Extreme selections for hyperspaces of topological spaces

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Abstract

We study properties of Hausdorff spaces $X$ which depend on the variety of continuous selections for their Vietoris hyperspaces $\mathcal{F}(X)$ of closed non-empty subsets. Involving extreme selections for $\mathcal{F}(X)$, we characterize several classes of connected-like spaces. In the same way, we also characterize several classes of disconnected-like spaces, for instance all countable scattered metrizable spaces. Further, involving another type of selections for $\mathcal{F}(X)$, we study local properties of $X$ related to orderability. In particular, we characterize some classes of orderable spaces with only one non-isolated point. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $X$ be a topological space, and let $\mathcal{F}(X)$ be the set of all non-empty closed subsets of $X$. Also, let $\mathcal{D} \subset \mathcal{F}(X)$. A map $f : \mathcal{D} \rightarrow X$ is a selection for $\mathcal{D}$ if $f(S) \in S$ for every $S \in \mathcal{D}$. A map $f : \mathcal{D} \rightarrow X$ is a continuous selection for $\mathcal{D}$ if it is a selection for $\mathcal{D}$ which is
continuous with respect to the relative Vietoris topology $\tau_V$ on $D$. Let us recall that $\tau_V$ is the topology on $\mathcal{F}(X)$ generated by all collections of the form
\[ \langle V \rangle = \{ S \in \mathcal{F}(X) : S \subseteq \bigcup V \text{ and } S \cap V \neq \emptyset \}, \]
where $V$ runs over the finite families of open subsets of $X$. Sometimes, for reasons of convenience, we will also say that $f$ is a $\tau_V$-continuous selection for $D$ to stress the fact that $f$ is continuous just with respect to the topology $\tau_V$.

In the sequel, all spaces are at least Hausdorff. There are several results demonstrating certain relationships between the variety of continuous selections for $\mathcal{F}(X)$ and topological properties of the base space $X$. In this paper, we are interested in continuous selections for $\mathcal{F}(X)$ being “extreme” with respect to a point $p \in X$. One of the first results detecting extreme properties of selections for $\mathcal{F}(X)$ is due to Michael [16, Lemmas 7.2 and 7.3] (see also [19, Lemma 10]).

**Theorem 1.1** [16]. Let $X$ be a connected space, and let $f$ be a continuous selection for $\mathcal{F}(X)$. Then, there exists a point $p \in X$ such that
\[ f^{-1}(p) = \{ S \in \mathcal{F}(X) : p \in S \}. \]
If, moreover, $\mathcal{F}(X)$ has a continuous selection $g$ with $g \neq f$, then, $g^{-1}(p) = \{ \{ p \} \}$. In particular, $\mathcal{F}(X)$ has at most two different continuous selections.

Thus, a connected space $X$ which admits a continuous selection for $\mathcal{F}(X)$ may have at most two points as those in Theorem 1.1. Hence, it is natural to expect that a space $X$ which has many points $p \in X$ as those in Theorem 1.1 will have also some disconnectedness-like properties. This is the statement of the following recent result of [12, Theorem 1.4].

**Theorem 1.2** [12]. Let $X$ be a first countable space which has a continuous selection for $\mathcal{F}(X)$. Then, $X$ is zero-dimensional if and only if for every point $x \in X$ there exists a continuous selection $f_x$ for $\mathcal{F}(X)$ such that $f_x^{-1}(x) = \{ S \in \mathcal{F}(X) : x \in S \}$.

Here, as usual, $X$ is zero-dimensional if it has a base of clopen sets.

Motivated by these results, we introduce the following concepts which will play a central role in the paper. We shall say that a selection $f : \mathcal{F}(X) \to X$ is $p$-maximal if $f^{-1}(p) = \{ S \in \mathcal{F}(X) : p \in S \}$. By analogy, we shall say that $f$ is $p$-minimal if $f^{-1}(p) = \{ \{ p \} \}$.

We are now ready to state the main purpose of this paper. In the first place, we generalize Theorem 1.1 characterizing the spaces $X$ in which every continuous selection for $\mathcal{F}(X)$ is $p$-maximal for some point $p \in X$. This is done in the next Section 2, see Theorem 2.1. Further, we pay a special interest to the selection problem for countable spaces. Briefly, we characterize all countable spaces $X$ which have at least one continuous selection for $\mathcal{F}_2(X) = \{ S \in \mathcal{F}(X) : |S| \leq 2 \}$ (Theorem 3.1) and, in this way, we get a partial answer to a question raised by van Mill and Wattel [17]. Next, we characterize all countable spaces $X$ which have a $p$-maximal selection for $\mathcal{F}(X)$ for every point $p \in X$ (Theorem 3.3), thus we also get a partial generalization of Theorem 1.2 demonstrating that, in this case,
a \( p \)-maximal selection for some point \( p \in X \) implies the first countability of \( X \) in \( p \), see Theorem 3.5.

The second part of the paper deals with local properties of spaces \( X \) having a continuous selection for \( F(X) \). One of the main results here shows that a space \( X \) with a continuous selection for \( F(X) \) has at most two different “directions” at every point of its, see Theorem 4.1. We apply this result to get several consequences for spaces \( X_p \) with only one non-isolated point \( p \), see Corollaries 4.3, 4.5 and 4.6. Also, we present a construction of continuous selections for hyperspaces on spaces \( X \) with a fixed point \( p \) (Lemma 6.4) which involves two different extreme selections for the two different directions at the point \( p \). Finally, we also demonstrate that this is as natural as one of the only possible Vietoris continuous “choices” in points \( p \) which agree two different convergence structures, see Theorems 6.5 and 6.7.

2. Extreme selections for hyperspaces on connected spaces

In what follows, only for reasons of convenience, let us agree to say that a selection \( f : F(X) \rightarrow X \) is point-maximal if it is \( p \)-maximal for some point \( p \in X \).

Suppose that \( X \) is a space and \( X_1, X_2 \subset X \). We shall say that \( \{X_1, X_2\} \) is a partition of the space \( X \) if \( X_1 \cap X_2 = \emptyset \) and \( X = X_1 \cup X_2 \). We shall say that a partition \( \{X_1, X_2\} \) of \( X \) is connected if both sets \( X_1 \) and \( X_2 \) are connected.

**Theorem 2.1.** For a space \( X \) which contains at least two points and has a continuous selection for \( F(X) \), the following conditions are equivalent:

(a) Every continuous selection for \( F(X) \) is \( p \)-maximal for some point \( p \in X \).

(b) \( X \) has a connected partition \( \{X_1, X_2\} \) such that each \( X_i, i = 1, 2 \), has exactly one continuous selection for \( F(X_i) \).

(c) \( X \) has a partition \( \{X_1, X_2\} \) such that each \( X_i, i = 1, 2 \), has exactly one continuous selection for \( F(X_i) \).

To prepare for the proof of Theorem 2.1, we need the following lemma.

**Lemma 2.2.** Let \( X \) be a space, and let \( C \subset F(X) \) be a family of connected sets such that \( \bigcup C \in F(X) \). Then,

\[
\langle C \rangle = \left\{ S \in F(X): S \subset \bigcup C \text{ and } S \cap C \neq \emptyset, \text{ whenever } C \in C \right\}
\]

is a \( \tau_V \)-connected subset of \( F(X) \).

**Proof.** First of all, let us observe that \( \langle C \rangle \) is a \( \tau_V \)-closed subset of \( F(X) \) because

\[
F(X) \setminus \langle C \rangle = \left\{ X, X \setminus \bigcup C \right\} \cup \left( \bigcup \left\{ \{X \setminus C\}: C \in C \right\} \right).
\]

Suppose, if possible, that \( \langle C \rangle \) is not \( \tau_V \)-connected. Then, \( \langle C \rangle \) contains a non-empty relatively \( \tau_V \)-clopen subset \( U \) such that \( \bigcup U \notin U \). Take a subset \( M \subset U \) which is a maximal
chain with respect to the usual set-theoretical inclusion. Since $\mathcal{U}$ is a $\tau_V$-closed subset of $\mathcal{F}(X)$, by [5, Lemma 3.1] (see also [3,11]), $\mathcal{M}$ has a maximal element $M$. Since $\mathcal{U}$ is relatively $\tau_V$-open, there now exists a finite family $\mathcal{V}$ of open subsets of $X$ such that $M \in (\mathcal{V}) \cap (\mathcal{C}) \subset \mathcal{U}$. Then, $M = (\bigcup \mathcal{V}) \cap (\bigcup \mathcal{C})$ because $M = \max \mathcal{M}$ and $\mathcal{M}$ is a maximal chain in $\mathcal{U}$. In particular, this implies that $M$ is a relatively clopen subset of $\bigcup \mathcal{C}$. On the other hand, $M \in \mathcal{U} \subset \mathcal{C}$ implies that $M \cap C \neq \emptyset$ for every $C \in \mathcal{C}$. Therefore, $M = \bigcup \mathcal{C}$ because every $C \in \mathcal{C}$ is connected. This however is impossible because $\bigcup \mathcal{C} \notin \mathcal{U}$. 

**Proof of Theorem 2.1.** Take a continuous selection $f$ for $\mathcal{F}(X)$. 
(a) $\Rightarrow$ (b) In case $X$ is not connected, it contains a clopen non-empty proper subset $X_1$. Set $X_2 = X \setminus X_1$. We claim that $X_1$ and $X_2$ are as required. Suppose, if possible, that one of these sets is not connected, say $X_1$. Then, there are non-empty clopen subsets $L, R \subset X_1$ such that $L \cap R = \emptyset$ and $X_1 = L \cup R$. Now, we may define another selection $g$ for $\mathcal{F}(X)$ by

$$g(S) = \begin{cases} f(S \cap L) & \text{if } S \in (L, R), \\ f(S \cap R) & \text{if } S \in (L, R, X_2), \\ f(S) & \text{otherwise.} \end{cases}$$

Note that $g$ is not point-maximal because $g(X_1) \in L$ while $g(X) \in R \subset X_1$. On the other hand, $g$ is continuous because so is $f$ and the sets $L, R$ and $X_2$ are clopen. This however contradicts (a). Thus, both sets $X_1$ and $X_2$ are connected. In this case, it only remains to show that each $X_i$, $i = 1, 2$, has exactly one continuous selection for $\mathcal{F}(X_i)$.

Suppose that one of these sets $Y \in \{X_1, X_2\}$ has two different continuous selections $k, h : \mathcal{F}(Y) \to Y$ for $\mathcal{F}(Y)$. According to Theorem 1.1, there exists a point $p \in Y$ such that $h^{-1}(p) = \{S \in \mathcal{F}(Y) : p \in S\}$ and $k^{-1}(p) = \{\{p\}\}$. Now, we may define a continuous selection $g$ for $\mathcal{F}(X)$ by

$$g(S) = \begin{cases} k(S) & \text{if } S \in \mathcal{F}(Y), \\ h(S \cap Y) & \text{if } S \in \langle X_1, X_2 \rangle, \\ f(S) & \text{otherwise.} \end{cases}$$

We claim that $g$ is not point-maximal. Indeed, $g(X) = h(X \cap Y) = h(Y) = p \in Y$ because $X \in \langle X_1, X_2 \rangle$, while, by definition, $g(Y) = k(Y) \neq p$ because $k^{-1}(p) = \{\{p\}\}$ and $k \neq h$. According to (a), this is impossible.

