# Convex Hulls of More-Part Sperner Families 

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#### Abstract

The convex hulls of more-part Sperner families is defined and studied. Corollaries of the results are some well-known theorems on 2 or 3-part Sperner families. Some methods are presented giving new theorems.


## 1. Introduction

In the last 25 years the well known Sperner Theorem [22] has been generalized in many directions. One of these directions are the so called more-part Sperner families.

The first result, proved independently by Katona [13] and Kleitman [18] is the next one.

2-part Sperner Theorem. Let $X=X_{1} \cup X_{2}, X_{1} \cap X_{2}=\varnothing$ and let $\mathscr{F}$ be a family of subsets of $X$ (i.e. $\mathscr{F} \subseteq 2^{X}$ ) such that $F_{1}, F_{2} \in \mathscr{F}$ and any of the next two conditions

$$
\begin{array}{ll}
F_{1} \cap X_{1}=F_{2} \cap X_{1} & \text { and } \tag{1}
\end{array} \quad F_{1} \cap X_{2} \subset F_{2} \cap X_{2}, ~ 子 \quad \text { and } \quad F_{1} \cap X_{2}=F_{2} \cap X_{2}
$$

imply $F_{1}=F_{2}$. Then $|\mathscr{F}| \leq\binom{\mathrm{n}}{\lfloor n / 2\rfloor}$ and this is the best upper bound.
If $X_{2}=\varnothing$ then this theorem gives the classic Sperner Theorem. Moreover, if $X_{1}$ and $X_{2}$ are non-empty then this theorem is a sharpening of it: the conditions of the 2-part Sperner theorem are weaker but the statement is the same.

Let $X=X_{1} \cup \cdots \cup X_{M}$ be a partition of $X . \mathscr{F} \subseteq 2^{X}$ is called an $M$-part Sperner family if $\mathscr{F}$ does not contain two members $E, F$ such that $E \cap X_{i} \subset F \cap X_{i}$ holds for some $i$ while $E \cap X_{j}=F \cap X_{j}$ for all $j \neq i$. A simple example of [13] shows that a 3-part Sperner family may contain more than $\binom{n}{\lfloor n / 2\rfloor}$ members. Papers [17] and [11] found additional conditions to ensure this upper bound. The question of the maximum sized $M$-part Sperner family is not really solved. [8] and [12] give good asymtotic results. (See also [21].) The very recent [9] is a very good survey of $M$-part Sperner type problems.

If $\mathscr{F}$ is a family of subsets, its profile is the vector $\left(p_{0}, p_{1}, \ldots, p_{n}\right)$ where $p_{i}$ is the number of $i$-element members of $\mathscr{F}$. Take the class of all families on $X$ satisfying a certain property. The profiles of the families belonging to this class form a set of points in $\mathbb{R}^{n+1}$ having integer components. In [2], [3] and [7] the extreme points of this set are determined for several classes of families.

The main aim of the present paper is to obtain analogous results for $M$-part Sperner families and other families given on the $M$-parts. Here the profile will be, however, an $M$-dimensional matrix whose $i_{1} i_{2} \ldots i_{M}$ th entry is the number of members of the family satisfying $\left|F \cap X_{j}\right|=i_{j}(1 \leq j \leq M)$. Our Theorem 2.1 determines the extreme points of the class of the $M$-part Sperner families.

In Section 3 we prove this theorem in a much more general form (Theorem 3.2). Namely, we show that the extreme points (matrices) of many classes of families on $M$ parts mimic the properties of the class. Griggs, Odlyzko and Shearer [12] proved that there are (sizewise) homogeneous families $\mathscr{F}$ maximizing $|\mathscr{F}|$ for $M$-part Sperner families. A consequence of Theorem 3.2 is that this is true for many other classes, too.

In Section 4 several known results are deduced as consequences of the theorems of Sections 2 and 3. Theorem 4.2 is the above mentioned 2-part Sperner theorem. Theorems 4.3 [20] and 4.6 [10] consider Sperner families of subsets meeting the first part (of 2 parts) in at least $l$ elements.

Let us remark that the families themselves are really investigated only until the extreme points are determined. To deduce the above mentioned consequences in Section 4, we deal with "numbers", only. The families can be forgotten.

