ASYMPTOTIC INDEPENDENCE AND ADDITIVE FUNCTIONALS

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Abstract

A strong approximation result is proved for the partial sum process of i.i.d. sequence of vectors having dependent components, where the components of the approximating process are independent. This result is applied for additive functionals of random walks in one and two dimensions.

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1. Introduction and main result

Let $(X_i, \tau_i)_{i=1}^{\infty}$ be a sequence of i.i.d. vectors, with nonnegative second component. The two components of the vector are *not* supposed to be independent. Let

$$S_n = \sum_{i=1}^{n} X_i, \quad \rho_n = \sum_{i=1}^{n} \tau_i.$$

In our paper [8] we proved, that under very mild conditions on X_i and τ_i, one can approximate (S_n, ρ_n), by (S'_n, ρ'_n) having the same marginal distributions, on such a way that (S'_n) and (ρ'_n) are independent. In fact we proved the following theorem.

Theorem 1A. *Let $(X_i, \tau_i)_{i=1}^{\infty}$ be a sequence of i.i.d. vectors, such that $\tau_i \geq 0$ and*

$$\mathbf{P}(|X_i| > x) < \frac{c}{x^{\beta}}, \qquad \mathbf{P}(\tau_i > x) \leq \frac{c}{x^{\alpha}} \tag{1.1}$$

for x large enough, where $0 < \alpha < 1$, $\beta > 2$ and $c > 0$ are constants.

Then on an appropriate probability space one can construct two independent copies

$\left(X_i^{(1)}, \tau_i^{(1)}\right)_{i=1}^{\infty}$ *and* $\left(X_i^{(2)}, \tau_i^{(2)}\right)_{i=1}^{\infty}$ *together with* $(X_i, \tau_i)_{i=1}^{\infty}$ *such that*

$$(S_n, \rho_n)_{n=1}^{\infty} \stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)})_{n=1}^{\infty} \qquad j = 1, 2 \tag{1.2}$$

$$\sup_{k \leq n} |\rho_k - \rho_k^{(1)}| = O\left(n^{1/\alpha^*}\right) \qquad \text{a.s.} \tag{1.3}$$

$$\sup_{k \leq n} |S_k - S_k^{(2)}| = O\left(n^{1/\beta^*}\right) \qquad \text{a.s.} \tag{1.4}$$

as $n \to \infty$, where $S_k^{(j)} = \sum_{i=1}^{k} X_i^{(j)}$, $\rho_k^{(j)} = \sum_{i=1}^{k} \tau_i^{(j)}$, $S_k = \sum_{i=1}^{k} X_i$, $\rho_k = \sum_{i=1}^{k} \tau_i$ and $\alpha^* > \alpha$, $\beta^* > 2$.

In the present paper the condition on τ_i will be replaced by the following one:

$$\mathbf{P}(\tau_i > x) \leq \frac{1}{h(x)},$$

where $h(x)$ is slowly varying at infinity. Thus Theorem 1A takes care of the regularly varying tail case, and the present paper deals with the case of slowly varying tail for τ_i. Our main result here is the following

Theorem 1.1. *Let $(X_i, \tau_i)_{i=1}^{\infty}$ be a sequence of i.i.d. vectors, such that $\tau_i \geq 0$ and*

$$\mathbf{P}(|X_i| > x) < \frac{c}{x^{\beta}}, \qquad \mathbf{P}(\tau_i > x) \leq \frac{1}{h(x)} \tag{1.5}$$

for x large enough, where $\beta > 2$, $c > 0$ and $h(x)$ is slowly varying at infinity, increasing with $\lim_{x\to\infty} h(x) = +\infty$.

Then on an appropriate probability space one can construct two independent copies $\left(X_i^{(1)}, \tau_i^{(1)}\right)_{i=1}^{\infty}$ and $\left(X_i^{(2)}, \tau_i^{(2)}\right)_{i=1}^{\infty}$ together with $(X_i, \tau_i)_{i=1}^{\infty}$ such that

$$(S_n, \rho_n)_{n=1}^{\infty} \stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)})_{n=1}^{\infty} \qquad j = 1, 2 \tag{1.6}$$

$$\sup_{k \leq n} |S_k - S_k^{(2)}| = O\left(n^{1/\beta^*}\right) \qquad \text{a.s.} \tag{1.7}$$

$$\sup_{k \leq n} |\rho_k - \rho_k^{(1)}| = O\left(h^*(n^{\alpha})\right) \qquad \text{a.s.} \tag{1.8}$$

as $n \to \infty$, where $S_k^{(j)} = \sum_{i=1}^{k} X_i^{(j)}$, $\rho_k^{(j)} = \sum_{i=1}^{k} \tau_i^{(j)}$, $S_k = \sum_{i=1}^{k} X_i$, $\rho_k = \sum_{i=1}^{k} \tau_i$, $\alpha < 1$, $\beta^* > 2$, and $h^*(\cdot)$ is the inverse of $h(\cdot)$.

Now a brief explanation of this whole problem is in order.

Let $\{U_n\}_{n=1}^{\infty}$ be a simple symmetric random walk on the line i.e. $U_n = \sum_{k=1}^{n} Y_k$, where the random variables Y_i, $i = 1, 2, ...$ are i.i.d. with $\mathbf{P}(Y_1 = +1) = \mathbf{P}(Y_1 = -1) = 1/2$.

Define the local time of the walk by

$$\xi(x, n) = \#\{k;\ 0 \leq k \leq n,\ U_k = x\}. \tag{1.9}$$

Theorem 1B. (Dobrushin [15]): *For any fixed integer $a \neq 0$*

$$\frac{\xi(a, n) - \xi(0, n)}{(4|a| - 2)^{1/2} n^{1/4}} \stackrel{\mathcal{D}}{\to} U\sqrt{|V|} \tag{1.10}$$

as $n \to \infty$, where U and V are two independent standard normal variables and $\stackrel{\mathcal{D}}{\to}$ denotes convergence in distribution.

In fact the above result is only a special case of Dobrushin's theorem, and it has several generalizations most of them for Brownian local time in one dimension. (See Borodin [3], Kasahara [20], Papanicolaou et al. [25], Yor [28], Csörgő and Révész [13], Csáki and Földes [7]).

Denote by ρ_n the n-th return time of the random walk to zero and $S_n = \xi(a, \rho_n) - n$, i.e. $X_i = \xi(a, \rho_i) - \xi(a, \rho_{i-1}) - 1$ and $\tau_i = \rho_i - \rho_{i-1}$. Then clearly here $(X_i, \tau_i)_{i=1}^{\infty}$ is an i.i.d. sequence of vectors but X_i and τ_i are *not* independent. However as Theorem 1B suggests it, they are asymptotically independent in distribution. Further results are on this matter in Kesten [22], Kasahara [21] and in Csáki et al. [5, 6].

