The longest segment in the complement of a packing

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1 Introduction

Let $K$ be a compact convex body in $\mathbb{R}^n$ not contained in a hyperplane, and denote the norm whose unit ball is $\frac{1}{2}(K - K)$ by $\|\cdot\|_K$. Given a translatively packing of $K$, we are interested in how long segments (with respect to $\|\cdot\|_K$) lie in the complement of the interiors of the translates. The main result of this note is showing the existence of a translatively packing such that the length of the longest segments avoiding it is only exponential in the dimension $n$ (see below). We start here with a lower bound showing that this bound is close to optimal for balls.

We show that any packing of the unit Euclidean ball $B^n$ avoids a segment of length exponential in $n$. It is a rather interesting question to find how long segments necessarily exist that avoid any packing of any convex, open body in $\mathbb{R}^n$. Our lower bound proof does not work for bodies allowing dense packings.

Let $|\cdot|$ denote the $n$-dimensional Lebesgue measure. Let us consider any packing of $B^n$, and denote the area and the packing density of the unit ball by $\kappa_n$ and $\delta(B^n)$, respectively. Choose a unit segment $s$, and denote the projection of $B^n$ into some hyperplane orthogonal to $s$ by $B^{n-1}$, and set

$$\lambda = \frac{\kappa_n}{3\kappa_{n-1}} \cdot \frac{1}{\delta(B^n)} \geq 2^{0.599n+o(n)}. \quad (1)$$

Here we used $\kappa_{n-1}/\kappa_n = O(\sqrt{n})$ and the estimate $\delta(B^n) \leq 2^{-0.599n+o(n)}$ of Kabatjanskiï & Levenstein [4]. The definition of the packing density yields that there exists a translate $Z$ of the cylinder $\lambda \cdot s + n \cdot B^{n-1}$ which is intersected
by at most
\[ |Z + B^n| \cdot \frac{\delta(B^n)}{\kappa_n} \leq (\lambda + 2)(n + 1)^{n-1}\kappa_{n-1} \cdot \frac{\delta(B^n)}{\kappa_n} = \frac{\lambda + 2}{3\lambda} \cdot (n + 1)^{n-1} < n^{n-1} \]
balls in the packing. The last inequality only holds for large enough \( n \) and follows from our estimate (1) on \( \lambda \). Therefore the total area of the projections of these balls into the base of \( Z \) is less than the area of the base. Thus there exist a point \( x \) not covered, meaning that the segment \( s' \) consisting of the points of \( Z \) mapped to \( x \) avoids all the balls of the packing. Clearly, the length of \( s' \) is \( \lambda \), thus Equation (1) provides a lower bound on the length of the longest segment avoiding all balls of the packing.

Slight modification of the argument above yields that for any lattice packing of equal balls, there exists a line avoiding all balls (see A. Heppes [3]). On the other hand, Ch. Zong conjectured that there exists a packing where the length of the longest segment in the complement is at most \( c^n \) for some constant \( c \). The paper M. Henk & Ch. Zong [2] constructed a packing where the segments in the complement have bounded length, although their method did not give any meaningful bound.

**Theorem 1** Let \( K \) a compact convex body in \( \mathbb{R}^n \) not contained in a hyperplane. Then there exists a periodic translative packing of \( K \) such that any segment of length \( c_0 n^2 \cdot \frac{|K - K|}{|K|} \) (with respect to \( \| \cdot \|_K \)) intersects the interior of some translate where \( c_0 \) is an absolute constant.

**Remark:** Note that the bound in the theorem is \( c_0 n^2 2^n \) for centrally symmetric bodies \( K \), while in the general case it is bounded by \( c_0 n^2 4^n \), since \(|K - K| \leq (2^n)^n \cdot |K|\) according to the celebrated result of C.A. Rogers & G. Shepard [6], and we have \((2^n)^n < 4^n\).