## 2. Definitions, Notations and the Main Result

In this section we introduce the necessary definitions, notations and formulate the main theorem. It will be analogous to [2] and [3].

Put $X=X_{1} \cup X_{2} \cup \cdots \cup X_{M}, X_{i} \cap X_{j}=\varnothing(i \neq j),\left|X_{i}\right|=n_{i},|X|=n$. Let $\mathscr{H} \subset 2^{X}$ be a family of subsets. The $M$-dimensional matrix $P(\mathscr{H}):=\left(p_{i_{1}}, \ldots, i_{M}\right) i_{j}=0, \ldots, n_{j}$, is called the profile-matrix of $\mathscr{H}$, where

$$
p_{i_{1}, \ldots, i_{M}}(\mathscr{H}):=\left|\left\{H \in \mathscr{H}: \forall_{j}\left|H \cap X_{j}\right|=i_{j}\right\}\right| .
$$

The matrix $P(\mathscr{H})$ can also be considered as a vector of $\left(n_{1}+1\right)\left(n_{2}+1\right) \ldots\left(n_{M}+1\right)$ components. If $M=1$ then we obtain the profile $P(\mathscr{H})$ of $\mathscr{H}$ (see [2] or [3]). In this way $P(\mathscr{H})$ is a point of the Euclidean space $\mathbb{R}^{\left(n_{1}+1\right) \ldots\left(n_{M}+1\right)}=\mathbb{R}^{N}$.

If $\alpha$ is a finite set in $\mathbb{R}^{N}$, the convex hull $\langle\alpha\rangle$ of $\alpha$ is the set of all convex linear combinations of the elements of $\alpha$. We say that $e \in \alpha$ is an extreme point of $\alpha$ iff $e$ is not a convex linear combination of elements of $\alpha$ different from $e$. The set of extreme points of $\alpha$ is $\varepsilon(\alpha)$. It is well-known that $\langle\alpha\rangle$ is equal to the set of all convex linear combinations of its extreme points. That is, the determination of the convex hull of a set is equivalent to finding its extreme points.

Suppose that the partition $X=X_{1} \cup \cdots \cup X_{M}$ is fixed. Let $\mathbb{A} \subset 2^{2 x}$ be a class of families of subsets of $X$. Denote by $\mu(\mathbb{A})$ the set of all profile-matrices of the families belonging to $\mathbb{A}$. $\left(\mu(\mathbb{A}) \subset \mathbb{R}^{N}\right.$ is obvious.)

A family $\mathscr{H}$ is called an $M$-part Sperner family if $E, F \in \mathscr{H}, E \cap X_{i} \subset F \cap X_{i}$, and $E \cap X_{j}=F \cap X_{j}$ for all $j \neq i$ (for all fixed $i$ ) imply $E=F$. Denote by $\mathbb{S}^{M}$ the family
of all $M$-part Sperner families for the given partition. (The notation should contain $X_{1}, \ldots, X_{M}$; they are omitted for sake of brevity.)

Theorem 2.1. The extreme points of $\left\langle\mu\left(\mathbb{S}^{M}\right)\right\rangle$ are the matrices $S(I)$ with the entries

$$
S_{i_{1}, \ldots, i_{M}}(I)= \begin{cases}0 & \text { if }\left(i_{1}, \ldots, i_{M}\right) \notin I \\ \binom{n_{1}}{i_{1}}\binom{n_{2}}{i_{2}} \cdots\binom{n_{M}}{i_{M}} & \text { otherwise },\end{cases}
$$

where $I$ is any set in $\left(0, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\} \times \cdots \times\left\{0, \ldots, n_{M}\right\}$ such that $\left(i_{1}, \ldots\right.$, $\left.i_{j}, \ldots, i_{M}\right) \in I,\left(i_{1}, \ldots, i_{j}^{\prime}, \ldots, i_{M}\right) \in I$ imply $i_{j}=i_{j}^{\prime}$.

## 3. Proof of the Main Theorem

The proof is analogous to those of [2] and [3]. First we formulate and prove a lemma similar to the cyclic permutation method (see [15]) then we prove our main theorem.

