# A tour of $M$-part $L$-Sperner families ${ }^{\text {* }}$ 

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#### Abstract

In this paper we investigate common generalizations of more-part and $L$-Sperner families. We prove a BLYM inequality for $M$-part $L$ Sperner families and obtain results regarding the homogeneity of such families of maximum size through the convex hull method. We characterize those $M$-part Sperner problems, where the maximum family size is the classical $\binom{n}{\lfloor n / 2\rfloor}$. We make a conjecture on the maximum size of $M$-part Sperner families for the case of equal parts of size $2^{\ell}-1$ and prove the conjecture in some special cases. We introduce the notion of $k$-fold $M$-part Sperner families, which specializes to the concept of $M$-part Sperner families for $k=1$, and generalize some $M$-part Sperner results to $k$-fold $M$-part Sperner families. We also approach the $M$-part Sperner problem from the viewpoints of graph product and linear programming, and prove the 2-part Sperner theorem using linear programming. This paper can be used as a survey, as in addition to the new results, problems and conjectures, we provide a number of alternative proofs, discuss at length a number of generalizations of Sperner's theorem, and for the sake of completeness, we give proofs to many simple facts that we use.


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## 1. History and exposition

One of the central issues in extremal set theory is Sperner's theorem [34] from 1928, and its generalizations, like the well-known BLYM-inequality. A Sperner family is a family of sets such that none of them is a proper subset of another. Sperner's theorem asserts that a Sperner family on an $n$-element set has at most $\binom{n}{[n / 2\rfloor}$ elements. G.O.H. Katona [21] and D.J. Kleitman [25] discovered independently and almost simultaneously that one can relax the condition of the Sperner theorem while keeping its conclusion. They relaxed the definition of Sperner families to more-part Sperner families, and showed that 2-part Sperner families still have at most $\binom{n}{\lfloor n / 2\rfloor}$ elements.

To be more formal, let $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ be a fixed partition of the underlying set $X$, where $\left|X_{i}\right|=n_{i}$, and $|X|=n_{1}+n_{2}+\cdots+n_{M}=n$. A family $\mathcal{F}$ of subsets of $X$ is called an $M$-part Sperner family if

$$
\forall E, F \in \mathcal{F}, \quad E \varsubsetneqq F \quad \Rightarrow \quad \forall i: F \backslash E \nsubseteq X_{i}
$$

As Sperner families are also $M$-part Sperner families for any $M$-partition of the underlying set, some $M$-part Sperner families reach the size $\binom{n}{\lfloor n / 2\rfloor}$. Around 1965-1966, as we mentioned above, G.O.H. Katona and D.J. Kleitman discovered:

Theorem 1.1. (See 2-part Sperner theorem G.O.H. Katona [21] and D.J. Kleitman [25].) The size of a two-part Sperner family cannot exceed $\binom{n}{\lfloor n / 2\rfloor}$.

Theorem 1.1 gives back Sperner's theorem as a special case when one $X_{i}$ is empty. G.O.H. Katona and D.J. Kleitman found several two-part Sperner families achieving this bound. It took 20 years to characterize all maximum size 2-part Sperner families. This work was completed by P.L. Erdős and G.O.H. Katona using the convex hull method [10]. In 1996, S. Shahriari [33] found an alternative way to describe all maximum size two-part Sperner families through chain decomposition.

In 2002, P.L. Erdős, Z. Füredi and G.O.H. Katona [12] came up with another way of doing the characterization, based on a combination of the permutation and convex hull methods. In 2007, H. Aydinian and P.L. Erdős [2] found the shortest and perhaps the last proof for the characterization of 2-part Sperner families.

In 1973, G.O.H. Katona recognized that there are 3-part Sperner families exceeding the expected $\binom{n}{(n / 2\rfloor}$ bound [23]. For this fact, we will give a new, very simple construction in Example 4.1 in Section 4.

The results cited above allowed the theory to develop in three directions: (a) finding conditions under which the $\binom{n}{\lfloor n / 2\rfloor}$ bound would still hold for $M$-part Sperner families; (b) find the actual maximum size of $M$-part Sperner families; (c) find analogues of the Sperner and 2-part Sperner theorem for other structures.
G.O.H. Katona [23], J. Griggs [16], and J. Griggs and D. Kleitman [18] gave extra conditions to achieve (a). In Section 4 we give a full characterization of the problems belonging to (a) (Theorem 4.5).

In Section 5 we introduce the notion of $k$-fold $M$-part Sperner families, which specializes to the concept of $M$-part Sperner families for $k=1$. We call a family $\mathcal{F}$ on an $M$-partitioned underlying set an $M$-part $k$-fold Sperner family if for every $E, F \in \mathcal{F}$ with $E \varsubsetneqq F$ we have that $F-E$ is not a subset of the union of any $k$ partition classes. We show that the size of a $k$-fold $M$-part Sperner family may exceed $\binom{n}{\lfloor n / 2 \downarrow}$, give a range of $k$ for which the maximum size of families is $\binom{n}{\lfloor n / 2\rfloor}$, and give close upper and lower bounds on the maximum size, when $k$ is outside this range. In a remarkable way, for a range of problems the maximum size $M$-part $k$-fold Sperner families are still exactly all subsets of size $\lfloor n / 2\rfloor$ or of size $\lceil n / 2\rceil$.

Doing (b) proved to be the most difficult. In 1987, P.L. Erdős and G.O.H. Katona [11] determined the maximum size of 3-part Sperner families under the condition that one class is a singleton, using the convex hull method. Even in this case, the size of some families exceeds $\binom{n}{\lfloor n / 2\rfloor}$.

Not having much success at exact results towards (b), researchers tried to bound the ratio of the maximum size of an $M$-part Sperner family and $\binom{n}{\lfloor n / 2\rfloor}$, making no assumptions on class sizes.

To answer this relaxed question is still hard. In 1980 J. Griggs [17] showed that this ratio is at most $2^{M-2}$, and later in 1985 Z. Füredi [13] and independently J.R. Griggs, A.M. Odlyzko, and J.B. Shearer [19] showed that the ratio is at most

$$
\begin{equation*}
(1+o(1)) \sqrt{M}, \tag{1}
\end{equation*}
$$

as $n \rightarrow \infty$; and also proved that this ratio, as $M \rightarrow \infty$, is

$$
\begin{equation*}
(1+o(1)) \sqrt{\frac{\pi M}{4 \log M}} . \tag{2}
\end{equation*}
$$

For (c), Mark Huber in 1994 used the convex hull method to extend the 2-part Sperner theorem to the poset of multisets [20], i.e. to the divisor lattice of a natural number.

Paul Erdős found another generalization of the Sperner theorem, which does not involve more parts:

Theorem 1.2. (See Paul Erdős [7].) Assume that $\mathcal{H}_{i}(i=1,2, \ldots, t)$ are Sperner families on the underlying set $X$, where $|X|=n$. If these families are pairwise disjoint, then

$$
\begin{equation*}
\sum_{i=1}^{t}\left|\mathcal{H}_{i}\right| \leqslant \sum_{i=0}^{t-1}\binom{n}{\left\lfloor\frac{n+1-t}{2}\right\rfloor+i} \tag{3}
\end{equation*}
$$

where the sum includes the largest binomial coefficients of the form $\binom{n}{i}$.
For $0<L \leqslant n$, a family $\mathcal{H}$ is called an $L$-Sperner family, if no $L+1$ sets in $\mathcal{H}$ form a chain for inclusion. A dual Dilworth theorem for posets, attributed to P. Erdős and G. Szekeres (see [26, Ex. 9.32b]), immediately implies that any $L$-Sperner family can be decomposed into the union of $L$ pairwise disjoint Sperner families, therefore Theorem 1.2 gives a sharp upper bound on the size of $L$-Sperner families.

In order to combine the two lines of generalization mentioned so far, assume that the underlying set $X$ is partitioned into $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$, where $\left|X_{i}\right|=n_{i}$. Introducing a new generalization, we say that a family $\mathcal{F}$ is $M$-part $\left(n_{1}, n_{2}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner, if there is no $i$ such that an increasing ( $L_{i}+1$ )-chain of $\mathcal{F}$, say $E_{1} \varsubsetneqq E_{2} \varsubsetneqq \cdots \varsubsetneqq E_{L_{i}+1}$ would have all its growth in a single $X_{i}$ only, i.e. we do not have $E_{L_{i}+1} \backslash E_{1} \subseteq X_{i}$. To avoid degeneracy in this definition, we always assume $0<L_{i} \leqslant n_{i}$. If all $L_{i}$ 's are equal, say $L_{i}=L$, then we speak about an $M$-part $L$-Sperner family.

In Section 3 we extend the convex hull method from $M$-part $L$-Sperner families to $M$-part $\left(n_{1}, n_{2}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner families.

A family of subsets of an $M$-partitioned underlying set is called homogeneous, if for any set, the sizes of its intersections with the partition classes already determine whether the set belongs to the family or not.

In 1986, Z. Füredi, J. Griggs, A.M. Odlyzko, and J.M. Shearer [14], and independently P.L. Erdős and G.O.H. Katona [9] discovered that there must exist homogeneous maximum size $M$-part $L$-Sperner families. We will slightly generalize this result in Section 3. It is an important question, under what conditions all maximum size $M$-part $L$-Sperner families are homogeneous, as the cases $L=1, M=1$ (strong Sperner theorem) and $M=2$ (see [10], Theorem 5.2 in this paper) may suggest this phenomenon. Homogeneous $M$-part $L$-Sperner families can be identified with a set of ordered $M$-tuples of numbers, which list the intersection sizes with classes that occur. We call the sets of ordered Mtuples of numbers corresponding to a homogeneous $M$-part $L$-Sperner family a transversal, and those of them that have the largest possible cardinality a full transversal. Theorem 5.2 of P.L. Erdős and G.O.H. Katona [10] shows that all maximum size 2-part Sperner families correspond to full transversals. However, some maximum size homogeneous $M$-part Sperner families do not correspond to a full transversal. Section 3 has several results and counterexamples on homogeneity and transversals.

Another early generalization of the Sperner theorem is the BLYM (see Bollobás (1965) [5], Lubell (1966) [27], Yamamoto (1954) [35] and Meshalkin (1963) [28]) inequality. If $P_{i}(\mathcal{F})$ denotes the number of $i$-element sets in the Sperner family $\mathcal{F}$ (the vector of these quantities will be called the profile of the family later), then

$$
\begin{equation*}
\sum_{i=0}^{n} \frac{P_{i}(\mathcal{F})}{\binom{n}{i}} \leqslant 1 \tag{4}
\end{equation*}
$$

which immediately implies Sperner's theorem, as the largest denominator is $\binom{n}{\lfloor n / 2\rfloor}$. Surprisingly, the BLYM inequality has a generalization into an identity, see R. Ahlswede and Z. Zhang [1]. Another notable generalization of BLYM is due to C. Bey [4]. The following joint generalizations of the BLYM inequality and Theorem 1.2 has been folklore and was first in print in [12].

Theorem 1.3. (Folklore, see Theorem 2.1 in [12].) Let $\mathcal{F}$ be an $L$-Sperner family on an $n$-element underlying set. Then

$$
\sum_{i=0}^{n} \frac{P_{i}(\mathcal{F})}{\binom{n}{i}} \leqslant L,
$$

with equality if and only if $\mathcal{F}$ contains every set of some $L$ distinct sizes.
For $L=1$, this theorem implies the so-called "strong" or "strict" Sperner theorem, which asserts that the ( $\left.\begin{array}{c}n \\ \lfloor n / 2\rfloor\end{array}\right)$ upper bound can be realized only with all $\lfloor n / 2\rfloor$ sized sets or with all $\lceil n / 2\rceil$ sized sets. Z. Füredi had a one-line "book proof" for the strong Sperner theorem, which was first published in [12].

In Section 6 we prove a BLYM inequality for $M$-part ( $n_{1}, \ldots, n_{M} ; L_{1}, L_{2}, \ldots, L_{M}$ )-Sperner families (Theorem 6.1), and obtain some conditions that imply the homogeneity of all maximum size families. However, we also exhibit non-homogeneous maximum size families for certain $M$-part Sperner problems, even for $L=1$. Our Theorem 6.2 is a direct generalization of the strong Sperner theorem.

In 1971, J. Schonheim in [31] generalized the $L$-Sperner problem of P. Erdős (and also the 2-part Sperner problem, see our earlier reference [20]) to the poset of multisets, i.e. the divisor lattice of a natural number. G.O.H. Katona [22] gave further generalization of the work of J. Schonheim [31].

In 1980, J. Griggs [17] showed that $L 2^{M-2}\binom{n}{(n / 2\rfloor}$ an upper bound on the maximum size of $M$-part L-Sperner families, where the maximization is over all possible class sizes. Later, in 1988, A. Sali [30] obtained an upper bound for antichain sizes in products of symmetric chain orders, which lowered the upper bound to $c L \sqrt{M}\binom{n}{\lfloor n / 2\rfloor}$ with an unspecified $c$.
Z. Füredi, J.R. Griggs, A.M. Odlyzko, and J.M. Shearer [14] determined the maximum size of 2part 2-Sperner families, $2\binom{n}{\lfloor n / 2\rfloor}$, where the maximization is extended over all class sizes, exactly, and found that any maximum size 2-part 2-Sperner family is the union of 2 disjoint 2-part Sperner families. They found that this result does not allow a pleasant direct generalization: $M \geqslant 3, M$-part $L$-Sperner families are not always unions of $L$ disjoint $M$-part 1 -Sperner families. To see this, take $X_{1}=\{1,2\}, X_{2}=\{3,4\}, X_{3}=\{5,6\}$, and consider the 3-part 2-Sperner family

$$
\{\emptyset,[2],[4],[6],\{6\},\{4,6\},\{2,4,6\},\{2,4,5,6\},\{2,3,4,5,6\}\},
$$

whose partition into two 3 -part 1 -Sperner families would yield a good 2-coloration of a 9 -cycle defined by the Hasse diagram of the above sets ordered by inclusion. Therefore they looked for asymptotic results for the maximum size of $M$-part $L$-Sperner families, where the maximization is extended again over all class sizes, and obtained for fixed $M, L$ and $n \rightarrow \infty$ the asymptotics $\phi_{L}(M)\binom{n}{\lfloor n / 2\rfloor}$; and as $M \rightarrow \infty$ the asymptotics $\phi_{L}(M) \sim L \sqrt{\frac{\pi M}{4 \log M}}$, generalizing (1) and (2). In Section 4 we provide a uniform bound (Theorem 4.6) that is very close to the asymptotic results of Z. Füredi, J.R. Griggs, A.M. Odlyzko, and J.M. Shearer [14], in particular, for $L=1$, to (1) and (2), providing a small constant for Sali's result.