In this paper just like in [8] we prove the asymptotic independence in strong sense, which, as we will show, proves to be useful to get strong theorems for Dobrushin type results.

The aim of this paper is to present a method which was developed by proving strong approximation results in similar situations (see [5, 6], Csáki and Csörgő [4], Csáki and Salminen [11], Csáki, Földes and Révész [9], Csáki and Földes [8]). The proof of our Theorem 1.1 is in Section 2. Having at hand our Theorem 1.1 and Theorem 1.A, we turn our attention to applications. We apply Theorem 1.1 to prove strong approximation results for additive functionals of random walks on $\mathbf{Z}^1$ and $\mathbf{Z}^2$. Theorem 1.1 is applied for random walks having slowly varying truncated Green functions, this is the content of Section 3. In Section 4 we apply our Theorem 1.A for the case of regularly varying truncated Green function. In our paper [9] we proved similar results for the local time difference of the simple symmetric planar random walk, which led us to the generalization of our method in [8] for the slowly varying case. Some consequences of our results are discussed in Section 5.

2. Proof of Theorem 1.1.

Before proving Theorem 1.1 we formulate two simple properties of slowly varying functions, and prove some lemmas.

Property 1: *If $h(x)$ is slowly varying at infinity, then for every $\kappa > 0$, there exist x_0, y_0 such that*

$$h(xy) \le h(x) y^{\kappa} \qquad \text{if } x > x_0, \quad y > y_0. \tag{2.1}$$

This property easily follows from Potter's theorem (see e.g. Bingham, Goldie and Teugels [2], Theorem 1.5.6 (i), pp. 25).

Property 2: *If $h(x)$ is slowly varying at infinity with $h(x) \uparrow +\infty$ and L_0 is so large that $1/h(x)$ is locally bounded in $[L_0, \infty)$, then*

$$\sum_{k=L_0}^{L} \frac{1}{h(k)} \le C \frac{L}{h(L)} \tag{2.2}$$

with some constant C.

Property 2 easily follows from [2], Proposition 1.5.8, pp. 26.

Now assume that $\left(X_i^{(j)}, \tau_i^{(j)}\right)_{i=1}^{\infty}$, $j = 1,2$ are two indepent copies of the sequence of vectors $(X_i, \tau_i)_{i=1}^{\infty}$ defined on our probability space. Considering the corresponding partial sums $\left(S_n^{(j)}, \rho_n^{(j)}\right)$ $n = 1, 2, ..., j = 1, 2$ we build up the process (S_n, ρ_n) as follows. Let $A_k = 2^k$ and $r_k = A_k - A_{k-1} = 2^{k-1}$, $k = 1, 2...$ We define the k-th block of our processes as

$$(S^{(j)}_{A_{k-1}+1}, \rho^{(j)}_{A_{k-1}+1}), \dots \quad (S^{(j)}_{A_k}, \rho^{(j)}_{A_k}) \qquad j = 1, 2.$$

Consider now $\tau_i^{(j)}$-s for $j = 1, 2$ within the k-th block. Fix $q < 1$. We will call $\tau_i^{(j)}$ and the whole pair $(X_i^{(j)}, \tau_i^{(j)})$ *large* if

$$\tau_i^{(j)} \geq h^*(r_k^q) \qquad j = 1, 2, \qquad A_{k-1} < i \leq A_k \tag{2.3}$$

where $h^*(\cdot)$ is the inverse of $h(\cdot)$. $\tau_i^{(j)}$ and the whole pair $(X_i^{(j)}, \tau_i^{(j)})$ will be called *small* if

$$\tau_i^{(j)} < h^*(r_k^q) \qquad j = 1, 2, \qquad A_{k-1} < i \leq A_k. \tag{2.4}$$

Denote by $\nu_k^{(j)}$, $\mu_k^{(j)}$ the number of large and small $\tau_i^{(j)}$, respectively in the k-th block. We create the k-th block of our new partial sum process

$$(S_{A_{k-1}+1}, \rho_{A_{k-1}+1}), \dots \quad (S_{A_k}, \rho_{A_k})$$

by piecing together vectors from the processes $^{(1)}$ and $^{(2)}$ according to the following rules.

In the k-th block we will have $\nu_k = \nu_k^{(1)}$ large pairs and $\mu_k = r_k - \nu_k^{(1)}$ small pairs. To achieve this goal let $(X_i, \tau_i) = (X_i^{(1)}, \tau_i^{(1)})$ for $A_{k-1} < i \leq A_k$ if $\tau_i^{(1)}$ is large. When $\tau_i^{(1)}$ happens to be small we replace the pair $(X_i^{(1)}, \tau_i^{(1)})$ by a small pair from process $^{(2)}$ whenever it is possible. More precisely if $\mu_k^{(2)} \geq \mu_k^{(1)}$ then each small pair from process $^{(1)}$ will be replaced by the first (first within the k-th block) $\mu_k^{(1)}$ small pairs from process $^{(2)}$. However, if $\mu_k^{(2)} < \mu_k^{(1)}$ then we can only replace the first (within the k-th block) $\mu_k^{(2)}$ small pairs by the corresponding small pairs from process $^{(2)}$, leaving the remaining small pairs from process $^{(1)}$ unaltered.

Piecing the blocks together we get our new process (S_n, ρ_n). First observe, that our process automatically satisfies (1.6) of our theorem by construction.

Lemma 2.1. *For any $q > 0$ and any integer $\ell \geq 1$*

$$\mathbf{E}\left(\tau_i^{(j)} I(\tau_i^{(j)} < h^*(r_\ell^q))\right) < C\frac{h^*(r_\ell^q)}{r_\ell^q}, \quad j = 1, 2, \quad A_{\ell-1} \leq i < A_\ell. \tag{2.5}$$

Proof. The proof directly follows from (2.2) and (1.5) as

$$\mathbf{E}\left(\tau_i^{(j)} I(\tau_i^{(j)} < h^*(r_\ell^q))\right) < L_0 + \sum_{k=L_0}^{[h^*(r_\ell^q)]} \frac{1}{h(k)} \le L_0 + C\frac{h^*(r_\ell^q)}{r_\ell^q} \le C\frac{h^*(r_\ell^q)}{r_\ell^q}, \tag{2.6}$$

where C is a constant. In the sequel we keep denoting uninteresting constant by C, the value of which might vary from line to line. □