We say that $\mathscr{L}$ is a product-chain of $X=X_{1} \cup \cdots \cup X_{M}$ if $\mathscr{L}=\left(x_{1}, \ldots, x_{n}\right)$ is a permutation of $X$ and $\left\{x_{i}: i=n_{1}+\cdots+n_{j-1}+1, \ldots, n_{1}+\cdots+n_{j}\right\}=X_{j}$, that is, if the elements of $X_{j}$ are consecutive in $\mathscr{L}$ for every $j$. Furthermore, we say that a subset $H \subset X$ is initial with respect to $\mathscr{L}$ if

$$
H \cap X_{j}=\left\{x_{n_{1}+\cdots+n_{j-1}+1}, \ldots, x_{n_{1}+\cdots+n_{j-1}+\left|H \cap X_{j}\right|}\right\}
$$

that is, equal to the set of the first $\left|H \cap X_{j}\right|$ elements of $X_{j}$. If $\mathscr{H} \subset 2$ then $\mathscr{H}(\mathscr{L})$ denotes those members of $\mathscr{H}$ which are initial with respect to $\mathscr{L}$. It is easy to see that the profile-matrix $P(\mathscr{H}(\mathscr{L}))$ is a 0,1 matrix.

Sometimes it is easier to use the set of places with entry 1 rather than the 0,1 matrix itself. Let $I$ be a subset of $\left\{0, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\} \times \cdots \times\left\{0, \ldots, n_{M}\right\}$. Then $T(I)$ denotes the matrix having 1 's in the place belonging to $I$ and 0 's otherwise:

$$
t_{i_{1}, \ldots, i_{M}}(I)= \begin{cases}1 & \text { if }\left(i_{1}, \ldots, i_{M}\right) \in I \\ 0 & \text { if }\left(i_{1}, \ldots, i_{M}\right) \notin I\end{cases}
$$

The definition of $S(I)$ given in Theorem 2.1 is analogous, but 1 is replaced by $\binom{n_{1}}{i_{1}}\binom{n_{2}}{i_{2}} \cdots\binom{n_{M}}{i_{M}}$.

If $\mathbb{A} \subset 2^{2^{x}}$, introduce the notation $\mathbb{A}(\mathscr{L})=\{\mathscr{H}(\mathscr{L}): \mathscr{H} \in \mathbb{A}\}$ for any productchain $\mathscr{L}$. Then, by definition, $\mu(\mathbb{A}(\mathscr{L}))=\{P(\mathscr{H}(\mathscr{L})): \mathscr{H} \in \mathbb{A}\}$. As $P(\mathscr{H}(\mathscr{L}))$ are 0,1 matrices, the set of extreme points $\varepsilon(\mathbb{A}(\mathscr{L}))=\mu(\mathbb{A}(\mathscr{L}))$.

The next lemma plays a fundamental role in the proof of Theorem 2.1 and is analogous to Theorem 4 (Blowing up the circle) of [3]. It shows an important connection between $\varepsilon(\mathbb{A})$ and $\varepsilon(\mathbb{A}(\mathscr{L}))$.

Lemma 3.1. (Blowing up the product-chain). Suppose that $\varepsilon(\mathbb{A}(\mathscr{L}))=\mu(\mathbb{A}(\mathscr{L}))$ does not depend on $\mathscr{L}$. Then

$$
\begin{equation*}
\mu(\mathbb{A}) \subseteq\langle\{S(I): T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))\}\rangle \tag{2}
\end{equation*}
$$

holds.

Proof. To prove (2) it is enough to show that for every $\mathscr{H} \in \mathbb{A}$ there are coefficients $\mu(I)\left(\mu(I) \geq 0, \sum_{T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))} \mu(I)=1\right)$ such that $P(\mathscr{H})=\sum_{T(I) \in \varepsilon\left(\mathbb{A}_{( }\left(\mathscr{L}_{)}\right)\right.} \mu(I) S(I)$.

