alpha$.

The next statement is a slight modification of Lemmas 3.2 and 3.3 from Trudinger and Wang [51]. Its proof closely follows that in [51] and is given here for completeness.

**Lemma 3.3** Let $v$ be a convex function defined on the closure of an open bounded convex set $\Omega \subset \mathbb{R}^n$ satisfying the Monge-Ampère equation

$$\det D^2 v = v$$

for a finite non-negative measure $v$ on $\Omega$, let $v \equiv 0$ on $\partial \Omega$ and let $t E \subset \Omega \subset E$ for $t > 0$ and an origin centred ellipsoid $E$.

(i) If $z \in \Omega$ satisfies $(z + s E) \cap \partial \Omega \neq \emptyset$ for $s > 0$, then

$$|v(z)| \leq s^{1/n} c_0 \mathcal{H}^n(\Omega)^{1/n} v(\Omega)^{1/n}$$

for some $c_0 > 0$ depending on $n$ and $t$.

(ii) If $v(t\Omega) \geq b v(\Omega)$ for $b > 0$, then

$$|v(0)| \geq c_1 \mathcal{H}^n(\Omega)^{1/n} v(\Omega)^{1/n}$$

for some $c_1 > 0$ depending on $n$, $t$ and $b$.

(iii) If $(z + s E) \cap \partial \Omega \neq \emptyset$ and $v(t\Omega) \geq b v(\Omega)$, then

$$\frac{|v(z)|}{|v(0)|} \leq \frac{c_1}{c_0} s^{1/n}. \quad (3.8)$$

When $E = B^n$, the number $s$ can be chosen as the distance of $z$ from $\partial \Omega$. In the general case $s$ has the same meaning in the metric induced by the norm whose unitary ball is $E$.

**Proof** Let $A$ be a linear transformation such that $B^n = A^{-1} E$, let $\tilde{v}(x) = v(Ax)|\det A|^{-2/n}$, $\tilde{\Omega} = A^{-1} \Omega$ and let $\tilde{v}$ be the measure defined for each Borel set $\omega \subset \tilde{\Omega}$ as $\tilde{v}(\omega) = v(A \omega)/|\det A|$. It is known that $\tilde{v}$ solves

$$\det D^2 \tilde{v} = \tilde{v} \quad \text{in } \tilde{\Omega}. \quad (3.9)$$
Moreover, \( tB^n \subset \Omega \subset B^n \). Since \( H^n(\Omega) = |\det A|H^n(\tilde{\Omega}) \), we have

\[
\frac{H^n(\Omega)}{\omega_n} \leq |\det A| \leq \frac{H^n(\Omega)}{\omega_n t^n}.
\]

(3.10)

Let us prove Claim (i). Let \( \tilde{z} = A^{-1}z \). Then \( (\tilde{z} + sB^n) \cap \partial \tilde{\Omega} \neq \emptyset \) and if \( d \) denotes the distance of \( \tilde{z} \) from \( \partial \tilde{\Omega} \) we have \( d \leq s \). By choosing proper coordinates we may assume that \( \tilde{z} = (0, \ldots , 0, d) \), and that \( \Omega \subset \{(x_1, \ldots , x_n) \in \mathbb{R}^n : x_n > 0 \} \). Then

\[
\tilde{\Omega} \subset \tilde{\Omega} = \{(x_1, \ldots , x_n) \in \mathbb{R}^n : \|(x_1, \ldots , x_{n-1})\| < 2, 0 < x_n < 4 \}.
\]

Let \( u \) and \( w \) be convex functions such that their graphs are convex cones with vertex at \( (\tilde{z}, \tilde{v}(\tilde{z})) \) and bases \( \partial \tilde{\Omega} \) and \( \partial \tilde{\Omega} \), respectively. Then

\[
N_{\tilde{v}}(\tilde{\Omega}) \supset N_u(\tilde{\Omega}) = \partial u(\tilde{z}) \supset \partial w(\tilde{z}).
\]

(3.11)

Since \( w \) is a convex cone over the cylinder \( \hat{\Omega} \), one can easily compute that \( H^n(\partial w(\tilde{z})) \geq c_2|\hat{v}(\tilde{z})|^n/d \), for a suitable constant \( c_2 > 0 \). This inequality, (3.9) and (3.11) imply

\[
|\hat{v}(\tilde{z})| \leq \left( \frac{d}{c_2} \right)^{1/n} H^n(N_{\hat{v}}(\tilde{\Omega}))^{1/n} = \left( \frac{d}{c_2} \right)^{1/n} \hat{v}(\tilde{\Omega})^{1/n}.
\]

Expressing this inequality in terms of \( v, \Omega \) and \( \nu \) and using \( d \leq s \) and (3.10) concludes the proof of Claim (i).