Lemma 2.2. *For any* $q^* > q > 0$

$$\sup_{\ell \le N} |\rho_\ell - \rho_\ell^{(1)}| \le h^*\left(N^{q^*}\right) \quad \text{a.s.} \tag{2.7}$$

if N *is big enough.*

Proof. First observe that for $A_{k-1} < N \le A_k$

$$\sup_{\ell \le N} |\rho_\ell - \rho_\ell^{(1)}| \le \sup_{\ell \le A_k} |\rho_\ell - \rho_\ell^{(1)}| \le \sum_{\ell=1}^{k} \sum_{j=1}^{2} \sum_{i=A_{\ell-1}+1}^{A_\ell} \tau_i^{(j)} I\left(\tau_i^{(j)} < h^*(r_\ell^q)\right). \tag{2.8}$$

Moreover, for any $p > 0$, by Lemma 2.1 and Markov's inequality

$$\begin{aligned} &\mathbf{P}\left(\sum_{j=1}^{2} \sum_{i=A_{\ell-1}+1}^{A_\ell} \tau_i^{(j)} I\left(\tau_i^{(j)} < h^*(r_\ell^q)\right) > r_\ell^p h^*(r_\ell^q)\right) \le \\ &r_\ell C\frac{h^*(r_\ell^q)}{r_\ell^q} \frac{1}{h^*(r_\ell^q) r_\ell^p} = C r_\ell^{1-p-q} = C 2^{(\ell-1)(1-p-q)}. \end{aligned} \tag{2.9}$$

Thus for any $p > 0$ for which $1 - p - q < 0$,

$$\sum_{\ell=1}^{\infty} 2^{(\ell-1)(1-p-q)} < \infty$$

implying by Borel-Cantelli lemma that for $\ell \ge \ell_0(\omega)$

$$\sum_{j=1}^{2} \sum_{i=A_{\ell-1}+1}^{A_\ell} \tau_i^{(j)} I\left(\tau_i^{(j)} < h^*(r_\ell^q)\right) \le r_\ell^p h^*\left(r_\ell^q\right) \qquad \text{a.s.} \tag{2.10}$$

Hence by (2.8) for k big enough

$$\sup_{\ell \le A_k} |\rho_\ell - \rho_\ell^{(1)}| \le \sum_{\ell=1}^{k} r_\ell^p h^*(r_\ell^q) \le k r_k^p h^*\left(r_k^q\right). \tag{2.11}$$

Select now an arbitrary $q^* > q$, and $p > 0$, such that $p + q > 1$ should hold. We show now that with this selection we have

$$kr_k^p h^*\left(r_k^q\right) < h^*\left(r_k^{q^*}\right), \tag{2.12}$$

for k big enough.

To see this, observe that for any $p' > p > 0$,

$$kr_k^p < r_k^{p'}$$

for k big enough.

Now applying $h(\cdot)$ on both sides of (2.12) and using Property 1, with $\kappa > 0$ small enough such that

$$q + p'\kappa \le q^*$$

should hold, we get

$$h\left(h^*\left(r_k^q\right)kr_k^p\right) \le r_k^q\left(kr_k^p\right)^\kappa \le r_k^q r_k^{p'\kappa} = r_k^{q+p'\kappa} \le r_k^{q^*}. \tag{2.13}$$

Thus (2.13), and hence (2.12) hold.

Consequently by (2.8),(2.11) and (2.12) for $A_{k-1} < N \le A_k$

$$\sup_{\ell \le A_k} |\rho_\ell - \rho_\ell^{(1)}| \le h^*\left(r_k^{q^*}\right) \le h^*\left(N^{q^*}\right) \tag{2.14}$$

proving our lemma. □

Lemma 2.3. *For any fixed $q > 0$*

$$\mathbf{P}\left(\nu_\ell^{(1)} + \nu_\ell^{(2)} > 5r_\ell^{1-q}\right) \le \exp\left(-r_\ell^{1-q}\right) \tag{2.15}$$

if ℓ is large enough.

Proof. Clearly $\nu_\ell^{(1)}$ and $\nu_\ell^{(2)}$ are both binomial with parameters r_ℓ and $p_\ell = \mathbf{P}(\rho_1 > h^*(r_\ell^q)) \le 1/r_\ell^q$ and independent, hence their sum is binomial with parameters $2r_\ell$ and p_ℓ. Now observe that for any $A > 0$

$$\begin{aligned}&\mathbf{P}(\nu_\ell^{(1)} + \nu_\ell^{(2)} > A) \le \exp(-A)\mathbf{E}(\exp(\nu_\ell^{(1)} + \nu_\ell^{(2)})) = \\ &\exp(-A)(1 + p_\ell(e-1))^{2r_\ell} \le \exp(-A)\exp(4p_\ell r_\ell) \le \\ &\exp(-A + 4r_\ell^{1-q}).\end{aligned} \tag{2.16}$$

Select now $A = 5r_\ell^{1-q}$ to get the lemma. □

Lemma 2.4. *For any $0 < q < 1$, and any $z > 0$ such that $\beta(z + q - 1) > 1$, we have for large N*

$$\sup_{\ell < N} |S_\ell - S_\ell^{(2)}| \le CN^z \qquad \text{a.s.} \tag{2.17}$$

Proof. Let $A_{k-1} \le N < A_k$. Define

$$M_\ell = \max\left(\max_{A_{\ell-1} \le i < A_\ell} \left(|X_i^{(1)}|\right), \max_{A_{\ell-1} \le i < A_\ell} \left(|X_i^{(2)}|\right) \right) \tag{2.18}$$

Observe now that

$$\sup_{1 \le \ell \le N} |S_\ell - S_\ell^{(2)}| \le \sup_{1 \le \ell \le A_k} |S_\ell - S_\ell^{(2)}| \le \sum_{\ell=1}^{k} 2(\nu_\ell^{(1)} + \nu_\ell^{(2)}) M_\ell. \tag{2.19}$$

Therefore by Lemma 2.3 and (1.5)

$$\begin{aligned} &\mathbf{P}(2(\nu_\ell^{(1)} + \nu_\ell^{(2)}) M_\ell > r_\ell^z) \le \\ &\exp(-r_\ell^{1-q}) + \mathbf{P}\left(M_\ell > \frac{1}{10} r_\ell^{z+q-1}\right) \le \\ &\exp(-r_\ell^{1-q}) + C r_\ell^{1-\beta(z+q-1)} = \\ &\exp(-2^{(\ell-1)(1-q)}) + C 2^{(\ell-1)(1-\beta(z+q-1))}. \end{aligned} \tag{2.20}$$