Consider the sum

$$
\begin{equation*}
\sum P(\{H\}) / n_{1}!n_{2}!\ldots n_{M}! \tag{3}
\end{equation*}
$$

for all pairs $(\mathscr{L}, H)$ where $\mathscr{L}$ is a product-chain, $H \in \mathscr{H}$ and $H$ is initial with respect the product-chain $\mathscr{L}$. We use double counting for this sum.

$$
\begin{align*}
\sum_{(\mathscr{L}, H)} P(\{H\}) / n_{1}!\ldots n_{M}! & =\sum_{\mathscr{L}} \frac{1}{n_{1}!\ldots n_{M}!} \sum_{\substack{\text { is initial } \text { with } \\
\text { respect to } \mathscr{L}}} P(\{H\})  \tag{4}\\
& =\sum_{\mathscr{L}} \frac{1}{n_{1}!\ldots n_{M}!} P(\mathscr{H}(\mathscr{L})) .
\end{align*}
$$

Observe that $P(\mathscr{H}(\mathscr{L})) \in \mu(\mathbb{A}(\mathscr{L}))=\varepsilon(\mathbb{A}(\mathscr{L}))$, therefore for any $\mathscr{L}$ there is a unique $I$ such that $T(I)=P(\mathscr{H}(\mathscr{L}))$. Collect the equal terms on the right hand side of (4):

$$
\begin{equation*}
\sum_{\mathscr{L}} \frac{1}{n_{1}!\ldots n_{M}!} P(\mathscr{H}(\mathscr{L}))=\sum_{T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))} \lambda(I) T(I) \tag{5}
\end{equation*}
$$

where $\lambda(I)$ is the proportion of the $n_{1}!\ldots n_{M}$ ! permutations such that $P(\mathscr{H}(\mathscr{L}))=$ $T(I)$. Therefore the sum of the coefficients $\lambda(I)$ is 1 . Hence we have

$$
\begin{equation*}
\sum_{(\mathscr{Q}, H)} P(\{H\}) / n_{1}!\ldots n_{M}! \tag{6}
\end{equation*}
$$

is a convex linear combination of the elements of $\varepsilon(\mathbb{A}(\mathscr{L}))$.
On the other hand, summing the other way around we obtain

$$
\begin{align*}
\sum_{(\mathscr{Q}, H)} P(\{H\}) / n_{1}!\ldots n_{M}! & =\sum_{H} \sum_{\mathscr{\mathscr { L }}} P(\{H\}) / n_{1}!\ldots n_{M}! \\
& =\sum_{H \in \mathscr{H}} \frac{i_{1}!\left(n_{1}-i_{1}\right)!\ldots i_{M}!\left(n_{M}-i_{M}\right)!}{n_{1}!\ldots n_{M}!} P(\{H\}) \tag{7}
\end{align*}
$$

where $\left|H \cap X_{j}\right|=i_{j}(1 \leq j \leq M) . i_{1}!\left(n_{1}-i_{1}\right)!\ldots i_{M}!\left(n_{M}-i_{M}\right)$ ! is the number of product-chains satisfying this condition and $H$ is initial with respect to these $\mathscr{L}$. The last sum is equal to

$$
\begin{equation*}
\sum_{H \in \mathscr{H}} \frac{P(\{H\})}{\binom{n_{1}}{i_{1}} \cdots\binom{n_{M}}{i_{M}}}=\left(\frac{p_{i_{1}, \ldots, i_{M}}(\mathscr{H})}{\binom{n_{1}}{i_{1}} \ldots\binom{n_{M}}{i_{M}}}\right)_{\forall j: i_{j}=0, \ldots, n_{j}} \tag{8}
\end{equation*}
$$

It follows by $(6),(7)$ and $(8)$ that the right hand side of $(8)$ is a convex linear combination of the elements of $\varepsilon(\mathbb{A}(\mathscr{L}))$. This implies that $P(\mathscr{H})$ is a convex linear combination of the matrices $S(I)$ such that $T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))$.

Lemma 3.1 becomes really useful if $S(I) \in \mu(\mathbb{A})$ holds for all $T(I) \in \varepsilon(\mathbb{A}(\mathscr{L}))$. Then, by (2), any element of $\mu(\mathbb{A})$ is a convex linear combination of them. We use this idea to prove Theorem 2.1.
Proof of Theorem 2.1. Lemma 3.1 will be applied with $\mathbb{A}=\mathbb{S}^{M}$.
The elements of $\mu\left(\mathbb{S}^{M}(\mathscr{L})\right)$ are 0,1 matrices. $I \subset\left\{0, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\} \times$ $\cdots \times\left\{0, \ldots, n_{M}\right\}$ is called a partial transversal if $\left(i_{1}, \ldots, i_{j}, \ldots, i_{M}\right) \in I$ and $\left(i_{1}, \ldots\right.$,
$\left.i_{j}^{\prime}, \ldots, i_{M}\right) \in I$ imply $i_{j}=i_{j}^{\prime}$. It is easy to see that $\mu\left(\mathbb{S}^{M}(\mathscr{L})\right)=\{T(I)$ :I is a partial transversal $\}$.