In Section 7 we give a conjecture on the maximum size of $M$-part Sperner families for the case of equal parts of size $2^{\ell}-1$; and show that if our conjecture holds, then for these problems all maximum size families are homogeneous. This is the first conjecture for an exact maximum value in a non-trivial instance of the $M$-part Sperner problem. We prove our conjecture for the special cases $\ell=1,2$ and bring the conjectured maximum value to closed form. We also approach the $M$-part Sperner problem from the viewpoints of graph product and linear programming, and give a new proof for the 2-part Sperner theorem, in the case of the partition classes of equal size, using linear programming.

A comprehensive survey of Sperner theory can be found in K. Engel's excellent book, [6].
We are indebted to Lincoln Lu and two anonymous referees for their useful comments.

## 2. Notation

We are going to use shorthand notations as $[n]=\{1, \ldots, n\},[i, j]=\{i, i+1, \ldots, j-1, j\}, 2^{X}$ for the power set of the set $X$, and $\binom{X}{i}$ for the set of all $i$-element subsets of a set $X$. We use the standard notation $\widehat{a}$ to indicate that the term $a$ under the hat is missing. For $L$-problems (in particular for $L=1$ ), we assume that the partition classes $X_{i}$ of the underlying set $X$ are ordered by size, i.e. $n_{1} \geqslant \cdots \geqslant n_{M}$. However, we deviate from this convention when we carry out additive operations on the $n_{i}$ numbers, like in (12), (13), (15).

For $M$-part $\left(n_{1}, n_{2}, \ldots, n_{M} ; L_{1}, L_{2}, \ldots, L_{M}\right)$-Sperner problems, we assume that the $X_{i}$ partition classes of the underlying set $X$ are ordered by

$$
\begin{equation*}
\frac{L_{1}}{n_{1}+1} \leqslant \frac{L_{2}}{n_{2}+1} \leqslant \cdots \leqslant \frac{L_{M}}{n_{M}+1} \tag{5}
\end{equation*}
$$

The first, simpler convention is actually a special case of this more complicated convention.

## 3. Transversals and the convex hull method

This section contains a natural generalization of the convex hull method of [8] and [9] for $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$ Sperner families. Let $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ be a partition of the $n$-element underlying set $X$, where $\left|X_{i}\right|=n_{i} \geqslant 1$ and $n_{1}+\cdots+n_{M}=n$. Let $\mathcal{F} \subseteq 2^{X}$ be a family of sets. The $M$-dimensional matrix $P(\mathcal{F}):=\left(p_{i_{1}, \ldots, i_{M}}\right): i_{j} \in\left[0, n_{j}\right]$ is called the profile-matrix of $\mathcal{F}$, if

$$
p_{i_{1}, \ldots, i_{M}}(\mathcal{F})=\left|\left\{F \in \mathcal{F}: \forall j\left|F \cap X_{j}\right|=i_{j}\right\}\right|
$$

This $P(\mathcal{F})$ can be viewed as a point of the Euclidean space $\mathbb{R}^{N}$, where $N=\left(n_{1}+1\right)\left(n_{2}+1\right) \cdots\left(n_{M}+1\right)$; and when it is convenient, we will consider it as a vector in this space as well.

Let $\alpha \subseteq \mathbb{R}^{N}$ be a finite point set. Let $\langle\alpha\rangle$ denote the convex hull of the point set, and $\varepsilon(\alpha)$ its extreme points. It is well known that $\langle\alpha\rangle$ is equal to the set of all convex linear combinations of its extreme points.

Let $\mathbb{A} \subseteq 2^{2^{X}}$ be a family of families of sets. Let $\mu(\mathbb{A})$ denote the set of all profile-matrices of the families in $\mathbb{A}$, i.e.

$$
\mu(\mathbb{A})=\{P(\mathcal{F}): \mathcal{F} \in \mathbb{A}\}
$$

Then the extreme points $\varepsilon(\mu(\mathbb{A}))$ are integer vectors and they are profile-matrices of families from $\mathbb{A}$.
The knowledge of $\varepsilon(\mu(\mathbb{A}))$ has tremendous impact on solving extremal problems for $\mathbb{A}$. For any linear objective function $w$ whose variables are the entries of profile-matrices, $w$ attains the value $\max \{w(p): p \in \mu(\mathbb{A})\}$ at some point of $\varepsilon(\mu(\mathbb{A}))$. Let $W \subseteq \varepsilon(\mu(\mathbb{A}))$ denote the set of extreme points, where the maximum is attained. The objective function can be, for example, the cardinality of the families of sets, since the cardinality of a family $\mathcal{F}$ is equal to the sum of all entries in its profilematrix, and therefore it is linear. A similar argument applies to the volume of the family, i.e. the total sum of the cardinalities of the elements in the family.

The following simple observations will be useful later:

## Lemma 3.1.

(i) The profile-matrix $P(\mathcal{F})$ of any family $\mathcal{F} \in \mathbb{A}$, which maximizes $w$, is a convex linear combination of profile-matrices of elements of $W$.
(ii) If a linear function $f$ vanishes on all profile-matrices in $W$, then in every point of $\mu(\mathbb{A})$ maximizing $w$, $f$ vanishes as well.
P.L. Erdős and G.O.H. Katona developed in [9] a general method to determine the extreme points in $\varepsilon(\mu(\mathbb{A}))$. To present their results, set $\Pi=\left[0, n_{1}\right] \times \cdots \times\left[0, n_{M}\right]$, and let $I$ denote a subset of $\Pi$. Let $T(I)$ denote the $M$-dimensional 0-1 matrix, in which

$$
T_{i_{1}, \ldots, i_{M}}(I)= \begin{cases}1, & \text { if }\left(i_{1}, \ldots, i_{M}\right) \in I, \\ 0, & \text { if }\left(i_{1}, \ldots, i_{M}\right) \notin I .\end{cases}
$$

Furthermore, let $S(I)$ be the $M$-dimensional matrix, in which

$$
S_{i_{1}, \ldots, i_{M}}(I)= \begin{cases}\binom{n_{1}}{i_{1}} \cdots\binom{n_{M}}{i_{M}}, & \text { if }\left(i_{1}, \ldots, i_{M}\right) \in I,  \tag{6}\\ 0, & \text { if }\left(i_{1}, \ldots, i_{M}\right) \notin I,\end{cases}
$$

and let

$$
I(\mathcal{F}):=\left\{\left(i_{1}, \ldots, i_{M}\right) \in \Pi: P_{i_{1}, \ldots, i_{M}}(\mathcal{F}) \neq 0\right\} .
$$

Recall that a family of subsets of an $M$-partitioned underlying set is called homogeneous, if for any set, the sizes of its intersections with the partition classes already determine whether the set belongs to the family or not. It is easy to see the equivalence of the following two definitions:
(i) a family $\mathcal{F} \subseteq 2^{X}$ is homogeneous if and only if $P(\mathcal{F})=S(I)$ for a certain set $I \subseteq \Pi$,
(ii) a family $\mathcal{F} \subseteq 2^{X}$ is homogeneous if and only if $P(\mathcal{F})=S(I(\mathcal{F}))$.

Homogeneous families will play an important role later in this paper.
We call an $I \subseteq \Pi$ a transversal, if there are no two elements of $I$ differing in exactly one coordinate. If $\mathcal{F}$ is a homogeneous $M$-part Sperner family, then $I(\mathcal{F})$ is a transversal. We call an $I \subseteq \Pi$ an $\left(L_{1}, L_{2}, \ldots, L_{M}\right)$-transversal, if there are no $L_{i}+1$ elements of $I$, any two of them differing from each other only in the $i$ th coordinate, for any fixed $i \in[M]$. If $\mathcal{F}$ is a homogeneous $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, L_{2}, \ldots, L_{M}\right)$-Sperner family, then $I(\mathcal{F})$ is an $\left(L_{1}, L_{2}, \ldots, L_{M}\right)$-transversal.

Lemma 3.2. If I is an $\left(L_{1}, \ldots, L_{M}\right)$-transversal, then for every $i \in[M]$

$$
\begin{equation*}
|I| \leqslant \frac{L_{i}}{n_{i}+1} \prod_{j=1}^{M}\left(n_{j}+1\right) \tag{7}
\end{equation*}
$$

Proof. The statement is almost trivial: deleting the $i$ th coordinate from the elements of $I$ leaves at most $\left(n_{1}+1\right) \cdots\left(n_{M}+1\right) /\left(n_{i}+1\right)$ distinct elements. None of these elements may have more than $L_{i}$ different pre-images, since otherwise $I$ would fail being an ( $L_{1}, \ldots, L_{M}$ )-transversal.

We call an $\left(L_{1}, \ldots, L_{M}\right)$-transversal $I$ (and also the corresponding homogeneous $M$-part ( $n_{1}, \ldots, n_{M}$; $L_{1}, \ldots, L_{M}$ )-Sperner family) full if $I$ shows equality in (7) for some $i$. In another paper with K. Engel [3] we prove that full transversals always exist.
P.L. Erdős and G.O.H. Katona [9] characterized the extreme points of the convex hull of the profilematrices of $M$-part $L$-Sperner families.

Theorem 3.3. (See [9, Theorem 3.6].) If $\mathbb{A}$ denotes the set of $M$-part $L$-Sperner families on $X$, then

$$
\epsilon(\mu(\mathbb{A}))=\{S(I): \text { I is an L-transversal }\} .
$$

Following the arguments of P.L. Erdős and G.O.H. Katona [9], it is not difficult to prove the following slightly more general result.

Theorem 3.4. Let $\mathbb{A}$ denote the set of $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner families on $X$. Then we have:

$$
\epsilon(\mu(\mathbb{A}))=\left\{S(I): \text { I is an }\left(L_{1}, \ldots, L_{M}\right) \text {-transversal }\right\} .
$$

Proof. We say that $\mathfrak{L}$ is a product-chain of $X$, if the ordered $n$-tuple $\mathfrak{L}=\left(x_{1}, \ldots, x_{n}\right)$ is a permutation of $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ such that

$$
X_{j}=\left\{x_{i}: i=n_{1}+\cdots+n_{j-1}+1, \ldots, n_{1}+\cdots+n_{j}\right\}
$$

that is $\mathfrak{L}$ is a juxtaposition of permutations of $X_{1}, X_{2}, \ldots, X_{M}$, in this order. Furthermore, we say that a subset $H \subseteq X$ is initial with respect to $\mathfrak{L}$, if for all $j=1,2, \ldots, M$ we have

$$
H \cap X_{j}=\left\{x_{n_{1}+\cdots+n_{j-1}+1}, \ldots, x_{n_{1}+\cdots+n_{j-1}+\left|H \cap X_{j}\right|}\right\},
$$

i.e. $H \cap X_{j}$ is an initial segment in the permutation of $X_{j}$. If $\mathcal{H} \subseteq 2^{X}$ then $\mathcal{H}(\mathfrak{L})$ denotes those members of $\mathcal{H}$ which are initial with respect to $\mathfrak{L}$. It is clear that the profile-matrix $P(\mathcal{H}(\mathfrak{L}))$ is a 0,1 matrix. Similarly, for $\mathbb{B} \subseteq 2^{2^{X}}$ let $\mathbb{B}(\mathfrak{L}):=\{\mathcal{H}(\mathfrak{L}): \mathcal{H} \in \mathbb{B}\}$. Then clearly $\varepsilon(\mu(\mathbb{B}(\mathfrak{L})))=\mu(\mathbb{B}(\mathfrak{L}))$ holds, in other words, all profile-matrices from $\mu(\mathbb{B}(\mathfrak{L}))$ are extreme points, since no $0-1$ vector is a convex linear combination of other $0-1$ vectors.

In order to find points defined by homogeneous families among the extremal points, the following two results were proved by P.L. Erdős and G.O.H. Katona in [9]:

Theorem 3.5 (Blowing up the product-chain). (See [9, Lemma 3.1].) Suppose that $\mathbb{B} \subseteq 2^{2^{X}}$ such that $\varepsilon(\mu(\mathbb{B}(\mathfrak{L})))=\mu(\mathbb{B}(\mathfrak{L}))$ does not depend on the choice of $\mathfrak{L}$. Then

$$
\begin{equation*}
\mu(\mathbb{B}) \subseteq\langle\{S(I): T(I) \in \varepsilon(\mu(\mathbb{B}(\mathfrak{L}))) \text { for some } I \subseteq \Pi\}\rangle \tag{8}
\end{equation*}
$$

holds, where $\langle\cdots\rangle$ denotes the convex hull.
Theorem 3.6. (See [9, Theorem 3.2].) Suppose that $\mathbb{B} \subseteq 2^{2^{X}}$ satisfies the following two conditions:

$$
\begin{align*}
& \text { the set } \varepsilon(\mu(\mathbb{B}(\mathfrak{L})))=\mu(\mathbb{B}(\mathfrak{L})) \quad \text { does not depend on } \mathfrak{L},  \tag{9}\\
& \text { for all } I \subseteq \Pi, \quad T(I) \in \mu(\mathbb{B}(\mathfrak{L})) \quad \text { implies } \quad S(I) \in \mu(\mathbb{B}) . \tag{10}
\end{align*}
$$

Then

$$
\begin{equation*}
\varepsilon(\mathbb{B})=\{S(I): T(I) \in \mu(\mathbb{B}(\mathfrak{L})) \text { for some } I \subseteq \Pi\} . \tag{11}
\end{equation*}
$$

To finish the proof of Theorem 3.4, recall that $\mathbb{A}$ is the set of $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$ Sperner families on $X$, and let $\mathfrak{L}$ be an arbitrary product-chain. By definition, there are no ( $L_{i}+1$ ) elements in $\mathbb{A}(\mathfrak{L})$ that pairwise differ in $X_{i}$ only. Therefore, for every $\mathcal{F} \in \mathbb{A}, I(\mathcal{F}(\mathfrak{L}))$ forms an $\left(L_{1}, \ldots, L_{M}\right)$-transversal. As $\mathbb{A}$ is invariant to renumbering the elements of $X_{i}, \mu(\mathbb{A}(\mathfrak{L}))$ does not depend on the choice of $\mathfrak{L}$, and condition (9) of Theorem 3.6 holds. Condition (10) also holds, since for any ( $L_{1}, \ldots, L_{M}$ )-transversal $I$ the homogeneous system $S(I)$ belongs to $\mu(\mathbb{A})$. Hence the conclusion of Theorem 3.6, equality (11), also holds, so the extreme points of $\mu(\mathbb{A})$ are the profile-matrices of homogeneous $M$-part ( $n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}$ )-Sperner families.

|  | 0 | 1 | 2 | 3 | 4 | 5 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | 1 |  |  |  |  | 1 |
| 1 |  | 25 | 50 |  | 25 |  |
| 2 |  |  | 100 | 100 | 50 |  |
| 3 |  | 50 | 100 | 100 |  |  |
| 4 |  | 25 |  | 50 | 25 |  |
| 5 | 1 |  |  |  |  | 1 |

Fig. 1. Maximum size homogeneous system without full transversal.