Under the conditions of our lemma, the last line (2.20) is convergent in ℓ, implying, that almost surely for large N

$$\begin{aligned} &\sup_{1 \le \ell < N} |S_\ell - S_\ell^{(2)}| \le \sum_{1 \le \ell \le A_k} |S_\ell - S_\ell^{(2)}| \le \\ &\sum_{\ell=1}^{k} 2^{(\ell-1)z} \le C 2^{kz} \le CN^z. \end{aligned} \tag{2.21}$$

□

Now we are ready to prove Theorem 1.1. To get (1.7) of our theorem by Lemma 2.4 we have to ensure that z can be selected to be less than $1/2$. Being $\beta > 2$, select a small enough $\epsilon > 0$ such that $1/\beta < 1/2 - 4\epsilon$, and select $q = 1 - 2\epsilon$, and $z = 1/2 - \epsilon$ to meet the conditions of Lemma 2.4 and we get (1.7) with the selection $1/\beta^* = z$. To ensure (1.8) as well, select $q^* = 1 - \epsilon$ in Lemma 2.2 and select $\alpha = q^*$. This completes the proof of Theorem 1.1. □

3. Recurrent random walk on $\mathbf{Z}^1$ and $\mathbf{Z}^2$ with slowly varying truncated Green function

Let $\{U_n\}_{n=1}^{\infty}$ be a recurrent symmetric random walk on the integer lattice $\mathbf{Z}^d$, $d = 1, 2$, i.e. $U_n = \sum_{k=0}^{n} Y_k$, where the random variables Y_i, $i = 1, 2, \dots$ are independent symmetric and identically distributed, taking values in $\mathbf{Z}^d$. We suppose that the law of Y_1 is not supported on a proper subgroup of $\mathbf{Z}^d$.

Define the local time of the walk by

$$\xi(x,n) = \#\{k;\ 0 < k \le n,\ U_k = x\}. \tag{3.1}$$

The transition probabilities of the walk are denoted by $p_n(x,y) = p_n(x-y)$, and the truncated Green function is defined as

$$g(n) = \sum_{k=0}^{n} p_k(0). \tag{3.2}$$

We can extend $g(t)$ to be a continuous monotone increasing function of $t \ge 0$, and we will denote the inverse of $g(t)$ by $g^*(t)$. Let

$$\rho_0 = 0, \qquad \rho_k = \inf\{n;\ n \ge \rho_{k-1},\ U_n = 0,\ \ k = 1, 2, \dots\} \tag{3.3}$$

then clearly $\{\rho_i - \rho_{i-1}\}_{i=1}^{\infty}$ are i.i.d. random variables.

Let

$$\gamma(n) = \mathbf{P}(\rho_1 > n).$$

Then considering the last return to the origin before time n we have

$$\sum_{k=0}^{n} \gamma(n-k) p_k(0) = 1, \tag{3.4}$$

where we used the strong Markov property and the fact, that if the last return before n is in $k (= 0, 1, 2 \dots n)$, then the next return must be later than $n - k$, hence it has the same probability as $\{\rho_1 > n - k\}$.

Consequently

$$\gamma(n) \sum_{k=0}^{n} p_k(0) \le 1, \tag{3.5}$$

implying that

$$\mathbf{P}(\rho_1 > n) \le 1/g(n). \tag{3.6}$$

For $d = 2$ a much more precise result than (3.6) is known (see [10], Lemma 3, where the above argument comes from).

We will use the following bound for the upper tail of $\xi(0,n)$ of Marcus and Rosen [24], Lemma 2.5.

$$\mathbf{P}\left(\frac{\xi(0,n)}{g(n)} \geq x\right) \leq C\sqrt{x+1}\, e^{-x} \quad \text{for all } x \geq 0,\ n \geq 1. \tag{3.7}$$

This result is true for $d = 1, 2$ but for $d = 2$ again much more is known, namely the limit distributon of $\xi(0,n)$ is (see e.g. Gantert and Zeitouni [17])

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\xi(0,n)}{g(n)} \geq x\right) = e^{-x}, \tag{3.8}$$

but for our purpose (3.7) will be enough. We make the following assumptions:

For $d = 1$ we suppose that

(a) *$p_n(0)$ is regularly varying at infinity with index minus one, which implies that $g(n)$ is slowly varying at infinity, and we also assume that*

$$\lim_{n\to\infty} g(n) = +\infty,$$

which is equivalent with the recurrence of the walk. (see [24], Proposition 3.1 for examples and details on these assumptions.)

(b)

$$\lim_{n\to\infty} \frac{g(n/\log\log g(n))}{g(n)} = 1. \tag{3.9}$$

If $d = 2$, then it is known (see e.g. [24], Proposition 2.14) that

$$p_n(0) \leq \frac{C}{n}, \qquad g(n) \leq C \log n. \tag{3.10}$$

For $d = 2$ we suppose that

(c) *$g(\cdot)$ is slowly varying at infinity and*

$$\lim_{n\to\infty} g(n) = +\infty.$$

This is satisfied if Y_1 is in the domain of attraction of a nondegenerate $\mathbf{R}^2$ valued Gaussian random variable. In the case $d=2$ (3.9) always holds true (see the proof of Theorem 1.1. in [24]).

Consequently for $d=1,2$ under the above assumptions according to [24] Theorem 1.3, we have

$$\limsup_{n\to\infty} \frac{\xi(0,n)}{g(n)\log\log g(n)} = 1 \qquad \text{a.s.} \tag{3.11}$$

First we prove the following

Lemma 3.1. *Let $\{U_n\}_{n=1}^{\infty}$ be a random walk on $\mathbf{Z}^1$ or $\mathbf{Z}^2$ satisfying the above conditions. For $K>0$, $\gamma>0$ let*

$$a_n = g^*\left((g(n))^K\right), \qquad b_n = g^*\left((g(n))^\gamma\right). \tag{3.12}$$

Then for any $\kappa>1$ we have

$$\sup_{a\le a_n}\left(\xi(0,a+[b_n])-\xi(0,a)\right) = O\left((g(n))^{\gamma\kappa}\right) \qquad \text{a.s.} \tag{3.13}$$

Proof. Let $n_k=[g^*(k)]$. Then clearly we have that

$$P(k) = \mathbf{P}\left(\sup_{a\le a_{n_k}}\left(\xi(0,a+[b_{n_k}])-\xi(0,a)\right) > (g(n_{k-1}))^{\gamma\kappa}\right) \le$$
$$\mathbf{P}\left(\sup_{i;\,\rho_i\le a_{n_k}}\left(\xi(0,\rho_i+[b_{n_k}])-\xi(0,\rho_i)\right) > (g(n_{k-1}))^{\gamma\kappa}\right) \le$$
$$\mathbf{P}\left(\sup_{i;\,\rho_i\le a_{n_k}}\left(\xi(0,\rho_i+[b_{n_k}])-\xi(0,\rho_i)\right) > (g(n_{k-1}))^{\gamma\kappa}, \xi(0,a_{n_k}) < (g(a_{n_k}))^{3/2}\right) +$$
$$\mathbf{P}\left(\xi(0,a_{n_k}) \ge (g(a_{n_k}))^{3/2}\right).$$