It does not depend on $\mathscr{L}$, therefore the lemma can be applied. It follows by (2) that any element of $\mu\left(\mathbb{S}^{M}\right)$ is a convex linear combination of the matrices $S(I)$ where $I$ are partial transversals.

For a partial transversal I take the family

$$
\mathscr{F}=\bigcup_{\left(i_{1}, \ldots, i_{M}\right) \in I}\left\{F:\left|F \cap X_{j}\right|=i_{j} \text { for all } 1 \leq j \leq M\right\} .
$$

which is obviously an $M$-Sperner family, that is, the matrices $S(I)$ are all in $\mu\left(\mathbb{S}^{M}\right)$. On the other hand, none of them is a convex linear combination of the other ones. Hence they are the extreme points of $\mu\left(\mathbb{S}^{M}\right)$.

Lemma 3.1 and the above (trivial) proof allow a much more general form:
Theorem 3.2. Suppose that $\mathbb{A} \subseteq 2^{2^{x}}$ satisfies the following two conditions:

$$
\begin{gather*}
\varepsilon(\mathbb{A}(\mathscr{L}))=\mu(\mathbb{A}(\mathscr{L})) \text { does not depend on } \mathscr{L}  \tag{9}\\
T(I) \in \mu(\mathbb{A}(\mathscr{L})) \text { implies } S(I) \in \mu(\mathbb{A}) \tag{10}
\end{gather*}
$$

Then

$$
\begin{equation*}
\varepsilon(\mathbb{A})=\{S(I): T(I) \in \mu(\mathbb{A}(\mathscr{L})) \tag{11}
\end{equation*}
$$

that is, the extreme points of $\mu(\mathbb{A})$ are obtained by replacing the 1's by $\binom{n_{1}}{i_{1}} \cdots\binom{n_{M}}{i_{M}}$ in the $i_{1}, \ldots, i_{M}$ th entries of the elements of $\mu(\mathbb{A}(\mathscr{L}))$.

In what follows, we show many consequences of this general theorem.
A family $\mathscr{F}$ on $X=X_{1} \cup \cdots \cup X_{M}$ is called homogeneous if $F \in \mathscr{F}$ implies $G \in \mathscr{F}$ for all $G \subset X$ satisfying $\left|F \cap X_{j}\right|=\left|G \cap X_{j}\right|$ (for all $1 \leq j \leq M$ ). Observe that the matrices $S(I)$ in the theorem are profile-matrices of homogeneous families belonging to $A$.

$$
|\mathscr{F}|=\sum_{i_{1}=0}^{n_{1}} \ldots \sum_{i_{M}=0}^{n_{M}} p_{i_{1}, \ldots, i_{M}}
$$

where $p_{i_{1}, \ldots . i_{M}}$ are the entries of the profile-matrix $P(\mathscr{F})$. This is a linear function of $p_{i_{1}, \ldots, i_{M}}$ so it assumes its maximum in one of the extreme points given in Theorem 3.2. As we mentioned, these points are profile-matrices of homogeneous families. Therefore the following statement is true:

Corollary 3.3. Suppose that $\mathbb{A}$ satisfies (9) and (10). Then there is a homogeneous $\mathscr{F} \subset \mathbb{A}$ maximizing $|\mathscr{F}|$ in $\mathbb{A}$.

Of course, the proof of the corollary above shows that it is true for any linear function of the variables $p_{i_{1}, \ldots, i_{M}}$.

Griggs, Odlyzko and Shearer [12] proved Corollary 3.3 for the special case when $\mathbb{A}$ is $\mathbb{S}^{M}$, the class of all $M$-part Sperner families. In [9], jointly with Füredi, they independently proved it for several other special cases. Of course, they used other means.