Corollary 3.7. There exist maximum size M-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner families that are homogeneous.

Which families are they? Lemma 3.1(i) guarantees that the profile-matrix of any maximum size $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner family is a convex linear combination of the profile-matrices of some maximum size homogeneous ones. However, maximum size homogeneous systems do not always correspond to full transversals.

Example 3.8. There are maximum size homogeneous systems that do not correspond to full transversals.

Consider the following example: $n_{1}=n_{2}=5$ and $L_{1}=L_{2}=3$. Full transversals contain 18 elements, but (one of) the maximum size homogeneous system's profile-matrix has only 16 entries (see Fig. 1).

## 4. A new construction and two new results on $\boldsymbol{M}$-part $\boldsymbol{L}$-Sperner families

As we mentioned, G.O.H. Katona [21,23] and D. Kleitman [25] observed that the conclusion of the Sperner theorem does not hold for $M$-part Sperner families for $M \geqslant 3$. Here we give a simple new construction to show this fact.

Example 4.1. Consider an $M$-partitioned set $X$ such that $M \geqslant 3,\left|X_{i}\right|=m$. Define a family of sets $\mathcal{F}(M, m)$ as

$$
\mathcal{F}(M, m)=\left\{E \subseteq X:|E| \equiv\left\lfloor\frac{M m}{2}\right\rfloor(\bmod m+1)\right\} .
$$

Then $\mathcal{F}(M, m)$ is an $M$-part Sperner family, such that $|\mathcal{F}(M, m)|$ exceeds the middle level.

Indeed, if $E \varsubsetneqq F$, then $|F \backslash E|=|F|-|E| \geqslant m+1$, and therefore $E$ and $F$ must differ on at least two partition classes. Furthermore, $\mathcal{F}(M, m)$ contains the whole middle level in $X$, i.e. $\binom{X}{\lfloor M m / 2\rfloor}$, and, in addition, it also contains sets of size $\left\lfloor\frac{M m}{2}\right\rfloor+m+1 \leqslant M m$, since $M>2$.

We obtain some new results below for the maximum size of $M$-part $L$-Sperner families using only elementary tools, i.e. without convex hull theory.

Let $f_{L}\left(n_{1}, \ldots, n_{M}\right)$ be the maximum size of an $M$-part $L$-Sperner family, with given partition classes of sizes $n_{1} \geqslant \cdots \geqslant n_{M} \geqslant 1$. For $L=1$ we omit the subscript.

Let $F_{L}(n ; M)$ denote the maximum size of an $M$-part $L$-Sperner family on an underlying set $X$ of size $n$. (Here we do not fix the $M$-partition or its class sizes.) For $L=1$, again, we omit the subscript.
J.R. Griggs and D. Kleitman [18] made the following useful observation, originally for $L=1$. For $i \in[M]$,

$$
\begin{equation*}
f_{L}\left(n_{1}, \ldots, n_{M}\right) \leqslant 2 f_{L}\left(n_{1}, \ldots, n_{i-1}, n_{i}-1, n_{i+1}, \ldots, n_{M}\right) \tag{12}
\end{equation*}
$$

We allow $n_{i}=1$ in (12), and in this case we delete the zeros from the argument of $f$ on the RHS of (12). Indeed, assume that $\mathcal{F}$ is a maximum size $M$-part $L$-Sperner family for a partition $n_{1}, n_{2}, \ldots, n_{M}$. Fix an $x \in X_{i}$, and consider $\mathcal{F}_{1}=\{F \in \mathcal{F}: x \notin F\}, \mathcal{F}_{2}=\{F \in \mathcal{F}: x \in F\}$, and $\mathcal{F}_{2}^{\prime}=\left\{F \backslash\{x\}: F \in \mathcal{F}_{2}\right\}$. Now $\mathcal{F}_{1}$ and $\mathcal{F}_{2}^{\prime}$ are both $M$-part $L$-Sperner families on the underlying set $X_{1} \cup \cdots \cup X_{i-1} \cup X_{i} \backslash\{x\} \cup X_{i+1} \cup \cdots \cup X_{M}\left((M-1)\right.$-part $L$-Sperner if $\left.n_{i}=1\right) ;\left|\mathcal{F}_{2}\right|=\left|\mathcal{F}_{2}^{\prime}\right|, \mathcal{F}=\mathcal{F}_{1} \cup \mathcal{F}_{2}$ is a disjoint union, and for $j=1,2$ we have $\left|\mathcal{F}_{j}\right| \leqslant f_{L}\left(n_{1}, \ldots, n_{i-1}, n_{i}-1, n_{i+1}, \ldots, n_{M}\right)$. Hence (12) holds.

Furthermore, we note that (12) is tight in the following important special case:
Lemma 4.2. For $M \geqslant 4$,

$$
f(\overbrace{n_{1}, n_{2}, 1, \ldots, 1}^{M})=f\left(n_{1}, n_{2}, 1\right) 2^{M-3} .
$$

Proof. By (12), we only have to prove $f\left(n_{1}, n_{2}, 1, \ldots, 1\right) \geqslant f\left(n_{1}, n_{2}, 1\right) 2^{M-3}$. This statement is vacuously true for $M=3$, and it follows for all $M \geqslant 4$ from $f\left(n_{1}, n_{2}, \ldots, n_{M-2}, 1,1\right) \geqslant 2 f\left(n_{1}, n_{2}, \ldots\right.$, $n_{M-2}, 1$ ), which we show here.

Assume that $\mathcal{F}$ is a maximum size $n_{1}, n_{2}, \ldots, n_{M-2}, 1(M-1)$-part Sperner family with $X_{M-1}=\{x\}$. Set $\mathcal{F}_{1}=\{F \in \mathcal{F}: x \notin F\}, \mathcal{F}_{2}=\{F \in \mathcal{F}: x \in F\}, \mathcal{F}_{1}^{\prime}=\left\{F \cup\{x\}: F \in \mathcal{F}_{1}\right\}, \mathcal{F}_{2}^{\prime}=\{F \backslash\{x\}:$ $\left.F \in \mathcal{F}_{2}\right\}, \mathcal{F}^{\prime}=\mathcal{F}_{1}^{\prime} \cup \mathcal{F}_{2}^{\prime}$. Take $X_{M}=\{y\}$, and the $M$-part Sperner family $\{F \cup\{y\}: F \in \mathcal{F}\} \cup \mathcal{F}^{\prime}$ on $X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M-1} \uplus X_{M}$. The cardinality of this family is $2 f\left(n_{1}, n_{2}, \ldots, n_{M-2}, 1\right)$.

Another well-known general observation is that for $M \geqslant 2$,

$$
\begin{equation*}
f_{L}\left(n_{1}, n_{2}, \ldots, n_{M}\right) \geqslant f_{L}\left(n_{1}, n_{2}, \ldots, n_{M-2}, n_{M-1}+n_{M}\right) . \tag{13}
\end{equation*}
$$

Indeed, any ( $M-1$ )-part $L$-Sperner family for the partition $X_{1} \uplus \cdots \uplus X_{M-2} \uplus\left(X_{M-1} \cup X_{M}\right)$ is also an $M$-part $L$-Sperner family for the partition $X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M-1} \uplus X_{M}$.
P.L. Erdős and G.O.H. Katona [11] solved the 3 -part Sperner problem in the special case $n_{3}=1$ exactly:

Lemma 4.3. (See [11].) If $n_{1}$ or $n_{2}$ is odd, then

$$
f\left(n_{1}, n_{2}, 1\right)=2\binom{n-1}{\left\lfloor\frac{n-1}{2}\right\rfloor},
$$

otherwise, if both of them are even,

$$
f\left(n_{1}, n_{2}, 1\right)=2\binom{n_{1}+n_{2}}{\frac{n_{1}+n_{2}}{2}}-\left(\binom{n_{1}}{\frac{n_{1}}{2}}-\binom{n_{1}}{\frac{n_{1}}{2}-1}\right)\left(\binom{n_{2}}{\frac{n_{2}}{2}}-\binom{n_{2}}{\frac{n_{2}}{2}-1}\right) .
$$

This gives us a characterization of the 3-part Sperner problems where one of the partition sizes is 1 and the maximum size of the family is $\binom{n}{(n / 2\rfloor}$ :

Lemma 4.4. $f\left(n_{1}, n_{2}, 1\right)=\binom{n}{\lfloor n / 2\rfloor}$ if and only if $n$ is even, i.e. $n_{1}+n_{2}$ is odd.
Proof. From Lemma 4.3 we have that if $n_{1}+n_{2}$ is odd, then $f\left(n_{1}, n_{2}, 1\right)=2\binom{n-1}{\left\lfloor\frac{n-1}{2}\right\rfloor}=\binom{n}{\lfloor n / 2\rfloor}$, and otherwise $f\left(n_{1}, n_{2}, 1\right)>\binom{n}{\lfloor n / 2\rfloor}$.

Now we are in a position to characterize the $M$-part Sperner problems, whose solution is the classic formula $\binom{n}{\left(n^{n} 2\right\rfloor}$.

Theorem 4.5. For $M \geqslant 3$, we have $f\left(n_{1}, \ldots, n_{M}\right)=\binom{n}{\lfloor n / 2\rfloor}$ if and only if $M=3, n_{3}=1$ and $n$ is even.

Proof. First assume that $M=3, n_{3}=1$ and $n$ even. Lemma 4.4 implies the required equality.
For the other direction of the equivalence, assume that $f\left(n_{1}, \ldots, n_{M}\right)=\binom{n}{\lfloor n / 2\rfloor}$. If $n_{1} \leqslant\lceil n / 2\rceil-1$, then $\lfloor n / 2\rfloor+\left(n_{1}+1\right) \leqslant n$, and the family

$$
\mathcal{F}=\left\{E \subseteq X:|E| \equiv\lfloor n / 2\rfloor \bmod \left(n_{1}+1\right)\right\}
$$

is an $M$-part Sperner family with $|\mathcal{F}|>\binom{n}{\lfloor n / 2\rfloor}$, contradicting the hypothesis that we have. Hence we must have $n_{1} \geqslant\lceil n / 2\rceil$.

The rest of the proof will take the form of a case analysis.
Case 0. $M=3$ and $n_{3}=1$. Lemma 4.4 handles this case.
Case 1. $M=3, n_{2} \geqslant n_{3} \geqslant 2$. Take $\mathcal{F}^{\prime}=\left(\begin{array}{c}X \\ \lfloor n / 2 \\ \hline\end{array}\right)$. Now we will transform $\mathcal{F}^{\prime}$ into a strictly bigger $M$-part Sperner family. This will contradict the hypothesis that we have.
$\mathcal{F}^{\prime}$ contains all those sets that contain $X_{2} \cup X_{3}$ as a subset and have exactly $\lfloor n / 2\rfloor-n_{2}-n_{3}$ elements from $X_{1}$. To obtain $\mathcal{F}$, remove all these sets from $\mathcal{F}^{\prime}$, but add all sets that contain $X_{2} \cup X_{3}$ and has exactly $\lfloor n / 2\rfloor-n_{3}+1$ elements from $X_{1}$. We removed $\binom{n_{1}}{\lfloor n / 2\rfloor-n_{2}-n_{3}}$ sets and put back $\binom{n_{1}}{\lfloor n / 2\rfloor-n_{3}+1}$ sets. Observe that $\mathcal{F}$ is a 3 -part Sperner family with the given parts: indeed, if two members of $\mathcal{F}$ have different cardinalities, then the difference in their sizes is $n_{2}+1$. Assume that one set is included in another, and the difference falls into a single class $X_{i}$. By the size of the difference this class must be $X_{1}$. But then, by our construction, the smaller set is no longer in $\mathcal{F}$. We are going to show that

$$
\begin{equation*}
\binom{n_{1}}{\lfloor n / 2\rfloor-n_{3}+1}>\binom{n_{1}}{\lfloor n / 2\rfloor-n_{2}-n_{3}}, \tag{14}
\end{equation*}
$$

which will conclude this case. Observe that $\lfloor n / 2\rfloor-n_{2}-n_{3}=\left\lfloor\frac{n_{1}+n_{2}+n_{3}}{2}\right\rfloor-n_{2}-n_{3} \leqslant \frac{n_{1}-n_{2}-n_{3}}{2}<\frac{n_{1}}{2}$. We have either

$$
\left\lfloor\frac{n}{2}\right\rfloor-n_{2}-n_{3}<\left\lfloor\frac{n}{2}\right\rfloor-n_{3}+1 \leqslant \frac{n_{1}}{2},
$$

when we have (14), or

$$
\left\lfloor\frac{n}{2}\right\rfloor-n_{2}-n_{3}<\frac{n_{1}}{2} \leqslant\left\lfloor\frac{n}{2}\right\rfloor-n_{3}+1 .
$$

In this alternative, (14) is equivalent to

$$
\frac{n_{1}}{2}-\left\lfloor\frac{n}{2}\right\rfloor+n_{2}+n_{3}>\left\lfloor\frac{n}{2}\right\rfloor-n_{3}+1-\frac{n_{1}}{2}
$$

which is equivalent to $n_{1}+n_{2}+2 n_{3}-1=n+n_{3}-1>2\lfloor n / 2\rfloor$. The last inequality can only fail for $n_{3}=1$ and $n$ even.