If the upper bound of Theorem 1 is improved to \( c^{n+o(n)} \) for some \( c < 2 \) for the ball, then (1) yields that \( \delta(B^n) \geq c^{-n+o(n)} \). Therefore such an improvement seems to be hard to prove. Actually, in order to improve on the classical lower bound \( \delta(B^n) \geq 2^{-n} \), it is sufficient to construct a packing such that any segment parallel to a given direction and having length of at least \( c^{n+o(n)} \), \( c < 2 \), intersects the interior of some of the balls.

Let us consider a consequence of Theorem 1. A cloud for the convex body \( K \) is defined as a packing of translates \( K \), which do not overlap \( K \), and any half line emanating from \( K \) intersects the interior of at least one translate. It was proved in K. Böröczky & V. Soltan [1] that there always exists a finite cloud. The same fact was verified independently by Ch. Zong [9], who gave the first reasonable upper bound for the cardinality of a cloud; namely, an upper bound of order \( n^{n^2} \). This bound was improved to \( c^{n^2} \)
independently by I. Talata (see [8]), and by I. Bárány and I. Leader (see Ch. Zong [10]). Here I. Talata [8] proved $2^{1.401n^2+o(n^2)}$ if $K$ is a ball, $3^{n^2+o(n^2)}$ if $K$ is centrally symmetric, and $6^{n^2+o(n^2)}$ in general. We can construct clouds of lower cardinality in the following way: we fix any translate of $K$ in the packing given by Theorem 1, and consider all translates in the packing, which are at most distance $c_0n^2 \frac{|K-K|}{|K|}$ from the fixed copy. We deduce

Corollary 1 For any centrally symmetric convex $K$ in $\mathbb{R}^n$, there exists a cloud by $2^{n^2+o(n^2)}$ translates. For a general convex body $K$, a cloud can be formed using $4^{n^2+o(n^2)}$ translates.

With respect to a lower bound, I. Talata [8] verified that a cloud of the unit ball always has at least $2^{0.59n^2+o(n^2)}$ elements. A lower bound with slightly weaker constant was independently obtained by I. Bárány (see Ch. Zong [10]).

Packing with high relative distance are also well studied. We say that the packing $\{x_i + K\}$ has relative distance $\rho > 2$ if it satisfies $||x_i - x_j||_K \geq \rho$ for $i \neq j$. Our arguments show that for such a packing, there exists a segment of length $c_1(n)\rho^n$ in the complement, and there exists such a packing where the length of any segment in the complement is at most

$$c_2(n)\rho^n \log \rho.$$ 

For clouds, it easy to see that at least $c_3(n)\rho^{n^2-n} \rho$ translates are needed for any cloud of relative distance $\rho$ (even if the source is only one point), and our argument yields such a family consisting of at most

$$c_4(n)\rho^{n^2-n}(\log \rho)^n$$

translates clouding $K$. Here $c_1(n)$, $c_2(n)$, $c_3(n)$ and $c_4(n)$ are positive constants depending only on the dimension $n$.

2 The proof of Theorem 1

Let $K$ be a compact convex body in $\mathbb{R}^n$ not contained in a hyperplane. All distances and lengths below are measured with respect to $\| \cdot \|_K$.

Our proof is probabilistic: we select random translates of $K$ for the packing and show that with high probability their collection satisfies the requirement of Theorem 1. More precisely, we consider a large enough compact factor $T^n$ of $\mathbb{R}^n$ and throw uniform random translates of $K$ into $T^n$ one by

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one. By keeping those that are disjoint from all earlier translates we obtain our periodic packing. Note that this method is not greedy, as our rule excludes a translate from the packing if it intersects some earlier translates even though all those translates may have been excluded themselves. This suboptimal rule is necessary to obtain independence between the configurations of regions far from each other.

The detailed argument is as follows. We set $c_0 = 10000$ and assume $n > 2$ for simplicity. According to the Minkowski-Hlawka theorem, there exists a lattice $\Lambda$ such that $\Lambda + 2c_0 n^2 \cdot 4^n (K - K)$ is a packing and

$$\det \Lambda \leq 2^n \cdot |2c_0 n^2 \cdot 4^n (K - K)|. \quad (2)$$

The condition on $\Lambda$ yields that if $\|x - y\|_K < 2c_0 n^2 4^n$ then the distance of images of $x$ and $y$ in the torus $T^n = \mathbb{R}^n / \Lambda$ is still $\|x - y\|_K$.

