Combinatorica 4 (1) (1984) 21—34

INTERSECTING SPERNER FAMILIES AND THEIR CONVEX HULLS

Péter L. ERDŐS, Peter FRANKL and Gyula O. H. KATONA

Dedicated to Paul Erdős on his seventieth birthday

Received 26 January 1983

Let $\mathscr{F}$ be a family of subsets of a finite set of n elements. The vector $(f_0, \dots, f_n)$ is called the profile of $\mathscr{F}$ where f_i denotes the number of i-element subsets in $\mathscr{F}$. Take the set of profiles of all families $\mathscr{F}$ satisfying $F_1 \not\subset F_2$ and $F_1 \cap F_2 \neq \emptyset$ for all $F_1, F_2 \in \mathscr{F}$. It is proved that the extreme points of this set in $\mathbf{R}^{n+1}$ have at most two non-zero components.

1. Definitions, results

1.1. *Convex hull of the Sperner families.* Let X be a finite set of n elements and $\mathscr{F}$ be a family of its subsets ($\mathscr{F} \subset 2^X$). Then $\mathscr{F}_k$ denotes the subfamily of the k-element subsets in $\mathscr{F}$: $\mathscr{F}_k = \{A\colon A \in \mathscr{F},\ |A| = k\}$. Its size $|\mathscr{F}_k|$ is denoted by f_k. The vector $(f_0, f_1, \dots, f_n)$ in the $(n+1)$-dimensional Euclidean space $\mathbf{R}^{n+1}$ is called the *profile* of $\mathscr{F}$.

If α is a finite set in $\mathbf{R}^{n+1}$, the *convex hull* $\langle\alpha\rangle$ of α is the set of all convex linear combinations of the elements of α. We say that $e \in \alpha$ is an extreme point of α iff e is not a convex linear combination of elements of α different from e. It is easy to see that $\langle\alpha\rangle$ is equal to the set of all convex linear combinations of its extreme points. That is, the determination of the convex hull of a set is equivalent to finding its extreme points.

$\mathscr{F}$ is a *Sperner-family* iff it contains no members A, B with $A \subset B$ *(Sperner-property)*. Consider the set σ of all profiles of the *Sperner-families*. The elements of σ can be perfectly characterized by a sequence of complicated inequalities (see [2], [3]). Sometimes it might be more useful to determine a small convex set containing σ. The best one of them is, of course, $\langle\sigma\rangle$. We find $\langle\sigma\rangle$ determining its extreme points (the extreme points of $\langle\alpha\rangle$ are briefly called extreme points of α):

Theorem 1. *The extreme points of the set σ of the profiles of the Sperner-families are*

$$(1) \qquad z = (0, 0, \dots, 0)$$

$$v_i = \Bigl(\underset{\frown}{0}, \underset{\frown}{0}, \dots, 0, \underset{\frown}{\binom{n}{i}}, 0, \dots, \underset{\frown}{0}\Bigr) \quad (0 \leqq i \leqq n).$$

$$\qquad\quad \hat{0}\ \ \hat{1}\ \ \dots \qquad \hat{i} \qquad \dots\ \hat{n}$$

AMS subject classification (1980): 05 C 35; 05 C 65, 52 A 20

Proof. We will show that this is nothing else but the well-known LYM-inequality ([8], [9], [12]):

$$\sum_{i=0}^{n} \frac{f_i}{\binom{n}{i}} \leqq 1. \tag{2}$$

We have to prove two statements:
(a) any element $(f_0, \ldots, f_n)$ is a convex combination of vectors of form (1),
(b) these latter ones are extreme points.

(a) means, by definition, that $(f_0, \ldots, f_n)$ is a linear combination of z and v_i with some non-negative coefficients $\lambda, \lambda_0, \lambda_1, \ldots, \lambda_n$ satisfying

$$\lambda + \sum_{i=0}^{n} \lambda_i = 1.$$

The choice $\lambda_i = f_i \Big/ \binom{n}{i}$ $(0 \leqq i \leqq n)$, $\lambda = 1 - \sum_{i=0}^{n} f_i \Big/ \binom{n}{i}$ satisfies these conditions by (2). Part (b) is also easy. z is an extreme point since all other elements of σ have non-negative coordinates with at least one positive one. Their convex combination cannot be z. On the other hand, if $\mathcal{F}$ is a Sperner-family then $|\mathcal{F}_i| \leqq \binom{n}{i}$ holds with equality only if $\mathcal{F}$ consists of all i-element subsets. Therefore, if $u \in \sigma$ then its i-th coordinate is $\leqq \binom{n}{i}$ with equality only for v_i. Hence v_i is an extreme point. ∎

1.2. *Intersecting Sperner-families.* A family is an *intersecting* family if $A, B \in \mathcal{F}$ implies $A \cap B \neq \emptyset$. A classical theorem concerning intersecting families is the

Erdős—Ko—Rado theorem [4]. *If $\mathcal{F}$ is an intersecting family of k-element $(k \leqq n/2)$ subsets of an n-element set then*

$$\max |\mathcal{F}| = \binom{n-1}{k-1}. \quad ∎$$

Let μ denote the set of profiles of the intersecting Sperner-families. There exist some inequalities in the literature trying to give good necessary conditions for the elements of μ. First Bollobás [1] proved

$$\sum_{1 \leqq i \leqq n/2} \frac{f_i}{\binom{n-1}{i-1}} \leqq 1 \tag{3}$$

later Greene, Katona and Kleitman [5] found

$$\sum_{1 \leqq i \leqq n/2} \frac{f_i}{\binom{n}{i-1}} + \sum_{n/2 < j \leqq n} \frac{f_j}{\binom{n}{j}} \leqq 1 \tag{4}$$

for any $(f_0, f_1, \ldots, f_n)$. Both inequalities are far from describing the convex hull of μ. The main aim of the present papei is to determine the convex hull or in other words the extreme points of μ.

