Studia Scientiarum Mathematicarum Hungarica 14 (1979), 461—469

ON A GENERALIZATION OF THE GAME GO-MOKU, II

by

L. CSIRMAZ and ZS. NAGY

Abstract

Two players, I and II play the following game. They pick alternately the points of a set A until either all elements of A have been chosen or α moves have been made. The first and every limit move (if any) is I's turn. I wins if he picks all elements of some set of the winning family $\mathscr{F}\subset P(A)$, otherwise the winner is II. If the elements of $\mathscr{F}$ are finite and I has a winning strategy, then I has a winning strategy in finitely many moves. The cases when the elements $\mathscr{F}$ are countable are discussed in details. Various consistency results are given for undetermined and determined games. Several interesting problems are stated.

We study here another possible generalization of this well-known Oriental game. This part of our paper can be read independently from the previous one [1], but we do not repeat here the motivation behind our concepts.

1. Definitions

Our set theoretical notation will be standard. Ordinal numbers are denoted by α, β, etc. cardinal numbers by $\varkappa$, λ. If A and B are sets, then ${}^AB$ denotes the family of functions from A to B, and, by definition, $[A]^\varkappa=\{X\subset A\colon |X|=\varkappa\}$, $[A]^{<\varkappa}=\{X\subset A\colon |X|<\varkappa\}$. In this paper the inclusion $A\subset B$ allows the sets A and B to be equal.

The game we are going to deal with is denoted by $(A, \mathscr{F})^\alpha$, and consists of the board A, the family of winning sets $\mathscr{F}\subset P(A)$, and the ordinal α. The game is played by the players I and II as follows. The players pick elements of A alternately, every element can be picked at most once. I starts and every limit step (if any) is I's turn. The game ends if either all elements of A have been chosen or if α moves have been made.

The winner is I if he picked all elements of some $X\in\mathscr{F}$, otherwise the winner is II.

The game $(A, \mathscr{F})$ denotes the game $(A, \mathscr{F})^{|A|^+}$. In this paper under the word "game" we always mean game of this type.

The notion of strategy, that of a play according to a strategy can be found in [6]. A game is undetermined if neither I nor II has a winning strategy, abbreviated as WS.

1980 *Mathematics Subject Classification.* Primary 04A20; Secondary 54A35.
Key words and phrases. Infinite games, Martin's axiom.

2. Basic results

We start with some trivial observations.

PROPOSITION 2.1. *If I has a WS in $(A, \mathscr{F})^\alpha$ and $A_1 \supset A, \mathscr{F}_1 \supset \mathscr{F}, \alpha_1 \geqq \alpha$ then I has a WS in $(A_1, \mathscr{F}_1)^{\alpha_1}$. Moreover, if II has a WS in $(A_1, \mathscr{F}_1)^{\alpha_1}$ then he has a WS in $(A, \mathscr{F})^\alpha$.*

Now suppose that I has a WS in $G=(A, \mathscr{F})$. Then there exists a least $\alpha \in \mathrm{On}$ such that I still has a WS in $(A, \mathscr{F})^\alpha$, and a least cardinal $\varkappa$, such that I still has a WS in $(A, \mathscr{F}')$ for some $\mathscr{F}' \in [\mathscr{F}]^\varkappa$. Let us say that $\alpha = \mathrm{ord}\,(G)$, and $\varkappa = \mathrm{card}\,(G)$.

PROPOSITION 2.2. *For every ordinal α there is a game G_1 and for every cardinal $\varkappa$ there is a game G_2 such that $\mathrm{ord}\,(G_1)=\alpha$ and $\mathrm{card}\,(G_2)=\varkappa$.*

PROOF. We construct a game $(A, \mathscr{F})^\alpha$ such that $|\mathscr{F}|=|\alpha|$, I has a WS in it but II has a WS in

(i) $(A, \mathscr{F})^\beta$ if $\beta<\alpha$;

(ii) $(A, \mathscr{F}')^\alpha$ if $\mathscr{F}' \subset \mathscr{F}$, $\mathscr{F}' \neq \mathscr{F}$.