Now observe that by the strong Markov property the first probability above can be overestimated by $(g(a_{n_k}))^{3/2}$ times the probability $\mathbf{P}\left(\xi(0,[b_{n_k}]) > (g(n_{k-1}))^{\gamma\kappa}\right)$. By the definition of n_k and (3.12) we have that

$$g(a_{n_k}) = g\left(g^*\left((g(n_k))^K\right)\right) = (g(n_k))^K \le \left(g\left(g^*(k)\right)\right)^K = k^K.$$

Combining this observation with (3.7) we conclude that

$$P(k) \le (g(a_{n_k}))^{3/2}\mathbf{P}\left(\xi(0,[b_{n_k}]) > (g(n_{k-1}))^{\gamma\kappa}\right) + \mathbf{P}\left(\xi(0,a_{n_k}) \ge (g(a_{n_k}))^{3/2}\right) \le$$
$$k^{3K/2}\exp\left(-(1-\delta)\frac{(k-1)^{\gamma\kappa}}{k^{\gamma}}\right) + \exp\left(-(1-\delta)k^{K/2}\right) \le$$
$$k^{3K/2}\exp\left(-(1-\delta^*)k^{\gamma(\kappa-1)}\right) + \exp\left(-(1-\delta)k^{K/2}\right). \tag{3.14}$$

for any $\delta^* > \delta > 0$ and k big enough.

According to the last line of (3.14) we get that $\sum P(k)$ is convergent for $\kappa > 1$ and we get (3.13) by Borel-Cantelli lemma combined with the usual monotonicity argument. □

We define the additive functional

$$Z_n = \sum_{k=0}^{n} f(U_k), \tag{3.15}$$

where $f(\cdot)$ is a real-valued function on $\mathbf{Z}^1$ or $\mathbf{Z}^2$. Define

$$X_i = Z_{\rho_i} - Z_{\rho_{i-1}} = \sum_{\rho_{i-1}+1}^{\rho_i} f(U_k) \tag{3.16}$$

Then clearly $\{X_i\}_{i=0}^{\infty}$ are i.i.d. random variables. We suppose that for some $\delta > 0$

$$\mathbf{E}\left((\sum_{k=1}^{\rho_1} |f(U_k)|)^{2+\delta} \right) < \infty. \tag{3.17}$$

For the expectation of X_1 we have

$$\bar{f} = \mathbf{E}(X_1) = \mathbf{E}\sum_{x\in\mathbf{Z}^d} \xi(x,\rho_1) f(x) = \sum_{x\in\mathbf{Z}^d} f(x) \tag{3.18}$$

since

$$\mathbf{E}(\xi(x,\rho_1)) = 1, \qquad x \in \mathbf{Z}^d \tag{3.19}$$

(see Spitzer [27], Auer [1]).

Theorem 3.1. *For any random walk* $\{U_n\}_0^{\infty}$ *on* $\mathbf{Z}^1$ *or* $\mathbf{Z}^2$ *satisfying our conditions* (a)-(b) *or* (c) *respectively, and for any real valued function* $f(\cdot)$ *satisfying* (3.17) *there exists a probability space where we can redefine* $\{U_n\}_0^{\infty}$ *together with its local time process* $\xi(0,n)$, *and with the corresponding additive functional Z_n on such a way that on the same probability space there is*

(i) a standard Wiener process $\{W(t), t \geq 0\}$

(ii) and a process

$$\{\xi^{(1)}(0,n)\ n = 0,1,2,...\} \stackrel{\mathcal{D}}{=} \{\xi(0,n)\ n = 0,1,2,...\}$$

such that $\{W(t), t \geq 0\}$ and $\{\xi^{(1)}(0,n)\ n = 0,1,2,...\}$ are independent and we have

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi^{(1)}(0,n)) + O(g^s(n)) \qquad \text{a.s.} \tag{3.20}$$

and

$$|\xi^{(1)}(0,n) - \xi(0,n)| = O(g^p(n)) \qquad \text{a.s.} \tag{3.21}$$

as $n \to \infty$, where $\sigma^2 = \mathbf{Var}(X_1)$, X_1 is defined by (3.16)*, $s < 1/2$, and $p < 1$.*

Proof of Theorem 3.1. Define

$$\tau_i = \rho_i - \rho_{i-1}, \quad i = 1,2,... \tag{3.22}$$

Now we want to apply our Theorem 1.1 for the sequence of vectors $(X_i, \tau_i)_{i=1}^{\infty}$. Clearly they satisfy the conditions (1.5) with $\beta = 2+\delta$, and $h(x) = g(x)$. Consequently for $S_n = \sum_{k=0}^n X_k$ and $\rho_n = \sum_{k=0}^n \tau_k$, we have, that on an appropriate probability space one can construct independent processes $(S_n^{(1)}, \rho_n^{(1)})$ and $(S_n^{(2)}, \rho_n^{(2)})$, such that

$$(S_n, \rho_n) \stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)}) \qquad n = 1,2... \qquad j = 1,2 \tag{3.23}$$

$$\sup_{k \leq n} |S_k - S_k^{(2)}| = O\left(n^{1/\beta^*}\right) \qquad \text{a.s.} \tag{3.24}$$

$$\sup_{k \leq n} |\rho_k - \rho_k^{(1)}| = O\left(g^*(n^\alpha)\right) \qquad \text{a.s.} \tag{3.25}$$

where $\beta^* > 2$, and $\alpha < 1$. Apply now the Komlós-Major-Tusnády theorem [23] (see also Csörgő and Révész [12], Theorem 2.6.6, pp. 108) for $S_N^{(2)} - N\bar{f}$ to get by (3.17), that

$$|S_N^{(2)} - N\bar{f} - \sigma W(N)| = O\left(N^{1/(2+\delta)}\right) \qquad \text{a.s.} \tag{3.26}$$

as $N \to \infty$, where $\sigma^2 = \mathbf{Var}(X_1)$ (the existence of which follows from (3.17)). Denote

$$\min(2+\delta, \beta^*) = 2 + \eta$$

to get from (3.24) and (3.26) that

$$|S_N - N\bar{f} - \sigma W(N)| = O\left(N^{1/(2+\eta)}\right) \qquad \text{a.s.} \tag{3.27}$$