Theorem 2. *The extreme points of the set μ of the profiles of intersecting Sperner families have at most two positive coordinates, more precisely, the extreme points are*

$$z = (0, 0, \ldots, 0),$$

$$v_j = \Bigg(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \ldots, \underset{\widehat{j}}{\binom{n}{j}}, \ldots, \underset{\widehat{n}}{0}\Bigg) \quad (n/2 < j \leqq n),$$

$$w_i = \Bigg(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \ldots, \underset{\widehat{i}}{\binom{n-1}{i-1}}, \ldots, \underset{\widehat{n}}{0}\Bigg) \quad (1 \leqq i \leqq n/2),$$

$$w_{ij} = \Bigg(\underset{\widehat{0}}{0}, \underset{\widehat{1}}{0}, \ldots, \underset{\widehat{i}}{\binom{n-1}{i-1}}, \ldots, \underset{\widehat{j}}{\binom{n-1}{j}}, \ldots, \underset{\widehat{n}}{0}\Bigg) \quad (1 \leqq i \leqq n/2, \quad i+j > n).$$

There is another way to describe the convex hull $\langle\mu\rangle$. Namely, we could list the hyperplanes bordering it. Some of them are trivial because they separate the positive orthant from the other ones, only. The next theorem presents a set of inequalities. The inequalities representing the non-trivial bordering hyperplanes are among them. Sometimes they are more applicable than the form given in Theorem 2. Anyway, we will deduce Theorem 2 from this theorem:

Theorem 3.

$$\sum_{1 \leqq i \leqq n/2} (1 - y_{n-i+1}) \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leqq n-1} y_j \frac{f_j}{\binom{n-1}{j}} \leqq 1 \tag{5}$$

for any $(f_0, f_1, \ldots, f_n) \in \mu$ and for any sequence $y_{\lfloor n/2 \rfloor+1} \geqq y_{\lfloor n/2 \rfloor+2} \geqq \ldots \geqq y_n \geqq 0$ satisfying

$$y_j \leqq 1 - \frac{j}{n} \quad (n/2 < j \leqq n). \tag{6}$$

Obseive that (5) gives (3) and (4) in the cases $y_{\lfloor n/2 \rfloor+1} = \ldots = y_n = 0$ and $y_j = 1 - j/n$ $(n/2 < j \leqq n)$, resp.

1.3. *Weighted extremal hypergraphs.* The classical theorem of Sperner [11] states that a Sperner-family on n elements cannot have more than $\binom{n}{n/2}$ members. The analogous question for intersecting Sperner-families was solved by Milner [10]. Their maximal size is $\binom{n}{\lfloor n/2 \rfloor + 1}$. Let $c(i)$ $(0 \leqq i \leqq n)$ be a given real function. We may need to maximize $\sum_{i=0}^{n} c(i) |\mathscr{F}_i|$, rather than $|\mathscr{F}| = \sum_{i=0}^{n} |\mathscr{F}_i|$, for a certain class of families $\mathscr{F}$. The solution of this question for Sperner-families was a folklore but it was formulated in [6]. We deduce it here from Theorem 1. (Earlier it was deduced from the equivalent (2).) Indeed, we have to maximize $\sum_{i=0}^{n} c(i) f_i$ for the elements of σ. The maximum is attained for at least one extreme point, hence

$$\max \sum_{i=0}^{n} c(i) f_i = \max \left\{0, \max_i c(i) \binom{n}{i}\right\}.$$

Analogously, Theorem 2 implies the next statement:

Theorem 4. *Given a real function $c(i)$ $(0 \leqq i \leqq n)$ $\max \sum c(i)|\mathscr{F}_i|$ for intersecting Sperner-families $\mathscr{F}$ is attained for a family containing members of at most two different sizes, more precisely, for families with profiles listed in Theorem 2.*

1.4. *An application of Theorem 2 for extremal problems for directed hypergraphs.* Let X be a finite set of n elements. A *directed hypergraph* on X is a set of different sequences $(x_1, \ldots, x_k)$ $(x_i \in X,\ x_i \neq x_j$ if $1 \leqq i, j \leqq k,\ i \neq j)$ where k can vary from 0 (empty sequence) to n. The sequences are the *edges* of the directed hypergraph. The first possible extremal problem is the following: what is the maximum number of edges in a directed hypergraph if it does not contain two different edges $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_l)$ such that $(x_1, \ldots, x_k)$ is a subsequence of $(y_1, \ldots, y_l)$ (that is, $x_i = y_{j_i}$, $1 \leqq j_1 < \ldots < j_k \leqq l$). We call these hypergraphs *directed Sperner-hypergraphs.*

Theorem 5. *The maximum number of edges of a directed Sperner-hypergraph on n elements is $n!$.*

Proof. If $x_1, \ldots, x_n$ is any permutation of the elements of X then a directed Sperner-hypergraph contains at most one edge from the sequence $(x_1), (x_1, x_2), \ldots, (x_1, x_2, \ldots$ $\ldots, x_n)$. Hence it cannot contain more than $n!$ edges. All the edges with n or $n-1$ elements, resp. give equality in the theorem. One can easily see that these constructions are the only ones. ∎

If D is a sequence of different elements then $s(D)$ denotes the set of its elements. We may call $s(D)$ the *undirected version* of D. The next theorem answers a problem similar to that of Theorem 5.