Case 2. $M \geqslant 4$. We are going to show that $f\left(n_{1}, n_{2}, \ldots, n_{M}\right)>\binom{n}{\lfloor n / 2\rfloor}$, and therefore we have nothing to prove. Assume for contradiction that $f\left(n_{1}, n_{2}, \ldots, n_{M}\right)=\binom{n}{\lfloor n / 2\rfloor}$, and we get by (13),

$$
\begin{equation*}
\binom{n}{\lfloor n / 2\rfloor}=f\left(n_{1}, n_{2}, \ldots, n_{M}\right) \geqslant f_{L}\left(n_{1}, n_{2}, \ldots, n_{M-2}, n_{M-1}+n_{M}\right) \geqslant\binom{ n}{\lfloor n / 2\rfloor} . \tag{15}
\end{equation*}
$$

Reducing $M$ by the repeated application of (15), joining the two smallest classes, we reach an $n_{1}^{\prime}, n_{2}^{\prime}, n_{3}^{\prime} 3$-part Sperner problem with $f\left(n_{1}^{\prime}, n_{2}^{\prime}, n_{3}^{\prime}\right)=\binom{n}{n / 2\rfloor}$. According to Case 1 , we must have $n_{3}^{\prime}=1$. As this 1 was not added to any other $n_{i}$ in the last application of (15), the only possibility is that $n_{2}^{\prime}=2$ was obtained as a sum of two 1 's. Hence, before a single application of (15), we had $M=4$ and $n_{2}=n_{3}=n_{4}=1$. We reach contradiction with (15): $f(n-3,1,1,1)=2 f(n-3,1,1)=$ $4\binom{n-2}{\left\lfloor\frac{n-2}{2}\right\rfloor}>\binom{n}{\lfloor n / 2\rfloor}$, where the equalities hold by Lemmata 4.2 and 4.3 , and the inequality holds as one of $n$ and $n-1$ must be odd.

Recall that $F_{L}(n ; M)$ is the largest size of an $M$-part $L$-Sperner family with unspecified class-sizes. Griggs [16] showed that

$$
\begin{equation*}
F_{L}(n ; M) \leqslant L 2^{M-2}\binom{n}{\lfloor n / 2\rfloor} . \tag{16}
\end{equation*}
$$

A. Sali [30] obtained a general Sperner-type bound for products of symmetric chain orders, which in the special case of the Boolean lattice says that there exists a constant $c>0$ such that

$$
\begin{equation*}
F_{L}(n ; M) \leqslant c L \sqrt{M}\binom{n}{\lfloor n / 2\rfloor} . \tag{17}
\end{equation*}
$$

This has been the best upper bound. No specific value for the constant $c$ was known. Z. Füredi, J. Griggs, A.M. Odlyzko, and J.M. Shearer [14] proved that for fixed $M$ and $L$, for $n \rightarrow \infty$, the asymptotic formula $F_{L}(n ; M) \sim \phi_{L}(M)\binom{n}{n}$ holds; and as $M \rightarrow \infty$, the asymptotic formula $\phi_{L}(M) \sim$ $L \sqrt{\frac{\pi M}{4 \log M}}$ holds.

We show next, through a series of simple observations, that one can take $c<1$ in (17), making a common generalization and strengthening of (1) and (17). Note that the asymptotic results of [14] cited above do not provide a uniform bound for all values of $n, M, L$.

Theorem 4.6. For every $n, M \geqslant 3$,

$$
F_{L}(n ; M) \leqslant \begin{cases}L \sqrt{\frac{M}{2}} e^{\frac{5}{18 n}}\binom{n}{\lfloor n / 2\rfloor}, & \text { if } n \text { even },  \tag{18}\\ L \sqrt{\frac{M(1+1 / n)}{2}} e^{\frac{5}{18(n+1)}}\binom{n}{\lfloor n / 2\rfloor}, & \text { if n odd; }\end{cases}
$$

i.e. $F_{L}(n ; M)<L \sqrt{M}\binom{n}{\lfloor n / 2\rfloor}$ in both cases.

Proof. Select $n_{1}, n_{2}, \ldots, n_{M}$ such that $f_{L}\left(n_{1}, \ldots, n_{M}\right)=F_{L}(n ; M)$. Start with the bound (12):

$$
\begin{equation*}
f_{L}\left(n_{1}, \ldots, n_{M}\right) \leqslant f_{L}\left(n_{1}, n_{2}\right) 2^{n-n_{1}-n_{2}} . \tag{19}
\end{equation*}
$$

Combine (19) with

$$
\begin{equation*}
f_{L}\left(n_{1}, n_{2}\right) \leqslant F_{L}\left(n_{1}+n_{2} ; 2\right) \leqslant L\binom{n_{1}+n_{2}}{\left\lfloor\frac{n_{1}+n_{2}}{2}\right\rfloor}, \tag{20}
\end{equation*}
$$

where we used (16) for the second estimate, to obtain

$$
\begin{equation*}
F_{L}(n ; M) \leqslant L\binom{n_{1}+n_{2}}{\left\lfloor\frac{n_{1}+n_{2}}{2}\right\rfloor} 2^{n-n_{1}-n_{2}} . \tag{21}
\end{equation*}
$$

Robbins' formula [29] gives uniformly valid and asymptotically tight bounds for the factorial:

$$
\begin{equation*}
\left(\frac{n}{e}\right)^{n} \sqrt{2 \pi n} \cdot e^{\frac{1}{12 n+1}}<n!<\left(\frac{n}{e}\right)^{n} \sqrt{2 \pi n} \cdot e^{\frac{1}{12 n}} . \tag{22}
\end{equation*}
$$

Little calculation from (22) gives that for all even integers $m \geqslant 1$,

$$
\begin{equation*}
e^{-\frac{5}{18 m}} \sqrt{\frac{2}{\pi}} \frac{2^{m}}{\sqrt{m}} \leqslant\binom{ m}{\frac{m}{2}} \leqslant \sqrt{\frac{2}{\pi}} \frac{2^{m}}{\sqrt{m}}, \tag{23}
\end{equation*}
$$

and for $m$ odd, from $\binom{m}{\frac{m-1}{2}}=\frac{1}{2}\binom{m+1}{\frac{m+1}{2}}$ and (23), we obtain

$$
e^{-\frac{5}{18(m+1)}} \sqrt{\frac{2}{\pi}} \frac{2^{m}}{\sqrt{m+1}} \leqslant\binom{ m}{\frac{m-1}{2}} \leqslant \sqrt{\frac{2}{\pi}} \frac{2^{m}}{\sqrt{m+1}}
$$

For $n$ even, apply (23) to the RHS of inequality (21) to get

$$
\begin{equation*}
F_{L}(n ; M) \leqslant L \frac{2^{n_{1}+n_{2}} \cdot 2^{n-n_{1}-n_{2}}}{\sqrt{n_{1}+n_{2}}} \leqslant L \frac{\sqrt{n}}{\sqrt{n_{1}+n_{2}}} e^{\frac{5}{18 n}}\binom{n}{\lfloor n / 2\rfloor} . \tag{24}
\end{equation*}
$$

Since $n_{1} \geqslant \cdots \geqslant n_{M}$, we have $\frac{n}{n_{1}+n_{2}} \leqslant \frac{M}{2}$, and hence the result follows. For $n$ odd, the last term in (24) changes to $L \frac{\sqrt{n+1}}{\sqrt{n_{1}+n_{2}}} e^{\frac{5}{18(n+1)}}\binom{n}{\lfloor n / 2\rfloor}$, and the inequality $\frac{n+1}{n_{1}+n_{2}} \leqslant \frac{M}{2}(1+1 / n)$ finishes the proof.

Note that Theorem 4.6 is pretty tight for $L=1$, as it is shown by (2).
Conjecture 4.7. For $L_{1}=\cdots=L_{M}$, the function $F_{L}(n, M)$ takes its maximum value when the numbers $n_{i}$ are as equal as possible, i.e. $n_{1}-n_{M} \leqslant 1$.

## 5. $\boldsymbol{k}$-fold $\boldsymbol{M}$-part Sperner property

Let us be given a partition of the underlying set $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ into nonempty parts, and $\mathcal{F} \subseteq 2^{X}$ be a family of sets. For a $k \geqslant 1$, we say that $\mathcal{F}$ is a $k$-fold $M$-part Sperner family, if for every $E, F \in \mathcal{F}, E \varsubsetneqq F$ implies that

$$
\left|\left\{i:(F \backslash E) \cap X_{i} \neq \emptyset\right\}\right|>k
$$

Thus for $k=1$ we get back the definition of $M$-part Sperner families. Let us denote by $g\left(n_{1}, \ldots, n_{M} ; k\right)$ the maximum size of a $k$-fold $M$-part Sperner family with class sizes $n_{1}, \ldots, n_{M}$, and

$$
G(n, M, k)=\max _{n_{1}, \ldots, n_{M}} g\left(n_{1}, \ldots, n_{M} ; k\right)
$$

with $\sum n_{i}=n$, i.e. unspecified class sizes.
Note that for $1 \leqslant k<M \leqslant n$, any usual (i.e. 1-part) Sperner family is a $k$-fold $M$-part Sperner family for any $M$-partition of $X$, thus

$$
G(n, M, k) \geqslant g\left(n_{1}, \ldots, n_{M} ; k\right) \geqslant F_{1}(n ; 1)=\binom{n}{\lfloor n / 2\rfloor} .
$$

As the following theorem shows, equality holds if, in addition, we have $M \leqslant 2 k$.

Theorem 5.1. For integers $k, n, M$ satisfying $1 \leqslant k<M \leqslant \min (2 k, n)$, and for any positive integers $n_{1}, \ldots, n_{M}$ such that $\sum n_{i}=n$, we have

$$
\begin{equation*}
g\left(n_{1}, \ldots, n_{M} ; k\right)=G(n, M, k)=\binom{n}{\lfloor n / 2\rfloor} \tag{25}
\end{equation*}
$$

and moreover, for $k \geqslant 2$ any maximum size family is $\binom{X}{\lfloor n / 2\rfloor}$ or $\binom{X}{\lfloor n / 2\rfloor}$.
Proof. If $k=1$, then $M=2$ and the statement is proved in Theorem 1.1. So assume $k \geqslant 2$.
The first part of the statement is almost trivial. Let $\mathcal{F}$ be a $k$-fold $M$-part Sperner family, given partition sizes $n_{1} \geqslant \cdots \geqslant n_{M}$. Denote $X^{\prime}=X_{1} \cup \cdots \cup X_{k}$ and $X^{\prime \prime}=X-X^{\prime}$. Note now that $\mathcal{F}$ is a 2part Sperner family with respect to the partition $X^{\prime} \cup X^{\prime \prime}$. Hence $|\mathcal{F}| \leqslant\binom{ n}{\lfloor n / 2\rfloor}$, by the 2-part Sperner theorem, Theorem 1.1. Thus we have $G(n, M, k) \leqslant\binom{ n}{\lfloor n / 2\rfloor}$, which proves Eq. (25).

Proof of characterization: Suppose that $\mathcal{F}$ is a maximum size $k$-fold $M$-part Sperner family on a partition $X=X_{1} \uplus \cdots \uplus X_{M}$ for some $2 \leqslant k<M \leqslant \min (2 k, n)$, and, contrary to our statement, $\mathcal{F} \neq$ $\binom{X}{\lfloor n / 2\rfloor}$ and $\mathcal{F} \neq\binom{ X}{[n / 2\rceil}$. As we have seen before, $\mathcal{F}$ can be considered as a 2-part Sperner family with a partition $X=X^{\prime} \cup X^{\prime \prime}$, with $\left|X^{\prime}\right|=N_{1}=n_{1}+\cdots+n_{k},\left|X^{\prime \prime}\right|=N_{2}=n-N_{1}, N_{1} \geqslant N_{2}$, where these new classes are unions of some original classes. Maximum size 2-part Sperner systems have been completely characterized: They are homogeneous families where the $N_{2}+1$ levels of the smaller partition class are paired with the largest $N_{2}+1$ levels of the larger partition class in a "well-paired" manner: larger level from one partition class is paired with larger level of the other partition class. To be more precise, recall that $I(\mathcal{F})=\left\{\left(\left|A \cap X^{\prime}\right|,\left|A \cap X^{\prime \prime}\right|\right): A \in \mathcal{F}\right\}$, then:

Theorem 5.2.(See P.L. Erdős, G.O.H. Katona [10].) Every maximum size 2-part Sperner family is a homogeneous family with full transversal $I(\mathcal{F})=\left\{(\varphi(i), i): i \in\left[0, N_{2}\right]\right\}$. Furthermore, $\mathcal{F}$ is "well paired" in the following sense: values of $\varphi(i)$ make an interval of $N_{2}+1$ numbers corresponding to the $N_{2}+1$ largest levels in $2^{\left[N_{1}\right]}$, and $\binom{N_{2}}{i}<\binom{N_{2}}{j}$ implies $\binom{N_{1}}{\varphi(i)} \leqslant\binom{ N_{1}}{\varphi(j)}$, for all $0 \leqslant i, j \leqslant N_{2}$.

Note that in particular Theorem 1.1 and the theorem cited above imply that $\varphi(i) \in\{\lfloor n / 2\rfloor-i$, $\left.\lceil n / 2\rceil-i, N_{1}-\lfloor n / 2\rfloor+i, N_{1}-\lceil n / 2\rceil+i\right\}$ for all $i \in\left[0, n_{2}\right]$, and that our maximum size family $\mathcal{F}$ is homogeneous.

Let us define a partial ordering between elements of $\Pi:=\left[0, N_{1}\right] \times\left[0, N_{2}\right]$. We say that $\left(i_{1}, j_{1}\right) \geqslant$ $\left(i_{2}, j_{2}\right)$ iff $i_{1} \geqslant i_{2}$ and $j_{1} \geqslant j_{2}$. Thus, our homogeneous family $\mathcal{F}$ is an antichain iff $I(\mathcal{F}) \subseteq \Pi$ is an antichain.