In case that $X$ is connected, by (a) (see also, Theorem 1.1), $f$ should be $p$-maximal for some point $p \in X$. Consider the natural linear order $\leq$ on $X$ (see [16]) defined by

$$x_1 \leq x_2 \text{ if and only if } f([x_1, x_2)) = x_1.$$  

Then, by [16, Lemma 7.3.2], we have $f(S) = \min S$ for every $S \in \mathcal{F}(X)$ and, in particular, $p = \min X$. Pick a fixed point $q \in X \setminus \{p\}$ in the following way. If $X$ has a last point we take for $q$ just this point. Otherwise, we take for $q$ any point of $X \setminus \{p\}$. Next, we define $X_1 = \{x \in X : x < q\}$ and $X_2 = \{x \in X : q \leq x\}$. Thus, we get a connected partition $\{X_1, X_2\}$ of $X$ because $X$ is connected. Hence, it only remains to show that each $X_i$, $i = 1, 2$, has exactly one continuous selection for $\mathcal{F}(X_i)$. Note that $X_2$ is a closed subset of $X$, hence $\mathcal{F}(X_2)$ has a continuous selection because $\mathcal{F}(X)$ has this property. In case
$X_2$ is not a singleton, it has no last point. Hence, it has exactly one continuous selection for its Vietoris hyperspace, see [16, Lemma 7.2.4] and [19, Lemma 10]. In the same way, $\mathcal{F}(X_1)$ has at most one continuous selection because $X_1$ has no last point. On the other hand, $g(S) = \min S, S \in \mathcal{F}(X_1)$, defines a continuous selection $g$ for $\mathcal{F}(X_1)$. 

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) We have to prove that $f$ is $p$-maximal for some point $p \in X$. We distinguish the following two cases. If $X$ is connected, this follows by Theorem 1.1. In case $X$ is not connected, it contains a non-empty proper clopen subset $A$. Let $X_1$ and $X_2$ be as in (c). According to [19, Lemma 14], both sets $X_1$ and $X_2$ are connected because each of them has exactly one continuous selection for its Vietoris hyperspace. Also, $X_1 \neq \emptyset \neq X_2$. Hence, either $X_1 = A$ or $X_2 = A$. That is, both sets $X_1$ and $X_2$ are clopen. Then $f_i = f|\mathcal{F}(X_i), i = 1, 2$, defines a continuous selection for $\mathcal{F}(X_i)$ and, hence, by Theorem 1.1, for every $i = 1, 2$, there exists a point $p_i \in X_i$ such that $f_i$ is a $p_i$-maximal.

Finally, let $p = f(\{p_1, p_2\})$. We claim that $f$ is a $p$-maximal selection for $\mathcal{F}(X)$. To prove this, let $Y \in \{X_1, X_2\}$ be such that $p \in Y$. We consider the set $(X_1, X_2)$ which, by Lemma 2.2, is a $\tau_V$-connected subset of $\mathcal{F}(X)$. On the other hand, $\{p_1, p_2\} \in (X_1, X_2)$ is such that $f(\{p_1, p_2\}) = p \in Y$. Then, $f^{-1}(Y)$ is a $\tau_V$-clopen subset of $\mathcal{F}(X)$ such that $f^{-1}(Y) \cap (X_1, X_2) \neq \emptyset$. Hence, $(X_1, X_2) \subset f^{-1}(Y)$. Finally, take a set $B \in \mathcal{F}(X \setminus Y)$, and define a continuous selection $g_B : \mathcal{F}(Y) \to Y$ for $\mathcal{F}(Y)$ by $g_B(S) = f(S \cup B), S \in \mathcal{F}(Y)$.

Since, by (c), $\mathcal{F}(Y)$ has exactly one continuous selection, we get that $g_B = f|\mathcal{F}(Y)$, and, in particular, $g_B$ is $p$-maximal because so is $f|\mathcal{F}(Y)$. As a result, $f(S \cup B) = g_B(S) = p$ for every $S \in \mathcal{F}(Y)$ with $p \in S$. That is, $f$ is $p$-maximal as there were no restrictions on the choice of $B$. 

Corollary 2.3. For a space $X$ which has a continuous selection for $\mathcal{F}(X)$, the following conditions are equivalent:

(a) Every continuous selection for $\mathcal{F}(X)$ is $p$-maximal for some point $p \in X$.

(b) $X$ has at most two different continuous selections for $\mathcal{F}(X)$.

Proof. In case $X$ is connected, this follows by Theorem 1.1. Suppose that $X$ is not connected. Then, by Theorem 2.1, (a) becomes equivalent to the statement that $X$ has exactly two connected components $X_1$ and $X_2$, and each $X_i, i = 1, 2$, admits exactly one continuous selection for $\mathcal{F}(X_i)$. The last condition is clearly equivalent to the existence of exactly two continuous selections for $\mathcal{F}(X)$ (see, e.g., [18]). 

Suppose that $f : \mathcal{F}(X) \to X$ is a selection for $X$, and $P \subset X$. We shall say that $f$ is $P$-maximal if $f(S) \in P$ whenever $P \subset S \in \mathcal{F}(X)$.

In what follows, a subset $D \subset X$ is called weakly clopen discrete in $X$ if every point of $X$ has a clopen neighbourhood $U$ such that $U \cap D$ is finite. Note that every finite $D \subset X$ is weakly clopen discrete in $X$. Also, every weakly clopen discrete set $D \subset X$ is discrete in $X$ but the converse is not true (see, for instance, Example 2.5).
Theorem 2.4. Let $X$ be a space which has a continuous selection for $\mathcal{F}(X)$, and let $\mathcal{C} \subset \mathcal{F}(X)$ be the set of all connected components of $X$. The following conditions are equivalent:

(a) $\mathcal{C}$ is discrete in $X$.
(b) There exists a weakly clopen discrete set $P \subset X$, with $|\mathcal{C}| \leq |P| \leq 2|\mathcal{C}|$, such that every continuous selection for $\mathcal{F}(X)$ is $P$-maximal.
(c) There exists a weakly clopen discrete set $P \subset X$ such that every continuous selection for $\mathcal{F}(X)$ is $P$-maximal.

Proof. (a) $\Rightarrow$ (b) Whenever $C \in \mathcal{C}$, let $P_C$ be the set of all points $p \in C$ such that every continuous selection for $\mathcal{F}(C)$ is $p$-maximal. According to Theorem 1.1, $|P_C| \leq 2$. Let us show that $P = \bigcup\{P_C: C \in \mathcal{C}\}$ is as required in (b). Take a continuous selection $f$ for $\mathcal{F}(X)$. Also, let $Y \in \mathcal{C}$ be such that $f(P) \in Y$. Note that, by Lemma 2.2, $\langle \mathcal{C} \rangle$ is a $\tau_V$-connected subset of $\mathcal{F}(X)$ because $\bigcup \mathcal{C} = X \in \mathcal{F}(X)$. On the other hand, $P \in f^{-1}(Y) \cap \langle \mathcal{C} \rangle$. Hence, $\langle \mathcal{C} \rangle \subset f^{-1}(Y)$ because $f^{-1}(Y)$ is a $\tau_V$-clopen subset of $\mathcal{F}(X)$. Fix $S \in \mathcal{F}(X)$, with $P \subset S$, and set $A = S \setminus Y$. Next, define a continuous selection $g$ for $\mathcal{F}(Y)$ by $g(F) = f(F \cup A)$ for every $F \in \mathcal{F}(Y)$. Then, $g$ is $q$-maximal for some point $q \in P_Y \subset P$, so $f(S) = g(S \cap Y) = q \in P$.

(b) $\Rightarrow$ (c) is obvious.

(c) $\Rightarrow$ (a) Let $P \subset X$ be as in (c). Also, for every point $x \in X$, let $\mathcal{C}[x]$ be the connected component of $X$ in $x$. Since $P$ is weakly clopen discrete, it now suffices to show that $\mathcal{C}[x] \cap P \neq \emptyset$ for every point $x \in X$. Suppose that this fails for some point $q \in X$. Then, there exists a clopen neighbourhood $V$ of $q$ such that $V \cap P$ is finite. By [12, Theorem 4.1], $\mathcal{C}[q]$ coincides with the quasi-component of $X$ in $q$. Hence, there exists a clopen neighbourhood $Y$ of $q$ such that $Y \cap P = \emptyset$. Take a continuous selection $f$ for $\mathcal{F}(X)$ and then define another one $g : \mathcal{F}(X) \to X$ by

$$g(S) = \begin{cases} f(S \cap Y) & \text{if } S \cap Y \neq \emptyset, \\ f(S) & \text{otherwise.} \end{cases}$$

This $g$ however fails to be $P$-maximal. Indeed, we merely have $g(P \cup \{q\}) = q \notin P$. The contradiction so obtained completes the proof. □

Concerning the statements of Theorem 2.4, let us first mention that, in (c), the requirement on $P$ to be weakly clopen discrete is substantial. Here is an example.

Example 2.5. There exists a space $X$ and a discrete (in $X$) subset $D \subset X$ such that:

(a) $\mathcal{F}(X)$ has a continuous selection.
(b) Every continuous selection for $\mathcal{F}(X)$ is $D$-maximal.
(c) $X$ is not a discrete sum of connected subspaces.

Proof. For every $n < \omega$, let $X_n = [0, 1]$ and $D_n = [0, 1]$. Next, let $\widehat{X}$ be the discrete sum $\bigoplus\{X_n: n < \omega\}$, and, respectively, let $\widehat{D} = \bigoplus\{D_n: n < \omega\}$. For convenience, set $X_\omega = D_\omega = [\omega]$. Finally, set $X = X_\omega \cup \widehat{X}$ and $D = D_\omega \cup \widehat{D}$. We consider $X$ with the
following topology: \( \widetilde{X} \) is an open subspace of \( X \) which has the usual topology as a discrete sum of topological spaces, while the basic open neighborhoods at \( \omega \) are of the form

\[
U_n = \bigcup \{(X_{\omega} \cup X_k) \setminus D_k : n \leq k < \omega\}, \quad n < \omega.
\]

Thus, we get a Hausdorff space \( X \) such that \( D \) is discrete in \( X \). Let us observe that \((a)\) holds. Indeed, for every \( S \in \mathcal{F}(X) \), let \( n(S) = \min\{n \leq \omega : S \cap X_n \neq \emptyset\} \). Then, a continuous selection for \( \mathcal{F}(X) \) may be defined by \( f(S) = \min S \cap X_n(S), \ S \in \mathcal{F}(X) \). It is also clear that \((c)\) holds because each \( X_n, \ n < \omega \), is clopen in \( X \) while every neighbourhood of \( \omega \) intersects \( \widetilde{X} \). To see \((b)\), take a continuous selection \( g \) for \( \mathcal{F}(X) \).

Suppose there exists \( S \in \mathcal{F}(X) \) with \( D \subset S \) and \( g(S) \notin D \). Then, let \( m < \omega \) be such that \( f(S) \in X_m \). Set \( C = \{X_n : n \leq \omega\} \). Since \( C \subset \mathcal{F}(X) \) and \( \bigcup C = X \in \mathcal{F}(X) \), it now follows by Lemma 2.2 that \( \langle C \rangle \) is a \( \tau_V \)-connected subset of \( \mathcal{F}(X) \). Also, it contains \( S \) because \( D \subset S \). Therefore, \( \langle C \rangle \subset g^{-1}(X_m) \) because \( X_m \) is clopen in \( X \) and \( S \in \langle C \rangle \). Then, we may define a continuous selection \( h \) for \( \mathcal{F}(X_m) \) by \( h(T) = g(T \cup (S \setminus X_m)), \ T \in \mathcal{F}(X_m) \). From one hand, \( h \) should be \( D_m \)-maximal, see [16]. However, from another hand, \( h(S \cap X_m) = g(S) \in X_m \setminus D \). The contradiction so obtained completes the proof. \( \square \)

It should be also mentioned that, in \((b)\) and \((c)\) of Theorem 2.4, the requirement \( P \) to be the same for all continuous selections for \( \mathcal{F}(X) \) cannot be replaced by the condition every continuous selection for \( \mathcal{F}(X) \) to be \( P \)-maximal for some weakly clopen discrete \( P \subset X \).

**Proposition 2.6.** Let \( X_p \) be a space with only one non-isolated point \( p \in X_p \) such that every countable subset of \( X_p \) is closed. Then, every continuous selection for \( \mathcal{F}(X_p) \) is \( Q \)-maximal for some finite \( Q \subset X_p \).

**Proof.** Let \( f \) be a continuous selection for \( \mathcal{F}(X_p) \). According to [2, Lemma 2.9], we have \( f(X_p) = q \neq p \). Then, by [9, Lemma 6.2], there exists a finite \( Q \subset X_p \) such that \( p \notin Q \) and \( f(S) = q \) for every \( S \in \mathcal{F}(X_p) \) with \( Q \subset S \). That is, \( f \) is \( Q \)-maximal. \( \square \)

**Theorem 2.4** should be compared with [18, Theorem 1]. Namely, by this theorem we have the following consequence.

**Corollary 2.7.** For a space \( X \) which has a continuous selection for \( \mathcal{F}(X) \), the following conditions are equivalent:

(a) \( X \) has finitely many continuous selections.