Theorem 6. *The maximum number of the edges of a directed Sperner-hypergraph $\mathscr{H}$ satisfying the additional property*

$$\nexists D, E \in \mathscr{H}\colon\ s(D) \cup s(E) = X$$

is $(n-1)!+1$.

Proof. Fix an element $x \in X$. The hypergraph consisting of (x) and of all the sequences of length $n-2$ made from $X-x$ satisfies the conditions of the theorem and has $(n-1)(n-2)!+1$ members. We have to prove that $|\mathscr{H}|$ cannot be more.

Let $\mathscr{M}$ denote the family of the maximal undirected versions of $\mathscr{H}$, that is, $\mathscr{M} = \{A\colon\ (A = s(D),\ D \in \mathscr{H}) \wedge \nexists E\colon\ (E \in \mathscr{H},\ s(E) \supset A,\ s(E) \neq A)\}$. In the next row we use Theorem 5:

$$|\mathscr{H}| = \sum_{D \in \mathscr{H}} 1 = \sum_{A \in \mathscr{M}} |\{D\colon D \in \mathscr{H}, s(D) \subset A\}| \leqq \sum_{A \in \mathscr{M}} |A|!. \tag{7}$$

$\mathcal{M}$ is obviously a Sperner-family and $A, B \in \mathcal{H}$ imply $A \cup B \neq X$. Let $\mathcal{M}^-$ denote the family of the complements of the members of $\mathcal{M}$. Then

$$\sum_{A \in \mathcal{M}} |A|! = \sum_{B \in \mathcal{M}^-} (n-|B|)! = \sum_{k=0}^{n} (n-k)! |(\mathcal{M}^-)_k|. \tag{8}$$

Here $\mathcal{M}^-$ is an intersecting Sperner-family. We may apply Theorem 4 with $c(i) = (n-i)!$. If we show that

$$\sum_{k=0}^{n} (n-k)! f_k \leqq (n-1)!+1 \tag{9}$$

for any extreme point listed in Theorem 2 then (7), (8) and (9) prove the theorem. It is sufficient to prove (9) for v_j $(n/2 < j \leqq n)$ and w_{ij} $(1 \leqq i \leqq n/2,\ i+j > n)$. If $(f_0, \dots, f_n) = v_j$ then we need the trivial inequality $(n-j)! \binom{n}{j} \leqq (n-1)!+1$. If $(f_0, \dots, f_n) = w_{ij}$ then the left hand side of (9) is $(n-i)! \binom{n-1}{i-1} + (n-j)! \binom{n-1}{j} = \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(n-j)}{j!}$ $\leqq \frac{(n-1)!}{(i-1)!} + \frac{(n-1)!(i-1)}{(n-i+1)!}$. If $i=1, 2$, then this quantity is $\leqq (n-1)!+1$. If $3 \leqq i \leqq n/2$ then $1/(i-1)! \leqq 1/2$ and $(i-1)/(n-i+1)! \leqq 1/2$ (the case $n \leqq 4$ should be checked separately) are trivial and imply (9). ▮

2. Proofs

2.1. *Theorem 3 for cyclic permutations.* We first prove Theorem 3. The method of cyclic permutations will be used. Let us fix a cyclic permutation of the elements of X and consider only those sets having consecutive elements in this cyclic permutation. These are called *consecutive sets*. The idea of the method is to prove the statement for a given cyclic permutation with the consecutive sets and then we prove the original statement by some counting argument listing all cyclic permutations [7]. So let us prove now the analogue of Theorem 3:

Lemma. *Let $\mathcal{G}$ be an intersecting Sperner-family of consecutive sets in a cyclic permutation of an n-element set and denote by g_i the number of i-element members of $\mathcal{G}$. The inequality*

$$\sum_{1 \leqq i \leqq n/2} (1 - y_{n-i+1}) \frac{g_i}{i} + \sum_{n/2 < j \leqq n-1} y_j \frac{g_j}{n-j} \geqq 1 \tag{10}$$

holds for any sequence $y_{\lfloor n/2 \rfloor + 1} \geqq \dots \geqq y_n \geqq 0$ satisfying

$$y_j \leqq 1 - \frac{j}{n} \quad (\lfloor n/2 \rfloor < j \leqq n). \tag{11}$$

Proof. Define $r = \min_{A \in \mathcal{G}} |A|$ and $s = n - \max_{A \in \mathcal{G}} |A|$. First we prove the lemma for $r-s \leqq 1$ (Part 1) then we prove it by induction on $r-s > 1$ (Part 2).

We will suppose in the future that

$$r \leqq n/2. \tag{12}$$

The opposite case $r>n/2$ is easy. Indeed, the Sperner-property implies that at most one member of $\mathcal{G}$ can start from one point of X. Therefore $|\mathcal{G}|=\sum_{\lfloor n/2\rfloor<j\leqq n-1} g_j\leqq n$ holds and hence (10) follows:

$$\sum_{\lfloor n/2\rfloor<j\leqq n-1} y_j\frac{g_j}{n-j}\leqq\sum_{\lfloor n/2\rfloor<j\leqq n-1}\left(1-\frac{j}{n}\right)\frac{g_j}{n-j}=\frac{1}{n}\sum_{\lfloor n/2\rfloor<j\leqq n-1} g_j\leqq 1.$$

Part 1. $r-s\leqq 1$. Let A_1 realize the size r, that is, $A_1\in\mathcal{G}$, $|A_1|=r$. Denote the elements of A_1 by $\alpha_1,\alpha_2,\ldots,\alpha_r$ in the order of the fixed cyclic permutation. Since $\mathcal{G}$ is a Sperner-family it can contain at most two sets with α_i as an endpoint or starting point (along the permutation) (Fig. 1). Let us denote them by E_i and S_i, resp. $\mathcal{G}$ is intersecting therefore if both E_i and S_{i+1} are defined then they must intersect "at their other end" (Fig. 2).