The next theorem presents the extreme points of the set of the profile-matrices
of the Sperner families ( $=1-$ Sperner families). That is, we consider the Spernercondition on $X$, but the profile-matrices are defined by the partition $X=X_{1} \cup \cdots \cup$ $X_{M} \cdot \mathbb{S}^{1}=\mathbb{S}$ denotes the set of all Sperner families on $X$. We call the set $I \subset$ $\left\{0, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\} \times \cdots \times\left\{0, \ldots, n_{M}\right\}$ an antichain iff $\left(i_{1}, \ldots, i_{M}\right),\left(j_{1}, \ldots\right.$, $\left.j_{M}\right) \in I$ and $i_{1} \leq j_{1}, \ldots, i_{M} \leq j_{M}$ imply $i_{1}=j_{1}, \ldots, i_{M}=j_{M}$.

Theorem 3.4.

$$
\varepsilon(\mathbb{S})=\{S(I): \text { I is an antichain }\} .
$$

The proof is analogous to that of Theorem 2.1.
This theorem can be generalized toward a theorem of Paul Erdös [1]. A family $\mathscr{F} \subset 2^{x}$ is called $k$-Sperner iff it contains no $k+1$ distinct members satisfying $F_{1} \subset F_{2} \subset \cdots \subset F_{k+1}$. Similarly, a set $I \subset\left\{0, \ldots, n_{1}\right\} \times \cdots \times\left\{0, \ldots, n_{M}\right\}$ is a $k$ antichain iff it contains no $k+1$ distinct elements $\left(i_{1}^{1}, \ldots, i_{M}^{1}\right), \ldots,\left(i_{1}^{k+1}, \ldots, i_{M}^{k+1}\right)$ such that $i_{j}^{1} \leq i_{j}^{2} \leq \cdots \leq i_{j}^{k+1}$ holds for all $j(1 \leq j \leq M)$.

## Theorem 3.5.

$$
\mu(k \text {-Sperner families })=\{S(I): I \text { is a } k \text {-antichain }\} .
$$

We need some more definitions for another variant of these theorems. The family $\mathscr{F} \subset 2^{X}$ is an $M$-part $k$-Sperner family iff there are no $k+1$ distinct members such that $F_{1} \subset F_{2} \subset \cdots \subset F_{k+1}$ and $F_{k+1}-F_{1} \subset X_{j}$ holds for some $j(1 \leq j \leq M)$. Analogously, a partial $k$-transversal $I$ contains no $k+1$ distinct members in $\left\{0, \ldots, n_{1}\right\} \times \cdots \times\left\{0, \ldots, n_{M}\right\}$ with $M-1$ equal components.

## Theorem 3.6.

$$
\mu(M \text {-part } k \text {-Sperner families })=\{S(I): I \text { is a partial } k \text {-transversal }\} .
$$

We mention 3 other theorems determining $\max \{|\mathscr{F}|: \mathscr{F} \in \mathbb{A}\}$ for some classes $A$ on 2 and 3 parts. 1) [14] gives a common generalization of the 2-part Sperner theorem and Paul Erdös's $k$-Sperner theorem. 2) [17] gives an additional condition to ensure the maximum to be $\binom{n}{\lfloor n / 2\rfloor}$. 3) [11] gives a nicer addition condition to ensure the same conclusion. In all these cases the extreme points can be easily determined by Theorem 3.2.

We developed Theorem 3.2 for classes of families on $M$ parts, however, for our great surprise, it gives new results for 1 part, too. Let $\mathbb{F}_{k}$ denote the class of families $\mathscr{F} \subset 2^{X}(|X|=n)$ such that $F_{1} \subset F_{2}, F_{1} \neq F_{2}$ imply $\left|F_{2}-F_{1}\right| \geq k$. max $\{|\mathscr{F}|$ : $\left.\mathscr{F} \in \mathbb{F}_{k}\right\}$ was determined for this class in [15]. Theorem 3.2 allows us to describe the extreme points, too.

Theorem 3.7. The extreme points of the set $\mu\left(\mathbb{F}_{k}\right)$ of the profile-vectors are of the form $\left(0, \ldots, 0,\binom{n}{i_{1}}, 0, \ldots, 0,\binom{n}{i_{2}}, 0, \ldots, 0,\binom{n}{i_{r}}, 0, \ldots, 0\right)$ where $r \geq 0$ and the non-zero components are separate with at least $k-1$ zeros.