(b) There exists a finite set \( P \subset X \) such that every continuous selection for \( \mathcal{F}(X) \) is \( P \)-maximal.

We conclude this section characterizing the locally connected spaces by means of extreme selections for their hyperspaces. To this end, let us make the following agreements. For a topological space \((X, T)\) we denote by \( \mathcal{F}(X, T) \) the set of all non-empty closed subsets of \((X, T)\), and by \( \tau_V(T) \) the corresponding Vietoris topology on \( \mathcal{F}(X, T) \). We consider the following relation of a partial order on the set of all possible Hausdorff
topologies on a set $X$. Namely for topologies $\mathcal{T}$ and $\mathcal{T}_0$ on $X$ we shall write that $\mathcal{T}_0 \ll \mathcal{T}$ if and only if

(i) $\mathcal{T}_0 \subset \mathcal{T}$;

(ii) $f|\mathcal{F}(X, \mathcal{T}_0)$ is a $\tau_{\mathcal{V}(\mathcal{T}_0)}$-continuous selection for $\mathcal{F}(X, \mathcal{T}_0)$, whenever $f$ is a $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selection for $\mathcal{F}(X, \mathcal{T})$.

We shall say that a space $X$ is orderable if there exists a linear order $<$ on $X$ such that the topology of $X$ coincides with the interval topology on $X$ generated by this order. Let us recall that the interval topology on $X$ is generated by all sets of the form

$$(-\infty, x) = \{z \in X : z < x\}, \quad (x, +\infty) = \{z \in X : x < z\}$$

and

$$(x, y) = \{z \in X : x < z < y\},$$

where $x, y \in X$ and $x < y$.

**Theorem 2.8.** For a topological space $(X, \mathcal{T})$ which has a $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selection for $\mathcal{F}(X, \mathcal{T})$, the following conditions are equivalent:

(a) $(X, \mathcal{T})$ is locally connected.

(b) There exists a weakly clopen discrete subset $P$ of $(X, \mathcal{T})$ such that every $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selection for $\mathcal{F}(X, \mathcal{T})$ is $P$-maximal, and $\mathcal{T}$ is the “$\ll$”-minimal topology on $X$ with this property.

**Proof.** (a) $\Rightarrow$ (b) The extreme property of the $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selections for $\mathcal{F}(X, \mathcal{T})$ follows by Theorem 2.4. Let us show that $\mathcal{T}$ is the “$\ll$”-minimal topology on $X$ with this property. Suppose a topology $\mathcal{T}_0$ on $X$ is as in (b), and $\mathcal{T}_0 \ll \mathcal{T}$. According to Theorem 2.4, the set $\mathcal{C}_0$ of all connected components of $(X, \mathcal{T}_0)$ is a clopen discrete cover of $(X, \mathcal{T}_0)$. Let $\mathcal{C}$ be the set of all connected components of $(X, \mathcal{T})$. Since, $\mathcal{T}_0 \subset \mathcal{T}$, we now have that $\mathcal{C}$ is a refinement of $\mathcal{C}_0$. Hence, every component $K \in \mathcal{C}_0$ of $(X, \mathcal{T}_0)$ is a discrete sum of components of $(X, \mathcal{T})$. We claim that $\mathcal{C} = \mathcal{C}_0$. Indeed, take $K \in \mathcal{C}_0$ and let $\mathcal{C}_K = \{C \in \mathcal{C} : C \subset K\}$. By Theorem 1.1, every $\tau_{\mathcal{V}(\mathcal{T}_0)}$-continuous selection for $\mathcal{F}(K, \mathcal{T}_0)$ is point-maximal and $\mathcal{F}(K, \mathcal{T}_0)$ has at most two $\tau_{\mathcal{V}(\mathcal{T}_0)}$-continuous selections. Since $\mathcal{T}_0 \ll \mathcal{T}$, this implies that $\mathcal{F}(K, \mathcal{T})$ has at most two different selections. Hence, by Theorem 2.1 and Corollary 2.3, $|\mathcal{C}_K| \leq 2$. Let us see that $|\mathcal{C}_K| = 1$. Suppose, in the opposite, that $|\mathcal{C}_K| = 2$. Take a $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selection $f$ for $\mathcal{F}(X, \mathcal{T})$, and let $\leq$ be the order-like relation on $X$ generated by $f$, see (1). Then, by [12, Lemma 3.3], we may assume that $\mathcal{C}_K = \{L, R\}$, where

$$x < y \quad \text{for every } x \in L \text{ and } y \in R. \quad (2)$$

Since both sets $L, R$ are clopen in $(X, \mathcal{T})$, there exists another $\tau_{\mathcal{V}(\mathcal{T})}$-continuous selection $g$ for $\mathcal{F}(X, \mathcal{T})$ such that

$$g|\mathcal{F}(L, \mathcal{T}) = f|\mathcal{F}(L, \mathcal{T}) \quad \text{and} \quad g|\mathcal{F}(R, \mathcal{T}) = f|\mathcal{F}(R, \mathcal{T}), \quad (3)$$

while the order-like relation $\preceq$ generated by $g$ has the property that

$$y < x \quad \text{for every } x \in L \text{ and } y \in R. \quad (4)$$
Now, we accomplish the proof in the following way. Since \(|C_k| = 2\) and \(F(K, T_0)\) has at most two different \(\tau_{V(T_0)}\)-continuous selections, \(T_0 \prec T\) implies that every \(C \in C_k\) has exactly one \(\tau_{V(T)}\)-continuous selection for \(F(C, T)\). Then, by [19, Theorem 5], both spaces \((L, T)\) and \((R, T)\) are orderable with respect to "\(\leq\)" and each of them is either a singleton or has no last point. On the other hand, both maps \(f|F(X, T_0)\) and \(g|F(X, T_0)\) are \(\tau_{V(T_0)}\)-continuous selections for \(F(X, T_0)\). Hence, by [19, Theorem 1], \((K, T_0)\) is compact because it has exactly two continuous selections for its hyperspace and, in particular, is orderable with respect to both "\(\leq\)" and "\(<\)". Since \(K = L \cup R\), we get by (2), (3) and (4) that both \(L\) and \(R\) must have last points. Hence, both \(L\) and \(R\) are singletons. Therefore, \(K\) is a connected subset of \((X, T_0)\) such that \(|K| = 2\) which is clearly impossible. The contradiction so obtained demonstrates that \(C = C_0\).

We finally prove that \(T = T_0\) showing that they coincide on the members of \(C\). Namely take \(C \in \mathcal{C}\). Since, by hypothesis, \(C\) is locally connected with respect to the topology \(T\), it follows by [19, Theorem 4] that \((C, T)\) is orderable with respect to the linear order \(\leq\) on it generated by \(f\). On the other hand, the topology \(T_0\) on \(C\) contains the interval topology generated by \(\leq\) because \((C, T_0)\) is connected, see [16]. Hence, \(T | C \subset T_0 | C\).

(b) \(\Rightarrow\) (a) Let \(T\) be as in (b). Then, by Theorem 2.4, there exists a discrete family \(C \subset F(X, T)\) of connected subsets of \(X\) such that \(X = \bigcup \mathcal{C}\). Take a \(\tau_{V(T)}\)-continuous selection \(f\) for \(F(X, T)\), and let \(\leq_{V}\) be the order-like relation on \(X\) generated by \(f\). Next, for every \(C \in \mathcal{C}\), let \(T_C\) be the interval topology on \(C\) generated by "\(\leq\)". Then, by a result of [16], \(T_C \subset T\) for every \(C \in \mathcal{C}\). Let us consider the topology \(T_0\) on \(X\) which has as a base the family \(\bigcup \{T_C: C \in \mathcal{C}\}\). As a result, we have \(T_0 \subset T\). Let us check that \(T_0 \prec T\). Take a \(\tau_{V(T)}\)-continuous selection \(g\) for \(F(X, T)\). To show that \(g|F(X, T_0)\) is \(\tau_{V(T_0)}\)-continuous, take \(S \in F(X, T_0)\) and let \(C_S = \{C \in \mathcal{C}: C \cap S \neq \emptyset\}\). Also, take \(Y \in C_S\) with \(s = g(S) \in Y\). Then, either \(s = \min S \cap Y\) or \(s = \max S \cap Y\). Indeed, by Lemma 2.2, \(\langle C_S \rangle\) is a \(\tau_{V(T)}\)-connected subset of \(F(X, T)\). On the other hand, \(S \in g^{-1}(Y) \cap \langle C_S \rangle\). Hence, \(\langle C_S \rangle \subset g^{-1}(Y) = \tau_{V(T)}\) clopen set. Then let \(A = S \cap Y \in F(X, T)\) and \(B = \min S \cap Y = g(S) = h_A(T)\). Next, we consider the \(\tau_{V(T)}\)-continuous selection \(h_A\) for \(F(Y, T)\) defined by \(h_A(T) = g(T \cup A)\) for every \(T \in F(Y, T)\). Then, by a result of [16], we have either \(h_A(T) = \min T\) for every \(T \in F(Y, T)\) or \(h_A(T) = \max T\) for every \(T \in F(Y, T)\). Hence, in particular, \(g(S) = h_A(B) = \max S \cap Y\) or \(g(S) = h_A(B) = \min S \cap Y\).

Finally, let us show that \(g\) is \(\tau_{V(T_0)}\)-continuous at \(S\). Suppose, for instance, that \(s = g(S) = \min S \cap Y\).

First, we will define a special basic \(\tau_{V(T_0)}\)-neighbourhood of \(B = S \cap Y\) in the following way:

(i) In case \(B\) is a singleton, take an arbitrary \(U \in T_0\) with \(s \in U \subset Y\) and then set \(\mathcal{V} = \{U\}\).

(ii) In case \(B\) is not a singleton, there exists a point \(y \in B\) with \(s \neq y\). Hence, \(s \prec y\) and there are intervals \(U, H \in T_0\) such that \(s \in U \subset Y\), \(y \in H \subset Y\), and \(x \prec t\) for every \(x \in U\) and \(t \in H\). In this case, take \(\mathcal{V} = \{U, H, (s, +\infty) \cap Y\}\).

Having already chosen such \(\mathcal{V}\) and \(U\), we proceed to the verification of \(\tau_{V(T_0)}\)-continuity of \(g\) in \(S\). Namely, if \(A = S \cap Y = \emptyset\) we merely have \(g(\mathcal{V}) \subset U\). Indeed, \(\mathcal{V} = \{U\}\).
implies \( f(T) \in T \subset U \) for every \( T \in \mathcal{V} \). In case \( B \) is not a singleton, (5) implies that 
\[ g(T) = \min T \] for every \( T \in \mathcal{V} \). Hence, by (ii), \( T \in \mathcal{V} \) implies 
\[ g(T) = \min T \in U \] because of the special choice of \( \mathcal{V} \).

Suppose now that \( A \neq \emptyset \). By the \( \tau_{V(T)} \)-continuity of \( g \), there exists a a finite family 
\( G \subset T \) of subsets of \( \bigcup \mathcal{C}_s \) such that \( S \in \langle \mathcal{G} \rangle \), \( g(\langle \mathcal{G} \rangle) \subset U \) and \( \{G \cap Y : G \in \mathcal{G}\} \) refines \( \mathcal{V} \).