Let $A=\{P_\beta, Q_\beta : \beta<\alpha\}$ and let the elements of $\mathscr{F}$ be $\{P_\gamma : \gamma \leqq \beta\} \cup \{Q_\beta\}$ for each $\beta<\alpha$ and the set $\{P_\gamma : \gamma<\alpha\}$. I can win only by picking the points P_γ in succession, so the game $(A, \mathscr{F})$ has all the properties required.

PROPOSITION 2.3. $\mathrm{card}\,(G)^+ \geqq \mathrm{ord}\,(G)$.

PROOF. For every $\beta < \mathrm{ord}\,(G)$ there is a play of length β which is not a win for I. Obviously, we may assume that II kills at least one winning set by his every move, i.e. $\mathrm{card}\,(G) \geqq |\beta|$. From this the proposition follows immediately.

REMARK. The proof shows that if $\mathrm{ord}\,(G)$ is not a cardinal number then $\mathrm{card}\,(G)^+ > \mathrm{ord}\,(G)$, and the game defined in the proof of 2.2 gives examples where $\mathrm{card}\,(G) < \mathrm{ord}\,(G)$.

PROBLEM. Is $\mathrm{card}\,(G)^+ > \mathrm{ord}\,(G)$ always true?

PROPOSITION 2.4. $2^{|\mathrm{ord}(G)|} \geqq \mathrm{card}\,(G)$.

PROOF. Let $\alpha = \mathrm{ord}\,(G)$, and $G^* = (A, \mathscr{F})^\alpha$. I has a WS in G^* so we may assume $\mathscr{F} \subset [A]^{\leqq|\alpha|}$. This strategy is function F from ${}^{\underset{\sim}{\alpha}}A = \cup\{{}^\beta A : \beta<\alpha\}$ to A, which gives the response of I to the series of moves $\langle b_\gamma : \gamma<\beta \rangle$ of II for every $\beta<\alpha$. Now there is a subset $B \subset A$, $|B| \leqq 2^{|\alpha|}$ which is closed under F, i.e. whenever $b_\gamma \in B$ for $\gamma<\beta$, then $F(\langle b_\gamma : \gamma<\beta \rangle) \in B$. This means that $F \restriction B$ is a I-WS in $(B, \mathscr{F} \cap P(B))^\alpha$. Therefore, by 2.1, I has a WS in $(A, \mathscr{F} \cap P(B))^\alpha$, i.e.

$$\mathrm{card}\,(G) \leqq |\mathscr{F} \cap P(B)| \leqq |{}^\alpha B| \leqq 2^{|\alpha|}.$$

REMARK. Let A be the points of a normal binary tree of height α, $\mathscr{F}$ be the family of the (maximal) branches. If $G=(A, \mathscr{F})$ then $\mathrm{ord}\,(G)=\alpha$ and $\mathrm{card}\,(G)=2^{|\alpha|}$. Therefore the inequality in 2.4 is sharp.

3. Games with finite winning sets

In this section we discuss the games with finite winning sets.

THEOREM 3.1. *Let $\mathscr{F}\subset[A]^{<\omega}$ and suppose I has WS in $G=(A,\mathscr{F})^{\alpha}$. Then there is an $\mathscr{F}_0\subset\mathscr{F}$, $|\mathscr{F}_0|<\omega$ and $n\in\omega$ such that I wins the game $(A,\mathscr{F}_0)^n$.*

PROOF. Let S be a I-WS, and let $\beta<\alpha$ be an ordinal. The sequence

$$s=\langle a_\gamma\colon \gamma<\beta\rangle\in{}^{\alpha}A$$

is a *partial play* according to S, if $a_\gamma=S(s\restriction\gamma)$ for every even ordinal γ. We identify S with the tree whose nodes are the partial plays (excluding the empty sequence), and $s<_S t$ iff s is a proper initial segment of t. We assume, that every branch in this tree has a highest node in some even level, and climbing on a branch I covers some element of $\mathscr{F}$ only at the last node (i.e., if I wins, the strategy ends).