Observe that in (3.26) the Wiener process is the one which was constructed to the process $^{(2)}$, hence independent from process $^{(1)}$. Clearly, (3.27) and (3.11) imply that for any $\epsilon_1 > 0$

$$S_{\xi(0,n)} - \bar{f}\xi(0,n) = \sigma W(\xi(0,n)) + O\left((g(n))^{\frac{1+\epsilon_1}{2+\eta}}\right) \qquad \text{a.s.} \tag{3.28}$$

As our next step, we want to get an almost sure upper bound for $|Z_n - S_{\xi(0,n)}|$. We have

$$\rho_{\xi(0,n)} < n \leq \rho_{\xi(0,n)+1} \tag{3.29}$$

and denoting

$$X_k^* = \sum_{i=\rho_k+1}^{\rho_{k+1}} |f(U_i)|,$$

$$|Z_n - S_{\xi(0,n)}| \leq \sum_{i=\rho_{\xi(0,n)}+1}^{\rho_{\xi(0,n)+1}} |f(U_k)| = X^*_{\xi(0,n)+1}. \tag{3.30}$$

By condition (3.17) for any $\epsilon_2 > 0$

$$\mathbf{P}\left(X_k^* > k^{\frac{1+\epsilon_2}{2+\eta}}\right) \leq \frac{C}{k^{1+\epsilon_2}}. \tag{3.31}$$

So by Borel-Cantelli lemma we have

$$X_k^* = O\left(k^{\frac{1+\epsilon_2}{2+\eta}}\right) \qquad \text{a.s.} \tag{3.32}$$

as $k \to \infty$, i.e. for any $\epsilon_3 > \epsilon_2 > 0$ we have

$$X^*_{\xi(0,n)} = O\left((g(n))^{\frac{1+\epsilon_3}{2+\eta}}\right) \qquad \text{a.s.} \tag{3.33}$$

(3.28), (3.30) and (3.33) imply that for any $\epsilon > \epsilon_3$

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi(0,n)) + O(g(n))^{\frac{1+\epsilon}{2+\eta}} \qquad \text{a.s.} \tag{3.34}$$

holds as well. To finish our proof we only have to show that on the right hand side of (3.34) we can replace $\xi(0,n)$ by $\xi^{(1)}(0,n)$ by the price of a posssible small increase in the order of the error term in the approximation.

To accomplish this goal we prove

Lemma 3.2. *There exists a $p<1$ such that*

$$|\xi^{(1)}(0,n)-\xi(0,n)|=O(g^p(n)) \qquad \text{a.s.} \tag{3.35}$$

Proof. First observe that $\rho^{(1)}_{\xi^{(1)}(0,n)}$ is the time of the last return to zero before time n of the walk $^{(1)}$, hence

$$\xi^{(1)}(0,n)=\xi^{(1)}(0,\rho^{(1)}_{\xi^{(1)}(0,n)}).$$

Also, since $\xi(0,\rho_k)=k$, we have

$$\xi^{(1)}(0,n)=\xi(0,\rho_{\xi^{(1)}(0,n)}).$$

As $\rho^{(1)}_{\xi^{(1)}(0,n)}\le n$, we see that

$$\begin{aligned}\xi^{(1)}(0,n)-\xi(0,n)\le \xi^{(1)}(0,\rho^{(1)}_{\xi^{(1)}(0,n)})-\xi(0,\rho^{(1)}_{\xi^{(1)}(0,n)})=\\ \xi(0,\rho_{\xi^{(1)}(0,n)})-\xi(0,\rho^{(1)}_{\xi^{(1)}(0,n)}).\end{aligned} \tag{3.36}$$

Now observe that by (3.11) $\xi^{(1)}(0,n)<(g(n))^{1+\psi}$ for any $\psi>0$ if n is big enough. Consequently by (3.25), for some $\alpha<1$ we have

$$|\rho_{\xi^{(1)}(0,n)}-\rho^{(1)}_{\xi^{(1)}(0,n)}|\le \sup_{i\le(g(n))^{(1+\psi)}}|\rho_i-\rho_i^{(1)}|\le g^*\left((g(n))^{(1+\psi)\alpha}\right) \qquad \text{a.s.} \tag{3.37}$$

Now apply Lemma 3.1 with $b_n=g^*((g(n))^{(1+\psi)\alpha})$ and $a_n=n$ (being $\rho^{(1)}_{\xi^{(1)}(0,n)}\le n$). Thus we get by (3.36) and (3.37) that for any $\kappa>1$

$$\xi^{(1)}(0,n)-\xi(0,n)=O((g(n)^{(1+\psi)\alpha\kappa}) \qquad \text{a.s.}$$

Repeat now the argument for $\xi(0,n)-\xi^{(1)}(0,n)$, and observe that we can select $\psi>0$ small enough, and $\kappa>1$ close enough to 1 such that

$$p=(1+\psi)\kappa\alpha<1$$

should hold, and we have the lemma. □

Using now the result of our lemma and Theorem 1.2.1 of [12] on the maximal increment of the Wiener process, we get from (3.34) that for any $1>p^*>p$

$$Z_n-\bar f\xi(0,n)=\sigma W(\xi^{(1)}(0,n))+O\left((g(n))^{p^*/2}\right)+O\left((g(n))^{\frac{1+\epsilon}{2+\eta}}\right) \qquad a.s. \tag{3.38}$$

Select now $\epsilon > 0$ small enough, that

$$\frac{1+\epsilon}{2+\eta} < \frac{1}{2}$$

should hold, then with

$$s = \max\left(\frac{1+\epsilon}{2+\eta}, \frac{p^*}{2}\right) < \frac{1}{2} \tag{3.39}$$

we get

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi^{(1)}(0,n)) + O(g^s(n)) \qquad \text{a.s.} \tag{3.40}$$

as $n \to \infty$. □

4. Recurrent random walk on $\mathbf{Z}^1$ in the domain of attraction of a stable law.

Let $\{U_n\}_{n=1}^{\infty}$ be a recurrent symmetric random walk on $\mathbf{Z}^1$, i.e. $U_n = \sum_{k=0}^{n} Y_k$, where Y_i, $i = 1, 2, \ldots$ are independent, symmetric and identically distributed, with $\mathbf{E}Y_1 = 0$. Suppose futhermore that
(d) Y_1 *is in the domain of attraction of a stable law of index* $1/(1-\alpha)$, *where* $0 < \alpha \le 1/2$.