For ease of argument, for any two sets $A, B \subseteq X$, define

$$
t(A, B):=\left|\left\{i: A \cap X_{i}=B \cap X_{i}\right\}\right| .
$$

Note that the fact that $\mathcal{F}$ is a $k$-fold $M$-part Sperner family can be restated as follows: for any $E, F \in \mathcal{F}$ such that $E \varsubsetneqq F$ we have $t(F \backslash E, \emptyset)=t(X \backslash(F \backslash E), X)=t(E, F)<M-k=: r$, where $1 \leqslant r \leqslant \min (k, n-k)$; an observation, which we will use frequently.

For simplicity we identify $2^{X}$ with $2^{X^{\prime}} \times 2^{X^{\prime \prime}}$, and use the notation $\mathbb{S}(i, j)=\binom{X^{\prime}}{i} \times\binom{ X^{\prime \prime}}{j}$.
We start with simple observations:
Claim 5.3. $\left(\left\lfloor N_{1} / 2\right\rfloor,\left\lceil N_{2} / 2\right\rceil\right)$ or $\left(\left\lceil N_{1} / 2\right\rceil,\left\lfloor N_{2} / 2\right\rfloor\right) \in I(\mathcal{F})$.
Proof. When $2 \nmid n$ or both $N_{1}, N_{2}$ are even, the claim directly follows from the well-pairing condition in Theorem 5.2. If $n$ is even and both $N_{1}$ and $N_{2}$ are odd, then the only extra possibility that Theorem 5.2 allows and we need to exclude is when ( $\frac{N_{1}-1}{2}, \frac{N_{2}-1}{2}$ ) and ( $\frac{N_{1}+1}{2}, \frac{N_{2}+1}{2}$ ) are both in $I(\mathcal{F})$. But in this case there are $E \varsubsetneqq F$ such that $E \in \mathbb{S}\left(\frac{N_{1}-1}{2}, \frac{N_{2}-1}{2}\right), F \in \mathbb{S}\left(\frac{N_{1}+1}{2}, \frac{N_{2}+1}{2}\right)$ and, since $k \geqslant 2$, we have $t(E, F) \geqslant M-|F \backslash E|=M-2=r+k-2 \geqslant r$, which is a contradiction.

Let

$$
\left(\ell_{1}, \ell_{2}\right) \in I(\mathcal{F}) \cap\left\{\left(\left\lfloor N_{1} / 2\right\rfloor,\left\lceil N_{2} / 2\right\rceil\right),\left(\left\lceil N_{1} / 2\right\rceil,\left\lfloor N_{2} / 2\right\rfloor\right)\right\} .
$$

Define $t_{0}:=\max t(F, \emptyset), t_{1}:=\max t(F, X)$, where the maximum is taken over all $F \in \mathbb{S}\left(\ell_{1}, l, 2\right)$. It is not hard to see that the following is true.

## Claim 5.4.

(i) If $C \subseteq E \varsubsetneqq F \subseteq D$, then $t(E, F) \geqslant t(C, D)=t(D-C, \emptyset)$.
(ii) There exists $(i, j) \in I(\mathcal{F})$ such that $(i, j)<\left(\ell_{1}, \ell_{2}\right)$ or $(i, j)>\left(\ell_{1}, \ell_{2}\right)$.
(iii) Suppose $(i, j)<\left(\ell_{1}, \ell_{2}\right)\left(r e s p\right.$. $\left.(i, j)>\left(\ell_{1}, \ell_{2}\right)\right)$. Then $\max t(F, E) \geqslant t_{0}\left(\operatorname{resp}\right.$. max $\left.t(F, E) \geqslant t_{1}\right)$ where $E \in \mathbb{S}(i, j), F \in \mathbb{S}\left(\ell_{1}, \ell_{2}\right), E \varsubsetneqq F$.
(iv) $t_{0}, t_{1} \geqslant\lfloor k / 2\rfloor+\lfloor r / 2\rfloor$.

Proof. To see (i), observe that if $C \subseteq E \varsubsetneqq F \subseteq D$, then $(D-C) \cap X_{i}=\emptyset$ implies that $E \cap X_{i}=F \cap X_{i}$. Therefore

$$
t(D-C, \emptyset)=\left|\left\{i:(D-C) \cap X_{i}=\emptyset\right\}\right| \leqslant\left|\left\{i: F \cap X_{i}=E \cap X_{i}\right\}\right|=t(E, F)
$$

(ii) follows from the fact that $I(\mathcal{F})$ is well paired and $\mathcal{F}$ is not a single level of $X$.

Assume for the rest that $(i, j)<\left(\ell_{1}, \ell_{2}\right)$ (the other direction is similar). For (iii), let $E \in \mathbb{S}(i, j)$, $F \in \mathbb{S}\left(\ell_{1}, \ell_{2}\right)$ with $E \varsubsetneqq F$, and use part (i) with $C=\emptyset$ and $D=F$. We get that $t(E, F) \geqslant t(F, \emptyset)$, which implies the statement.

To see (iv), note that $\sum_{s=1}^{[k / 2\rceil} n_{s} \geqslant \ell_{1}$ and $\sum_{s=1}^{[r / 2\rceil} n_{k+s} \geqslant \ell_{2}$. This implies that we can use part (i) with $C=E=\emptyset$ and $D=\left(\bigcup_{s=1}^{[r / 2\rceil} X_{s}\right) \cup\left(\bigcup_{s=1}^{[s / 2]} X_{k+s}\right)$, and an appropriate $F \in \mathbb{S}\left(\ell_{1}, \ell_{2}\right)$. Now

$$
t_{0} \geqslant t(F, \emptyset) \geqslant t(D, \emptyset)=\left|\left\{i: D \cap X_{i}=\emptyset\right\}\right|=\lfloor k / 2\rfloor+\lfloor r / 2\rfloor .
$$

Observe now that if $t_{0} \geqslant r$ and $(i, j)<\left(\ell_{1}, \ell_{2}\right)$ for some $(i, j) \in I(\mathcal{F})$, or $t_{1} \geqslant r$ and $(i, j)>\left(\ell_{1}, \ell_{2}\right)$ for some $(i, j) \in I(\mathcal{F})$, then we are done, since we have an $E, F \in \mathcal{F}$ such that $E \varsubsetneqq F$ and $t(E, F) \geqslant r$, which is a contradiction. If $r<k$ or $r=k$ is even, then $t_{0}, t_{1} \geqslant r$. Thus, we have to concern ourselves only with the case when $M=2 k$ and $k \geqslant 3$ is odd, and the relevant (perhaps both) of $t_{0}, t_{1}$ is equal to $k-1$, which we assume for the rest of this proof.

Assume that there are $\left(i_{0}, j_{0}\right) \in I(\mathcal{F})$ such that $\left(i_{0}, j_{0}\right)<\left(\ell_{1}, \ell_{2}\right)$ (the other case can be proved similarly). We will use the following notation for brevity:

$$
\begin{aligned}
& \alpha_{1}=n_{1}+n_{2}+\cdots+n_{\frac{k-1}{2}} \\
& \beta_{1}=n_{1}+n_{2}+\cdots+n_{\frac{k-1}{2}}+n_{\frac{k+1}{2}}=\alpha_{1}+n_{\frac{k+1}{2}} \\
& \alpha_{2}=n_{k+1}+n_{k+2}+\cdots+n_{\frac{3 k-1}{2}} \\
& \beta_{2}=n_{k+1}+n_{k+2}+\cdots+n_{\frac{3 k-1}{2}}+n_{\frac{3 k+1}{2}}=\alpha_{2}+n_{\frac{3 k+1}{2}},
\end{aligned}
$$

and for $i=1,2$,

$$
A_{i}=\bigcup_{j=(i-1) k+1}^{(i-1) k+\frac{k-1}{2}} X_{i} \quad \text { and } \quad B_{i}=\bigcup_{j=(i-1) k+1}^{(i-1) k+\frac{k+1}{2}} X_{i}=A_{i} \cup X_{(i-1) k+\frac{k+1}{2}} .
$$

This gives in particular that $\left|A_{i}\right|=\alpha_{i}$ and $\left|B_{i}\right|=\beta_{i}$.
Claim 5.5. . The following statements hold:
(i) $\alpha_{1}<\ell_{1} \leqslant \beta_{1}$ and $\alpha_{2}<\ell_{2} \leqslant \beta_{2}$.
(ii) $i_{0}<n_{\frac{k-1}{2}} \leqslant \alpha_{1}$ and $j_{0}<n_{\frac{k k-1}{2}} \leqslant \alpha_{2}$.
(iii) $\beta_{1}-\ell_{1}^{2} \geqslant \ell_{2}-\alpha_{2}$ or $\ell_{1}-\alpha_{1} \geqslant \beta_{2}-\ell_{2}$.
(iv) If $\beta_{1}-\ell_{1} \geqslant \ell_{2}-\alpha_{2}$ then $\left(\ell_{1}+\ell_{2}-\alpha_{2}, \alpha_{2}\right) \in I(\mathcal{F})$.
(v) If $\ell_{1}-\alpha_{1} \geqslant \beta_{2}-\ell_{2}$ then $\left(\alpha_{1}, \ell_{1}+\ell_{2}-\alpha_{1}\right) \in I(\mathcal{F})$.

Let $F \in \mathbb{S}\left(\ell_{1}, \ell_{2}\right)$ and we will have $E \varsubsetneqq F$.
Proof. (i): Suppose $E \in \mathbb{S}\left(i_{0}, j_{0}\right)$ (since ( $\left.i_{0}, j_{0}\right)<\left(\ell_{1}, \ell_{2}\right)$, this can be done). Since $n_{1} \geqslant n_{2} \geqslant \cdots \geqslant n_{k}$ and $N_{i}=n_{(i-1) k+1}+\cdots+n_{i k}$, we must have $\ell_{1} \leqslant\left\lceil N_{1} / 2\right\rceil \leqslant \beta_{1}$ and $\ell_{2} \leqslant\left\lceil N_{2} / 2\right\rceil \leqslant \beta_{2}$. If $\ell_{1} \leqslant \alpha_{1}$, then we can assume that $F \subseteq A_{1} \cup B_{2}$. We use Claim 5.4(i) with $C=\emptyset$ and $D=A_{1} \cup B_{2}$, to obtain

$$
t(E, F) \geqslant t\left(A_{1} \cup B_{2}, \emptyset\right)=\frac{k+1}{2}+\frac{k-1}{2}=k,
$$

which is a contradiction. The case $\ell_{2} \leqslant \alpha_{2}$ can be handled similarly.
(ii): Suppose $E \in \mathbb{S}\left(i_{0}, j_{0}\right)$. If $i_{0} \geqslant n_{\frac{k-1}{2}}$, then we can assume that $X_{(k-1) / 2} \subseteq E \subseteq F \subseteq B_{1} \cup B_{2}$. We use Claim 5.4(i) with $C=X_{(k-1) / 2}$ and $D=B_{1} \cup B_{2}$, to obtain

$$
t(E, F) \geqslant t\left(\left(B_{1} \cup B_{2}\right)-X_{(k-1) / 2}, \emptyset\right)=\frac{k+1}{2}+\frac{k-1}{2}=k
$$

which is a contradiction. The case $j_{0} \leqslant n_{\frac{3 k+1}{2}}$ can be handled similarly.
(iii): $\beta_{1}-\ell_{1}<\ell_{2}-\alpha_{2}$ and $\ell_{1}-\alpha_{1}<\beta_{2}-\ell_{2}$ together imply that $n_{\frac{k+1}{2}}=\beta_{1}-\alpha_{1}<\beta_{2}-\alpha_{2}=n_{\frac{3 k+1}{2}}$, a contradiction.
(iv): Since $I(\mathcal{F})$ is well paired and $\left(\ell_{1}, \ell_{2}\right) \in I(\mathcal{F})$, for every $j \in\left[N_{2}\right]$ either $\left(\ell_{1}+\ell_{2}-j, j\right) \in I(\mathcal{F})$ or ( $\left.\ell_{1}-\left(\ell_{2}-j\right), j\right) \in I(\mathcal{F})$. Thus, if $\beta_{1}-\ell_{1} \geqslant \ell_{2}-\alpha_{2}$ and (iv) is false, then we must have ( $\ell_{1}-\ell_{2}+$ $\left.\alpha_{2}, \alpha_{2}\right) \in I(\mathcal{F})$. Now, $\left(\ell_{1}-\ell_{2}+\alpha_{2}, \alpha_{2}\right)<\left(\ell_{1}, \ell_{2}\right)$, thus we can assume that $E \in \mathbb{S}\left(\ell_{1}-\ell_{2}+\alpha_{2}, \alpha_{2}\right)$, $A_{2} \subseteq E$ and $F \subseteq B_{1} \cup B_{2}$. We use Claim 5.4(i) with $C=A_{2}$ and $D=B_{1} \cup B_{2}$, to obtain

$$
t(E, F) \geqslant t\left(B_{1} \cup X_{(3 k+1) / 2}, \emptyset\right)=\frac{k-1}{2}+\frac{2(k-1)}{2}>k
$$

which is a contradiction. The case (v) can be handled similarly.
To finish the proof of Theorem 5.1, observe that by part (iii) of Claim 5.5 we have $\beta_{1}-\ell_{1} \geqslant \ell_{2}-\alpha_{2}$ or $\ell_{1}-\alpha_{1} \geqslant \beta_{2}-\ell_{2}$.