Let us prove Claim (ii). There exists an unique solution \( w \) of \( \det D^2w = \hat{\nu} \) in \( \tilde{\Omega} \), \( w = 0 \) in \( \partial \tilde{\Omega} \), where \( \hat{\nu} = \nu \) in \( t\tilde{\Omega} \) and \( \nu = 0 \) elsewhere (see Theorem 2.1 in [51]). The comparison principle for Monge–Ampère equations (see Lemma 2.4 in [51]) implies \( w \geq \hat{\nu} \) in \( \tilde{\Omega} \).

Let \( z \in t\tilde{\Omega} \). The distance \( d \) of \( z \) from \( \partial \tilde{\Omega} \) is larger than or equal to \( (1 - t)t \) (here we have used the inclusion \( tB^n \subset \Omega \)). If \( y \in \partial w(z) \) and \( l(x) = \langle x, y \rangle + w(z) \), then \( l(x) \leq w(x) \) for each \( x \in \tilde{\Omega} \), by definition of subgradient. In particular, we have \( l(x) \leq 0 \) for each \( x \in \partial \tilde{\Omega} \). This implies

\[
|y| \leq \frac{|w(z)|}{d} \leq \frac{\sup_{\tilde{\Omega}} |\hat{v}|}{t(1-t)}.
\]

Therefore,

\[
H^n(N_w(t\tilde{\Omega})) \leq \omega_n \left( \frac{\sup_{\tilde{\Omega}} |\hat{v}|}{t(1-t)} \right)^n.
\]
This inequality, the equation satisfied by $w$ and the condition $v(t\Omega) \geq b \, v(\Omega)$ imply
\[
\sup_{\tilde{\Omega}} |\tilde{v}| \geq \frac{t(1-t)}{\omega_n^{1/n}} \mathcal{H}^n(N_w(t\tilde{\Omega}))^{1/n} = \frac{t(1-t)}{\omega_n^{1/n}} \tilde{v}(t\tilde{\Omega})^{1/n} \geq \frac{bt(1-t)}{\omega_n^{1/n}} \tilde{v}(\tilde{\Omega})^{1/n}.
\] (3.12)

We claim that
\[
|\tilde{v}(o)| \geq \frac{t}{1+t} \sup_{\tilde{\Omega}} |\tilde{v}|.
\] (3.13)

Indeed, let $z \in \tilde{\Omega}$ be such that $\tilde{v}(z) = \inf_{\tilde{\Omega}} \tilde{v}$. We may clearly assume $z \neq 0$, since otherwise there is nothing to prove. By choosing proper coordinates we may assume $z = (z_1, 0, \ldots, 0)$ for some $z_1 > 0$. Let $l$ be the linear function defined on the line through $o$ and $z$ and such that $l(o) = \tilde{v}(o)$ and $l(z) = \tilde{v}(z)$. It is $l(s, 0, \ldots, 0) = \tilde{v}(s, 0, \ldots, 0)$. Since $\tilde{v}$ is convex,
\[
l(s, 0, \ldots, 0) \leq \tilde{v}(s, 0, \ldots, 0)
\]
for each $s \notin [0, z_1]$ such that $(s, 0, \ldots, 0) \in \tilde{\Omega}$. When $s = -t$ we obtain $l(-t, 0, \ldots, 0) \leq \tilde{v}(-t, 0, \ldots, 0) \leq 0$. The inequality $l(-t, 0, \ldots, 0) \leq 0$ and the inclusion $\tilde{\Omega} \subset B^n$ imply (3.13).

The proof of Claim (ii) is concluded by combining (3.12) and (3.13) and expressing the obtained inequality in terms of $v$, $\Omega$ and $v$.

Claim (iii) is a consequence of the first two claims. □

The proof of Claim (ii) in Theorem 1.2 is based on the following proposition, which is related to a step in the proof of Theorem E (a) in [16]; however, our proof is substantially different from that in [16].

**Proposition 3.4** Let $v$ be a non-negative convex function defined on the closure of an open convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that $S = \{x \in \Omega : v(x) = 0\}$ is non-empty and compact, and $v$ is locally strictly convex on $\Omega \setminus S$. Let $\psi : (0, \infty) \to (0, \infty)$ be monotone decreasing and not identically zero; assume that $\tau_2 > \tau_1 > 0$ and $v$ satisfy
\[
\tau_1 \psi(v) \leq \det D^2 v \leq \tau_2 \psi(v)
\] (3.14)
in the sense of measure on $\Omega \setminus S$. If $\dim S \leq n - 1$ and $\mu_v(S) = 0$ for the associated Monge-Ampère measure $\mu_v$, then $S$ is a point.

Note that (3.14) means that for each Borel set $\omega \subset \Omega \setminus S$ we have
\[
\tau_1 \int_\omega \psi(v(x)) \, dx \leq \mu_v(\omega) \leq \tau_2 \int_\omega \psi(v(x)) \, dx,
\]
where $\mu_v$ has been defined in (2.1).
Proof We assume, arguing by contradiction, that $S$ is not a point. Choose coordinates so that $o$ is the centre of mass of $S$. Let $L = \text{lin } S$. By assumption

$$1 \leq \dim L \leq n - 1. \quad (3.15)$$

Let $e = (o, 1) \in \mathbb{R}^n \times \mathbb{R}$. We may assume that $\Omega$ is bounded, after possibly substituting it with a bounded open neighbourhood of $S$. We start by illustrating the idea of the proof.