Now let s be a node of S, $S\restriction s$ is the subtree above (and including) s. If the level of s is even then it induces a subgame $G\restriction s$ as follows. Let $s=\langle a_\gamma\colon\gamma<\beta\rangle$, β even,

$$A'=A-\{a_\gamma\colon \gamma<\beta\}$$

$$\mathscr{F}'=\big\{X-\{\alpha_\gamma\colon\gamma<\beta\}\colon X\in\mathscr{F} \text{ and } X\cap\{a_\gamma\colon\gamma<\beta \text{ and } \gamma \text{ is odd}\}=\emptyset\big\}.$$

Then $G\restriction s=(A',\mathscr{F}')^{\alpha-\beta}$. Obviously, $S\restriction s$ is a I-WS for $G\restriction s$, and if we replace $S\restriction s$ by any I-WS for $G\restriction s$, the resulting tree is a I-WS for the game G.

We shall prove the existence of a I-WS in which all of I's moves belong to the same finite subset of the board. The existence of this strategy implies the statement. The proof is by induction on the height of the tree S which will be denoted by h.

Case 1. $h=0$. The statement is true because I wins by his (unique) move.

Case 2. $0<h<\omega$. h is even, so $h\geqq 2$. Let $\{s_\gamma\}$ be the set of nodes at level 2. Because height $(S\restriction s_\gamma)\leqq h-2$, we may apply the induction assertion for the game $G\restriction s_\gamma$. Changing the subtrees $S\restriction s_\gamma$ to these strategies, we get that in every $S\restriction s_\gamma$ only finitely many points are engaged to I. In particular, let $B\subset A$, $|B|<\omega$ be the set of I-engaged points in $S\restriction s_0$. Now for every γ, if $s_\gamma(1)\notin B$ then replace $S\restriction s_\gamma$ by $S\restriction s_0$. The resulting tree S^* is good because B is finite and therefore only finitely many $S\restriction s_\gamma$ remain unchanged. S^* is not necessarily a I-WS because it may require I to pick the same element twice, but it can be turned into a strategy easily (Fig. 1).

Case 3. $h>\omega$ is a successor. Let $\beta<h$ be the maximal limit ordinal below h and let $s\in S$ be a node of height β. By Case 2 we may assume that $S\restriction s$ is a good strategy, i.e. there is a finite $\mathscr{F}_0\subset\mathscr{F}$ such that I covers one of them totally. $\cup\mathscr{F}_0$ is finite and there is a $\gamma<\beta$ such that the elements of this finite set picked by I during the first β moves, were picked before the γ-th move. Let $s\restriction\gamma=t$, and then we may replace $S\restriction t$ by $S\restriction s$ (Fig.2).

This transformation can be done for the remaining nodes of height β, and we get finally a tree S^* of height $\leqq\beta$ which is a I-WS, and the induction assertion can be applied.

Case 4. h is a limit. Let the node s be a *-node if the height of the tree $S\restriction s$ is h. For example the root of S is a *-node and the predecessors of a *-node are *-nodes,

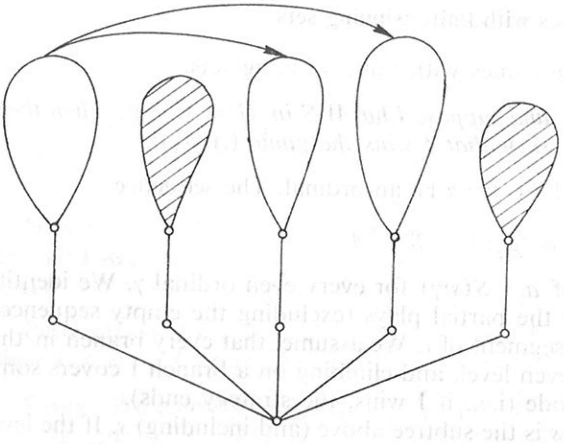

Fig. 1

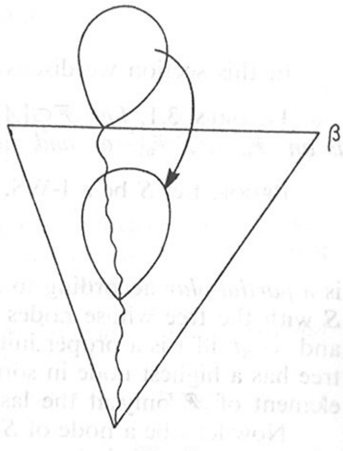

Fig. 2

too. Suppose that in some branch of S the limit s of *-nodes is not a *-node. Then height $(S{\restriction}s)<h$, and by the induction assertion, $S{\restriction}s$ can be supposed to be good. Then, as in Case 3, this $S{\restriction}s$ can be lowered in place of some *-node of this branch.