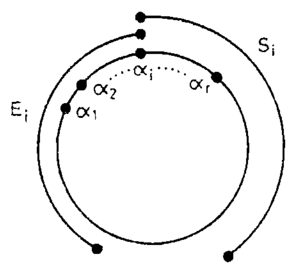

Fig. 1

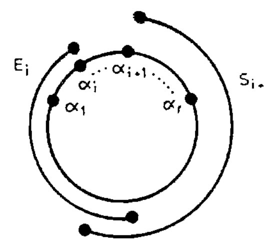

Fig. 2

This implies

(13) $$|E_i|+|S_{i+1}|>n.$$

Introduce the notation

$$w(j)=\begin{cases}\dfrac{1-y_{n-j+1}}{j} & \text{if}\quad 1\leqq j\leqq n/2\\[2ex] \dfrac{y_j}{n-j} & \text{if}\quad n/2<j\leqq n-1.\end{cases}$$

We shall prove the inequality

(14) $$w(|E_i|)+w(|S_{i+1}|)\leqq\frac{1}{r}$$

in several cases where $w(|E_i|)$ and $w(|S_{i+1}|)$ are considered to be 0 if E_i and S_{i+} are not defined, resp.:

a) (14) is trivial if none of E_i and S_{i+1} is defined.

b) If one of them is defined, only (say E_i), and it has a size $\leqq n/2$ then

$$w(|E_i|)=\frac{1-y_{n-|E_i|+1}}{|E_i|}\leqq\frac{1}{r}$$

follows from $|E_i|\geqq r$ and $y_{n-|E_i|+1}\geqq 0$.

c) If one of them (say E_i) is defined, only, and it has a size $>n/2$ then

$$w(|E_i|) = \frac{y_{|E_i|}}{n-|E_i|} \leqq \frac{1}{n} \leqq \frac{1}{r}$$

follows by $y_{|E_i|} \leqq \frac{n-|E_i|}{n}$ (see (11)).

d) If both of them are defined and their sizes are $>n/2$ then $w(|E_i|)$, $w(|S_{i+1}|) \leqq 1/n$ follow like before. Hence (14) is an easy consequence of (12).

e) Suppose now that both E_i and S_{i+1} are defined and one of them (say E_i) has a size $\leqq n/2$. It follows by (13) that $|S_{i+1}| > n/2$. Then we can prove the weaker inequality

$$w(|E_i|)+w(|S_{i+1}|) \leqq \frac{1}{r}+\frac{y_{n-r+1}}{r(r-1)} \tag{15}$$

instead of (14).

$$w(|E_i|) = \frac{1-y_{n-|E_i|+1}}{|E_i|} \leqq \frac{1-y_{n-|E_i|+1}}{r} \tag{16}$$

is a consequence of the definition of r. (13) and the monotonity of y's imply

$$y_{|S_{i+1}|} \leqq y_{n-|E_i|+1}. \tag{17}$$

By the definition of s we have $n-|S_{i+1}| \geqq s \geqq r-1$. Hence and from (17) we obtain

$$w(|S_{i+1}|) = \frac{y_{|S_{i+1}|}}{n-|S_{i+1}|} \leqq \begin{cases} \dfrac{y_{n-|E_i|+1}}{r} & \text{if} \quad n-|S_{i+1}| \geqq r \\ \dfrac{y_{n-r+1}}{r-1} & \text{if} \quad n-|S_{i+1}| = r-1. \end{cases}$$

The sum of (16) and this inequality gives (14) in the first case while in the second case we use $y_{n-|E_i|+1} \geqq y_{n-r+1}$ before the summation:

$$w(|E_i|)+w(|S_{i+1}|) \leqq \frac{1-y_{n-r+1}}{r}+\frac{y_{n-r+1}}{r-1} = \frac{1}{r}+\frac{y_{n-r+1}}{r(r-1)}.$$

(15) is proved.

As any member of $\mathcal{G}$ meets A_1 and no other member can contain it, the possible members of $\mathcal{G}$ are $A_1, E_1, S_2, E_2, S_3, \ldots, E_{r-1}, S_r$ (some of them might be undefined). Hence, applying (15) we obtain the inequality

$$\sum_{A\in\mathcal{G}} w(|A|) \leqq w(|A_1|)+(r-1)\frac{1}{r}+\frac{y_{n-r+1}}{r} = \frac{1-y_{n-r+1}}{r}+\frac{r-1}{r}+\frac{y_{n-r+1}}{r} = 1$$

what is nothing else but the desired (10). We have proved the lemma for $r-s \leqq 1$.