Set \( \mathcal{C}_A = \mathcal{C}_S \setminus \{Y\} \), \( Z = \bigcup \mathcal{C}_A \) and \( W = \{G \in \mathcal{G} : G \cap Z \neq \emptyset\} \). Next, for every \( C \in \mathcal{C}_A \) pick a finite subset \( Q_C \subset C \) such that \( Q_C \in \{W \in \mathcal{W} : C \cap W \neq \emptyset\} \). Also, for every \( W \in \mathcal{W} \) pick a fixed \( C_W \in \mathcal{C}_A \) such that \( C_W \cap W \neq \emptyset \), and then let \( C_0 = C_A \setminus \{C_W : W \in \mathcal{W}\} \). Finally, define
\[ U = \begin{cases} \mathcal{V} \cup \mathcal{C}_A \cup \{\bigcup C_0\} & \text{if } \bigcup C_0 \neq \emptyset, \\ \mathcal{V} \cup \mathcal{C}_A & \text{otherwise.} \end{cases} \]

We claim that \( g(\langle \mathcal{U} \rangle) \subset U \). To prove this, we apply the same trick as above. Namely, take \( T \in \langle \mathcal{U} \rangle \) and then set \( \mathcal{C}_T = \{C \in \mathcal{C}_A : C \cap T \neq \emptyset\} \), \( Q_T = \bigcup \{Q_C : C \in \mathcal{C}_T\} \) and \( R = T \cap A \). Then, note that \( L = B \cup Q_T \in \mathcal{G} \) because \( Q_T \subset \bigcup \mathcal{V} \) and \( Q_T \cap W \supset Q_C \cap W \neq \emptyset \) for every \( W \in \mathcal{W} \). Hence, we have \( g(L) \in U \subset Y \). On the other hand, \( Q_T \in \langle \mathcal{C}_T \rangle \) while, by Lemma 2.2, \( \langle \mathcal{C}_T \rangle \) is \( \tau_{V(T)} \)-connected set. Then, \( g^{-1}(Y) \subset \langle \mathcal{C}_T \rangle \) because \( g^{-1}(Y) \) is \( \tau_{V(T)} \)-clopen. Now, we use the following "connected" way to our set \( T \). Namely, for every \( F \in \langle \mathcal{C}_T \rangle \) we have a \( \tau_{V(T)} \)-continuous selection \( k_F \) for \( F \in \mathcal{Y}(Y, T) \) defined by \( k_F(D) = g(D \cup F) \), \( D \in \mathcal{F}(Y, T) \). Since \( \mathcal{F}(Y, T) \) has at most two different \( \tau_{V(T)} \)-continuous selections, the set
\[ \{k_F(B) : F \in \langle \mathcal{C}_T \rangle\} = \{g(F \cup B) : F \in \langle \mathcal{C}_T \rangle\} \]
is at most two-point set. On the other hand, it is connected because so is \( \langle \mathcal{C}_T \rangle \). Hence, it is a singleton and, more precisely, it is just equal to \( \{s\} \) because \( g(L) \in U \) and \( L \cap Y = B \), see (5). Thus, in particular, \( g(R \cup B) = s \). This reduces our proof to one of the cases (i) and (ii) of the choice of \( \mathcal{V} \). Namely, the map \( k_R \) is a \( \tau_{V(T)} \)-continuous selection for \( F(Y, T) \). If \( \mathcal{V} = \{U\} \) then we immediately have \( g(T) = k_R(T \cap Y) \in T \cap Y \subset U \). If \( \mathcal{V} \) is as in (ii), by (5), we have \( k_R(B) = \min B \). Hence, \( k_R(D) = \min D \) for every \( D \in \mathcal{F}(Y, T) \). Then, because of the special choice of the elements of \( \mathcal{V} \) we have \( g(T) = k_R(T \cap Y) = \min T \cap Y \in U \). Thus, \( g \) is \( \tau_{V(T)} \)-continuous.

To finish the proof, it only remains to observe that, by (b), \( T_0 \ll T \) implies that \( T_0 = T \). Indeed, by Theorem 2.4, the topology \( T_0 \) is just like in (b) with respect to the \( \tau_{V(T_0)} \)-continuous selections for \( \mathcal{F}(X, T_0) \). On the other hand, \( T_0 \) is a locally connected topology on \( X \) (see, for instance, [12, Theorem 4.1]). \( \Box \)

3. Continuous selections and countable spaces

In this section, we first describe all countable spaces which have a selection for their hyperspaces. To this end, for a space \( X \), let
\[ \mathcal{F}_2(X) = \{S \in \mathcal{F}(X) : |S| \leq 2\}. \]

In what follows, we shall say that a space \( X \) is weakly orderable if there exists a linear order \( \prec \) on \( X \) such that, for any point \( x \in X \), the sets \((-\infty, x)\) and \((x, +\infty)\) are open (i.e., if the topology of \( X \) contains the corresponding interval topology on \( X \) generated by \( \prec \)).
Our first result concerns selections for \( F_2(X) \) and provides a partial answer to [17, Question].

**Theorem 3.1.** Let \( X \) be a countable space. Then, \( X \) is weakly orderable if and only if \( F_2(X) \) has a continuous selection.

**Proof.** The case \( X \) is weakly orderable is settled in [17]. Suppose \( F_2(X) \) has a continuous selection \( f \), and let \( T \) be the topology of \( X \). Also, let \( \leq \) be the order-like relation on \( X \) generated by \( f \), see (1). Note that \( \leq \) may fail to be a transitive order on \( X \).

In what follows, we shall say that a subset \( B \subset X \) is an *interval* in \( X \) if there are points \( x, y \in X \) such that \( x \leq y \) and

\[
(x, y) = \{ z \in X : x < z < y \} \subset B \subset [x, y] = \{ z \in X : x \leq z \leq y \}.
\]

Also, to any interval \( B \) in \( X \) we shall associate points \( \ell(B), r(B) \in X \) such that

\[
\ell(B) \leq r(B) \quad \text{and} \quad (\ell(B), r(B)) \subset B \subset [\ell(B), r(B)].
\]

Now, set \( B = \{ B \in T : B \text{ is an interval in } X \} \). Thus, we get a countable family \( B \) because the set \( \{ \ell(B), r(B) : B \in B \} \subset X \) is countable. Let us show that, for every point \( z \in X \), there exists a subset \( B(z) \subset B \) such that

1. \( \bigcap B(z) = \{ z \} \),
2. whenever \( B \in B(z) \), there is \( B' \in B(z) \) with \( [\ell(B), r(B)] \cap [\ell(B'), r(B')] \subset B \).

To this end, take a non-isolated point \( z \in X \). We have the following two cases:

1. Both sets \((x < z < y) \}= \{x \in X : x < z \} \) and \([z, +\infty) = \{x \in X : z \leq x \} \) are not open. In this case, set \( B(z) = \{ (x, y) : x < y \in (x, z) \} \). Then, \( B(z) \subset B \) and clearly (i) holds if \( B(z) \neq \emptyset \). To show that \( B(z) \) is as required in (i) and (ii), let \( x, y \in X \) be such that \( x < y \in (x, z) \) (it is allowed \( x = -\infty \) or \( y = +\infty \)). First, we show that there exists \( x' \in (x, z) \) with \( x' < y \). Suppose in the opposite that \( y < x' \) for every \( x' \in (x, z) \). Then, we have \((x, y) \cap (\infty, z) = \emptyset \) and, as a consequence, \([z, +\infty) = (x, y) \cup (z, +\infty) \)

In particular, this implies that \([z, +\infty) \) is open which is impossible. Thus, there exists a point \( x' \in (x, z) \) with \( x' < y \). We apply the same arguments to get that there exists a point \( y' \in (z, y) \cap (x', y) \cap (x, z) \) with \( x' < y' \)

2. One of the sets \((\infty, z) \) or \([z, +\infty) \) is open, say this is \((\infty, z) \). Then, for every \( x \in (\infty, z) \), the set \( (x, z) \) is open. Indeed, \( (x, z) = (x, +\infty) \cap (\infty, z) \) for every \( x \in (\infty, z) \). Then, \( B(z) = \{ (x, z) : x < z \} \) is as required because \( x' \in (x, z) \) implies \( (x, z) \cap [x', z] = (x, z) \cap [x', z] \subset (x, z) \). Finally, for an isolated point \( z \in X \) we set \( B(z) = \{ z \} \).
Now, let $B_0 = \bigcup\{\bigcap B' : B' \subset B \text{ is finite}\}$. Then, $B_0$ is a countable family of open subnets of $(X, T)$ which is a base for some topology $T_0$ on $X$. Note that, by (1) and (2), $(-\infty, x), (x, +\infty) \in T_0$ for any point $x \in X$. Hence for every two points $x, y \in X$, with $x \leq y$, the interval $[x, y]$ is closed in $(X, T_0)$. Therefore, by (i) and (ii), $T_0$ is a regular topology on $X$. According to the Urysohn’s metrization theorem [20], this implies that $(X, T_0)$ is metrizable. Since it is also separable and zero-dimensional, it is orderable [15]. That is, $X$ is weakly orderable because $T_0 \subset \mathcal{T}$ which completes the proof. \hfill \Box

Our next result demonstrates the difference between the countable spaces $X$ which have a continuous selection for $\mathcal{F}_2(X)$ and those which have a continuous selection for $\mathcal{F}(X)$.

**Theorem 3.2.** Let $X$ be a countable space which has a continuous selection for $\mathcal{F}_2(X)$. Then, it is a scattered space.

**Proof.** In fact, we just repeat the proof of [9, Theorem 6.1] having in mind that our space $X$ may fail to be regular. Namely, suppose that $X$ is not a scattered space. Hence, it contains a closed subset $Z \in \mathcal{F}(X)$ which is dense in itself. By hypothesis, there exists a continuous selection $f$ for $\mathcal{F}(Z)$ because $\mathcal{F}(X)$ has this property. We restrict our attention only to $Z$ showing that this is impossible. Let $\mathcal{T}$ be the topology of $Z$. As it was shown in Theorem 3.1, $\mathcal{T}$ contains a metrizable topology $T_0$ on $Z$. Hence, in particular, $(Z, \mathcal{T})$ is totally disconnected because $(Z, T_0)$ is zero-dimensional as a countable metrizable space. Hence, we may arrange the same construction as in the proof of [9, Theorem 6.1]. Briefly, let $Z = \{z_1, z_2, \ldots\}$. Since $Z$ is dense in itself, any non-empty clopen subset of $Z$ has no isolated points. On the other hand, it is totally disconnected. Then, repeating precisely the proof of [9, Theorem 6.1], for every $n \geq 0$ there is a non-empty finite subset $F_n$ of $Z$ and a clopen subset $A_n \subset Z$ such that

(i) $F_n \subset F_{n+1} \subset A_{n+1} \subset A_n$,

(ii) $f(S) \neq z_{n+1}$, whenever $S \in \mathcal{F}(Z)$ with $F_{n+1} \subset S \subset A_{n+1}$.

Set $B = \bigcap\{A_n : n = 1, 2, \ldots\}$. By (i), we get a non-empty closed subset $B$ of $Z$ such that $B \neq F_n \subset B \subset A_n$ for every $n \geq 1$. Hence, by (ii), $f(B) \neq z_n$ for every $n$ which is clearly impossible. \hfill \Box

Theorem 3.2 should be compared with a result of [13] that a regular space $X$ is hereditarily Baire provided it has a continuous selection for $\mathcal{F}(X)$. Note that there are countable non-regular spaces $X$ with continuous selections for $\mathcal{F}(X)$, while there are countable scattered spaces $X$ without any continuous selection for $\mathcal{F}(X)$, see Examples 3.6 and 3.7.

The rest of the section is devoted to the following further characterization of a class of countable spaces by continuous selections for their hyperspaces.

**Theorem 3.3.** For a countable space $X$ the following two conditions are equivalent:

(a) $X$ is a metrizable scattered space.

(b) For every point $p \in X$ there exists a continuous $p$-maximal selection for $\mathcal{F}(X)$. 

### Note

- **1** Objects and operations in the text are denoted with a combination of a letter, a number, and a symbol or a combination of these. For example, $B_0 = \bigcup\{\bigcap B' : B' \subset B \text{ is finite}\}$. This notation is used to emphasize the relationship between different elements or operations. However, in some cases, the notation might be more extensive or complex than necessary for the context. The goal is to ensure that the notation is clear, concise, and comprehensible. 

- **2** The use of symbols or mathematical expressions in the text is consistent with the conventions of the field. For example, $\mathcal{T}$ represents a topology, $\mathcal{F}_2$ represents the hyperspace of a collection of two subsets of a given set $X$, and $\mathcal{F}(X)$ represents the hyperspace of subsets of $X$. These notations are standard in topology and related fields, and their use helps in maintaining a clear and consistent exposition of the results. 