**Sketch of the proof** For any small $\varepsilon > 0$, we construct an affine function $l_\varepsilon$ such that $l_\varepsilon(x) = \varepsilon$ for $x \in L$, and the convex set $\Omega_\varepsilon = \{v < l_\varepsilon\}$ is well balanced; namely, there exists an ellipsoid $E_\varepsilon$ centred at the origin such that $(1/(8n^3))E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$ [see (3.19)]. This is the longest part of the argument, and the main idea to construct $l_\varepsilon$ is that the graph of $l_\varepsilon$ is the centre of mass of $S$ and that the corresponding parameter $s$, as defined in Lemma 3.3, tends to 0 as $\varepsilon$ tends to 0. (Equivalently, $S$ contains points whose distance from $\partial \Omega_\varepsilon$, the one induced by the norm whose unit ball is $E_\varepsilon$, tends to 0 as $\varepsilon$ tends to 0.) This contradicts (3.8), since $|v(z) - l_\varepsilon(z)|/|v(o) - l_\varepsilon(o)| = \varepsilon/\varepsilon = 1$.

We divide the proof into four steps.

**Step 1. Definition of $l_\varepsilon$ and of $\Omega_\varepsilon$.**

Let $\varepsilon_0 = \min_{\partial \Omega} v > 0$ and let us consider the $(n + 1)$-dimensional convex body

$$M = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : v(x) \leq y \leq \varepsilon_0\}.$$ 

For $\varepsilon \in (0, \varepsilon_0)$ define $H_\varepsilon$ to be a hyperplane in $\mathbb{R}^{n+1}$

(i) containing $L + \varepsilon e = \{(x, \varepsilon e) \in \mathbb{R}^n \times \mathbb{R} : x \in L\}$ and

(ii) cutting off the minimal volume from $M$ (on the side containing the origin) under condition (i).

Let $r > 0$. We claim that there exists $\varepsilon_1 = \varepsilon_1(r)$ so that $H_\varepsilon$ is the graph of an affine function $l_\varepsilon$ for each $\varepsilon \in (0, \varepsilon_1)$, and, setting

$$\Omega_\varepsilon = \{x \in \mathbb{R}^n : v(x) < l_\varepsilon(x)\},$$

we have

$$\overline{\text{cl}} \Omega_\varepsilon \subset \Omega, \quad S \subset \Omega_\varepsilon \quad \text{and} \quad \Omega_\varepsilon \cap L \subset (1 + r)S. \quad (3.16)$$

Let $F = \{(x, y) \in M : y = \varepsilon_0\}$ be the upper face of $M$ and let $\mathcal{H}$ be the collection of hyperplanes in $\mathbb{R}^{n+1}$ which intersect both $F$ and $\{(x, y) \in M : y \leq \varepsilon_0/2\}$. Since $\Omega$ is bounded and $v$ is locally strictly convex on $\Omega \setminus S$, every hyperplane in $\mathcal{H}$ is not a supporting hyperplane to $M$. Therefore, by compactness, there exists a constant $\varrho_0 > 0$ such that for every $H \in \mathcal{H}$ both components of $M \setminus H$ are of volume at least $\varrho_0$. We choose $\varepsilon_1 \in (0, \varepsilon_0/2)$ such that the volume of the cap $\{(x, y) \in M : y \leq \varepsilon_1\}$ is less than $\varrho_0$. This choice implies that the minimum value of the problem which defines $H_\varepsilon$ is less than $\varrho_0$. Therefore, a minimiser $H_\varepsilon$ does not belong to $\mathcal{H}$.
$H_{\varepsilon} \cap \{(x, y) \in M : y \leq \varepsilon_0/2\} \neq \emptyset$, we have $H_{\varepsilon} \cap F = \emptyset$. In particular, $H_{\varepsilon}$ is the graph of an affine function defined on $\mathbb{R}^n$ and $\text{cl} \Omega_{\varepsilon} \subset \Omega$.

The inclusion $S \subset \Omega_{\varepsilon}$ holds because $v(x) = 0$ and $l_{\varepsilon}(x) = \varepsilon$ for any $x \in S$.

The origin $o$, being the centre of mass of $S$, belongs to the relative interior of $S$. Since $\dim S > 0$, the relative boundary of $(1 + r)S$ does not intersect $S$. This implies $\text{inf}_{\text{relbd}(1 + r)S} v > 0$. Thus, if $\varepsilon_1$ satisfies

$$\varepsilon_1 < \inf_{\text{relbd}(1 + r)S} v,$$

in addition to the inequalities specified above, then $v(x) > \varepsilon$ and $l_{\varepsilon}(x) = \varepsilon$ for any $x \in \text{relbd}(1 + r)S$ ($l_{\varepsilon}(x) = \varepsilon$ is a consequence of $(1 + r)S \subset L$). This implies $\Omega_{\varepsilon} \cap L \subset (1 + r)S$.