Similarly, if no successor of a *-node is a *-node, then we apply the induction assertion on these successors and just as in Case 2 we may replace $S{\restriction}s$ by a good subtree.

Therefore applying these steps sufficiently many times, we may achieve a I-WS tree S^* in which the limits of *-nodes are *-nodes and every *-node has a *-node successor. It means that if there are *-nodes in S^* then there is a cofinal branch of *-nodes. But this is impossible, because a I-WS has no cofinal branch (of limit lenght). Therefore the root of S^* is not a *-node, i.e. height $(S^*)<h$, and we are done.

THEOREM 3.2. *Let* $F\subset[A]^{<\omega}$, *then the game* $(A, \mathscr{F})^\alpha$ *is determined.*

PROOF. Suppose that I has no WS. Then II can make a move such that I still has no WS. This strategy is a WS for II. We have only to check that at limit moves I still has no WS. If he has, then, by the previous theorem, he has WS in some finite part of the game, therefore he has a WS *before* this move, a contradiction.

4. Games with countable boards

While we have a nice compactness theorem for finite winning sets, we cannot hope for one in general, as the following example shows.

EXAMPLE 4.1. Let $\mathscr{F}\subset[\omega]^\omega$, $|\mathscr{F}|=2^\omega$ be the (maximal) branches of a binary tree of height ω, such that $\bigcup\mathscr{F}=\omega$. Then I has a WS in $(\omega, \mathscr{F})$ but II wins the games $(\omega, \mathscr{F}')$ with $\mathscr{F}'\subsetneqq\mathscr{F}$, and $(A, P(A)\cap\mathscr{F})$ with $A\subsetneqq\omega$.

There exist undetermined games. The following example is due to RALPH McKENZIE.

THEOREM 4.2. *Let* $U \subset P(\omega)$ *be a non-trivial ultrafilter on* ω. *The game* (ω, U) *is undetermined.*

PROOF. Suppose first that II has a WS. At the end of the game either I or II (but not both) pick all the points of some element of U, therefore this strategy ensures II to cover an element of U. But also I can play by this strategy, and so he too covers an element of U, a contradiction.

Now assume that I has a WS, and let them play three instances of this game, see the figure. Let the first move of I be i_0. II may manage, that after the first ω moves every square is occupied and in every column (of three squares) except for the i_0-th one, at least one square belongs to him. If I has played by his strategy in each of the rows then the sets of squares occupied by I in the rows are elements of U, i.e. their intersection is infinite, a contradiction.

0	1	2	.	.	i_0	.	.	.
I		II			I			
		I			I	II		
	II	I			I			

Fig. 3

In both examples the set of winning sets has cardinality 2^ω. This cannot be improved as the following theorem shows.

THEOREM 4.3. *Martin's axiom implies that if* $\mathscr{F} \subset [\omega]^\omega$, $|\mathscr{F}| < 2^\omega$ *then II has a WS in* $(\omega, \mathscr{F})$.

PROOF. In fact, we show that II may kill all of the winning sets in the first ω steps.

Let $\pi_k(x) = \min(x, k)$ for $x, k \in \omega$, and in general let

$$\pi_k(\langle x_0, \ldots, x_{i-1}\rangle) = \langle \pi_k(x_0), \ldots, \pi_k(x_{i-1})\rangle.$$