Part 2. Suppose now that $t=r-s>1$ and that the lemma is proved for smaller values of $r-s$. A subfamily $A_1, \ldots, A_b$ of $\mathcal{G}$ is called a *block* if $|A_1| = \ldots = |A_b| = n-s$ and there are consecutive elements $\alpha_0, \alpha_1, \ldots, \alpha_{b+n-s}$ (in this order along the given cyclic permutation) such that

$$A_i = \{\alpha_i, \ldots, \alpha_{i+n-s-1}\} \in \mathcal{G} \quad (1 \leqq i \leqq b)$$

but

$$\{\alpha_0, \alpha_1, \ldots, \alpha_{n-s-1}\} \notin \mathcal{G}, \quad \{\alpha_{b+1}, \ldots, \alpha_{b+n-s}\} \notin \mathcal{G}.$$

We have to distinguish two cases:

2a. $b \leqq s$ for any block in $\mathcal{G}$.

Define the family $\mathcal{G}^* = \{B: (|B| = n-s-1) \wedge (B \text{ consecutive}) \wedge (\exists A: A \in \mathcal{G}, |A| = n-s, A \supset B)\}$. As $\mathcal{G}$ is a Sperner-family $\mathcal{G} \cap \mathcal{G}^* = \emptyset$ follows. Let $\mathcal{G}' = (\mathcal{G} - \mathcal{G}_{n-s}) \cup \mathcal{G}^*$. It is easy to see that $\mathcal{G}'$ is a Sperner-family. On the other hand it is intersecting: $A \cap B \neq \emptyset$ $(A, B \in \mathcal{G}')$ is non-trivial only when one of them (say A) is an element of $\mathcal{G}^*$. Then $|A| = n-s-1$, $|B| \geqq r$ and $r-s > 1$ imply $|A|+|B| > n$, that is, $A \cap B \neq \emptyset$.

We will need the inequality

$$|\mathcal{G}_{n-s}|(s+1) \leqq |\mathcal{G}^*|s. \tag{18}$$

Let $\mathcal{G}_{n-s}$ be divided into blocks of lengths $b_1, \ldots, b_u$ where

$$\sum_{j=1}^{u} b_j = |\mathcal{G}_{n-s}| \leqq us \tag{19}$$

by the suppositions of this case. The block of length b_j induces b_j+1 members into $\mathcal{G}^*$. No element of $\mathcal{G}^*$ comes from two different blocks. Thus $|\mathcal{G}^*| = \sum_{j=1}^{u}(b_j+1)$.

(19) implies $(s+1)\left(\sum_{j=1}^{u} b_j\right) \leqq s \sum_{j=1}^{u}(b_j+1)$ what is nothing else but (18).

The inequality

$$y_{n-s}\frac{g_{n-s}}{s} \leqq y_{n-s-1}\frac{|\mathcal{G}^*|}{s+1} \tag{20}$$

follows by (18) and $y_{n-s} \leqq y_{n-s+1}$. (Observe that $r-s > 1$ implies $n-s-1 > n/2$ unless $r = (n/2)+1$ which is excluded by (12).) We needed (20) for proving that the left hand side of (10) is not less for $\mathcal{G}'$ than for $\mathcal{G}$:

$$\sum_{r \leqq i \leqq n/2}(1-y_{n-i+1})\frac{g_i}{i} + \sum_{n/2 < j \leqq n-s} y_j \frac{g_j}{n-j} \tag{21}$$

$$\leqq \sum_{r \leqq i \leqq n/2}(1-y_{n-i+1})\frac{g_i}{i} + \sum_{n/2 < j < n-s} y_j \frac{g_j}{n-j} + y_{n-s-1}\frac{|\mathcal{G}^*|}{s+1}.$$

The largest sets in $\mathcal{G}'$ have sizes $n-s-1$, thus $s' = s+1$, $t' = r-(s+1) < t$. We may apply the induction hypothesis: (10) holds for $\mathcal{G}'$. Consequently, it also holds for $\mathcal{G}$ by (21). Case $2a$ is settled.

2b. $\mathcal{G}$ contains a block with $b > s$.

Choose an $A_1 \in \mathcal{G}$ with $|A_1| = r$. Let the elements of X be $\alpha_1, \alpha_2, \ldots, \alpha_n$ following the cyclic permutation and suppose that $A_1 = \{\alpha_1, \ldots, \alpha_r\}$. We can list all $(n-s)$-element consecutive sets meeting but not containing A_1:

$$\{\alpha_2, \ldots, \alpha_{n-s+1}\}, \{\alpha_3, \ldots, \alpha_{n-s+2}\}, \ldots, \{\alpha_s, \ldots, \alpha_{n-1}\},$$

$$\{\alpha_{s+1}, \ldots, \alpha_n\}, \{\alpha_{s+2}, \ldots, \alpha_n, \alpha_1\}, \ldots, \{\alpha_{r+1}, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{r-s}\},$$

$$\{\alpha_{r+2}, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{r-s+1}\}, \ldots, \{\alpha_{r+s}, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{r-1}\}.$$

In each of the first and third rows there are $s-1$ sets. On the other hand, the union of A_1 and any set in the middle row is X. It follows from the supposition of this case that some $s+1$ consecutive sets of the above sequence belong to $\mathcal{G}$. One of them belongs to the middle row. Call this set $A_2=\{\alpha_u, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{u-s-1}\}$. Summarizing:

(22) $$|A_1| = r, \quad |A_2| = n-s$$

(23) $$A_1 \cup A_2 = X$$

(24) *any point of* $X-A_2$ *is either a starting point or an endpoint of a set* $A\in\mathcal{G}$, $|A|=n-s$.

It is easy to check that we can have one more assumption:

(25) $A_1\cap A_2$ *is a union of two non-empty intervals* $I=\{\alpha_1, \ldots, \alpha_{u-s-1}\}$ *and* $J=\{\alpha_u, \ldots, \alpha_r\}$.