- **3** The text includes examples and theorems that are illustrative of the concepts discussed. For instance, **Theorem 3.2** states that if a countable space has a continuous selection for its hyperspace of two-element subsets, then it is a scattered space. This theorem is an important result that highlights the relationship between the existence of continuous selections and the scattered property of a space. 

- **4** The proofs of the theorems are detailed and provide a rigorous foundation for the conclusions. For example, the proof of **Theorem 3.2** involves constructing a closed subset of $Z$ that is dense in itself and showing that it cannot have a continuous selection, thereby proving the scattered property of the space. 

- **5** The text includes references to previous works, such as [9, Theorem 6.1] and [13], which are cited to provide context and support the results presented. These references are essential for readers who wish to explore the background and the historical development of the concepts discussed. 

- **6** The notation used in the text is consistent with the conventions of the field. For example, $[x, y]$ represents an interval, and $\mathcal{F}(X)$ denotes the hyperspace of subsets of $X$. These notations are standard in topology and related fields, and their use helps in maintaining a clear and consistent exposition of the results.
To prepare for the proof of Theorem 3.3, we need the following two results which may have some independent interest.

**Lemma 3.4.** Let \( X \) be a space, and let \( p \in X \) be such that \( \mathcal{F}(X) \) admits a continuous \( p \)-maximal selection. Then, for every neighbourhood \( V \) of \( p \) there exists a neighbourhood \( U \) of \( p \) with \( \overline{U} \subset V \).

**Proof.** Let \( f \) be a continuous \( p \)-maximal selection for \( \mathcal{F}(X) \), and let \( V \) be a neighbourhood of \( p \). In case \( V = X \), merely take \( U = V \). Otherwise, set \( F = X \setminus V \). Since \( f \) is \( p \)-maximal and \( F \) is a non-empty closed subset which does not contain \( p \), we now have \( f(F) \neq f(F \cup \{ p \}) = p \). Since \( X \) is Hausdorff, there exist open sets \( W_F \) and \( W_p \) such that \( f(F) \in W_F \), \( p \in W_p \) and \( W_F \cap W_p = \emptyset \). Since \( f \) is continuous, this implies that \( f^{-1}(W_F) \) and \( f^{-1}(W_p) \) are \( \tau^*_V \)-open subsets of \( \mathcal{F}(X) \) such that \( F \in f^{-1}(W_F) \), \( F \cup \{ p \} \in f^{-1}(W_p) \) and \( f^{-1}(W_F) \cap f^{-1}(W_p) = \emptyset \). Hence, there exist finite families \( W_F \) and \( W_p \) of open subsets of \( X \) such that
\[
F \in \langle W_F \rangle, \quad F \cup \{ p \} \in \langle W_p \rangle \quad \text{and} \quad \langle W_F \rangle \cap \langle W_p \rangle = \emptyset.
\] (6)
Note that there is \( W \in W_p \) such that \( p \in W \) and \( F \cap W = \emptyset \). Then, the set
\[
\mathcal{U} = \{ W \in W_p : W \cap F = \emptyset \}
\] (7)
is such that
\[
p \in \bigcap \mathcal{U} \subset X \setminus \bigcup W_F.
\] (8)
Indeed, suppose there is a point \( q \in (\bigcap \mathcal{U}) \cap (\bigcup W_F) \). Then, \( F \cup \{ q \} \in \langle W_F \rangle \) because \( F \in \langle W_F \rangle \). However, by (7), we also have \( F \cup \{ q \} \in \langle W_p \rangle \) because \( F \cup \{ p \} \in \langle W_p \rangle \). This contradicts (6), so (8) holds. Then, in this case, we may take \( U = \bigcap \mathcal{U} \). \( \square \)

**Theorem 3.5.** Let \( X \) be a separable space, and let \( p \in X \) be such that \( \mathcal{F}(X) \) admits a continuous \( p \)-maximal selection. Then, \( X \) is first countable at \( p \).

**Proof.** Let \( f \) be a continuous \( p \)-maximal selection for \( \mathcal{F}(X) \). Suppose in the opposite that \( X \) is not first countable at \( p \). We are going to get a contradiction. For the purpose, note that there exists a countable subset \( Z \subset X \setminus \{ p \} \) which is dense in \( X \). Let \( \mathcal{K}(Z) \) denotes the non-empty finite subsets of \( Z \). Since \( f \) is \( p \)-maximal, we have \( f(\alpha \cup \{ p \}) = p \neq \alpha \) for every \( \alpha \in \mathcal{K}(Z) \). Since \( X \) is Hausdorff and \( f \) is continuous, for every \( \alpha \in \mathcal{K}(Z) \) there exists a neighbourhood \( V(\alpha) \) of \( p \) and a finite family \( \mathcal{V}(\alpha) \) of open subsets of \( X \) such that
\[
\alpha \in \langle \mathcal{V}(\alpha) \rangle, \quad \alpha \cap V(\alpha) = \emptyset \quad \text{and} \quad f(\langle \mathcal{V}(\alpha) \rangle, V(\alpha)) \subset V(\alpha).
\] (9)
Next, by Lemma 3.4, for every \( \alpha \in \mathcal{K}(Z) \) there exists a neighbourhood \( W(\alpha) \) of \( p \) such that
\[
W(\alpha) \subset V(\alpha).
\] (10)
Thus, we get a family \( \{ W(\alpha) : \alpha \in \mathcal{K}(Z) \} \) of neighborhoods of \( p \) such that
\[
\bigcap \{ W(\alpha) : \alpha \in \mathcal{K}(Z) \} = \{ p \}.
\] (11)
Indeed, suppose if possible that there is a point \( q \in \bigcap \{ W(\alpha); \alpha \in K(Z) \} \) with \( q \neq p \). Since \( f \) is a continuous \( p \)-maximal selection for \( F(X) \), this implies the existence of open sets \( U_q \) and \( U_p \) such that

\[
q \in U_q, \quad p \in U_p, \quad U_q \cap U_p = \emptyset \quad \text{and} \quad f(U_q, U_p) \subset U_p.
\]  

(12)

Next, take an \( \alpha_p \in K(Z) \) with \( \alpha_p \subset U_p \) which is possible because \( Z \) is dense in \( X \). Then, by (12), \( f(\alpha_p \cup \{ q \}) \subset U_p \) while, by (9), \( f(\alpha_p \cup \{ q \}) = q \in U_q \) because, by (10), \( q \in W(\alpha_p) \subset V(\alpha_p) \). The contradiction so obtained demonstrates that (11) holds.

Now, on the other hand, \( K(Z) \) is countable because so is \( Z \). On the other hand, by assumption, \( X \) is not first countable. As a result, the family \( \{ W(\alpha); \alpha \in K(Z) \} \) is not a local base at \( p \) in \( X \). Let \( \preceq \) be an order on \( K(Z) \) as that of the first infinite ordinal \( \omega \). Then, by Lemma 3.4, there exists a neighbourhood \( U \) of \( p \) such that

\[
\bigcap \{ W(\beta); \beta \preceq \alpha \} \setminus U \neq \emptyset \quad \text{for every} \ \alpha \in K(Z).
\]  

(13)

Therefore, for every \( \alpha \in K(Z) \), we may take \( z_\alpha \in Z \) with \( z_\alpha \in \bigcap \{ W(\beta); \beta \preceq \alpha \} \setminus U \). In this way, we get a closed discrete subset \( Z_0 = \{ z_\alpha; \alpha \in K(Z) \} \) of \( X \). Indeed, suppose if possible that there exists \( q \in \overline{B} \setminus B \) for some \( B \subset Z_0 \). Then \( B \subset X \setminus U \) implies \( q \neq p \). On the other hand, according to (13), \( \alpha \in K(Z) \) implies

\[
W(\alpha) \cap Z_0 \supset \{ z_\beta; \alpha \preceq \beta \},
\]

and therefore \( B \subset \{ z_\beta \in B; \beta \prec \alpha \} \cup W(\alpha) \). Since \( \{ z_\beta \in B; \beta \prec \alpha \} \) is finite, we have

\[
q \in \overline{B} \subset \{ z_\beta \in B; \beta \prec \alpha \} \cup W(\alpha).
\]

That is, \( q \in \overline{W(\alpha)} \) for every \( \alpha \in K(Z) \) which, by (11), finally implies \( q = p \). This is however impossible because \( q \neq p \), so \( Z_0 \) is discrete in \( X \).

We now accomplish the proof in the following way. Take an \( \alpha_0 \in K(Z) \), and then set \( \alpha_1 = \alpha_0 \cup \{ z_{\alpha_0} \} \). Next, for every \( n \), define \( \alpha_{n+1} = \alpha_n \cup \{ z_{\alpha_n} \} \). Thus, we get an increasing sequence \( \{ \alpha_n; n < \omega \} \subset K(Z) \) such that

\[
f(\alpha_{n+1}) = z_{\alpha_n} \neq \alpha_n \quad \text{for every} \ \ n < \omega.
\]  

(14)

Indeed, merely note that, by (9) and (10), \( \alpha_{n+1} = \alpha_n \cup \{ z_{\alpha_n} \} \in (V(\alpha_n), V(\alpha_n)) \) and therefore, according to (9) once again, we get just the statement of (14) because \( z_{\alpha_n} \in V(\alpha_n) \) while \( \alpha_n \cap V(\alpha_n) = \emptyset \). The sequence so constructed has also the property that

\[
A = \bigcup \{ \alpha_n; n < \omega \} = \alpha_0 \cup \{ z_{\alpha_n}; n < \omega \} \subset \alpha_0 \cup Z_0.
\]

That is, \( A \subset X \) is closed because \( Z_0 \) is discrete in \( X \) and \( \alpha_0 \) is finite. Then, \( f(A) \in A \) and therefore \( f(A) \in \alpha_n \) for some \( m < \omega \). Let us now recall the special way we have chosen the points \( z_{\alpha_n} \in W(\alpha) \) using the order \( \preceq \) on \( K(Z) \). According to this, we have that \( \{ k < \omega; \alpha_k \preceq \alpha_n \} \) is finite because, by (14), \( \{ \alpha_n; n < \omega \} \subset K(Z) \) is strictly increasing. Hence, there exists an \( \ell < \omega \) such that \( \alpha_m \preceq \alpha_n \) for every \( n > \ell \). This finally implies \( f(A) = \lim_{n \to \infty} f(\alpha_n) = \lim_{n \geq \ell} f(\alpha_n) \in \overline{W(\alpha_m)} \subset V(\alpha_m) \) which, comparing with (9), leads us to the promising contradiction. \( \square \)
Proof of Theorem 3.3. \( (a) \Rightarrow (b) \) Let \( X \) be as in (a). Then, \( X \) is a separable which implies that \( X \) is an absolute \( G_δ \)-set [14]. Hence, in particular, \( X \) is completely metrizable. Also, it is zero-dimensional as a countable space. Therefore, by a result of [4,9], \( F(X) \) admits a continuous selection. Finally, by Theorem 1.2, \( F(X) \) has a continuous \( p \)-maximal selection for every point \( p \in X \).

\((b) \Rightarrow (a)\) Let \( X \) be as in (b). By Theorem 3.2, \( X \) is a scattered space. By Theorem 3.5, \( X \) is first countable. Hence, it is second countable. On the other hand, by Lemma 3.4, \( X \) is a regular space. Therefore, by the Urysohn’s metrization theorem [20], \( X \) is also metrizable. \( \square \)

We conclude this sections with two examples related to the selection problem for countable spaces.

Example 3.6. There exists a non-regular countable Hausdorff space \( X \) which has a continuous selection for \( F(X) \).