We throw points $x_1, x_2, \ldots$ into $T^n$ independently with uniform distribution with respect to the Lebesgue measure. We color an $x_i$ red if $\|x_j - x_i\|_K > 2$ holds for any $j < i$, or in other words, if $x_i + K$ is disjoint from any $x_j + K$ for $j < i$. For a measurable $A \subset T^n$, denote the probability that $A$ contains no red point by $P(A)$.

**Lemma 1** Let $A, B \subset T^n$ be measurable such that the diameter of $B$ is less than 2, and there exist translates $y_i + B \subset A$, $i = 1, \ldots, N$ with $\|y_i - y_j\|_K \geq 6$ for $i \neq j$. Then

$$P(A) \leq \left( 1 - \frac{|B|}{|K - K|} \right)^N.$$

**Proof:** First we calculate the probability that $B$ contains a red point. The probability that $x_i$ lands in $B$ and it is colored red is

$$P_i = |B| \cdot (1 - |K - K|)^{i-1}. \quad (3)$$

Since the diameter of $B$ is less than 2, only at most one $x_i \in B$ is colored red, and we deduce that

$$1 - P(B) = \sum_{i \geq 1} P_i = \frac{|B|}{|K - K|}.$$

Now the sets $y_i + B - (K - K)$ are disjoint, and hence the events that $y_i + B$ contains no red point, $i = 1, \ldots, N$, are independent. Each of these events have equal probability as calculated in (3), hence the lemma follows. Q.E.D.
According to C.A. Rogers [5], there exists a covering \( \{ z + \frac{1}{n} K | z \in Z \} \) of \( T^n \) whose density is at most \( n \ln n + n \ln \ln n + 4n \). Therefore we deduce by (2) that

\[
|Z| \leq (n \ln n + n \ln \ln n + 4n) \cdot n^n \cdot \frac{\det \Lambda}{|K|} \leq 2^{10n^2}.
\]

Let \( S \) be the family of segments in \( T^n \) whose length is between \( c_0 n^2 \cdot \frac{|K-K|}{|K|} - 1 \) and \( c_0 n^2 \cdot \frac{|K-K|}{|K|} + 1 \), and the endpoints are chosen from \( Z \). Clearly, \( \#S \leq (\#Z)^2 \leq 2^{20n^2} \).

Now Lemma 1 can be applied to \( A = s_k - (1 - \frac{2}{n})K \) with \( B = -(1 - \frac{2}{n})K \) and \( N = \lfloor c_0 n^2 \frac{|K-K|}{|K|} \rfloor \). We deduce that the probability \( P_0 \) that there exists an \( s \in S \) such that \( s - (1 - \frac{2}{n})K \) contains no red point is

\[
P_0 \leq \#S \cdot \left(1 - \frac{|B|}{|K-K|}\right)^N \leq 2^{20n^2} \left(1 - \left(1 - \frac{2}{n}\right)^n \frac{|K|}{|K-K|}\right)^N < 1.
\]

Therefore there exists a sequence \( x_1, x_2, \ldots \) such that for any \( s \in S \), the set \( s - (1 - \frac{2}{n})K \) contains a red point. Denote the family of red points by \( r_1, \ldots, r_m \).

Now \( \Lambda + \{ r_1 + K, \ldots, r_m + K \} \) is a periodic translatively packing in \( \mathbb{R}^n \). Let us consider a segment \( s_0 = aa' \) with length \( c_0 n^2 \frac{|K-K|}{|K|} \). Embedding \( a \) and \( a' \) into \( T^n \), there exist points \( z, z' \in Z \) with \( a \in z + \frac{1}{n} K, a \in z' + \frac{1}{n} K \). We now have \( s = zz' \in S \) and \( s \subset s_0 - \frac{1}{n} K \). We have seen that that there exists some \( r_i \in s - (1 - \frac{2}{n})K \subset s_0 - (1 - \frac{1}{n})K \), and hence \( s \) intersects the interior of \( r_i + K \). In turn, we conclude Theorem 1.

References