We shall prove the following statement:

(26) *there are at most* $r-s+1$ *members of* $\mathcal{G}$ *containing* $X-A_2$.

Let $A\neq A_1$ be a member of $\mathcal{G}$ satisfying $A\supset X-A_2$. One of the endpoints of A must be in $I\cup J\cup\{\alpha_{u-1}, \alpha_{u-s}\}$ otherwise one of the conditions $A\supset X-A_2$, $A\not\subset A_2$, $A\not\supset A_1$ would be violated. Moreover, if both endpoints of A are in $I\cup J \cup\{\alpha_{u-1}, \alpha_{u-s}\}$ then they are both either in $I\cup\{\alpha_{u-s}\}$ or in $J\cup\{\alpha_{u-1}\}$. Let $e(A)$ denote the endpoint of A being in $I\cup J\cup\{\alpha_{u-1}, \alpha_{u-s}\}$ if there is only one. If there are two such endpoints let $e(A)$ denote the one being "closer" to $X-A_2$, that is, the endpoint with larger index in $I\cup\{\alpha_{u-s}\}$ and with smaller index in $J\cup\{\alpha_{u-1}\}$. It is easy to check that $e(A)$ is an injection and that $e(A)$ cannot be α_1 or α_r. Therefore $e(A)$ can have at most $|I\cup J|=|A_1\cap A_2|=r-s$ different values. Consequently, the number of sets $A\neq A_1$, $A\supset X-A_2$, $A\in\mathcal{G}$ is at most $r-s$. Including A we obtain the bound (26).

Let us show now that

(27) $$A\in\mathcal{G} \quad \textit{implies} \quad w(|A|) \leqq (1-y_{n-r+1})/r.$$

If $|A|\leqq n/2$ then it is sufficient to substitute $|A|\geqq r$ and $y_{n-|A|+1}\geqq y_{n-r+1}$ into the definition of $w(|A|)$. If $|A|>n/2$ then $y_{|A|}\leqq 1-\dfrac{|A|}{n}$, $r\leqq n/2$ and $y_{n-r+1}\leqq 1-\dfrac{n-r+1}{n}$ lead to

$$w(|A|) = \frac{y_{|A|}}{n-|A|} \leqq \frac{1}{n} \leqq \frac{1}{r}\,\frac{n-r+1}{n} \leqq \frac{1}{r}(1-y_{n-r+1}).$$

(27) is proved.

If $A\in\mathcal{G}$ but $A\cup A_2\neq X$, $A\neq A_2$ then one of the endpoints of A must be in $X-A_2$ (otherwise either $A\cup A_2=X$ or $A\subset A_2$ would follow). Since no member of $\mathcal{G}$ contains another one, any point of $X-A_2$ is an endpoint (starting point) of at most one member of $\mathcal{G}$. Altogether there are $2(s-1)$ such sets $A\in\mathcal{G}$, $A\cup A_2\neq X$, $A\neq A_2$. $s-1$ of them are of size $n-s$ by (24). For the rest we can use (27):

(28) $$\sum_{\substack{A\in\mathcal{G}\\ A\cup A_2\neq X\\ A\neq A_2}} w(|A|) \leqq (s-1)\frac{1-y_{n-r+1}}{r} + (s-1)\frac{y_{n-s}}{s}.$$

Hence we obtain the next upper bound for the left hand side of (10):

$$\sum_{A\in\mathcal{G}} w(|A|) = \sum_{\substack{A\in\mathcal{G}\\ A\cup A_2=X}} w(|A|)+w(|A_2|)+\sum_{\substack{A\in\mathcal{G}\\ A\cup A_2\neq X\\ A\neq A_2}} w(|A|)$$

$$\leqq (r-s+1)\frac{1-y_{n-r+1}}{r}+\frac{y_{n-s}}{s}+(s-1)\frac{1-y_{n-r+1}}{r}+(s-1)\frac{y_{n-s}}{s}$$

$$= 1-y_{n-r+1}+y_{n-s}$$

where (26), (27) and (28) are used. $r-s>1$ implies $n-r+1<n-s$ and therefore $y_{n-r+1}>y_{n-s}$. Indeed, we obtained

$$\sum_{A\in\mathcal{G}} w(|A|) \leqq 1-y_{n-r+1}+y_{n-s} \leqq 1. \quad \blacksquare$$

2.2. *Proof of Theorem 3 using the cyclic permutations.* Let $\mathcal{F}$ be a family with profile $(0, f_1, f_2, \ldots, f_{n-1}, 0)$. The following function will be defined for any cyclic permutation $\mathcal{C}$ of X and for any $A\subset X$:

$$w(\mathcal{C}, A) = \begin{cases} w(|A|) & \text{if } A\in\mathcal{F} \text{ and } A \text{ is consecutive in } \mathcal{C},\\ 0 & \text{otherwise.}\end{cases}$$

We will evaluate the sum $\sum_{\mathcal{C},A} w(\mathcal{C}, A)$ in two different ways: first fixing A, running $\mathcal{C}$ and then in the opposite order.