This theorem was independently discovered and proved by our-friend P. Frankl [6], too.

Kleitman [19] observed, that the result maximizing $|\mathscr{F}|$ in $\mathbb{F}_{k}$ can be generalized for other classes. This can be done here, again. If $A \subset 2^{2^{x}}$ is a class of families (no partition of $X$ ) and $A$ is described by conditions of type $\left(\nexists F_{1}, \ldots, F_{m} \in \mathscr{F}\right.$ such that $F_{1} \subset F_{2} \subset \cdots \subset F_{m}$ and $\left|F_{m}-F_{m-1}\right|, \ldots,\left|F_{2}-F_{1}\right|$ are in a certain set) then Theorem 3.2 determines the extreme points, therefore $\max |\mathscr{F}|$ can also be calculated.

## 4. Applications: Old Theorems as Consequences

In this section we will prove some old results using the previous theorems.
Actually, Theorem 2.1 is nothing else but the determination of the possible constructions maximizing any given linear function of the entries $p_{i_{1}, \ldots, i_{M}}$ of the profile matrix. However, the number of possible maximal constructions is too large for most of the functions. It is very hard to compare them to determine the really best one.

The most interesting case, of course, when the linear function is just the sum $\sum p_{i_{1}, \ldots, i_{M}}$, that is, when $|\mathscr{F}|$ has to be maximized. If $M=2$ the above comparison can be done by the following lemma.

Lemma 4.1. Suppose that $a_{1} \geq \cdots \geq a_{u} \geq 1$ and $b_{1} \geq \cdots \geq b_{v} \geq 1$ are integers. Choose entries from the matrix $\left(a_{i} \cdot b_{j}\right)_{1 \leq i \leq u, 1 \leq j \leq v}$, at most one from each row and each column. Their sum cannot exceed

$$
\sum_{i=1}^{\min \{u, 0\}} a_{i} \cdot b_{i} .
$$

Proof. The statement is proved by induction on $\min \{u, v\}$. Let $I \subset\{1, \ldots, u\} \times$ $\{1, \ldots, v\}$ be a set containing at most one element in each row and each column. We shall show that there is an $I^{\prime}$ containing $(1,1)$ and satisfying

$$
\begin{equation*}
\sum_{(i, j) \in I} a_{i} \cdot b_{j} \leq \sum_{(i, j) \in I} a_{i} \cdot b_{j} . \tag{12}
\end{equation*}
$$

Suppose first that $I$ contains no element with 1 as a component. Then $I^{\prime}=$ $I \cup\{(1,1)\}$ satisfies the requirements. If $(1, x) \in I$ but $(y, 1) \notin I$ for all $y$ then choose $I^{\prime}=$ $(I-\{(1, x)\}) \cup\{(1,1)\}$. Finally, if $(1, x) \in I(y, 1) \in I$, then let $I^{\prime}=(I-\{(1, x),(y, 1)\}) \cup$ $\{(1,1),(y, x)\}$.

To verify (12) take the difference of the two sides:

$$
\sum_{(i, j) \in I} a_{i} \cdot b_{j}-\sum_{(i, j) \in I} a_{i} \cdot b_{j}=a_{1} b_{1}+a_{y} b_{x}-a_{1} b_{x}-a_{y} b_{1}=\left(a_{1}-a_{y}\right)\left(b_{1}-b_{x}\right) \geq 0 .
$$

This proves that $(1,1) \in I$ can be supposed. The induction hypothesis, applied with $a_{2}, \ldots, a_{u}$ and $b_{2}, \ldots, b_{v}$, proves the desired result.