If $\beta_{1}-\ell_{1} \geqslant \ell_{2}-\alpha_{2}$, then by parts (ii) and (iv) of Claim 5.5 we have $j_{0}<\alpha_{2}<\ell_{2} \leqslant \beta_{2}$ and $i_{0}<\alpha_{1}<\ell_{1}<\ell_{1}+\ell_{2}-\alpha_{2}<\beta_{1}$, and we can choose $E \in \mathbb{S}\left(i_{0}, j_{0}\right)$ and $F \in \mathbb{S}\left(\ell_{1}+\ell_{2}-\alpha_{2}, \alpha_{2}\right)$ such that $E \varsubsetneqq F \subseteq B_{1} \cup A_{2}$. We use Claim 5.4(i) with $C=\emptyset$ and $D=B_{1} \cup A_{2}$, to obtain

$$
t(E, F) \geqslant t\left(B_{1}, \cup A_{2}, \emptyset\right)=\frac{k+1}{2}+\frac{k-1}{2}=k,
$$

which is a contradiction.
On the other hand if $\ell_{1}-\alpha_{1} \geqslant \beta_{2}-\ell_{2}$, then we get a contradiction similarly using parts (ii) and (v) of Claim 5.5. Thus, we must have that, contrary to our initial assumption, $\mathcal{F} \in\left\{\binom{X}{\lfloor n / 2},\binom{X}{[n / 27}\right\}$. This finishes the proof of Theorem 5.1.

Contrasting Theorem 5.1, it is not always the case that a $k$-fold $M$-part Sperner family has at most $\binom{n}{\lfloor n / 2\rfloor}$ elements. We follow the ideas of Example 4.1. Let $N:=n_{1}+\cdots+n_{k}, n=n_{1}+\cdots+n_{M}$, and assume $N<\lfloor n / 2\rfloor$. Then we can define the family $\mathcal{F}$ as follows:

$$
\mathcal{F}=\{E \subseteq X:|E| \equiv\lfloor n / 2\rfloor \bmod (N+1)\} .
$$

It is easy to see that $\mathcal{F}$ is indeed a $k$-fold $M$-part Sperner family with more than $\binom{n}{\lfloor n / 2\rfloor}$ elements. The following Theorem 5.6 sets an upper bound on $G(n, M, k)$ in the range that is the relative complement of the range of Theorem 5.1.

Theorem 5.6. For given natural numbers $k, n, M(2 k<M \leqslant n)$ we have

$$
G(n, M, k) \leqslant \begin{cases}\sqrt{\frac{M}{2 k}} \frac{5}{18 n}\binom{n}{\lfloor n / 2\rfloor}, & \text { if n even, }  \tag{26}\\ \sqrt{\frac{M(1+1 / n)}{2 k}} e^{\frac{5}{18(n+1)}}\binom{n}{\lfloor n / 2\rfloor}, & \text { if } n \text { odd. }\end{cases}
$$

Proof. Assume $\mathcal{F}$ is a $k$-fold $M$-part Sperner family with $M>2 k$, and $N_{1}:=n_{1}+\cdots+n_{2 k}$. Then the very same argument as in Theorem 4.6, together with equality (25), gives

$$
|\mathcal{F}| \leqslant G\left(N_{1}, 2 k, k\right) 2^{n-N_{1}}=\binom{N_{1}}{\left\lfloor N_{1} / 2\right\rfloor} 2^{n-N_{1}} .
$$

The rest of the proof goes along the same lines as the proof of Theorem 4.6.

How good is the upper bound in Theorem 5.6? We have the following lower bound, which is off by a $\sqrt{k}$ factor:

Theorem 5.7. Fixing $M, k$, and letting $n \rightarrow \infty$, there are $k$-fold $M$-part Sperner families of size

$$
G(n, M, k)=\Omega\left(\binom{n}{\lfloor n / 2\rfloor} \frac{\sqrt{M}}{k}\right) .
$$

Proof. We are going to use equal partition sizes $n_{1}=\cdots=n_{M}=m$, thus $n=m M$. To simplify the calculations we assume that $\sqrt{m} / 2$ is an integer. The condition $n \rightarrow \infty$ is equivalent to $m \rightarrow \infty$. We will use the notation $N=\left[\frac{m}{2}-\frac{\sqrt{m}}{2}, \frac{m}{2}+\frac{\sqrt{m}}{2}\right]$. Define $I=\left\{\left(x_{1}, \ldots, x_{M}\right): \forall i, x_{i} \in N\right\}$ and the homogeneous family $\mathcal{B}=\left\{A: \forall i,\left|A \cap X_{i}\right| \in N\right\}$, i.e. $P(\mathcal{B})=S(I)$. The asymptotic normality of binomial coefficients imply that for $m \rightarrow \infty$,

$$
\sum_{i \in N}\binom{m}{i}=\Theta\left(2^{m}\right)
$$

and we easily obtain that $|\mathcal{B}|=\Theta\left(2^{n}\right)=\Theta\left(\left(_{m M / 2}^{m M}\right) \sqrt{m M}\right)$.
For all $\ell \in N$, set $\varphi(\ell)=\ell+\sqrt{m} / 2-m / 2$, such that $0 \leqslant \varphi(\ell) \leqslant \sqrt{m}$. Define for $j \in[0, k(\sqrt{m}+1)]$,

$$
I_{j}=\left\{\left(x_{1}, \ldots, x_{M}\right) \in I: \sum_{i=1}^{M} \varphi\left(x_{i}\right) \equiv j \bmod k(\sqrt{m}+1)+1\right\}
$$

Define $\mathcal{B}_{j}=\left\{A:\left(\left|A \cap X_{1}\right|, \ldots,\left|A \cap X_{M}\right|\right) \in I_{j}\right\}$, i.e. $\mathcal{B}_{j}$ is the homogeneous family with $P\left(\mathcal{B}_{j}\right)=I_{j}$. It is clear that $\bigcup_{j} I_{j}=I$ and hence $\bigcup_{j} \mathcal{B}_{j}=\mathcal{B}$.

We claim that every $\mathcal{B}_{j}$ is a $k$-fold $M$-part Sperner family. Indeed, if $U, V \in \mathcal{B}_{j}$ and $U \subset V$, then $|U-V|=|U|-|V| \geqslant k(\sqrt{m}+1)+1$, and in any single part $U$ and $V$ can differ in at most $\sqrt{m}$ elements. Therefore they must differ on at least $k+1$ parts.

Finally, at least one $\mathcal{B}_{j}$ is at least as large as their average size, i.e.

$$
\max _{i}\left|\mathcal{B}_{i}\right| \geqslant \frac{|\mathcal{B}|}{k(\sqrt{m}+1)+1}=\Omega\left(\binom{n}{\lfloor n / 2\rfloor} \frac{\sqrt{M}}{k}\right) .
$$

## 6. BLYM and homogeneity results on $M$-part ( $L_{1}, \ldots, L_{M}$ )-Sperner families

### 6.1. M-part generalization of the BLYM inequality

The next result is a direct $M$-part generalization of Theorem 1.3:
Theorem 6.1. Let $\mathcal{F}$ be an $M$-part $\left(L_{1}, \ldots, L_{M}\right)$-Sperner family on the underlying set $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ with $\left|X_{i}\right|=n_{i}$ for $i \in[M]$. Then for every $k$ we have

By (5), the best upper bound in (27) is obtained for $k=1$.
Proof. Take an arbitrary $k$ and $F \subseteq X \backslash X_{k}$, and assume $f_{i}=\left|F \cap X_{i}\right|$ for $i \neq k$. Define $\mathcal{F}^{\prime}(F):=\left\{E \subseteq X_{k}\right.$ : $F \cup E \in \mathcal{F}\}$. Then $\mathcal{F}^{\prime}(F)$ is a "classical", i.e. 1-part $L_{k}$-Sperner family, and therefore Theorem 1.3 gives

$$
\begin{equation*}
\sum_{E \in \mathcal{F}^{\prime}(F)} \frac{1}{\binom{n_{k}}{|E|}} \leqslant L_{k} . \tag{28}
\end{equation*}
$$

From this we can write for any fixed $f_{1}, \ldots, f_{k-1}, f_{k+1}, \ldots, f_{M}$,

$$
\sum_{\substack{F: F \subseteq X \backslash X_{k} \\\left|F \cap X_{i}\right|=f_{i}, i \neq k}} \sum_{E \in \mathcal{F}^{\prime}(F)} \frac{1}{\binom{n_{k}}{|E|} \prod_{i: i \neq k}\binom{n_{i}}{f_{i}}} \leqslant L_{k} .
$$

Finally, summing up:

$$
\begin{equation*}
\sum_{f_{1}=0}^{n_{1}} \cdots \sum_{f_{k}=0}^{\widehat{n_{k}}} \cdots \sum_{f_{M}=0}^{n_{\substack{F: F \subseteq X \backslash X_{k} \\\left|F \cap X_{i}\right|=f_{i}, i \neq k}} \sum_{E \in \mathcal{F}^{\prime}(F)} \frac{1}{\binom{n_{k}}{|E|} \prod_{i: i \neq k}\binom{n_{i}}{f_{i}}} \leqslant L_{k} \prod_{i=1, i \neq k}^{M}\left(n_{i}+1\right), ~} \tag{29}
\end{equation*}
$$

and (29) is clearly equivalent to (27).

### 6.2. Results on homogeneity

Theorem 6.2. If equality holds everywhere in (5), i.e. $\frac{n_{i}}{L_{i}+1}$ is constant for $i=1,2, \ldots, M$, the RHS of (27) simplifies to $\frac{L_{k}}{n_{k}+1} \prod_{i=1}^{M}\left(n_{i}+1\right)$ for every $k$. In this special case equality in (27),

$$
\begin{equation*}
\sum_{i_{1}, \ldots, i_{M}} \frac{P_{i_{1}, \ldots, i_{M}}(\mathcal{F})}{\binom{\mu}{i_{1}} \cdots\binom{\mu}{i_{M}}}=\frac{L_{k}}{n_{k}+1} \prod_{i=1}^{M}\left(n_{i}+1\right) \tag{30}
\end{equation*}
$$

implies that the family is homogeneous. In particular, the conclusion holds for $n_{1}=n_{2}=\cdots=n_{M}=\mu$ and $L_{1}=L_{2}=\cdots=L_{M}=L$.

Proof. In the described case of equality, we have equality in (28), and by Theorem 1.3, $L_{k}$ full levels provide the E's. For any $k$, any subset of $X \backslash X_{k}$, can be extended with nothing or with everything from any particular level of $X_{k}$ to obtain an element of $\mathcal{F}$. This amounts to the homogeneity of $\mathcal{F}$.

Example 6.3. The conclusion of Theorem 6.2 may fail, even for 2-part Sperner families, if they satisfy (30), but they are not of maximum size.

Take $L=1, X=X_{1} \cup X_{2}$ with $X_{1}=\{1,2,3\}, X_{2}=\{4,5\}$, thus $n_{1}=3$ and $n_{2}=2$. In this setup the RHS of inequality (27) is $n_{2}+1=3$. Consider now the following 2 -part Sperner family

$$
\mathcal{F}:=\{\{1\},\{2\},\{3\},\{4\},\{1,2,3,5\},\{1,2,4,5\},\{1,3,4,5\},\{2,3,4,5\}\} .
$$

$\mathcal{F}$ is not homogeneous, as $\{4\} \in \mathcal{F}$ but $\{5\} \notin \mathcal{F}$. Still, equality holds in (27):

$$
\sum_{i, j} \frac{P_{i j}(\mathcal{F})}{\binom{3}{i}\binom{2}{j}}=\frac{1}{\binom{3}{0}\binom{2}{1}}+\frac{1}{\binom{3}{3}\binom{2}{1}}+\frac{3}{\binom{3}{1}\binom{2}{0}}+\frac{3}{\binom{3}{2}\binom{2}{2}}=3=n_{2}+1
$$

Unfortunately, the following tempting conjecture is false:
"Conjecture" 6.4. If a maximum size M-part $\left(L_{1}, \ldots, L_{M}\right)$-Sperner family $\mathcal{F}$ on the underlying set $X=X_{1} \uplus$ $X_{2} \uplus \cdots \uplus X_{M}$ with $\left|X_{i}\right|=n_{i}$ for $i \in[M]$ satisfies (27) with equality for $k=1$, then $\mathcal{F}$ is homogeneous.

Support for this conjecture are the cases $M=1$ with any $L$ (Theorem 1.3), $M=2, L=1$ (in [2]), and whatever is covered by Theorem 6.2, including arbitrary $M, L$ with equal class sizes. However,

Claim 6.5. "Conjecture" 6.4 is false, even for $L=1$.

Example A. Consider the following example: $M=2, n_{1}=3, n_{2}=2, X_{1}=\{1,2,3\}, X_{2}=\{4,5\}$, and $L_{1}=L_{2}=3$. The homogeneous family $\mathcal{F}=\mathcal{F}(I)$ with the transversal

$$
I=\{(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(0,0),(0,1),(0,2)\}
$$

is of maximum size and equality (27) holds for $\mathcal{F}$. To obtain $\mathcal{F}^{\prime}$, remove one out of two elements of type $(0,1)$, say $\{5\}$, and add the element $\{1,2,3,5\}$ of type $(3,1)$, which was not in $\mathcal{F}$. The new family $\mathcal{F}^{\prime}$ is 2-part 3 -Sperner; it is not homogeneous but has maximum size. Moreover, equality in (27) holds for $\mathcal{F}^{\prime}$ as well.

Example B. An even stronger example with $M=3$ and $L=1$ is as follows. Take $n_{1}=2 k+1$ sufficiently large, and $n_{2}=n_{3}=2$. Take $\mathcal{F}=\mathcal{F}(I)$, where $I$ is the transversal:

$$
\begin{aligned}
I= & \{(k, 1,1),(k+1,1,0),(k-1,1,2),(k+1,0,1),(k-1,0,0), \\
& (k, 0,2),(k-1,2,1),(k, 2,0),(k+1,2,2)\} .
\end{aligned}
$$

We claim that $\mathcal{F}$ is an optimal 3-part Sperner family with $L=1$. (This is not difficult to check with brute force.) Consider now the subfamily $E$ corresponding to ( $k-1,1,2$ ).

Partition $E$ into $E_{1} \cup E_{2}$ where $E_{1}=\binom{[2 k+1]}{k-1} \times\{2 k+2\} \times\{2 k+4,2 k+5\}$ and $E_{2}=\binom{[2 k+1]}{k-1} \times$ $\{2 k+3\} \times\{2 k+4,2 k+5\}$.