$$\sum_{\mathcal{C},A} w(\mathcal{C}, A) = \sum_{A\in\mathcal{F}} w(|A|)|A|!(n-|A|)! \tag{29}$$

follows from the fact that there are $|A|!(n-|A|)!$ cyclic permutations in which A is consecutive. On the other hand

$$\sum_{\mathcal{C},A} w(\mathcal{C}, A) = \sum_{\mathcal{C}} \sum_{\substack{A:\, A\in\mathcal{F}\\ A \text{ cons. in } \mathcal{C}}} w(|A|)$$

can be written. Here the last sum is $\leqq 1$ by the Lemma. Consequently

$$\sum_{\mathcal{C},A} w(\mathcal{C}, A) \leqq (n-1)!. \tag{30}$$

Comparing the right hand sides of (29) and (30)

$$\sum_{A\in\mathcal{F}} \frac{w(|A|)}{\dfrac{(n-1)!}{|A|!(n-|A|)!}} \leqq 1$$

can be obtained. Substituting the definition of $w(|A|)$ this inequality gives an equivalent form of (5). $\blacksquare$

2.3. *Proof of Theorem 2 using the duality theorem of linear programming.* 1. First we prove that *if* $(f_0, f_1, \ldots, f_n)\in\mu$ *then there is a convex combination* $(g_0, g_1, \ldots, g_n)$ *of* z, v_j $(n/2<j\leqq n)$ *and* w_{ij} $(1\leqq i\leqq n/2,\ i+j>n)$ *satisfying* $g_j\geqq f_j$ $(0\leqq j\leqq n)$.

Let $u_{\lfloor n/2 \rfloor+1}, \dots, u_{n-1}, u_n$ be a sequence of non-negative reals such that

(31) $$u_j \leqq 1-\frac{j}{n} \quad (n/2 < j \leqq n).$$

Then the sequence

$$y_j = \max_{k \geqq j} u_k \quad (n/2 < j \leqq n)$$

will be monotonic and preserves property (31) (e.g. (6)). On the other hand $u_j \leqq y_j$ $(n/2 < j \leqq n)$ holds, consequently (5) is true for these y and it implies

(32) $$\sum_{1 \leqq i \leqq n/2} \Big(1 - \max_{n-i+1 \leqq j \leqq n} u_j\Big) \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leqq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \leqq 1.$$

Suppose that $u_i \leqq 1 - \max_{n-i+1 \leqq j \leqq n} u_j$ $(1 \leqq i \leqq n/2)$ or equivalently

(33) $$u_i + u_j \leqq 1 \quad (\text{for all } 1 \leqq i \leqq n/2,\ n-i+1 \leqq j \leqq n).$$

Then we can substitute u_i in the place of $1-\max u_j$ in (32). We conclude that

$$\sum_{1 \leqq i \leqq n/2} u_i \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leqq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \leqq 1$$

holds under conditions (31) and (33). The above statement can be formulated in terms of linear programming:

$$\max \left(\sum_{1 \leqq i \leqq n/2} u_i \frac{f_i}{\binom{n-1}{i-1}} + \sum_{n/2 < j \leqq n-1} u_j \frac{f_j}{\binom{n-1}{j}} \right) \leqq 1$$

under constraints (33) and

(34) $$u_j \frac{n}{n-j} \leqq 1 \quad (n/2 < j \leqq n-1) \quad u_n \leqq 0, \quad u_j \geqq 0 \quad (n/2 \leqq j \leqq n).$$

Consider the dual problem. We associate the variables μ_j with constraints (34) and v_{ij} with (33):

(35) $$\min \Big(\sum_{n/2 < j \leqq n-1} \mu_j + \sum_{1 \leqq i \leqq n/2} \sum_{n-i+1 \leqq j \leqq n} v_{ij} \Big) \leqq 1$$

under the constraints

$$\sum_{n/2<j\leq n-1} \nu_{ij} \geqq \frac{f_i}{\binom{n-1}{i-1}} \quad (1 \leqq i \leqq n/2), \tag{36}$$

$$\sum_{1\leq i\leq n/2} \nu_{ij}+\mu_j\frac{n}{n-j} \geqq \frac{f_j}{\binom{n-1}{j}} \quad (n/2 < j \leqq n-1), \tag{37}$$

$$\sum_{1\leq i\leq n/2} \nu_{in}+\mu_n \geqq 0 \quad \text{and} \tag{38}$$

$$\nu_{ij} \geqq 0, \quad \mu_j \geqq 0 \quad (1 \leqq i \leqq n/2,\ n/2 < j \leqq n).$$

(38) is superfluous, (36) and (37) can be rewritten in the forms

$$\sum_{n-i+1\leq j\leq n-1} \nu_{ij}\binom{n-1}{i-1} \geqq f_i \quad (1 \leqq i \leqq n/2,\ n-i+1 \leqq j \leqq n) \tag{39}$$

and

$$\sum_{1\leq i\leq n/2} \nu_{ij}\binom{n-1}{j}+\mu_j\binom{n}{j} \geqq f_j \quad (n/2 < j \leqq n-1). \tag{40}$$

Let us concise (39) and (40) into a vectorial form:

$$\sum_{1\leq i\leq n/2} \nu_{ij}w_{ij}+\sum_{n/2<j\leq n-1} \mu_j v_j \geqq (f_1, f_2, \ldots, f_{n-1}) \tag{41}$$

(where w_{ij} and v_j are truncated; their first and last coordinates are omitted). We obtained that under constraint (41) (35) has a solution $\leqq 1$. In other words, there are non-negative ν's and μ's satisfying (41) with a sum $\leqq 1$. (41) can be easily completed with the 0th and nth coordinates: 1) $f_0=0$ since $\emptyset$ cannot be a member of an intersecting family ($\emptyset\cap\emptyset=\emptyset$); 2) if $f_n=0$ then the situation is the same; if $f_n=1$ then $f_0=\ldots=f_{n-1}=0$ by the Sperner-property and hence $\mu_n=1$ is suitable.