We define $\xi(x,n)$, ρ_k, τ_k, $g(n)$ and Z_n as in Section 3. It is known (see Jain and Pruitt [19]) that under condition (i) we have for $\tau_1 = \rho_1$ that

$$\mathbf{P}(\tau_1 > n) \sim \frac{1}{n^{\alpha} L(n)}, \quad n \to \infty. \tag{4.1}$$

where $L(n)$ is slowly varying at infinity. We remark that the above condition also implies that

$$g(n) \sim n^{\alpha} L^*(n) \tag{4.2}$$

where $L^*(n)$ is also slowly varying at infinity. (see [19]). Furthermore, we have under our condition (d) (see Jain and Pruitt [18]) that for any $\epsilon_1 > 0$

$$\xi(0,n) = O(n^{\alpha(1+\epsilon_1)}) \qquad \text{a.s.} \tag{4.3}$$

In fact more exact LIL type results are also given in [18]. Define now $f(\cdot)$ and X_i, $i = 1.2...$ as in in Section 3. We will prove the following

Theorem 4.1. *For any random walk $\{U_n\}_0^\infty$ on $\mathbf{Z}^1$ satisfying our condition* (d) *and for any real valued function $f(\cdot)$ satisfying* (3.17) *there exists a probability space where we can redefine $\{U_n\}_0^\infty$ together with its local time process $\xi(0,n)$, and with the corresponding additive functional Z_n on such a way that on the same probability space there are*

(i) a standard Wiener process $\{W(t), t \geq 0\}$

(ii) and a process

$$\{\xi^{(1)}(0,n),\ n = 0,1,2,...\} \stackrel{\mathcal{D}}{=} \{\xi(0,n),\ n = 0,1,2,...\}$$

such that $\{W(t), t \geq 0\}$ and $\{\xi^{(1)}(0,n),\ n = 0,1,2,...\}$ are independent and we have

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi^{(1)}(0,n)) + O(n^{\alpha s}) \qquad \text{a.s.} \tag{4.4}$$

and

$$|\xi^{(1)}(0,n) - \xi(0,n)| = O(n^{\alpha p}) \qquad \text{a.s.} \tag{4.5}$$

as $n \to \infty$, where $\sigma^2 = \mathbf{Var}(X_1)$, $s < 1/2$, and $p < 1$.

Proof of Theorem 4.1. Considering the sequence of random vectors $\{X_i, \tau_i\}_{i=1}^\infty$, where X_i, τ_i are defined by (3.16) and by (3.22) respectively, it is easy to see, that under the conditions of our theorem the conditions of Theorem 1.A are met. Hence we can find an appropriate probability space such that on this space there exist three sequences of random vectors $\{X_i^{(j)}, \tau_i^{(j)}\}$ $i = 1,2...$ and $j = 1,2$ with $\{X_i, \tau_i\}$ $i = 1,2...$ such that for the corresponding partial sum processes we have

$$(S_n, \rho_n) \stackrel{\mathcal{D}}{=} (S_n^{(j)}, \rho_n^{(j)}) \qquad n = 1,2... \qquad j = 1,2 \tag{4.6}$$

$$\sup_{k \leq n} |S_k - S_k^{(2)}| = O\left(n^{1/\beta^*}\right) \qquad \text{a.s.} \tag{4.7}$$

$$\sup_{k \leq n} |\rho_k - \rho_k^{(1)}| = O\left(n^{1/\alpha^*}\right) \qquad \text{a.s.} \tag{4.8}$$

as $n \to \infty$, where $\alpha^* > \alpha$ and $\beta^* > 2$ and the processes $(S_n^{(1)}, \rho_n^{(1)})$ $(S_n^{(2)}, \rho_n^{(2)})$ are independent. Apply now Komlós-Major-Tusnády [23] theorem and (4.7) as in the previous section to get

$$|S_N - N\bar{f} - \sigma W(N)| = O\left(N^{1/(2+\eta)}\right) \qquad \text{a.s.}, \tag{4.9}$$

where

$$\min(2+\delta, \beta^*) = 2 + \eta.$$

Following further the argument of the previous section we get by (4.3) that

$$S_{\xi(0,n)} - \bar{f}\xi(0,n) = \sigma W(\xi(0,n)) + O\left(n^{\frac{\alpha(1+\epsilon_1)}{2+\eta}}\right) \qquad \text{a.s.} \tag{4.10}$$

Now let X_k^* as in Section 3, then as $X_k^* = O\left(k^{\frac{1+\epsilon_2}{2+\eta}}\right)$ a.s. (see (3.32)), we get by (4.3)

$$X_{\xi(0,n)}^* = O\left(n^{\frac{\alpha(1+\epsilon_3)}{2+\eta}}\right) \qquad \text{a.s.} \tag{4.11}$$

for any $\epsilon_3 > \epsilon_2 > 0$. Thus by (4.10) and (4.11) and (3.30) we have that

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi(0,n)) + O\left(n^{\frac{\alpha(1+\epsilon)}{2+\eta}}\right) \qquad \text{a.s.} \tag{4.12}$$

for any $\epsilon > \epsilon_3$. Clearly one can select ϵ_1 , ϵ_2 ϵ_3 and ϵ small enough such that

$$\frac{1+\epsilon}{2+\eta} < \frac{1}{2} \tag{4.13}$$

should hold. Just like in the previous section, to finish our proof we have to replace $\xi(0,n)$ by $\xi^{(1)}(0,n)$, and we need the following

Lemma 4.1. *There exists a* $0 < p < 1$ *such that*

$$|\xi^{(1)}(0,n) - \xi(0,n)| = O(n^{\alpha p}) \qquad \text{a.s.} \tag{4.14}$$

as $n \to \infty$.

Proof. Proceed like in Lemma 3.2, and based on (4.8) and (4.3) we get similarly to (3.37) that for any $\epsilon_1 > 0$

$$|\rho_{\xi^{(1)}(0,n)} - \rho^{(1)}_{\xi^{(1)}(0,n)}| \leq \sup_{i \leq n^{\alpha(1+\epsilon_1)}} |\rho_i - \rho_i^{(1)}| = O\left(n^{\frac{\alpha(1+\epsilon_1)}{\alpha^*}}\right) \qquad \text{a.s.} \tag{4.15}$$

Now we need a result from Jain and Pruitt [19] again about the increments of $\xi(0,n)$. According to their Theorem 6.1 for any $t_n = n^q$, where $q < 1$

$$\sup_{k+t_n \leq n} (\xi(k+t_n) - \xi(k)) = O\left(n^{\alpha q(1+\gamma)}\right) \tag{4.16}$$

for any $\gamma > 0$.