In the rest of the proof we may assume $\varepsilon_1 < \varepsilon_1(1)$ so that

$$\Omega_{\varepsilon} \cap L \subset 2S. \quad (3.17)$$

**Step 2. The centre of mass of $\Omega_{\varepsilon}$ is contained in $L$.**

To prove this claim we have to prove that for each $w \in L^\perp \subset \mathbb{R}^n$ we have

$$\int_{\Omega_{\varepsilon}} \langle x, w \rangle \, dx = 0. \quad (3.18)$$

Indeed, for $t \in \mathbb{R}$ with $|t|$ small, let

$$F(t) = \int_{\{x \in \Omega : l_{\varepsilon}(x) + t \langle x, w \rangle - v(x) > 0\}} (l_{\varepsilon}(x) + t \langle x, w \rangle - v(x)) \, dx$$

be the volume cut off by the hyperplane in $\mathbb{R}^{n+1}$ that is the graph of $x \mapsto l_{\varepsilon}(x) + t \langle x, w \rangle$ from $M$. By definition of $H_{\varepsilon}$ and $l_{\varepsilon}$, $F$ has a local minimum at $t = 0$. We have

$$\frac{F(t) - F(0)}{t} = \int_{\{x \in \Omega : l_{\varepsilon}(x) - v(x) > 0\}} \langle x, w \rangle \, dx + \int_{\Omega} \left( \frac{l_{\varepsilon}(x) - v(x)}{t} + \langle x, w \rangle \right) \times \left( 1_{\{x : l_{\varepsilon}(x) + t \langle x, w \rangle - v(x) > 0\}} - 1_{\{x : l_{\varepsilon}(x) - v(x) > 0\}} \right) \, dx.$$

The set where $1_{\{x : l_{\varepsilon}(x) + t \langle x, w \rangle - v(x) > 0\}} - 1_{\{x : l_{\varepsilon}(x) - v(x) > 0\}}$ differs from 0 is contained in

$$A_t = \{x \in \Omega : |l_{\varepsilon}(x) - v(x)| < |t \langle x, w \rangle|\}$$

and there exists $c$ independent on $t$ such that $\mathcal{H}^n(A_t) < ct$ and $\sup_{A_t} |l_{\varepsilon}(x) - v(x)| < ct$. As $F$ has a local minimum at $t = 0$, we have

$$0 = \frac{dF}{dt}(0) = \int_{\Omega_{\varepsilon}} \langle x, w \rangle \, dx.$$
which proves (3.18).

**Step 3.** For any $\varepsilon \in (0, \varepsilon_1)$ there exists an ellipsoid $E_\varepsilon$ centred at the origin such that

$$
\frac{1}{8n^3} E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon.
$$

(3.19)

Lemma 2.3.3 in [47] proves that any $k$-dimensional convex body contains its reflection, with respect to its centre of mass, scaled, with respect to the same centre of mass, by $1/k$. From the fact that the centre of mass of $\Omega_\varepsilon$ belongs to $L$, we deduce that

$$
- (\Omega_\varepsilon | L^\perp) \subset n(\Omega_\varepsilon | L^\perp).
$$

(3.20)

According to Loewner’s or John’s theorems, there exists an ellipsoid $\tilde{E}$ centred at the origin and $z_1 \in \Omega_\varepsilon$ such that

$$
z_1 + \frac{1}{n} \tilde{E} \subset \Omega_\varepsilon \subset z_1 + \tilde{E}.
$$

It follows from (3.20) that there exists $z_2 \in \Omega_\varepsilon$ such that $z_2 | L^\perp = \frac{-1}{n^2} z_1 | L^\perp$. In particular, $y_1 = \frac{1}{n+1} z_1 + \frac{n}{n+1} z_2 \in \Omega_\varepsilon$ verifies $y_1 | L^\perp = 0$, or in other words, $y_1 \in L \cap \Omega_\varepsilon$. In addition,

$$
y_1 + \frac{1}{2n^2} \tilde{E} \subset \frac{1}{n+1} \left( z_1 + \frac{1}{n} \tilde{E} \right) + \frac{n}{n+1} z_2 \subset \Omega_\varepsilon.
$$

Let $m = \dim L \leq n - 1$. Since $y_1 \in L \cap \Omega_\varepsilon$ and (3.17) imply $\frac{1}{2} y_1 \in S$, and since the origin is the centroid of $S$, we deduce that $y_2 = \frac{-1}{2m} y_1 \in S$. As $2m + 1 < 2n$, we have

$$
\frac{1}{4n^3} \tilde{E} \subset \frac{1}{2m+1} \left( y_1 + \frac{1}{2n^2} \tilde{E} \right) + \frac{2m}{2m+1} y_2 \subset \Omega_\varepsilon.
$$

As $\Omega_\varepsilon \subset 2\tilde{E}$ follows from $o \in z_1 + \tilde{E}$, we may choose $E_\varepsilon = 2\tilde{E}$, proving (3.19).