Multiplying all ν_{ij} and μ_j with the appropriate constant ($\geqq 1$) their sum will be equal to 1 as desired.

2. In the first part of the proof we proved that there is a convex combination $(g_0, \ldots, g_n)$ of the vectors v_j and w_{ij} for any given $(f_0, \ldots, f_n)$ such that

$$g_i \geqq f_i \ (0 \leqq i \leqq n). \tag{42}$$

Choose $(g_0, \ldots, g_n)$ maximizing the number of coordinates with equality in (42). Suppose that this number is $<n+1$ and $g_t>f_t$. The vector $(g_0, \ldots, g_{t-1}, 0, g_{t+1}, \ldots, g_n)$ is also a convex combination of the vectors z, v_j, w_i, w_{ij}: we have to change the tth coordinate of each vector for 0; the set $z, v_j, w_i, w_{i,j}$ is closed under this operation. (This is the first place where the vectors z and w_i are used.) $(g_0, \ldots$ $\ldots, g_{t-1}, f_t, g_{t+1}, \ldots, g_n)$ is a convex combination of $(g_0, \ldots, g_{t-1}, 0, g_{t+1}, \ldots, g_n)$ and $(g_0, \ldots, g_n)$ (since $0\leqq f_t\leqq g_t$), therefore it is a convex combination of z, v_j, w_i

and w_{ij}. This new vector $(g_0, \ldots, g_{t-1}, f_t, g_{t+1}, \ldots, g_n)$ has more common coordinates with $(f_0, \ldots, f_n)$ than $(g_0, \ldots, g_n)$ does. This contradiction leads to the statement that $(f_0, \ldots, f_n)$ *itself is a convex combination of the vectors* z, v_j, w_i and w_{ij}.

3. In the second part of the proof we proved that only the vectors listed in Theorem 2 can be extreme points of μ. Now we have to verify that they are really extreme points. This is trivial for z.

It is easy to construct an intersecting Sperner-family with profile w_i $(1 \leqq i \leqq n/2)$: take all the i-element subsets containing a fixed element of the ground set. On the other hand, the Erdős—Ko—Rado theorem implies that if $(f_0, \ldots, f_n) \in \mu$ then $f_i \leqq \binom{n-1}{i-1}$. Hence if w_i is a convex combination of some vectors from μ then they all must have $f_i = \binom{n-1}{i-1}$. Similarly, their other coordinates are necessarily 0. The only such vector is w_i. One can see in the same way that v_j $(n/2 < j \leqq n)$ is in μ and it is an extreme point of μ.

The construction of an intersecting Sperner-family with profile w_{ij}: take all i-element subsets containing a fixed element x and all j-element subsets not containing x. Suppose that w_{ij} is a convex combination of some elements of μ. As above, all of them must have $\binom{n-1}{i-1}$ in the ith coordinate. The only intersecting Sperner-family with $\binom{n-1}{i-1}$ i-element sets is the above construction of all i-element subsets containing x. No j-element set can contain x. It would then contain an i-element set as a subset. Hence $(f_0, \ldots, f_n) \in \mu$ and $f_i = \binom{n-1}{i-1}$ imply $f_j \leqq \binom{n-1}{j}$. Therefore all the vectors in the convex combination must have $\binom{n-1}{j}$ as jth coordinate. Like above, the other coordinates are 0. w_{ij} is the only such vector, therefore it is really an extreme point of μ.

References

[1] B. Bollobás, Sperner systems consisting of pairs of complementary subsets, *J. Combinatorial Theory A* **15** (1973), 363—366.

[2] G. F. Clements, A minimization problem concerning subsets of a finite set, *Discrete Math.* **4** (1973), 123—128.

[3] D. E. Daykin, J. Godfrey and A. J. W. Hilton, Existence theorems for Sperner families, *J. Combinatorial Theory A* **17** (1974), 245—251.

[4] P. Erdős, Chao Ko and R. Rado, Intersection theorems for systems of finite sets, *Quart. J. Math. Oxford (2)* **12** (1961), 313—318.

[5] C. Greene, G. O. H. Katona and D. J. Kleitman, Extensions of the Erdős—Ko—Rado theorem, *SIAM* **55** (1976), 1—8.

[6] G. O. H. Katona, Two applications of Sperner type theorems, *Periodica Math. Hungar.* **3** (1973), (3) 19—26.

[7] G. O. H. Katona, A simple proof of Erdős—Ko—Rado theorem, *J. Combinatorial Theory B* **13** (1972), 183—184.

[8] D. Lubell, A short proof of Sperner's Lemma, *J. Combinatorial Theory* **1** (1966), 299.

[9] L. D. Meshalkin, A generalization of Sperner's theorem on the number of subsets of a finite set, *Teor. Verojatnost. i Primen.* **8** (1963), 219—220 (in Russian).

[10] E. C. MILNER, A combinatorial theorem on systems of sets, *J. London Math. Soc.* **43** (1968), 204—206.
[11] E. SPERNER, Ein Satz über Untermenge einer endlichen Menge, *Math. Z.* **27** (1928), 544—548.
[12] K. YAMAMOTO, Logarithmic order of free distributive lattices, *J. Math. Soc. Japan* **6** (1954), 343—353.

Péter L. Erdős
Institute of Mathematics and
Computer Science
K. Marx University of Economics
Budapest, Pf. 482, H-1828, Hungary

Peter Frankl
CNRS
15 Quai Anatole France
75007 Paris, France

Gyula O. H. Katona
Mathematical Institute of the
Hungarian Academy of Sciences
Budapest, Pf. 428, H-1395, Hungary