Let ${}^{\underline{n}}\omega = \bigcup\{{}^i\omega \colon i < n\}$ and $f \colon {}^{\underline{n}}\omega \to \omega \ (n \in \omega)$ be a partial strategy for II, i.e. $f(\langle x_0, \ldots, x_{i-1}\rangle) \notin \{x_0, \ldots, x_{i-1}\}$, etc., such that $f = f \circ \pi_k$ for some $k \in \omega$. Obviously, the set P of these strategies is countable, therefore the partial ordering $f_1 \leqq f_2$ iff $f_1 \supset f_2$ for $f_1, f_2 \in P$ satisfies c.c.c. Given $F \in \mathscr{F}$, the subset

$$D_F = \{f \in P \colon (\forall g \in {}^\omega\omega)(\exists i \in \omega) f(g \restriction i) \in F\}$$

is dense, and by Martin's axiom there is a chain $G \subset P$ such that $G \cap D_F \neq \emptyset$ for every $F \in \mathscr{F}$. Now $\bigcup G$ is the strategy whose existence was stated.

On the other hand we have the following

THEOREM 4.4. Con $(ZFC + 2^\omega = \omega_2 +$ "*there is an* $\mathscr{F} \subset [\omega]^\omega$, $|\mathscr{F}| = \omega_1$ *such that II has no WS in the game* $(\omega, \mathscr{F})$").

PROOF. It is well-known that $2^\omega=\omega_2$ is consistent with the existence of a nontrivial ultrafilter generated by ω_1 elements [2]. Let $\mathscr{F}\subset[\omega]^\omega$, $|\mathscr{F}|=\omega_1$ be the set of generators, and suppose that $\mathscr{F}$ is closed under finite intersections. We claim that II has no WS in $(\omega,\mathscr{F})$. Indeed, otherwise after the game the set X of the points occupied by II intersects every element of $\mathscr{F}$. Similarly, I can also play by this strategy therefore the set $\omega-X$ of the points occupied by I intersects every element of $\mathscr{F}$ as well. But either X or $\omega-X$ is an element of the ultrafilter, i.e. contains some $F\in\mathscr{F}$ and then the other one cannot have a common point with F.

PROBLEM. Is it consistent that $2^\omega=\omega_2$ and for some $\mathscr{F}\subset[\omega]^\omega, |\mathscr{F}|=\omega_1$ I has a WS in $(\omega,\mathscr{F})$?

5. Large games

So far we have dealt with games on countable boards, let us take a step ahead. For boards of cardinality ω_1 we have some results similar to that of Section 4.

EXAMPLE 5.1. Let $\mathscr{F}\subset[\omega_1]^\omega$, $|\mathscr{F}|=2^\omega$ be the (maximal) branches of a normal Aronszajn tree such that $\bigcup\mathscr{F}=\omega_1$. Then I has a WS in $(\omega_1,\mathscr{F})$ but II wins the games $(\omega_1,\mathscr{F}')$ with $\mathscr{F}'\subsetneqq\mathscr{F}$ and $(A, P(A)\cap\mathscr{F})$ with $A\subsetneqq\omega_1$.

A version of Theorem 4.4 is true in this case.

THEOREM 5.2. Con $(ZFC+2^\omega=\omega_2+$"*there is an* $\mathscr{F}\subset[\omega_1]^\omega$, $|\mathscr{F}|=\omega_1$ *such that I wins the game* $(\omega_1,\mathscr{F})$").

PROOF. The result Con $(ZFC+2^\omega=\omega_2+\clubsuit)$ is from [3], where $\clubsuit$ is the following combinatorial principle:

"There exists a sequence $\langle S_\alpha:\alpha<\omega_1,\alpha$ is limit$\rangle$ such that $\bigcup S_\alpha=\alpha$ and for every $X\in[\omega_1]^{\omega_1}$, there exists a limit $\alpha<\omega_1$ such that $S_\alpha\subset X$."

The family $\mathscr{F}=\{S_\alpha\}$ evidently works.

PROBLEM. Is it true (in ZFC) that there is an $\mathscr{F}\subset[\omega_1]^\omega$, $|\mathscr{F}|=\omega_1$ such that $(\omega_1,\mathscr{F})$ is a win for I?