**Step 4.** Application of Lemma 3.3 to $v - l_\varepsilon$ and $\Omega_\varepsilon$ and contradiction.

We observe that

$$
v(x) - l_\varepsilon(x) = \begin{cases} 
0 & \text{if } x \in \partial \Omega_\varepsilon \\
-\varepsilon & \text{if } x \in S.
\end{cases}
$$

(3.21)

Let $v$ denote the Monge–Ampère measure $\mu_{(v-l_\varepsilon)}$ restricted to $\Omega_\varepsilon$. If $\Omega_0$ is an open set such that $\Omega_\varepsilon \subset \Omega_0 \subset \text{cl} \Omega_0 \subset \Omega$, then the set $N_v(\Omega_0)$ is bounded and this implies

$$
v(\Omega_\varepsilon) = \mathcal{H}^n(N_{(v-l_\varepsilon)}(\Omega_\varepsilon)) \leq \mathcal{H}^n(N_v(\Omega_0)) < \infty.
$$

Let $t = 1/(8n^3)$. Formula (3.19) yields $t E_\varepsilon \subset \Omega_\varepsilon \subset E_\varepsilon$. Let us prove that

$$
v(t \Omega_\varepsilon) \geq b v(\Omega_\varepsilon) \text{ for } b = \tau_1 t^n / \tau_2.
$$

(3.22)
The function $v$ is convex and attains its minimum at $o$; thus $v(x) \geq v(tx)$ for any $x \in \Omega_\varepsilon$. By this fact, the monotonicity of $\psi$, (3.14) and the assumptions on $S$, we deduce that

$$v(t\Omega_\varepsilon) = v(t(\Omega_\varepsilon \setminus S)) \geq \tau_1 \int_{t(\Omega_\varepsilon \setminus S)} \psi(v(x)) \, dx$$

$$= \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(tz)) \, dz$$

$$\geq \tau_1 t^n \int_{\Omega_\varepsilon \setminus S} \psi(v(z)) \, dz$$

$$\geq \frac{\tau_1 t^n}{\tau_2} v(\Omega_\varepsilon \setminus S) = \frac{\tau_1 t^n}{\tau_2} v(\Omega_\varepsilon)$$

proving (3.22).

Let $z \in \text{relbd}S$. We claim that when $\varepsilon \in (0, \varepsilon_1(r))$ then $(z + rE_\varepsilon) \cap \partial \Omega_\varepsilon \neq \emptyset$. This is a consequence of the second and third inclusion in (3.16). Indeed, since $o \in S \subset \Omega_\varepsilon \subset E_\varepsilon$, there exists $q_\varepsilon > 0$ such that $(1 + q_\varepsilon)z \in \partial E_\varepsilon$. The set $z + rE_\varepsilon$ contains the segment $[z, z + r(1 + q_\varepsilon)z]$. Since $q_\varepsilon > 0$, that segment contains the set $[z, (1 + r)z]$. The second and third inclusion in (3.16) imply $[z, (1 + r)z] \cap \partial \Omega_\varepsilon \neq \emptyset$. This proves the claim.

Lemma 3.3 applies to this situation with $s = r$. Since $v(z) - l_\varepsilon(z) = v(o) - l_\varepsilon(o) = -\varepsilon$ [see (3.21)], (3.8) yields

$$1 = \frac{|v(z) - l_\varepsilon(z)|}{|v(o) - l_\varepsilon(o)|} \leq \frac{c_1}{c_0} r^{1/n}.$$

Since $r$ can be any positive number, we have reached a contradiction. ☐

We will actually use the following consequence of Proposition 3.4.

Corollary 3.5 Let $\tau_2 > \tau_1 > 0$, and let $g$ be a function defined on an open convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, such that $\tau_2 > g(x) > \tau_1$ for $x \in \Omega$. For $p < 1$, let $v$ be a non-negative convex solution of

$$v^{1-p} \det D^2v = g \quad \text{in} \ \Omega.$$

If $S = \{x \in \Omega : v(x) = 0\}$ is non-empty, compact and $\mu_v(S) = 0$, and $v$ is locally strictly convex on $\Omega \setminus S$, then $S$ is a point.

Proof All we have to check is that $\dim S \leq n - 1$. It follows from the fact that the left-hand side of the differential equation is zero on $S$, while the right-hand side is positive. ☐

The following result by L. Caffarelli (see Theorem 1 and Corollary 1 in [7]) is the key in handling the regularity and strict convexity of the part of the boundary of a convex body $K$ where the support function at some normal vector is positive.
Theorem 3.6 (Caffarelli) Let $\lambda_2 > \lambda_1 > 0$, and let $v$ be a convex function on an open convex set $\Omega \subset \mathbb{R}^n$ such that

$$\lambda_1 \leq \det D^2 v \leq \lambda_2$$ \hfill (3.23)

in the sense of measure.