Replace now $E_{2}$ by the family $G=\binom{[2 k+1]}{k+2} \times\{2 k+3\} \times\{2 k+4,2 k+5\}$, obtaining a new 3-part Sperner family $F^{*}$ which is not homogeneous. Clearly $|\mathcal{F}|=\left|\mathcal{F}^{*}\right|$ and $\mathcal{F}^{*}$ satisfies (27) with equality. (Note that we could have done similar replacement for $(k-1,2,1)$ or for both $(k-1,1,2)$ and $(k-1,2,1)$. The idea in general is simple: if in an optimal homogeneous family $F$, one of two symmetric levels is not used then we can do a similar replacement.)

The following Claim may be useful for positive results on homogeneity.
Claim 6.6. Assume that an $X=X_{1} \uplus X_{2} \uplus \cdots \uplus X_{M}$ with $\left|X_{i}\right|=n_{i}$ for $i \in[M]$ satisfies (5) with equalities, and that every homogeneous maximum size $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner family satisfies equality (30). Then every maximum size $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner family is homogeneous.

Proof. Assume that $\mathcal{F}$ is a maximum size $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner family on $X$. The cardinality of any family is the sum of the entries of its profile vector, $|\mathcal{F}|=\sum_{i} \mathcal{P}_{i}(\mathcal{F})$. Due to Theorem 3.4 the extreme points of the $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner polytope are profile vectors of homogeneous families, and therefore $\mathcal{P}(\mathcal{F})=\sum_{\ell} \lambda_{\ell} \mathcal{P}\left(\mathcal{H}_{\ell}\right)$. We have $|\mathcal{F}|=\sum_{i} \mathcal{P}_{i}(\mathcal{F})=\sum_{i} \sum_{\ell} \lambda_{\ell} \mathcal{P}_{i}\left(\mathcal{H}_{\ell}\right)=$ $\sum_{\ell} \lambda_{\ell}\left|\mathcal{H}_{\ell}\right|$. As $\mathcal{F}$ is of maximum size, so are all $\mathcal{H}_{\ell}$ 's. By the assumption, all $\mathcal{P}\left(\mathcal{H}_{\ell}\right)$ 's satisfy (30). By linearity, so does $\mathcal{P}(\mathcal{F})$, and Theorem 6.2 implies the homogeneity of $\mathcal{F}$.

## 7. An exact conjecture and some possible tools to prove it

7.1. Main conjecture-the case $L=1, n_{i}=2^{\ell}-1$

We will describe a conjecture for $M$-part Sperner families, where the partition sizes are all equal and of the form $2^{\ell}-1$ for some natural number $\ell$. This in particular means that the number of levels within a single partition class is a power of 2 .

Assume that $a, b$ are natural numbers written in binary representation. We define the number $a \oplus b$ as follows: write $a \oplus b$ in binary representation as well, and make the coefficient of $2^{i}$ equal to 1 , if the coefficient of $2^{i}$ was 1 in exactly one of the binary representations of $a$ and $b$, otherwise 0 . With other words, consider the binary representations of $a, b$ as vectors in $\mathbb{Z}_{2}^{k}$ for some sufficiently large $k$, and let $a \oplus b$ be the number whose binary representation is the sum of these two vectors in $\mathbb{Z}_{2}^{k}$. (This operation is also known as NIM addition.)

For example $\left(1+2+2^{3}\right) \oplus\left(1+2^{2}+2^{3}+2^{4}\right)=2+2^{2}+2^{4}$. Clearly $\oplus$ defines a group structure on $\mathbb{N}$, where 0 is the identity, every element has order 2 . For every $\ell$, the set [ $0,2^{\ell}-1$ ] is a subgroup.

For natural numbers $j, t$, let $|j \cap t|$ denote the number of powers of 2 that occur in the binary representation of both $j$ and $t$. For example, $|8 \cap 4|=0$, but $|7 \cap 19|=2$.

Let $\Delta_{\ell}$ be the set of permutations on $\left[0,2^{\ell}-1\right]$ that put the numbers $\binom{2_{\ell}^{\ell}-1}{\pi_{\ell}(i)}$ into decreasing order. The only freedom in such a permutation is whether $\binom{2^{\ell}-1}{k}$ or $\binom{2^{\ell}-1}{2^{\ell}-1-k}$ comes first, i.e. $\pi \in \Delta_{\ell}$ precisely when $\{\pi(2 i), \pi(2 i+1)\}=\left\{2^{\ell-1}+i, 2^{\ell-i}-1-i\right\}$ for each $i \in\left[0,2^{\ell-1}-1\right]$.

The group structure immediately implies that for any choice of $\ell, M$, permutations $\pi_{1}, \ldots, \pi_{M} \in$ $\Delta_{\ell}$, and $j \in\left[0,2^{\ell}-1\right]$,

$$
\begin{align*}
I_{j}\left(\pi_{1}, \ldots, \pi_{M}\right)=\left\{\left(x_{1}, x_{2}, \ldots, x_{M}\right):\right. & x_{i} \in\left[0,2^{\ell}-1\right], \text { and } \\
& \left.\left(\pi_{1}\right)^{-1}\left(x_{1}\right) \oplus \cdots \oplus\left(\pi_{M}\right)^{-1}\left(x_{M}\right)=j\right\} \tag{31}
\end{align*}
$$

is a full transversal, which corresponds to an $M$-part Sperner family. We define

$$
\begin{aligned}
& \mathfrak{I}_{j}^{(M)}=\left\{I_{j}\left(\pi_{1}, \ldots, \pi_{M}\right): \pi_{i} \in \Delta_{\ell}\right\}, \\
& \mathfrak{F}_{j}^{(M)}=\left\{\mathcal{F}: I(\mathcal{F}) \in \mathfrak{I}_{j}^{(M)}\right\} .
\end{aligned}
$$

It is easy to see that for any fixed $j$, any two elements of $\mathfrak{F}_{j}^{(M)}$ have the same size. We leave it to the reader to verify that for all $i \in\left[0,2^{\ell-1}-1\right]$ we have $\mathfrak{I}_{2 i}^{(M)}=\mathfrak{I}_{2 i+1}^{(M)}$, and also that for all $\mathcal{F} \in \mathfrak{F}_{j}^{(M)}$ we have that for all $\pi \in \Delta_{\ell}$,

$$
\begin{equation*}
|\mathcal{F}|=\sum_{i_{1} \oplus i_{2} \oplus \cdots \oplus i_{M}=j} \prod_{t=1}^{M}\binom{2^{\ell}-1}{\pi\left(i_{t}\right)} . \tag{32}
\end{equation*}
$$

Let us see a few simple examples.
If $\ell=1$, then the common size of the partition classes is 1 . The elements of $\Delta_{1}$ are the two permutations on the set $\{0,1\}, \mathfrak{I}_{0}^{(M)}=\mathfrak{I}_{1}^{(M)}$ has two elements: the set of vectors whose coordinates sum to an even number and the set of vectors whose coordinates sum to an odd number. $\mathfrak{F}_{0}^{(M)}=$ $\mathfrak{F}_{1}^{(M)}=\{\{A:|A|$ is even $\},\{A:|A|$ is odd $\}\}$; so for $\mathcal{F} \in \mathcal{F}_{0}^{(M)}$ we have $|\mathcal{F}|=2^{M-1}$.

If $M=1$, then for $j \in\left[0,2^{\ell-1}-1\right]$ we have that $\mathfrak{I}_{2 j}^{(1)}=\mathfrak{I}_{2 j+1}^{(1)}=\left\{\left\{\left(2^{\ell-1}+j\right)\right\},\left\{\left(2^{\ell-1}-1-j\right)\right\}\right\}$, and $\mathfrak{F}_{2 j}^{(1)}=\mathfrak{F}_{2 j+1}^{(2)}=\left\{\left(\begin{array}{c}X_{1}{ }^{\ell-1}+j\end{array}\right),\left(\begin{array}{c}{ }_{2^{\ell-1}-1-j}^{X_{1}}\end{array}\right)\right\}$, in particular for $\mathcal{F} \in \mathfrak{F}_{2 j}^{(1)}$ we have $|\mathcal{F}|=\left(\begin{array}{c}2^{\ell \ell-1}+j\end{array}\right)$.

We formulated a conjecture for the maximum size $M$-part Sperner family, if all parts have size $2^{\ell}-1$ :

Conjecture 7.1. For every $M \geqslant 1$ and every $n_{1}=\cdots=n_{M}=2^{\ell}-1$,
(i) If $\mathcal{F} \in \mathfrak{F}_{0}^{(M)}$, then $\mathcal{F}$ is a maximum size $M$-part Sperner family.
(ii) If $\mathcal{F}$ is a homogeneous $M$-part Sperner family and $\mathcal{F} \notin \mathfrak{F}_{0}^{(M)}$, then $\mathcal{F}$ is not of maximum size.
(iii) Furthermore, $F\left(M\left(2^{\ell}-1\right) ; M\right)$, in which the class sizes are not given, is realized by $\mathcal{F} \in \mathfrak{F}_{0}^{(M)}$.

We have some limited evidence for this conjecture. For all $M$, it holds almost trivially for $\ell=1$ (see Section 7.2), and we have a proof (see Theorem 7.3) that it holds for $\ell=2$ as well. For $M=1$ it is the so-called strong Sperner theorem, and for $M=2$ it follows from Theorem 5.2. We verified using computer that the conjecture holds for $M=3, \ell=3$. This conjecture is far from robust and depends on having a product of binomial coefficients as weight in (6). Should we change the weight $S_{i_{1}, \ldots, i_{M}}(I)=\prod_{t=1}^{M}\left(\begin{array}{c}\left.2_{\pi\left(i_{t}\right)}^{\ell}\right)\end{array}\right)$ to $\prod_{t=1}^{M} a_{i_{t}}$ in (32), examples show that the transversal given in $\Im_{0}^{(M)}$ may fail to maximize $\sum_{\left(i_{1}, \ldots, i_{M}\right) \in I} \prod_{t=1}^{M} a_{i_{t}}$, even if $a_{k}$ is a decreasing function of $k$ and repeats every entry. We believe (iii), as this is the case for $M=2$.

Let $\mathcal{F} \in \mathfrak{F}_{k}^{(M)}$. It is not difficult to bring $|\mathcal{F}|$ into closed form-assuming $\ell$ is fixed and $M$ varies.

Theorem 7.2. For each $k \in\left[0,2^{\ell}-1\right]$, let $\mathcal{F}_{k} \in \mathfrak{F}_{k}^{(M)}$ and set $s_{k}^{(M)}=\left|\mathcal{F}_{k}\right|$. Then we have that for any $\pi \in \Delta_{\ell}$,

$$
s_{k}^{(M)}=\frac{1}{2^{\ell}} \sum_{j=0}^{2^{\ell}-1}(-1)^{|j \cap k|}\left(\sum_{i=0}^{2^{\ell}-1}(-1)^{|j \cap i|}\binom{2^{\ell}-1}{\pi(i)}\right)^{M}
$$

or alternatively

$$
s_{2 j}^{(M)}=s_{2 j+1}^{(M)}=2^{M-\ell} \sum_{h=0}^{2^{\ell-1}-1}(-1)^{|j \cap h|}\left(\sum_{i=0}^{2^{\ell-1}-1}(-1)^{|h \cap i|}\binom{2^{\ell}-1}{2^{\ell-1}+i}\right)^{M} .
$$

Proof. Let $\mathfrak{a}=\left(a_{0}, a_{1}, \ldots, a_{2^{\ell}-1}\right)$ be an arbitrarily fixed complex sequence, and set $\mathfrak{c}^{(M)}(\mathfrak{a})=\left(c_{0}^{(M)}(\mathfrak{a})\right.$, $\left.c_{1}^{(M)}(\mathfrak{a}), \ldots, c_{2^{\ell}-1}^{(M)}(\mathfrak{a})\right)$, where

$$
c_{j}^{(M)}(\mathfrak{a})=\sum_{i_{1} \oplus i_{2} \oplus \cdots \oplus i_{M}=j} \prod_{s=1}^{M} a_{i_{s}} .
$$

Recall that the Fourier transform of the sequence $\mathfrak{a}$ over $\mathbb{Z}_{2}^{2 \ell}$ is the sequence $\mathfrak{A}=\left(A_{0}, A_{1}\right.$, $\ldots, A_{2^{\ell}-1}$ ), where $A_{j}=\sum_{i=0}^{2^{\ell}-1}(-1)^{\mid i \cap j} a_{i}$, and the inverse Fourier transform is described by $a_{k}=\frac{1}{2^{\ell}} \sum_{j=0}^{2^{\ell}-1}(-1)^{|j n k|} A_{j}$. Also recall that the $j$ th component of the Fourier transform of the convolution $\mathfrak{c}^{(M)}(\mathfrak{a})$ is the $M$ th power of the $j$ th component of the Fourier transform of $\mathfrak{a}$. Hence, applying inverse Fourier transform to the Fourier transform of $\mathfrak{c}^{(M)}(\mathfrak{a})$, we obtain

$$
\begin{equation*}
c_{k}^{(M)}=\frac{1}{2^{\ell}} \sum_{j=0}^{2^{\ell}-1}(-1)^{|j \cap k|}\left(\sum_{i=0}^{2^{\ell}-1}(-1)^{|j \cap i|} a_{i}\right)^{M} \tag{33}
\end{equation*}
$$

Formula (33) can be further simplified somewhat if $a_{2 i}=a_{2 i+1}=b_{i}$ for $i \in\left[0,2^{\ell-1}-1\right]$. Namely, terms in (33) with odd $j$ are all zero, as

$$
\sum_{i=0}^{2^{\ell}-1}(-1)^{|(2 t+1) \cap i|} a_{i}=\sum_{i=0}^{2^{\ell-1}-1}\left((-1)^{|(2 i) \cap(2 t+1)|} b_{i}+(-1)^{|(2 i+1) \cap(2 t+1)|} b_{i}\right)=0
$$

Terms in (33) with even $j$ also simplify, as

$$
(-1)^{|(2 i) \cap(2 t)|} b_{i}+(-1)^{|(2 i+1) \cap(2 t)|} b_{i}=2(-1)^{|i n t|} b_{i}
$$

and hence for $j=2 t$,

$$
\left(\sum_{i=0}^{2^{\ell}-1}(-1)^{|j \cap i|} a_{i}\right)^{M}=2^{M}\left(\sum_{i=0}^{2^{\ell-1}-1}(-1)^{|t \cap i|} b_{i}\right)^{M}
$$

Thus, for $j \in\left[0,2^{\ell-1}-1\right]$, we have

$$
c_{2 j}^{(M)}=c_{2 j+1}^{(M)}=2^{M-\ell} \sum_{h=0}^{2^{\ell-1}-1}(-1)^{|j \cap h|}\left(\sum_{i=0}^{2^{\ell-1}-1}(-1)^{|h \cap i|} b_{i}\right)^{M} .
$$

Note that by (32), the actual $s_{k}^{(M)}$ that we want to compute is obtained in this way as $c_{k}^{(M)}$ from $a_{i}=$ $\binom{2^{\ell}-1}{\pi(i)}$ and $b_{i}=\binom{2^{\ell}-1}{2^{\ell-1}+i}$. It is easy to derive special cases: for $\ell=2$ we have $s_{0}^{(M)}=2^{M-2}\left(4^{M}+2^{M}\right)$; and for $\ell=3$ we have $s_{0}^{(M)}=2^{M-3}\left(64^{M}+20^{M}+48^{M}+8^{M}\right)$.