If the cardinality of the board and that of the family of the winning sets do not exceed $\varkappa$, and the length of the game is $<\varkappa^+$ then the strategies can be formulated in $V_\varkappa$, therefore we have

PROPOSITION 5.3. *Let $\varkappa$ be a weakly compact cardinal, $\mathscr{F}\subset[\varkappa]^\omega$, $|\mathscr{F}|=\varkappa$, $\alpha<\varkappa^+$. If I has a WS for $(\varkappa,\mathscr{F})^\alpha$ then there are $\mathscr{F}'\subset\mathscr{F}$, $|\mathscr{F}'|<\varkappa$ and $\lambda<\varkappa$ such that I wins $(\varkappa,\mathscr{F}')^\lambda$.*

The following construction is an unpublished result of A. HAJNAL.

THEOREM 5.4. *Let $V=L$, $\varkappa>\omega$, $\varkappa$ regular and not weakly compact. Then there exists an $\mathscr{F}\subset[\varkappa]^\omega$ such that I wins $(\varkappa,\mathscr{F})^\varkappa$, but*

(i) *II wins $(\varkappa,\mathscr{F}')$ if $\mathscr{F}'\subset\mathscr{F}$, $|\mathscr{F}'|<\varkappa$;*

(ii) *I has no WS in $(\varkappa,\mathscr{F})^\lambda$ if the regular cardinal $\lambda<\varkappa$.*

PROOF. (ii) follows from (i) and from the following lemma.

LEMMA 5.5. *Let* $\lambda\geqq\omega$ *be any regular cardinal, and suppose that I has WS in* $(A, \mathcal{F})^{\lambda}$. *Then there is a* $B\subset A$, $|B|\leqq 2^{\underset{\smile}{\lambda}}=\sum_{\mu<\lambda}2^{\mu}$ *such that I still has WS in* $(B, P(B)\cap\mathcal{F})^{\lambda}$.

PROOF of the lemma. A I-WS is a function from ${}^{\underset{\smile}{\lambda}}A=\bigcup\{{}^{\alpha}A:\alpha<\lambda\}$ to A, choose B as the closure of any point of A with respect to this function.

As for (i) of Theorem 5.4, let $S\subset\varkappa$ be a stationary set consisting of ω-limits only such that for every limit $\xi<\varkappa$, $S\cap\xi$ is not stationary in ξ but S is stationary in $\varkappa$, and let $\langle X_\alpha:\alpha\in S\rangle$ be a $\diamondsuit_S$-sequence [4].

If we fix for every $\alpha\in S$ such that $\bigcup X_\alpha=\alpha$ a cofinal subset F_α of X_α of type ω, the family $\mathcal{F}$ of these F_α's will do.

Indeed, let I pick the elements of some closed unbounded set C of $\varkappa$. (He can do it by simply picking the smallest unpicked element above the set of previously picked ones.) Then, by $\diamondsuit_S$, the set $\{\alpha\in S: X_\alpha=C\cap S\cap\alpha\}$ is stationary in $\varkappa$, and so is its intersection with C, i.e.

$$A=\{\alpha\in S\cap C:\ X_\alpha=C\cap S\cap\alpha\}$$

is stationary, too. Now for some $\alpha\in A$, $\bigcup X_\alpha=\alpha$ otherwise there would be a regressive function on A which is impossible. But $\alpha\in S$, therefore α is an ω limit and $F_\alpha\subset X_\alpha\subset\subset C\cap S$. So F_α is covered by I, i.e. I wins the game.

Now let $\xi<\varkappa$ and $\mathcal{F}_\xi=\{F\in\mathcal{F}: F\subset\xi\}$. We claim that II wins the game $(\xi, \mathcal{F}_\xi)$, hence (i) follows. Instead of this we prove the following statement. Let $\alpha<\beta<\varkappa$, α and β be limit ordinals, $\alpha,\beta\notin S$, and $\mathcal{F}_{\alpha,\beta}=\{F\cap(\alpha,\beta): F\in\mathcal{F}$ and $\alpha<\bigcup F<\beta\}$. Obviously, the elements of $\mathcal{F}_{\alpha,\beta}$ are countably infinite sets, and $\mathcal{F}_{0,\xi}=\mathcal{F}_\xi$. II wins the games $(\beta-\alpha, \mathcal{F}_{\alpha,\beta})$, this will be proven by induction on $\beta-\alpha$.