Proof. Let \( X = (\omega + 1) \times (\omega + 1) \). As a topological space, we consider \( X \) endowed with the topology generated by the product one plus an additional open set, namely the complement of \( \{(k, \omega) : k < \omega\} \) in \( X \). Thus, we get a Hausdorff space \( X \) which is not regular at \( (\omega, \omega) \). Next, let \( \pi_1, \pi_2 : X \to \omega + 1 \) be the natural projections. Then, a continuous selection \( f \) for \( F(X) \) may be defined by setting for every \( S \in F(X) \) that

\[ f(S) = \left( \min \pi_1(S), \min \pi_2(S \cap \pi_1^{-1} \left( \min \pi_1(S) \right)) \right). \]

In fact, \( f \) is a continuous \((\omega, \omega)\)-minimal selection for \( F(X) \). \( \square \)

Example 3.7 [10]. There exists a countable, first countable scattered Hausdorff space \( X \) without any continuous selection for \( F(X) \).

4. Approaching a point via continuous selections

In this section we are interested in the possible directions to a point \( p \in X \) in a space \( X \) which has a continuous selection for \( F(X) \). Towards this end, we will use a cardinal invariant \( sa(p, X) \) of a space \( X \) in a non-isolated point \( p \in X \), called as a selection approaching number of \( X \) in \( p \), and defined by:

\[ sa(p, X) = \min \{ \kappa : p \in \overline{A} \text{ for some } A \subset X \setminus \{p\} \text{ with } |A| \leq \kappa \} . \]

For technical reasons only, we set \( sa(p, X) = 0 \) provided \( p \) is an isolated point of \( X \).

Note that \( sa(p, X) \) is very similar to the tightness \( t(p, X) \) of \( X \) in \( p \), and always we have \( sa(p, X) \leq t(p, X) \) while the converse is not true. For instance, Arhangel’skii [1] has constructed normal spaces \( X \) and \( Y \) such that \( t(p, X) = t(q, Y) \leq \omega \) for every \( p \in X \) and \( q \in Y \), while \( t((p, q), X \times Y) > \omega \) for some \( (p, q) \in X \times Y \). However, we always have \( sa((p, q), X \times Y) \leq \max \{ sa(p, X), sa(q, Y) \} \).
In the sequel, to every selection $f$ for $\mathcal{F}_2(X)$ and a point $p \in X$ we associate the following sets:

$$L_f(p) = \{ x \in X : f(\{x, p\}) = x \}$$
and

$$R_f(p) = \{ x \in X : f(\{x, p\}) = p \}.$$  

Sometimes, for convenience, we will write $L(p)$ instead of $L_f(p)$ and $R(p)$ instead of $R_f(p)$, respectively. Relying on these sets, we consider a possible “measure” of a left approach to $p$ as $\lambda(p) = sa(p, L(p))$ and a right approach to $p$ as $\rho(p) = sa(p, R(p))$.

Finally, let us agree to call the open sets of a space $X$ also as $G_0$-ones.

**Theorem 4.1.** Let $X$ be a space, $f$ be a continuous selection for $\mathcal{F}_2(X)$, and let $p \in X$. Then $p$ is a $G_\lambda(p)$-set in $L(p)$ and a $G_\rho(p)$-set in $R(p)$.

**Proof.** Our proof is based on a modification of an excellent technique of Eric K. van Douwen [8]. First according to the definition of the sets $L(p)$ and $R(p)$, the continuity of $f$ and the fact that $X$ is Hausdorff, for every point $z \in X \setminus \{p\}$ we can find open subsets $L(z), R(z) \subset X$ such that $z \in L(z), p \in R(z)$ and $L(z) \cap R(z) = \emptyset.$ (15)

$$f(\{L(z), R(z)\}) \subset L(z)$$ provided $z \in L(p), (16)$$

and

$$f(\{L(z), R(z)\}) \subset R(z)$$ provided $z \in R(p).$ (17)

We now find ourselves in one of the following two situations:

(L) Suppose, if possible, that $p$ is not $G_\lambda(p)$ in $L(p)$. By hypothesis, $p \in \bar{A}$ for some $A \subset L(p) \setminus \{p\}$ with $|A| = \lambda(p)$ because $\lambda(p) = sa(p, L(p))$. Note that, by (15),

$$p \in \bigcap \{R(z) : z \in A\}.$$

Since $p$ is not $G_\lambda(p)$ in $L(p)$ while $|A| = \lambda(p)$, this implies the existence of a point $y \in \bigcap \{R(z) \cap L(p) : z \in A\}$ with $y \neq p$. Now, we shall choose another special point. Namely, by (15), $p \in R(y)$. Hence, there exists $z \in R(y) \cap A$ because $p \in \bar{A}$. Thus, we get a two-point set $\{y, z\}$ of $L(p) \setminus \{p\}$ such that, by (15),

$$\{y, z\} \in [L(y), R(y)] \cap [L(z), R(z)]$$ because, by construction,$$

y \in R(z) \quad \text{and} \quad z \in R(y).$$ (19)

Let us try to recognize the point $t = f(\{y, z\}) \in \{y, z\}$. To do that, set $s = \{y, z\} \setminus \{t\}$. By (18) and (16), we have $t \in L(s)$ while, by (19), $t \in R(s)$. That is, $L(s) \cap R(s) \neq \emptyset$ which contradicts (15).

(R) Suppose now that $p$ is not $G_\rho(p)$ in $R(p)$. Next, take a subset $A \subset R(p) \setminus \{p\}$ such that $p \in \bar{A}$ and $|A| = \rho(p) = sa(p, R(p))$. Then, by (15), we get again that

$$p \in \bigcap \{R(z) : z \in A\}.$$
So, just like before, we can find a two points sets \( \{ y, z \} \subset \mathcal{R}(p) \setminus \{ p \} \) satisfying (18) and (19). Let us try to identify the point \( t = f(y, z) \in \{ y, z \}. \) In this case, (18) and (17) imply \( t \in \mathcal{R}(y) \cap \mathcal{R}(z) \subset \mathcal{R}(t) \) while, by (15), \( t \in \mathcal{L}(t). \) Hence, \( \mathcal{L}(t) \cap \mathcal{R}(t) \neq \emptyset \) which again contradicts (15).

The final contradiction so obtained completes the proof. ☐

We now list some possible consequence of this result.

**Corollary 4.2.** Let \( X \) be a space, and let \( p \in X \) be such that \( t(p, X) \leq \omega \). Also, let \( \mathcal{F}_2(X) \) have a continuous selection. Then, \( p \) is a \( G_δ \)-set in \( X \).

**Corollary 4.3.** Let \( X_p \) be a space with only one non-isolated point \( p \in X_p \), and let \( t(X_p) \leq \omega \). The following conditions are equivalent:

(a) \( \mathcal{F}(X_p) \) has a continuous selection.
(b) \( \mathcal{F}_2(X_p) \) has a continuous selection.
(c) \( \{ p \} \) is a \( G_δ \)-set in \( X_p \).

To prepare for our next consequences, we need the following proposition.

**Proposition 4.4.** Let \( X \) be a space which has a continuous selection \( f \) for \( \mathcal{F}(X) \), and let \( T \in \mathcal{F}(X) \) be such that \( f(T) \) is a non-isolated point of \( T \). Then, \( T \) contains a non-trivial convergent sequence.

**Proof.** Let \( A = T \setminus \{ f(T) \} \). Note that for every finite \( F \subset A \) there exists a non-empty finite \( S \subset A \setminus F \) with \( f(F \cup S) \in S \). Indeed, suppose that this fails for some finite \( F \subset A \). Hence, for every finite \( S \subset A \) we have that \( f(F \cup S) \in F \). Since \( \overline{A} = T \), the set \( T \) is an accumulating point of \( \{ F \cup S : S \subset A \) is finite\}. Therefore, \( f(T) \in F \subset A \). However this is impossible because \( f(T) \notin A \). Having already established this, we may complete the proof as follows. By induction, we may construct a sequence \( \{ F_n : n < \omega \} \) of non-empty finite subsets of \( A \) such that, for every \( n < \omega \),

\[
F_n \subset F_{n+1} \quad \text{and} \quad f(F_{n+1}) \in F_{n+1} \setminus F_n.
\]

Then, \( \{ F_n : n < \omega \} \) is an increasing sequence in \( \mathcal{F}(T) \), hence it is \( \tau_Y \)-convergent to \( F = \bigcup \{ F_n : n < \omega \} \). Thus, \( f(F) = \lim_{n \to \infty} F_n \). According to (20), \( \{ f(F_n) : n < \omega \} \) is a non-trivial convergent sequence. ☐

**Corollary 4.5.** Let \( X_p \) be a space with only one non-isolated point such that \( \mathcal{F}(X_p) \) has a continuous \( p \)-maximal selection. Then, \( \{ p \} \) is a \( G_δ \)-set in \( X_p \).

**Proof.** The condition about \( p \)-maximality together with Proposition 4.4 imply that for any \( A \subset X_p \setminus \{ p \} \), with \( p \in \overline{A} \), there exists a countable \( C \subset A \) such that \( p \in \overline{C} \). Hence, \( t(X_p) \leq \omega \) and we may apply Corollary 4.3 to complete the proof. ☐
Corollary 4.6. Let \( X_p \) be a space with only one non-isolated point \( p \). If there exists a continuous selection \( f \) for \( F(X_p) \) such that \( f(T) = p \) for every \( T \subset X_p \) with \( T = T\setminus\{p\} \), then \( X_p \) is a Fréchet–Urysohn space.

5. Weakly orderable spaces with only one non-isolated point

The results of this section should be compared with some results of [7,6].

A family \( U \) of subsets of \( X \) is nested if \( U \subset V \) or \( V \subset U \) for every \( U, V \in U \). In our next considerations, we shall say that a space \( X \) is \( p \)-orderable, where \( p \in X \), if \( X \) has an open nested base at \( p \).

The following observation suggests the right place of the \( p \)-orderable spaces, it is also the main result of this section.

Theorem 5.1. Let \( X_p \) be a space with only one non-isolated point \( p \). Then \( X_p \) is orderable so that \( p \) is the last or, respectively, the first point of \( X_p \) if and only if it is \( p \)-orderable.

To prepare for the proof, we need the following results about discrete spaces.

Proposition 5.2. Let \( S \) be an infinite discrete space. Then the topology of \( S \) is generated by a linear order on \( S \) so that \( S \) has neither a minimal nor a maximal element with respect to this order.

Proof. Let \( |S| = \kappa \). Since \( \kappa \cdot \omega = \kappa \), there exists a disjoint cover \( \{ S_\alpha : \alpha < \kappa \} \) of \( S \) such that \( |S_\alpha| = \omega, \alpha < \kappa \). For every \( \alpha < \kappa \) we consider two disjoint subsets \( S_0^\alpha \) and \( S_1^\alpha \) of \( S_\alpha \) such that \( S_\alpha = S_0^\alpha \cup S_1^\alpha \) and \( |S_0^\alpha| = |S_1^\alpha| = \omega \). Next, whenever \( i = 0, 1 \), let \( \prec_{(\alpha,i)} \) be a well-ordering on \( S_\alpha \) as that of the first infinite ordinal \( \omega \). Then, define an order \( \prec_\alpha \) on \( S_\alpha \) by letting for \( s_0 \neq s_1 \in S_\alpha \) that \( s_0 \prec_\alpha s_1 \) if and only if

(a) \( s_0, s_1 \in S_0^\alpha \) and \( s_1 \prec_{(\alpha,0)} s_0 \), or

(b) \( s_0, s_1 \in S_1^\alpha \) and \( s_0 \prec_{(\alpha,1)} s_1 \), or

(c) \( s_0 \in S_0^\alpha \) and \( s_1 \in S_1^\alpha \).

Clearly, this is a linear order on \( S_\alpha \) which agrees with the topology of \( S_\alpha \). Also, with respect to this order, \( S_\alpha \) has neither a minimal nor a maximal element. Finally, for every \( s \in S \), let \( \alpha(s) < \kappa \) be such that \( s \in S_{\alpha(s)} \). Then, define a linear order \( \prec \) on \( S \) by setting for \( s \neq t \in S \) that \( s \prec t \) if and only if

\[ \alpha(s) < \alpha(t) \] or \[ \alpha(s) = \alpha(t) \text{ and } s \prec_{\alpha(s)} t. \]

This order \( \prec \) satisfies all our requirements. \( \square \)

The following consequence of Proposition 5.2 will be found also useful.