Theorem 7.3. Part (i) of Conjecture 7.1 holds for every $M \geqslant 1$ with $\ell=2$.
Proof. Note that by Theorem 3.4 it is enough to prove the conjecture for homogeneous families. Let $s_{j}^{(M)}=|\mathcal{F}|$ for $\mathcal{F} \in \mathfrak{F}_{j}^{(M)}$. Theorem 7.2 gives

$$
s_{j}^{(M)}= \begin{cases}4^{M-1}\left(2^{M}+1\right), & \text { if } j \in\{0,1\}, \\ 4^{M-1}\left(2^{M}-1\right), & \text { if } j \in\{2,3\} .\end{cases}
$$

The above equations imply that for $M \geqslant 2$ we have

$$
\begin{equation*}
4 s_{j}^{(M-1)}+2^{3(M-1)}=s_{j}^{(M)} . \tag{34}
\end{equation*}
$$

To prove the theorem, we do induction on $M$ (the base case $M=1$ is clear). Assume that the size of any homogeneous ( $M-1$ )-part Sperner family is at most $s_{0}^{(M-1)}$ and let $\mathcal{E}$ be a homogeneous $M$-part Sperner family. Let $\pi \in \Delta_{2}$, and for $i \in[0,3]$ let $\mathcal{E}_{i}=\left\{A-X_{M}: A \in \mathcal{E},\left|A \cap X_{M}\right|=\pi(i)\right\}$. Then the $\mathcal{E}_{i}$ 's are disjoint ( $M-1$ )-part Sperner families, so $\sum_{i}\left|\mathcal{E}_{i}\right| \leqslant 2^{3(M-1)}$ and $\left|\mathcal{E}_{i}\right| \leqslant s_{0}^{(M-1)}$. Since the sets $\mathcal{E}_{i}^{\prime}=\left\{A \cup B: A \in \mathcal{E}_{i}, B \in\binom{X_{M}}{\pi(i)}\right\}$ partition $\mathcal{E}$, we have $|\mathcal{E}| \leqslant 3\left|\mathcal{E}_{0}\right|+3\left|\mathcal{E}_{1}\right|+\left|\mathcal{E}_{2}\right|+\left|\mathcal{E}_{3}\right|=$ $2\left(\left|\mathcal{E}_{0}\right|+\left|\mathcal{E}_{1}\right|\right)+\left(\left|\mathcal{E}_{0}\right|+\left|\mathcal{E}_{1}\right|+\left|\mathcal{E}_{2}\right|+\left|\mathcal{E}_{3}\right|\right) \leqslant 4 s_{0}^{(M-1)}+2^{3(M-1)}=s_{0}^{(M)}$.

We hope that the linear programming approach of Section 7.2 would help, but so far we could not succeed.

Perhaps Conjecture 7.1 holds in a more general way, when not all parts are equal, but $n_{i}=2^{\ell_{i}}-1$. In this case the conditions for (31) would change to $x_{i} \in\left[0,2^{\ell_{i}}-1\right]$ and $\pi_{i} \in \Delta_{\ell_{i}}$. For example, we know that for $n_{1}=2 t+1$ odd and $n_{2}=\cdots=n_{M}=3$, the analogue of (31) with $x_{1} \in[0,2 t]$, and for $i>1, x_{i} \in[0,3]$, is optimal. However, for $n_{1}=n_{2}=n_{3}=2$, the natural analogue of (31) gives 20 sets, while it is easy to construct 22 from the following transversal: $(1,1,1),(0,0,1),(2,2,1),(0,1,2)$, $(2,0,2),(1,2,2),(2,1,0),(1,0,0),(0,2,0)$.

### 7.2. Graph theoretical and optimization reformulations

In this subsection we describe a graph theoretical reformulation of the $M$-part 1-Sperner problem. For that end, let $F$ and $G$ be undirected graphs. Their Cartesian or box product is $F \square G$ where $V(F \square G)=V(F) \times V(G)$ and $(u v, x y) \in E(F \square G)$ iff $u=x$ and $(v, y) \in E(G)$ or $(u, x) \in E(F)$ and $v=y$. It is easy to see that in the Cartesian product of $M$ graphs the vertices are $M$-vectors of vertices, and two $M$-vector is connected in the product if $M-1$ coordinates of the vectors are equal with each other, while the $M$ th coordinate forms an edge in the corresponding graph.

Consider now the graphs $H_{1}, \ldots, H_{M}$ where $H_{i}$ is the comparability graph of the Boolean algebra $\mathcal{B}_{\left|X_{i}\right|}$ : its vertices are all the subsets in $X_{i}$ (there are $2^{\left|X_{i}\right|}$ vertices) and two are connected if one subset contains the other one. It is easy to check that the independent subsets in the Cartesian product of those graphs correspond to the $M$-part Sperner systems in the underlying set $\bigcup X_{i}$.

Take, for example, $n_{1}=\cdots=n_{M}=1$. In this case $H_{i}=P_{2}$, and the Cartesian product of the $H_{i}$ 's is the $M$-dimensional hypercube. The $M$-dimensional hypercube is bipartite and its independence number is $2^{M-1}$, so $f\left(n_{1}=1, \ldots, n_{M}=1\right)=2^{M-1}$.

If we restrict our interest to homogeneous $M$-part Sperner families, as we can do by Theorem 3.4, the graph theoretic problem simplifies to finding a maximum weight independent set in the Cartesian product of weighted paths $P_{n_{i}+1}, i=1,2, \ldots, M$; where the weights on the path vertices are $\binom{n_{i}}{0},\binom{n_{i}}{1}, \ldots,\binom{n_{i}}{n_{i}}$ in this order, and the weight of a vertex in the product graph is the product of the weights of its components.

A good survey about this problem for the Cartesian product is S. Klavžar [24]. So far results on the independence number of Cartesian product graphs have not yielded new results in Sperner theory. The maximum size of the independent subsets in the Cartesian (or actually any graph) product is a very hard problem in general.

Furthermore, looking for the maximum size of $M$-part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{m}\right)$-Sperner families for a given partition of the underlying set $X$, we want an $I \subset \Pi$, such that for every $j=1,2, \ldots, M$ :

$$
\begin{equation*}
\forall i_{1}, i_{2}, \ldots, \widehat{i_{j}}, \ldots, i_{M}, \quad\left|\left\{k:\left(i_{1}, i_{2}, \ldots, i_{j-1}, k, i_{j+1}, \ldots, i_{M}\right) \in I\right\}\right| \leqslant L_{j} \tag{35}
\end{equation*}
$$

and want to maximize $\sum_{i_{1}, i_{2}, \ldots, i_{M}} S_{i_{1}, i_{2}, \ldots, i_{M}}(I)$. Note that for every fixed $j$, the collection of I's satisfying (35) makes the independent sets of a matroid over the underlying set $\Pi$. So our optimization problem is a weighted matroid intersection problem. The general weighted matroid intersection problem is polynomially solvable for $M=2$, and is NP-hard for $M \geqslant 3$ [15]. (This could be another reason contributing to the difficulty with $M \geqslant 3$.) In particular, we can compute in polynomial time the maximum size of 2-part ( $n_{1}, n_{2} ; L_{1}, L_{2}$ )-Sperner families.

It is easy to formulate an integer program that finds the maximum size of $M$-part Sperner families. For each $\left(i_{1}, \ldots, i_{M}\right) \in \Pi$ we define

$$
\begin{aligned}
& \mathcal{F}_{i_{1}, i_{2}, \ldots, i_{M}}=\left\{F \in \mathcal{F}:\left|F \cap X_{j}\right|=a_{i_{j}}\right\}, \\
& f_{i_{1}, i_{2}, \ldots, i_{M}}=\left|\mathcal{F}_{i_{1}, i_{2}, \ldots, i_{M}}\right|, \\
& c_{i_{1}, \ldots, i_{M}}=\prod_{j=1}^{M}\binom{n_{j}}{i_{j}}, \\
& x_{i_{1}, i_{2}, \ldots, i_{M}}=\frac{f_{i_{1}, i_{2}, \ldots, i_{M}}}{c_{i_{1}, \ldots, i_{M}}} .
\end{aligned}
$$

Clearly, for homogeneous families, finding the maximum size is equivalent to solving the following integer program:

$$
\begin{align*}
& x_{i_{1}, \ldots, i_{M}} \in\{0,1\}, \\
& \forall k, \forall j \neq k, i_{j} \in\left[0, n_{j}\right], \quad \sum_{j=0}^{n_{k}} x_{i_{1}, \ldots, i_{k-1}, j, i_{k+1}, \ldots, i_{M}} \leqslant 1 . \\
& \text { Maximize } \quad f=\sum_{\left(i_{1}, \ldots, i_{M}\right) \in \Pi} c_{i_{1}, \ldots, i_{M}} x_{i_{1}, \ldots, i_{M}} . \tag{36}
\end{align*}
$$

Our conjecture is equivalent to a conjecture on what an optimal solution $f^{\text {opt }}$ to this integer program is. Moreover, if $f^{\text {opt }}$ is an optimal solution for the real relaxation as well (i.e. when we only require that $0 \leqslant x_{i_{1}, \ldots, i_{M}} \leqslant 1$ obtaining our primal linear program), then we might be able to prove the optimality by finding a solution $g^{\text {opt }}$ to the following dual problem:

For each $k=1,2, \ldots, M$ and $\left(i_{1}, \ldots, \widehat{i_{k}}, \ldots, i_{M}\right)$ define variables $y_{i_{1}, \ldots, \hat{i}_{k}, \ldots, i_{M}}^{(k)}$, and have the dual problem (see Schrijver [32] for the theory of duality):

$$
\begin{align*}
& 0 \leqslant y_{i_{1}, \ldots, \hat{i}_{k}, \ldots, i_{M}}^{(k)}, \\
& \forall\left(i_{1}, \ldots, i_{M}\right) \in \Pi, \quad \sum_{k=1}^{M} y_{i_{1}, \ldots, \hat{i}_{k}, \ldots, i_{M}}^{(k)} \geqslant c_{i_{1}, \ldots, i_{M}} .  \tag{37}\\
& \text { Minimize } \quad g=\sum_{k=1}^{M} \sum_{i_{1}=1}^{n_{1}} \cdots \sum_{i_{k-1}=1}^{n_{k-1}} \sum_{i_{k+1}=1}^{n_{k+1}} \ldots \sum_{i_{M}=1}^{n_{M}} y_{i_{1}, \ldots, \hat{i_{k}}, \ldots, i_{M}}^{(k)} \tag{38}
\end{align*}
$$

with the property that $f^{\text {opt }}=g^{o p t}$. Clearly, if the optimal solutions to the integer-valued problems are not optimal solutions in the primal linear extension, (or if our conjecture for the primal integer program is incorrect), then this approach will not work. However, we have the following reason to hope to use linear programming for Conjecture 7.1.

Proof of Theorem 1.1. (When $n_{1}=n_{2}$.) Use the feasible solution $y_{i}^{2}=\frac{1}{2}\binom{n_{1}}{i}^{2}$ and $y_{j}^{1}=\frac{1}{2}\binom{n_{2}}{j}^{2}$. The crucial condition of the dual problem turns into the following trivial quadratic inequality

$$
\frac{1}{2}\binom{n_{1}}{i}^{2}+\frac{1}{2}\binom{n_{2}}{j}^{2}=y_{i}^{2}+y_{j}^{1} \geqslant c_{i, j}=\binom{n_{1}}{i}\binom{n_{2}}{j}
$$

As $n_{1}=n_{2}=\frac{n}{2}$, we obtain

$$
g=\sum_{i=1}^{n_{1}} y_{i}^{2}+\sum_{j=1}^{n_{2}} y_{j}^{1}=\frac{1}{2}\binom{2 n_{1}}{n_{1}}+\frac{1}{2}\binom{2 n_{2}}{n_{2}}=\binom{n}{\lfloor n / 2\rfloor}
$$

It is not difficult to generalize the primal and dual linear/integer programming problems to $M$ part $\left(n_{1}, \ldots, n_{M} ; L_{1}, \ldots, L_{M}\right)$-Sperner families. In the primal problem we have to require explicitly $x_{i_{1}, \ldots, i_{M}} \leqslant 1$, which came for free earlier, and change 1 in the RHS of (36) to $L_{k}$. In the dual problem we have to introduce extra non-negative variables $w_{i_{1}, \ldots, i_{M}} \geqslant 0$, change $c_{i_{1}, \ldots, i_{M}}$ in the RHS of (37) to $c_{i_{1}, \ldots, i_{M}}-w_{i_{1}, \ldots, i_{M}}$, and in (38), change the objective function to

$$
g=\left(\sum_{i_{1}=1}^{n_{1}} \ldots \sum_{i_{M}=1}^{n_{M}} w_{i_{1}, \ldots, i_{M}}\right)+\sum_{k=1}^{M} \sum_{i_{1}=1}^{n_{1}} \ldots \sum_{i_{k-1}=1}^{n_{k-1}} \sum_{i_{k+1}=1}^{n_{k+1}} \ldots \sum_{i_{M}=1}^{n_{M}} L_{k} y_{i_{1}, \ldots, \hat{i}_{k}, \ldots, i_{M}}^{(k)}
$$

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