If $\beta-\alpha$ is countable, then $\mathcal{F}_{\alpha,\beta}$ is countable, too. (Different elements of $\mathcal{F}_{\alpha,\beta}$ have different suprema.) Let II sort the elements of $\mathcal{F}_{\alpha,\beta}$ in order type ω and at his i-th move pick an element of the i-th set.

If $\beta-\alpha\geqq\omega_1$ then $S\cap\beta$ is not stationary in β, i.e. there is a strictly increasing continuous sequence $\langle x_\xi:\xi\leqq\gamma\rangle$ such that $x_0=\alpha$, $x_\gamma=\beta$, $x_{\xi+1}-x_\xi<\beta-\alpha$ for $\xi<\gamma$ and $x_\xi\notin S$. Now if $F\in\mathcal{F}_{\alpha,\beta}$ then $\bigcup F\in S$, i.e. $x_\xi<\bigcup F<x_{\xi+1}$ for some $\xi<\gamma$, therefore for some $F'\in\mathcal{F}_{x_\xi, x_{\xi+1}}$, $F'\subset F$ and $|F-F'|<\omega$. Because $x_{\xi+1}-x_\xi<\beta-\alpha$, II can play independently in each of these intervals by his previously defined strategy.

By this theorem, if $V=L$ then for every not weakly compact $\varkappa$ there exists an $\mathcal{F}\subset[\varkappa]^\omega$ such that card $((\varkappa,\mathcal{F})^\varkappa)=\varkappa$.

PROBLEM. Is it consistent that for every $\mathcal{F}\subset[\omega_2]^\omega$, card $((\omega_2,\mathcal{F})^{\omega_2})<\omega_2$?

6. A topological game

Let (x,τ) be a Hausdorff topological space, $\tau\subset P(X)$ be the family of open sets in X. The game (X,τ) is the open-dense game. I wins if he covers an open subset of X, and II wins if he covers a dense subset of X. If X contains an isolated point then I has a WS, he has only to pick that point. However, I. JUHÁSZ made the following observation.

THEOREM 6.1. *Suppose X is a locally compact Hausdorff space with no isolated points. Then II has a WS in the game (X, τ).*

PROOF. Let $\mathcal{G} \subset \tau$ be the family of open sets G which have the following property. Every non-empty open subset of G has cardinality equal to that of G. $\mathcal{G}$ is evidently a π-base, take a maximal disjoint subfamily $\mathcal{G}^*$ of G. $\bigcup \mathcal{G}^*$ is dense in X therefore II can play independently in each element G of $\mathcal{G}^*$, because if II wins all of these games then he wins the whole game, too.

Now if $G \in \mathcal{G}^*$ then G is locally compact (endowed with the subspace topology), therefore the weight of G is at most $|G|$, see [5]. Let the base $\{G_\alpha : \alpha < |G|\}$, $\emptyset \neq G_\alpha \subset G$ witness this assertion, then $|G_\alpha| = |G|$ because $G \in \mathcal{G}^* \subset \mathcal{G}$. Then, at his α-th move (in G) II can choose an element of G_α, which ensures him the win.

On the other hand we have succeeded in proving the following theorems.

THEOREM 6.2. *Assume CH. There is a 0-dimensional Hausdorff topology τ on ω such that the game (ω, τ) is undetermined.*

PROOF. Let S_α^{I} and S_α^{II} for $\alpha < \omega_1$ be the possible strategies of I and II on ω respectively. We define the sets B_α, τ_α, X_α and Y_α for $\alpha < \omega_1$ by induction on α such that

(i) $B_\alpha \subset [\omega]^\omega$, $|B_\alpha| = \omega$ is a base for a T_2-, 0-dimensional topology on ω, and $B_\beta \subset B_\gamma$ if $\beta < \gamma$.

(ii) τ_α is the topology induced by B_α.

(iii) $X_\alpha, Y_\alpha \subset \omega$, X_α is dense in τ_β if $\beta < \alpha$, and is open in τ_β if $\beta > \alpha$. Moreover $Y_\alpha \cap Z$ is infinite for every $Z \in B_\beta$ if $\alpha, \beta < \omega_1$.