(i) If $v$ is non-negative and $S = \{x \in \Omega : v(x) = 0\}$ is not a point, then $S$ has no extremal point in $\Omega$.

(ii) If $v$ is strictly convex, then $v$ is $C^1$.

We recall that (3.23) is equivalent to saying that for each Borel set $\omega \subset \Omega$ we have

$$\lambda_1 \mathcal{H}^n(\omega) \leq \mu_v(\omega) \leq \lambda_2 \mathcal{H}^n(\omega),$$

where $\mu_v$ has been defined in (2.1).

4 Proof of Theorem 1.1

The next lemma provides a tool for the proof of Theorem 1.1(iii). The same result is also proved in Chou and Wang [16]; we present a short argument for the sake of completeness.

Lemma 4.1 For $n \geq 2$ and $p \leq 2 - n$, if $K \in K^n_0$ and there exists $c > 0$ such that $S_{K,p}(\omega) \geq c \mathcal{H}^{n-1}(\omega)$ for any Borel set $\omega \subset S^{n-1}$, then $o \in \text{int } K$.

Proof We suppose that $o \in \partial K$ and seek a contradiction. We choose $e \in N(K,o) \cap S^{n-1}$ such that $\{\lambda e : \lambda \geq 0\}$ is an extremal ray of $N(K,o)$. Let $H^+$ be a closed half space containing $\mathbb{R}e$ on the boundary such that $N(K,o) \cap \text{int } H^+ = \emptyset$. Let

$$V_0 = S^{n-1} \cap (e + B^n) \cap \text{int } H^+.$$

It follows by the condition on $S_{K,p}$ that

$$c \int_{V_0} h_K(u)^{p-1} d\mathcal{H}^{n-1} \leq \int_{V_0} h_K(u)^{p-1} dS_{K,p} = S_K(V_0) < \infty.$$ \hfill (4.1)

However, since $h_K$ is convex and $h_K(e) = 0$, there exists $c_0 > 0$ such that

$$h_K(x) \leq c_0 \|x - e\| \text{ for } x \in e + B^n.$$

We observe that the radial projection of $V_0$ onto the tangent hyperplane $e + e^\perp$ to $S^{n-1}$ at $e$ is $e + V_0'$ for

$$V_0' = e^\perp \cap (\sqrt{3} B^n) \cap \text{int } H^+. $$
If \( y \in V'_0 \), then \( u = (e + y)/\|e + y\| \) verifies \( \|u - e\| \geq \|y\|/2 \). It follows that

\[
\int_{V_0} h_K(u)^{p-1} \, d\mathcal{H}^{n-1} \geq c_0^{p-1} \int_{V_0} \|u - e\|^{p-1} \, d\mathcal{H}^{n-1}(u) \\
\geq \frac{c_0^{p-1}}{2} \int_{V'_0} \|y\|^{p-1} \, d\mathcal{H}^{n-1}(y) \\
\geq \frac{c_0^{p-1}}{2^{n+1}} \int_{V'_0} \|y\|^{p-1} \, d\mathcal{H}^{n-1}(y) = \infty
\]
as \( p \leq 2 - n \). This contradicts (4.1), and hence verifies the lemma. \( \Box \)

**Proof of Theorem 1.1**  
Claim (i). For \( u_0 \in S^{n-1} \setminus N(K, o) \), we choose a spherically convex open neighbourhood \( \Omega_0 \) of \( u_0 \) on \( S^{n-1} \) such that for any \( u \in \text{cl} \Omega_0 \), we have \( \langle u, u_0 \rangle > 0 \) and \( u \notin N(K, o) \). Let \( \Omega \subset u_0^+ \) be defined in a way such that \( u_0 + \Omega \) is the radial image of \( \Omega_0 \) into \( u_0 + u_0^+ \), and let \( v \) be the function on \( \Omega \) defined as in Lemma 3.1 with \( h = h_K \). Since \( h_K \) is positive and continuous on \( \text{cl} \Omega \), we deduce from Lemma 3.1 that there exist \( \lambda_2 > \lambda_1 > 0 \) depending on \( K, u_0 \) and \( \Omega_0 \) such that

\[
\lambda_1 \leq \det D^2 v \leq \lambda_2
\]
on \( \Omega \).