Corollary 5.3. Let \( S \) be a non-empty discrete space. Then the topology of \( S \) is generated by a linear order on \( S \) so that \( S \) has a maximal element with respect to this order.
Proof. If $S$ is finite, then there is nothing to prove. If $S$ is infinite, then $S = S_0 \cup S_1$ where $S_0$ and $S_1$ are disjoint and infinite, and $|S_1| = \omega$. Let $\prec_1$ be a well-ordering on $S_1$ generated by the order of the first infinite ordinal. By Proposition 5.2, the topology of $S_0$ is generated by a linear order $\prec_0$ on $S_0$ so that $S_0$ has neither a minimal nor a maximal element with respect to $\prec_0$. Finally, define a linear order $\prec$ on $S$ by setting for different points $s_0, s_1 \in S$ that $s_0 \prec s_1$ if and only if
(a) $s_0, s_1 \in S_0$ and $s_0 \prec_0 s_1$, or
(b) $s_0, s_1 \in S_1$ and $s_1 \prec_1 s_0$, or
(c) $s_0 \in S_0$ and $s_1 \in S_1$.
This order is as required. \qed

Finally, we need also the following simple observation.

**Proposition 5.4.** Let $\mathcal{N}$ be a nested family of subsets of a space $X$. Then, $\mathcal{N}$ contains a subfamily $\mathcal{B}$ which is well-ordered with respect to the reverse set-theoretical inclusion and is cofinal in $\mathcal{N}$.

**Proof.** Note that $\mathcal{N}_0 = \{V \in \mathcal{N} : V \subseteq U_0\}$ is a cofinal subset of $\mathcal{N}$, where $U_0 \in \mathcal{N}$ is a fixed element of $\mathcal{N}$. Then, let $\mathcal{A}$ be the set of all subset $\alpha \subseteq \mathcal{N}_0$ which are well-ordered with respect to the reverse inclusion and $U_0 \in \alpha$. Consider the natural partial order on $\mathcal{A}$ given by the set-theoretical inclusion. Clearly, $\mathcal{A} \neq \emptyset$ and every linearly ordered part of $\mathcal{A}$ has its union still in $\mathcal{A}$. Then, by the Zorn Lemma, $\mathcal{A}$ has at least one maximal element, say $\mathcal{B}$. This $\mathcal{B}$ is as required. Indeed, suppose in the contrary that there exists $U \in \mathcal{N}_0$ such that $\mathcal{B} \setminus U \neq \emptyset$ for every $\mathcal{B} \in \mathcal{B}$. Since $\mathcal{B} \subseteq \mathcal{N}_0$ and $\mathcal{N}_0$ is nested, we now get that $U$ is a proper subset of $\mathcal{B}$ for every $\mathcal{B} \in \mathcal{B}$. Hence, $\mathcal{B} \cup \{U\} \in \mathcal{A}$ which contradicts the maximality of $\mathcal{B}$. \qed

**Proof of Theorem 5.1.** Let $X_p$ be orderable so that $p$ is, say, the last point of $X_p$. Otherwise we can consider the inverse order on $X_p$, thus to reduce the situation to the previous one. Let $\mathcal{A}$ be the set of those subsets of $X_p \setminus \{p\}$ which are well-ordered with respect to the order of $X_p$. We define a partial order $\prec$ on $\mathcal{A}$ by letting for $A, B \in \mathcal{A}$ that $A \prec B$ if and only if $A \subset B$ and $a < b$ for every $a \in A$ and $b \in B \setminus A$. Note that $\bigcup A_0 \in \mathcal{A}$ for every “$<$”-chain $A_0$ in $\mathcal{A}$. Also, $\mathcal{A}$ is non-empty. So, by the Zorn Lemma, $\mathcal{A}$ has at least one maximal element, say $A \in \mathcal{A}$. This $A$ constitutes a well-ordered net which is convergent to the point $p$. If not, there certainly exists a point $x \in X_p$ such that $a < x < p$ for every $a \in A$. In this case, however, we will get that $A \cup \{x\} \in \mathcal{A}$ which is impossible. We now accomplish the proof setting $B_a = \{x \in X_p : a < x\}$ for every $a \in A$. Then, $\mathcal{B} = \{B_a : a \in A\}$ is an open nested base at $p$, hence $X_p$ is $p$-orderable.

Suppose that $X_p$ is $p$-orderable. Then, by Proposition 5.4, $X_p$ has an open base $\mathcal{B}_p$ at $p$ which is well-ordered with respect to the reverse inclusion. We distinguish the following two cases:

(1) There exists $B_1 \in \mathcal{B}_p$ such that $B_1 \setminus B$ is finite for every $B \in \mathcal{B}_p$. In this case, set $B_1 = \{B \in \mathcal{B}_p : B \subset B_1\}$. Next, note that $|B_1| = \omega$. Indeed, $B_1$ is infinite because it is a
neighbourhood of $p$. On the other hand, it is countable because $B_1$ is well-ordered and $B_1 \setminus B$ is finite for every $B \in B_1$. Then, let $<_1$ be a linear order on $B_1$ as that of $\omega + 1$ and with respect to which $p$ is the last point of $B_1$. Note that this order is compatible with the topology of $B_1$. Let $B_0 = X_p \setminus B_1$. Since $B_0$ is a discrete space, by Corollary 5.3, the topology of $B_0$ is generated by a linear order $<_{0}$ so that it has a maximal element provided $B_0 \neq \emptyset$. Finally, define a linear order $<$ on $X_p$ by letting for every two different points $x_0, x_1 \in X_p$ that $x_0 < x_1$ if and only if

$$x_0, x_1 \in B_i \quad \text{and} \quad x_0 <_i x_1, \quad i = 0, 1, \quad \text{or} \quad x_i \in B_1, \quad i = 0, 1.$$ 

Since $B_1$ has a minimal element, this order preserves the discrete structure on the isolated points of $X_p$. Therefore, it agrees with the topology of $X_p$. So, in this case $X_p$ is orderable.

(2) For every $B \in B_p$ there exists $B' \in B_p$ such that $B \setminus B'$ is infinite. In this case, we may define a nice sequence of elements of $B_p$. Namely, for every neighbourhood $A$ of $p$

$$b(A) = \min\{B \in B_p : B \subset A \text{ and } A \setminus B \text{ is infinite}\}.$$ 

Next, by transfinite induction we define a cofinal sequence $\{B_\alpha : \alpha < \kappa\} \subset B_p$ such that

$$B_0 = b(X_p) \quad \text{and} \quad B_\alpha = b\left(\bigcap\{B_\beta : \beta < \alpha\}\right) \quad \text{provided } \alpha > 0.$$ 

In this way, in fact, we get again a base at the point $p$ and since this point is non-isolated, the ordinal $\kappa$ is limit. Set $S_0 = X_p \setminus B_0$ and $S_\alpha = \bigcap\{B_\beta : \beta < \alpha\} \setminus B_\alpha$ provided $\alpha > 0$. Another very important property of this sequence is that for every $\alpha < \kappa$ the set $S_\alpha$ is an infinite discrete space. Then, by Proposition 5.2, for every $\alpha < \kappa$ there exists a linear order $<_\alpha$ on $S_\alpha$ compatible with the topology of $S_\alpha$ and with respect to which $S_\alpha$ has neither a minimal nor a maximal element. Set, for convenience, $S_\kappa = \{p\}$. Then, for every point $x \in X_p$, let $\alpha(x) = \min\{\alpha \leqslant \kappa : x \in S_\alpha\}$. Finally, define a linear order on $X_p$ such that for every two different points $x, y \in X_p$ we have $x < y$ if and only if

$$\alpha(x) < \alpha(y) \quad \text{or} \quad \alpha(x) = \alpha(y) \quad \text{and} \quad x \prec_\alpha y.$$ 

This order defines the same topology at $p$ while all other points remain isolated. So, it is as required in this case. $\square$

An immediate consequence of Theorem 5.1 is the following result.

**Corollary 5.5.** Let $X_p$ be a space with only one non-isolated point $p$. Then $X_p$ is weakly orderable so that $p$ is the last or, respectively, the first point of $X_p$ if and only if there exists a nested family $B$ of neighborhoods of $p$ such that $\{p\} = \bigcap B$.

Corollary 5.5 suggests the following class of weakly orderable space. We shall say that $X_p$ is weakly $p$-orderable if $X_p$ is weakly orderable so that $p$ is the last or, respectively, the first point of $X_p$. 
6. Selections for weakly orderable spaces with only one non-isolated point

First, let us consider the case of \(p\)-minimal and \(p\)-maximal selections for \(p\)-orderable spaces. To prepare for this, we need some preliminary observations. Let \(X\) be a space and let \(Z \in \mathcal{F}(X)\). Set

\[
\mathcal{F}_X(Z) = \{ F \in \mathcal{F}(X) : F \cap Z \neq \emptyset \}.
\]

**Proposition 6.1.** Let \(X\) be a topological space, \(Z \in \mathcal{F}(X)\), and \(p \in Z\) be such that \(Z \setminus \{p\}\) is open. Define a map \(\varphi : \mathcal{F}_X(Z) \to (\mathcal{F}(Z), \tau_V)\) by \(\varphi(F) = (F \cap Z) \cup \{p\}\) for every \(F \in \mathcal{F}_X(Z)\). Then, \(\varphi\) is continuous with respect to the relative Vietoris topology on \(\mathcal{F}_X(Z)\) as a subspace of \(\mathcal{F}(X)\).

**Proof.** Let \(\langle V \rangle\) be a basic \(\tau_V\)-neighbourhood of \(\varphi(F)\) in \(\mathcal{F}(Z)\), and \(V_0 = \{ V \in \mathcal{V} : V \cap (F \setminus \{p\}) \neq \emptyset \}\). Then, \(U_0 = \{ V \setminus \{p\} : V \in \mathcal{V}_0 \}\) is a family of non-empty open subsets of \(X\) because \(Z \setminus \{p\}\) is open. Set \(V_0 = \bigcap\{ V \in \mathcal{V} : p \in V \}\), and take an open set \(U_0 \subseteq X\) such that \(U_0 \cap Z = V_0\). Next, let \(U_1 = U_0\) if \(p \notin F\) and \(U_1 = U_0 \cup \{U_0\}\) otherwise. Finally, define

\[
U = \begin{cases} U_1 & \text{if } F \subset Z, \\ U_1 \cup \{X \setminus Z\} & \text{otherwise.} \end{cases}
\]

Then \(\langle U \rangle\) is a neighbourhood of \(F\) in \(\mathcal{F}(X)\) such that \(\varphi(S) \in \langle U \rangle\) for every \(S \in \langle U \rangle \cap \mathcal{F}_X(Z)\). Indeed, take an \(S \in \langle U \rangle \cap \mathcal{F}_X(Z)\). According to the definition of \(U\), we now have \(S \cap Z \subseteq \bigcup V\) and \(S \cap V \neq \emptyset\) for every \(V \in \mathcal{V}_0\). Therefore, \(\varphi(S) = (S \cap Z) \cup \{p\} \in \langle V \rangle\) because \(p \in V_0\). \(\square\)

**Corollary 6.2.** Let \(X\) be a space, and let \(Z \in \mathcal{F}(X)\) be a clopen discrete subset of \(X\). Then, \(\mathcal{F}_X(Z)\) has a continuous selection.

**Proof.** Take a point \(p \in Z\), and let “\(<\)” be a well-ordering on \(Z\) such that \(p\) is the last point of \(Z\). Then, \(g(S) = \min S, S \in \mathcal{F}(Z)\), defines a continuous \(p\)-minimal selection for \(\mathcal{F}(Z)\). On the other hand, by Proposition 6.1, \(\varphi(F) = (F \cap Z) \cup \{p\}\), \(F \in \mathcal{F}_X(Z)\), defines a continuous map \(\varphi : \mathcal{F}_X(Z) \to (\mathcal{F}(Z), \tau_V)\). So, \(f = g \circ \varphi : \mathcal{F}_X(Z) \to Z\) is a continuous map which is a selection for \(\mathcal{F}_X(Z)\) because \(g\) is a \(p\)-minimal selection for \(\mathcal{F}(Z)\). \(\square\)

Finally, we also need the following two results.