(iv) If I plays by the strategy S_α^{I} then II has a counterplay such that he covers Y_α;

(v) if II plays by the strategy S_α^{II} then I has a counterplay such that he covers X_α.

By these conditions the topology $\tau = \bigcup \{\tau_\alpha : \alpha < \omega_1\}$ satisfies the requirements of the theorem.

Now let τ_{-1} be the topology of the dense linear ordering without endpoints on ω (i.e. the subspace topology of the rationals) and let B_{-1} be a countable base for it.

Suppose we have defined $B_\beta, \tau_\beta, X_\beta, Y_\beta$ for $\beta < \alpha$, and let $B = \bigcup \{B_\beta : \beta < \alpha\}$ and $\tau = \bigcup \{\tau_\beta : \beta < \alpha\}$. Observe that τ is generated by the countable base B, and there is no isolated point in τ. Therefore II has a counterplay against the α-th strategy S_α^{I} of I playing which he covers such a set Y_α which has the following property. For every $Z \in B$, $Z \cap Y_\alpha$ is infinite, in particular Y_α is dense in τ.

Similarly, I has a counterplay against the strategy S_α^{II} of II playing which he covers such a set X_α which has the following property. The sets $X_\alpha \cap Z \cap Y_\beta$ and $(\omega - X_\alpha) \cap Z \cap Y_\beta$ are infinite for each $Z \in B$ and $\beta \leqq \alpha$. Finally, let $B_\alpha = \{X_\alpha \cap Z, (\omega - X_\alpha) \cap Z : Z \in B\}$. The validity of the conditions (i)—(v) for α can be checked easily.

THEOREM 6.3. *There is a 0-dimensional Hausdorff topology τ on ω without isolated points such that I wins the game (ω, τ).*

PROOF. By Zorn's lemma there is a maximal 0-dimensional T_2-topology τ on A without isolated points where $|A| = \omega$. We claim that there is no $X \subset A$ such that both X and $A-X$ are dense in τ. Assuming the claim false, we find that $\tau \cup \{X, A-X\}$ constitutes a subbase for a 0-dimensional T_2-topology σ on A. Of course $\sigma \supset \tau$, $\sigma \neq \tau$ and τ is maximal, therefore σ contains an isolated point $p \in A$. It means that for some $G \in \tau$, either $G \cap X = \{p\}$ or $G \cap (A-X) = \{p\}$. Let $q \in G$ and $G' \in \tau$ be such that $q \in G'$ and $p \notin G'$. Then $\emptyset \neq G' \cap G \in \tau$ and either $G' \cap G \cap X$ or $G' \cap G \cap (A-X)$ is empty which contradicts to the denseness of X and $A-X$.

Let (A', τ') be a disjoint copy of (A, τ) and let I and II play on the topological sum of these spaces. By the previous remark, II cannot cover dense subsets in both of these spaces if I plays as follows. If II picked a point in A, I picks the same point in A', and if II picked a point in A', I picks the same point in A.

REFERENCES

[1] AJTAI, M., CSIRMAZ, L. and NAGY, Zs., On a generalization of the game Go-moku, I, *Studia Sci. Math. Hungar.* **14** (1979), 209—226.

[2] BAUMGARTNER, J. and LAVER, L., Iterated perfect set forcing, *Ann. Math. Logic.* **17** (1979), 271—288.

[3] DEVLIN, K. J., Variations on $\Diamond$, *J. Symbolic Logic* **44** (1979), 51—58. *MR* **80c**: 03050.

[4] JENSEN, R. B., The fine structure of the constructible hierarchy. With a section by Jack Silver, *Ann. Math. Logic* **4** (1972), 229—308; erratum ibid. **4** (1972), 443. *MR* **46** #8834.

[5] JUHÁSZ, I., *Cardinal function in topology*, Amsterdam, 1975.

[6] MARTIN, D. A., Borel determinacy, *Ann. of Math.* (2) **102** (1975), 363—371. *MR* **53** #7785.

(Received July 2, 1980)

MTA MATEMATIKAI KUTATÓ INTÉZETE
REÁLTANODA U. 13—15.
H—1053 BUDAPEST
HUNGARY

81-4202 — Szegedi Nyomda — Felelős vezető: Dobó József igazgató