Clearly as $\alpha^* > \alpha$ in (4.15) we can take ϵ_1 small enough that $q = \frac{\alpha(1+\epsilon_1)}{\alpha^*} < 1$ should hold. Hence by (4.16) we have from (3.36), and (4.15) that

$$\xi^{(1)}(0,n) - \xi(0,n) \le \xi(0, \rho_{\xi^{(1)}(0,n)}) - \xi(0, \rho^{(1)}_{\xi^{(1)}(0,n)}) = O\left(n^{\alpha q(1+\gamma)}\right) \qquad \text{a.s.} \tag{4.17}$$

Repeating this argument for $\xi(0,n) - \xi^{(1)}(0,n)$, and selecting $\gamma > 0$ small enough such that $p = q(1+\gamma) < 1$ should hold, we get the lemma. □

Using again Theorem 1.2.1 of [12] we get from (4.12) and Lemma 4.1 that

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi^{(1)}(0,n)) + O(n^{\alpha p^*/2}) + O(n^{\frac{\alpha(1+\epsilon)}{2+\eta}}) \qquad \text{a.s.} \tag{4.18}$$

for any $1 > p* > p > 0$. Hence taking

$$s = \max\left(\frac{1+\epsilon}{2+\eta}, \frac{p^*}{2}\right) < \frac{1}{2} \tag{4.19}$$

we get

$$Z_n - \bar{f}\xi(0,n) = \sigma W(\xi^{(1)}(0,n)) + O(n^{\alpha s}) \qquad \text{a.s.} \tag{4.20}$$

Lemma 4.1 and (4.20) give our theorem. □

5. Applications

In this section we will obtain some limit theorems for random walks on $\mathbf{Z}^1$ or $\mathbf{Z}^2$ based on our approximation theorems. These results in case of simple symmetric walk on the plane were presented in [9]. In this section we will suppose that

The random walk on $\mathbf{Z}^1$ *or* $\mathbf{Z}^2$*, the real valued funcion* $f(\cdot)$*, and the corresponding additive functional satisfy the conditions of our* Theorem 3.1*, and the local time* $\xi(0,n)$ *of the walk has the limiting distribution formulated in* (3.8).

Remark. If the walk is on $\mathbf{Z}^2$ then our conditions in Theorem 3.1 ensure (3.8), if it is on $\mathbf{Z}^1$ then the extra condition that $p(0,n) < C/n$ is enough for (3.8) to hold, (see [17]) that is, if the walk is in the domain of attraction of a Cauchy random variable. Clearly this condition is not much more restrictive than our original condition (a) in Section 3.

It follows from (3.20) that the limiting distribution of

$$\frac{Z_n - \bar{f}\xi(0,n)}{\sigma\sqrt{g(n)}}$$

should be the same as that of

$$\frac{W(\xi^{(1)}(0,n))}{\sqrt{g(n)}},$$

Obviously we have from (3.8) that

$$\frac{W(\xi^{(1)}(n))}{\sqrt{g(n)}} = \frac{W(\xi^{(1)}(n))}{\sqrt{\xi^{(1)}(n)}} \sqrt{\frac{\xi^{(1)}(n)}{g(n)}} \xrightarrow{\mathcal{D}} U\sqrt{\mathcal{E}}$$

as $n \to \infty$, where U is a standard normal r.v. and $\mathcal{E}$ is an exponential r.v. with parameter 1, U and $\mathcal{E}$ are independent. The independence of U and $\mathcal{E}$ is a straightforward consequence of our Theorem 3.1, which provides the idependence of $W(\cdot)$ and $\xi^{(1)}(\cdot)$ hence by normalizing the first factor by $\sqrt{\xi^{(1)}(\cdot)}$ we achieve the independence of U and $\mathcal{E}$. One can obtain similarly

$$\frac{|W(\xi^{(1)}(n))|}{\sqrt{g(n)}} \xrightarrow{\mathcal{D}} |U|\sqrt{\mathcal{E}},$$

$$\frac{\sup_{k\le n} W(\xi^{(1)}(k))}{\sqrt{g(n)}} \xrightarrow{\mathcal{D}} |U|\sqrt{\mathcal{E}},$$

and

$$\frac{\sup_{k\le n} |W(\xi^{(1)}(k))|}{\sqrt{g(n)}} \xrightarrow{\mathcal{D}} T\sqrt{\mathcal{E}}$$

as $n \to \infty$, where T has the distribution of $\sup_{s\le 1} |W(s)|$ and is independent of $\mathcal{E}$.

Furthermore it is easy to see that the distribution of $U\sqrt{\mathcal{E}}$ is two-sided exponential with parameter $\sqrt{2}$, i.e. its density function is

$$k(x) = \sqrt{\frac{1}{2}} e^{-|x|\sqrt{2}} \qquad -\infty < x < \infty. \tag{5.1}$$

The distribution of $|U|\sqrt{\mathcal{E}}$ is exponential with parameter $\sqrt{2}$. Finally using a well-known formula for the distribution of T, we get by straightforward calculations that (cf. [9])

$$H(x) = \mathbf{P}(T\sqrt{\mathcal{E}} \le x) = 1 - \frac{1}{\cosh(x\sqrt{2})} \tag{5.2}$$

Hence we have the following limiting distributions:

Theorem 5.1.

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{Z_n - \bar{f}\xi(0,n)}{\sigma\sqrt{g(n)}} \le x\right) = \int_{-\infty}^{x} k(u)\,du, \tag{5.3}$$

where $k(x)$ is given by (5.1).

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{|Z_n - \bar{f}\xi(0,n)|}{\sigma\sqrt{g(n)}} \le x\right) = 1 - \exp(-x\sqrt{2}), \tag{5.4}$$

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{\sup_{1\le k\le n} |Z_k - \bar{f}\xi(0,k)|}{\sigma\sqrt{g(n)}} \le x\right) = H(x). \tag{5.5}$$

Turning now to the problem of strong limit theorems for the additive functionals we can apply our Theorem 3.1 together with the second order LIL result of Marcus and Rosen ([24], Theorem 1.3) to get the conclusion that

$$\begin{aligned}\limsup_{n\to\infty} \frac{Z_n - \bar{f}\xi(0,n)}{\sigma\sqrt{g(n)}\log\log g(n)} &= \limsup_{n\to\infty} \frac{W(\xi^{(1)}(0,n))}{\sqrt{g(n)}\log\log g(n)} \\ &= \limsup_{n\to\infty} \frac{\xi(a,n) - \xi(0,n)}{\sigma_a\sqrt{g(n)}\log\log g(n)} = \frac{1}{\sqrt{2}}\end{aligned}$$

where σ_a is a constant (see [24]).

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