**Lemma 6.3.** Let \(X_p\) be a space with only one non-isolated point \(p\) which is weakly \(p\)-orderable. Then, \(\mathcal{F}(X_p)\) has a continuous \(p\)-minimal selection.

**Proof.** By Corollary 5.5 and Proposition 5.4, there exists a family \(\mathcal{B}_p\) of neighborhoods of \(p\) which is well-ordered with respect to the reverse inclusion and \(\{p\} = \bigcap \mathcal{B}_p\). For every \(B \in \mathcal{B}_p\) let \(Z_B = X_p \setminus B\). Next for every \(F \in \mathcal{F}(X_p)\), with \(F \neq \{p\}\), set

\[
B(F) = \min \{ B \in \mathcal{B}_p : F \cap Z_B \neq \emptyset \}.
\]
Finally, for every $B \in \mathcal{B}_p$ let $Z_B = \{ F \in \mathcal{F}(X_p) : B(F) = B \}$. Note that this defines a disjoint family $\{ Z_B : B \in \mathcal{B}_p \}$ of $\tau_V$-open subsets $Z_B$ of $\mathcal{F}(X_p)$ such that

$$Z_B \subset \mathcal{F}_{X_p}(Z_B), \quad B \in \mathcal{B}_p.$$  \hfill (21)

and

$$\mathcal{F}(X) \setminus \bigcup \{ Z_B : B \in \mathcal{B}_p \} = \{ \{ p \} \}. \hfill (22)$$

Since each $Z_B$ is discrete in $X_p$, by Corollary 6.2, for every $B \in \mathcal{B}_p$, there exists a continuous selection $f_B : \mathcal{F}_{X_p}(Z_B) \to Z_B$ for $\mathcal{F}_{X_p}(Z_B)$. Finally, define a map $f : \mathcal{F}(X_p) \to X_p$ by $f(F) = p$ if $F = \{ p \}$ and $f(F) = f_B(F)$ otherwise. In this way, we get a selection for $\mathcal{F}(X_p)$ which is $p$-minimal. So, it only remains to show that it is continuous. By (21), $f|_{Z_B} = f_B|_{Z_B}$ for every $B \in \mathcal{B}_p$. Hence, by (22), it suffices to show the continuity of $f$ at the point $\{ p \}$ of $\mathcal{F}(X_p)$. However, every selection is continuous at the singletons which completes the proof.  \hfill $\square$

**Lemma 6.4.** Let $X$ be a space, and let $Z, Y \in \mathcal{F}(X)$ be such that $Z \cup Y = X$ and $Z \cap Y = \{ p \}$ for some point $p \in X$. Suppose that $\mathcal{F}(Z)$ has a continuous $p$-minimal selection while $\mathcal{F}(Y)$ has a continuous $p$-maximal selection. Then, $\mathcal{F}(X)$ has a continuous selection.

**Proof.** Let $\ell$ be a continuous $p$-maximal selection for $\mathcal{F}(Y)$, and let $r$ be a continuous $p$-minimal selection for $\mathcal{F}(Z)$. Note that

$$\mathcal{F}_X(Z) \cup \mathcal{F}(Y) = \mathcal{F}(X) \quad \text{and} \quad \mathcal{F}_X(Z) \cap \mathcal{F}(Y) = \{ F \in \mathcal{F}(Y) : p \in F \}.$$  

Also, $\mathcal{F}_X(Z)$ and $\mathcal{F}(Y)$ are closed in $\mathcal{F}(X)$. Then, let $\psi : \mathcal{F}_X(Z) \to \mathcal{F}(Z)$ be defined by $\psi(F) = (F \cap Z) \cup \{ p \}$, $F \in \mathcal{F}_X(Z)$. Hence, by Proposition 6.1, the composition $\rho = r \circ \psi : \mathcal{F}_X(Z) \to Z$ is a continuous selection for $\mathcal{F}_X(Z)$ because $r$ is a $p$-minimal selection for $\mathcal{F}(Z)$. Note that $\rho(F) = p$ for every $F \in \mathcal{F}_X(Z)$ with $F \cap Z = \{ p \}$. Since $\ell$ is a $p$-maximal selection for $\mathcal{F}(Y)$, this finally implies that

$$\rho|_{\mathcal{F}_X(Z) \cap \mathcal{F}(Y)} = \ell|_{\mathcal{F}_X(Z) \cap \mathcal{F}(Y)}.$$  

Then, the map $f : \mathcal{F}(X) \to X$, defined by $f|_{\mathcal{F}_X(Z)} = \rho$ and $f|_{\mathcal{F}(Y)} = \ell$, is a continuous selection for $\mathcal{F}(X)$.  \hfill $\square$

Motivated by Lemma 6.4, we shall say that a pair $\{ Z, Y \}$ of closed subset of $X$ is a $p$-section of $X$, where $p \in X$, if $Z \cup Y = X$ and $Z \cap Y = \{ p \}$.

We are now ready to prove the following theorem.

**Theorem 6.5.** Let $X_p$ be a linear ordered topological space with only one non-isolated point $p$. Then, $\mathcal{F}(X_p)$ has a continuous selection if and only if $X_p$ has a $p$-section $\{ Z_p, Y_p \}$ such that $Z_p$ is $p$-orderable while $Y_p$ is first countable.
Proof. Let \( \{Z_p, Y_p\} \) be a \( p \)-section of \( X_p \), where \( Z_p \) is \( p \)-orderable and \( Y_p \) is first countable. Then, by Lemma 6.4, \( \mathcal{F}(X_p) \) has a continuous selection because, by Lemma 6.3, \( \mathcal{F}(Z_p) \) has a continuous \( p \)-minimal selection while, by Theorem 1.2, \( \mathcal{F}(Y_p) \) has a continuous \( p \)-maximal selection.

Suppose now that \( \mathcal{F}(X_p) \) has a continuous selection. Further, let \( \leq \) be the linear order on \( X_p \), and let

\[
L_p = \{ x \in X_p : x \leq p \} \quad \text{and} \quad R_p = \{ x \in X_p : p \leq x \}.
\]

Note that both \( L_p \) and \( R_p \) are \( p \)-orderable. Hence, by Theorem 5.1 and Proposition 5.4, \( L_p \) has an open base \( L_p \) at the point \( p \) which is well-ordered with respect to the reverse inclusion and \( |L_p| = \chi(L_p) \). In the same way, \( R_p \) has an open base \( R_p \) at the point \( p \) which is well-ordered with respect to the reverse inclusion and \( |R_p| = \chi(R_p) \). We distinguish the following two cases:

1. \( \chi(L_p) = \chi(R_p) \). Since \( L_p \) and \( R_p \) have one and the same cardinality, it follows that the space \( X_p \) admits an open nested base \( B_p \) at \( p \). Hence, by definition, it is \( p \)-orderable and, in this case, we may merely take \( Z_p = X_p \) and \( Y_p = \{ p \} \).

2. Suppose now that \( \chi(L_p) \neq \chi(R_p) \), say \( \chi(L_p) > \chi(R_p) \). Next, take a continuous selection \( f \) for \( \mathcal{F}(X_p) \). Note that \( L_p \) does not contain any non-trivial convergent sequence because \( \chi(L_p) > \omega \). Therefore, by Proposition 4.4,

\[
f(L) \neq p, \quad \text{for every} \quad L \in L_p.
\]

Since \( f \) is \( \tau_V \)-continuous, this implies that, for every \( L \in L_p \), there exists an \( R_L \in R_p \) with

\[
f(L \cup R_L) \neq R_L. \tag{23}
\]

Now, let \( R_L = \{ R_L : L \in L_p \} \). Next, for every \( R \in R_L \), let \( L_R = \{ L \in L_p : R = R_L \} \). Since \( |L_p| \) is a regular cardinal and \( |R_L| \leq |R_p| < |L_p| \), there exists \( T \in R_L \) with \( |L_T| = |L_p| \).

Hence, \( L_T \) is a cofinal part of \( L_p \) and, in particular, a well-ordered base at \( p \) in \( L_p \). This finally implies that

\[
f(T) = \lim_{L \in L_T} f(L \cup T).
\]

According to (23), we now have \( f(T) = p \). In case \( p \) is a non-isolated point of \( R_p \) this implies, by Proposition 4.4, that \( R_p \) contains a non-trivial convergent sequence. On the other hand, by Lemma 6.3, \( \mathcal{F}(R_p) \) has a continuous (\( p \)-minimal) selection. Hence, by Theorem 4.1, \( p \) will be a \( G_\delta \)-point of \( R_p \). That is, always \( R_p \) is first countable because it is \( p \)-orderable. Hence we may take \( Z_p = L_p \) and \( Y_p = R_p \) which completes the proof. \( \square \)

By Theorem 6.5, we now get the following immediate consequence.

**Corollary 6.6.** Let \( X_p \) be a linear ordered topological space with only one non-isolated point \( p \), and let \( \chi(p, X_p) \leq \omega_1 \). Then, \( \mathcal{F}(X_p) \) admits a continuous selection.

We complete this paper with the following slight improvement of Theorem 6.5.
Theorem 6.7. Let $X_p$ be a space with only one non-isolated point $p \in X_p$, and let $\{Z_p, Y_p\}$ be a $p$-section of $X_p$ such that $Z_p$ is $p$-orderable while $Y_p$ is countable. Then, $\mathcal{F}(X_p)$ has a continuous selection if and only if one of the spaces $Z_p$ or $Y_p$ is first countable.

Proof. Note that $X_p$ is weakly $p$-orderable provided $Z_p$ is first countable. Hence, in this case, $\mathcal{F}(X_p)$ has a continuous selection by Lemma 6.3. In case $Y_p$ is first countable, $\mathcal{F}(Y_p)$ has a $p$-maximal selection by Theorem 1.2. Then, by Lemmas 6.3 and 6.4, $\mathcal{F}(X_p)$ has a continuous selection too. Thus, it remains to prove the converse. For this purpose let $f$ be a continuous selection for $\mathcal{F}(X_p)$ and $Z_p$ be not first countable. To show that $Y_p$ is first countable it now suffices, by Theorem 3.5, to show that $\mathcal{F}(Y_p)$ has a continuous $p$-maximal selection. Suppose this fails. Hence, $f|\mathcal{F}(Y_p)$ is not $p$-maximal. Since the non-empty finite subsets of $Y_p$ are dense in $\mathcal{F}(Y_p)$, this implies that every neighbourhood $V$ of $p$ contains a finite non-empty subset $F \subset Y_p \setminus \{p\}$ with

$$f(F \cup \{p\}) \in F.$$  \hfill (24)

Since $Z_p$ is $p$-orderable, by Proposition 5.4, it has an uncountable open base $\mathcal{B}$ at $p$ which is well-ordered with respect to the reverse inclusion and $|\mathcal{B}|$ is a regular cardinal. Hence, in particular, $Z_p$ does not contain any non-trivial convergent sequence. Therefore, by Proposition 4.4, $f(B) \in Z_p \setminus \{p\}$ for every $B \in \mathcal{B}$. Thus, by (24), the $\tau_V$-continuity of $f$ implies the existence of a non-empty finite $F \subset Y_p \setminus \{p\}$ with

$$f(B \cup F) \in Z_p \quad \text{and} \quad f(F \cup \{p\}) \in F_B.$$  \hfill (25)

Let $\mathcal{F} = \{F_B: B \in \mathcal{B}\}$, and let $B_{F} = \{B \in \mathcal{B}: F_B = F\}$. Since $\mathcal{F}$ is countable because so is $Y_p$, this defines a countable partition $\{B_F: F \in \mathcal{F}\}$ of $\mathcal{B}$. Since $\mathcal{B}$ is well-ordered while $|\mathcal{B}|$ is a regular cardinal, there now exists a finite $F_0 \in \mathcal{F}(Y_p)$ such that $B_0 = B_{F_0}$ is still a local base at $p$. For this particular $F_0$ we have, by (25), that $f(F_0 \cup \{p\}) \in F_0$. Hence, there exists $B_0 \in B_0$ such that $f(B \cup \{p\} \cup F_0) \in F_0$. However, by (25), $f(B_0 \cup F_0) \in Z_p$. The contradiction so obtained completes the proof.  \hfill $\blacksquare$

References