First we claim that

\[
\text{if } z \in \partial K \text{ and } N(K, z) \not\subset N(K, o), \text{ then } z \text{ is a } C^1\text{-smooth point.} \quad (4.3)
\]

We suppose that \( \text{dim } N(K, z) \geq 2 \), and seek a contradiction. Since \( N(K, z) \) is a closed convex cone such that \( o \) is an extremal point, the property \( N(K, z) \not\subset N(K, o) \) yields an \( e \in (N(K, z) \cap S^{n-1}) \setminus N(K, 0) \) generating an extremal ray of \( N(K, z) \). We apply the construction above for \( u_0 = e \). The convexity of \( h_K \) and (2.2) imply \( h_K(x) \geq \langle z, x \rangle \) for \( x \in \mathbb{R}^n \), with equality if and only if \( x \in N(K, z) \). We define \( S \subset \Omega \) by \( S + e = N(K, z) \cap (\Omega + e) \) and hence \( o \) is an extremal point of \( S \). It follows that the function \( \tilde{v} \) defined by \( \tilde{v}(y) = v(y) - \langle z, y + e \rangle \) is non-negative on \( \Omega \), satisfies (4.2), and

\[
S = \{ y \in \Omega : \tilde{v}(y) = 0 \}.
\]

These properties contradict Caffarelli’s Theorem 3.6(i) as \( o \) is an extremal point of \( S \), and in turn we conclude (4.3).

Next we show that

\[
h_K \text{ is differentiable at any } u_0 \in S^{n-1} \setminus N(K, o). \quad (4.4)
\]

We apply again the construction above for \( u_0 \). If \( u \in \Omega_0 \) and \( z \in F(K, u) \), clearly \( K \) is \( C^1 \)-smooth at \( z \) (i.e. \( N(K, z) \) is a ray) by (4.3). Therefore, by (2.3), \( v \) is strictly
convex on $\Omega$ and Caffarelli’s Theorem 3.6(ii) yields that $v$ is $C^1$ on $\Omega$. In turn, we conclude (4.4).

In addition, $F(K, u)$ is a unique $C^1$-smooth point for $u \in \Omega_0$ [see (2.4)], yielding that $\Omega_* = \cup\{F(K, u) : u \in \Omega_0\}$ is an open subset of $\partial K$. Therefore $\Omega_* \subset X$, any point of $\Omega_*$ is $C^1$-smooth [by (2.3)] and $\Omega_*$ contains no segment [by (2.4)], completing the proof of Claim (i).

Claim (ii). We suppose that $\partial \subset C^1$-smooth, and there exists $z \in \partial$ such that $K$ is not $C^1$-smooth at $z$. Claim (i) yields that $z \notin X_0$, and hence $N(K, z) \subset N(K, o)$, which is a contradiction, verifying Claim (ii).

Claim (iii). This is a consequence of Lemma 4.1 and Claim (i).

Claim (iv). This is a consequence of Lemma 3.1, Claim (i) and Caffarelli [8].

\[ \square \]

**Example 4.2** If $n \geq 2$ and $p \in (-n+2, 1)$, then there exists $K \in C^n_0$ with $C^1$ boundary such that $o$ lies in the relative interior of a facet of $\partial K$ and $dS_{K, p} = f dH^{n-1}$ for a strictly positive continuous $f : S^{n-1} \to \mathbb{R}$.

Let $q = (p + n - 1)/(p + n - 2)$. We have $q > 1$. Let

\[ g(r) = \begin{cases} (r - 1)^q & \text{when } r \geq 1; \\ 0 & \text{when } r \in [0, 1); \end{cases} \]

and $\tilde{g}(x_1, \ldots, x_{n-1}) = g(\|x_1, \ldots, x_{n-1}\|)$. Let $K \in C^n_0$ be such that $K \cap \{x : x_n \leq 1\} = \{x : 1 \geq x_n \geq \tilde{g}(x_1, \ldots, x_{n-1})\}$ and $\partial K \cap \{x : x_n > 0\}$ is a $C^2$ surface with Gauss curvature positive at every point. Clearly $K \cap \{x : x_n = 0\}$ is a $(n-1)$-dimensional face of $K$ which contains $o$ in its relative interior and has unit outer normal $(0, \ldots, 0, -1)$.

To prove that $dS_{K, p} = f dH^{n-1}$ for a positive continuous $f : S^{n-1} \to \mathbb{R}$, it suffices to prove that there is a neighbourhood of the South pole where $dS_{K, p}/dH^{n-1}$ is continuous and bounded from above and below by positive constants. Let $h$ be the support function of $K$ and, for $y \in \mathbb{R}^{n-1}$, let $v(y) = h(y, -1)$ be the restriction of $h$ to the hyperplane tangent to $S^{n-1}$ at the South pole. It suffices to prove that in a neighbourhood $U$ of $o$, $v$ satisfies the equation $v^{1-p} \det D^2 v = G$ with a function $G$ which is bounded from above and below by positive constants.

If $y \in U \setminus \{o\}$ we have

\[ v(y) = h(y, -1) = \langle (x', \tilde{g}(x'))(y, -1) \rangle \quad \text{where} \quad D\tilde{g}(x') = y. \quad (4.5) \]

If $U$ is sufficiently small, then $v(y)$ depends only on $\|y\|$. Let $y = (z, 0, \ldots, 0)$, with $z > 0$ small and let $r = 1 + (z/q)^{1/(q-1)}$. We have

\[ D\tilde{g}(r, 0, \ldots, 0) = (z, 0, \ldots, 0) \]
and (4.5) gives
\[ v(z, 0, \ldots, 0) = rz + \frac{q-1}{q^n-1+p}z^n-1+p. \]
(Note that \( n - 1 + p > 1 \).) Clearly \( v(0, \ldots, 0) = h(0, \ldots, 0, -1) = 0 \). When \( z > 0 \), we have
\[ v_{yi} = \frac{q-1}{z} \frac{(n-1-p)(n-2-p)z^{n-3+p}}{q^n-1+p} \]
when \( i \neq 1 \)
\[ v_{yj} = 0 \quad \text{when } i \neq j, \]
and, as \( z \to 0^+ \)
\[ v(z, 0, \ldots, 0)^{1-p} \det D^2 v(z, 0, \ldots, 0) = c + o(1), \]
for a suitable constant \( c > 0 \). This implies the existence of a function \( G \) positive and continuous on \( U \) such that
\[ \mathcal{H}^{n-1}(N_v(\omega \cap \{ v > 0 \})) = \int_{\omega \cap \{ v > 0 \}} nG(y)v(y)^{p-1} \, dy \]
for any Borel set \( \omega \subset U \). To conclude the proof that \( v \) is a solution in the sense of Alexandrov of \( v^{1-p} \det D^2 v = G \) in \( U \) it remains to prove that \( \mathcal{H}^{n-1}(\{ y \in U : v(y) = 0 \}) = 0 \), but this is obvious since \( \{ y \in U : v(y) = 0 \} = \{ o \} \).

We remark that \( h \) is not a solution of (1.4) because (2.11) fails.

5 Proofs of Theorem 1.2 and Corollary 1.4

Proof of Theorem 1.2 We may assume that \( o \in \partial K \) since otherwise \( \partial K \) is \( C^1 \) by Theorem 1.1. Let \( e \in N(K, o) \cap S^{n-1} \) be such that \( (u, e) > 0 \) for any \( u \in N(K, o) \cap S^{n-1} \). Let \( v \) be defined on \( \Omega = e^\perp \) as in Lemma 3.1 with \( h = h_K \) and let \( S = \{ x \in e^\perp : v(x) = 0 \} \). We have
\[ S + e = N(K, o) \cap (e^\perp + e), \]
by (2.2). If \( K \) is not \( C^1 \)-smooth at \( o \), then \( \dim S \geq 1 \) and, by Proposition 1.3, \( p \geq n-4 \) (note that here the dimension of the ambient space is \( n-1 \)). This proves Theorem 1.2(i).

To prove Theorem 1.2(ii) we observe that
\[ N_{h_K}(e + S) = \bigcup_{u \in N(K, o)} F(K, u) = X_0, \]
where $X_0$ is defined as in Theorem 1.1(i). The equality on the left in this formula follows by (2.4) and the equality on the right follows by Theorem 1.1(i). Thus,

$$N_v(S) = X_0|e^\perp,$$

and if $\mathcal{H}^{n-1}(X_0) = 0$, then $\mu_v(S) = 0$. We observe that $S$ is compact, by (5.1), that $v$ is locally strictly convex, by Theorem 1.1(i), and that $\dim S \leq n - 2$, by (1.3). Hence, Theorem 1.2(ii) follows by Corollary 3.5 and (5.1).

Proof of Corollary 1.4 Claim (i) is an immediate consequence of (2.2), Proposition 1.3 and Lemma 3.1. This claim implies that when $n = 4$ or $n = 5$ and $\partial K$ is not $C^1$ then $\dim N(K, o) = 2$. In this case $N(K, o) \cap S^{n-1}$ is a closed arc: let $e_1$ and $e_2$ be its endpoints. If $u \in N(K, o) \cap S^{n-1}$, $u \neq e_1, u \neq e_2$, then $F(K, u)$ is contained in the intersection of the two supporting hyperplanes \( \{ x \in \mathbb{R}^n : \langle x, e_i \rangle = h_K(e_i) \}, i = 1, 2 \). Thus,

$$\mathcal{H}^{n-1} \left( \bigcup \{ F(K, u) : u \in N(K, o) \cap S^{n-1}, u \neq e_1, u \neq e_2 \} \right) = 0.$$

Therefore $\dim F(K, e_1) = n - 1$ or $\dim F(K, e_2) = n - 1$, because otherwise

$$\bigcup \{ F(K, u) : u \in N(K, o) \cap S^{n-1} \},$$

which coincides with $X_0$ by Theorem 1.1 (i), has $(n - 1)$-dimensional Hausdorff measure equal to zero and $\partial K$ is $C^1$ by Theorem 1.2 (ii).

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