Apply this lemma with_

$$
\begin{aligned}
& \binom{n_{1}}{\left\lceil\frac{n_{1}}{2}\right\rceil} \geq\binom{ n_{1}}{\left\lceil\frac{n_{1}}{2}\right\rceil-1} \geq\binom{ n_{1}}{\left\lceil\frac{n_{1}}{2}\right\rceil+1} \geq\binom{ n_{1}}{\left\lceil\frac{n_{1}}{2}\right\rceil-2} \geq \ldots \text { and } \\
& \left(\left\lfloor\begin{array}{c}
n_{2} \\
\left\lfloor\frac{n_{2}}{2}\right. \\
\rfloor
\end{array}\right) \geq\binom{ n_{2}}{\left\lfloor\frac{n_{2}}{2}\right\rfloor^{\prime}+1} \geq\left(\left\lfloor\frac{n_{2}}{n_{2}}\right\rfloor^{2}-1\right) \geq\left(\left\lfloor\frac{n_{2}}{2}\right\rfloor^{n_{2}}+2\right) \geq \ldots\right.
\end{aligned}
$$

and place of the $a$ 's and $b$ 's, respectively. (12) is equal to

$$
\begin{aligned}
& \left(\begin{array}{c}
n_{1} \\
{\left[\frac{n_{1}}{2}\right.} \\
\rceil
\end{array}\right)\left(\left\lfloor\begin{array}{c}
n_{2} \\
\frac{n_{2}}{2} \\
\hline
\end{array}\right)+\left(\left[\begin{array}{c}
n_{1} \\
\left.\frac{n_{1}}{2}\right\rceil-1
\end{array}\right)\left(\left\lfloor\begin{array}{c}
n_{2} \\
n_{2} \\
\hline
\end{array}\right)+1\right)+\left(\left[\frac{n_{1}}{2}\right\rceil+1\right)\left(\left\lfloor\frac{n_{2}}{2}\right]^{n_{2}}-1\right)+\cdots\right.\right. \\
& \quad=\sum_{i}\binom{n_{1}}{i}\left(\left\lceil\frac{n_{1}}{2}\right\rceil+\left\lfloor\frac{n_{2}}{2}\right\rfloor-i\right)=\left(\left\lfloor\frac{n_{1}+n_{2}}{2}\right\rfloor\right)
\end{aligned}
$$

in this case.
Let $\mathscr{F} \in \mathbb{S}^{2}\left(X=X_{1} \cup X_{2},\left|X_{1}\right|=n_{1},\left|X_{2}\right|=n_{2}, n_{1}+n_{2}=n\right)$. By Theorem 2.1 (or [12]) we know that

$$
|\mathscr{F}| \leq \max _{I} \sum_{(i, j \in I}\binom{n_{1}}{i}\binom{n_{2}}{j}
$$

where the maximum is over all partial transversals $I$. By Lemma 4.1 and the above calculations this sum is at most $\left(\begin{array}{c}n \\ \frac{n}{2} \\ \rfloor\end{array}\right)$ :

Theorem 4.2. (Kleitman [18], Katona [13])

$$
\max \left\{|\mathscr{F}|: \mathscr{F} \in \mathbb{S}^{2}\right\}=\binom{n}{\frac{n}{2}} .
$$

Now, let $X$ be again partitioned $X_{1} \cup X_{2}$ and denote by $\mathbb{S}(l)$ the set of Sperner families $\mathscr{F} \subset 2^{X}\left(F_{1}, F_{2} \in \mathscr{F}, F_{1} \neq F_{2} \Rightarrow F_{1} \notin F_{2}\right)$ satifying the additional condition

$$
F \in \mathscr{F} \Rightarrow\left|F \cap X_{1}\right| \geq l .
$$

Theorem 4.3. (Lih [20]) If $0<n_{1} \leq n_{2}$ then

$$
\max \{|\mathscr{F}|: \mathscr{F} \in \mathbb{S}(1)\}=\binom{n}{\left[\frac{n}{2}\right]}-\binom{n_{2}}{\left[\begin{array}{c}
n \\
\frac{n}{2}
\end{array}\right]} .
$$

Proof. If $\mathscr{F} \in \mathbb{S}(1)$ then its profile-matrix $P(\mathscr{F})=\left(p_{i j}\right)$ has only zeros in the 0th row. Hence $|\mathscr{F}|=\sum_{i=1}^{n_{1}} \sum_{j=0}^{n_{2}} p_{i j}$ follows. The latter sum is a linear combination of the entries of $P(\mathscr{F})$. It can be maximum only for the extreme points determined by Theorem 3.4. Therefore, we have to find

$$
\max _{I} \sum_{(i, j) \in I}\binom{n_{1}}{i}\binom{n_{2}}{j}
$$

where the maximum runs over all antichains $I \subset\left\{1, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\}$. As an antichain is necessarily a partial transversal, we may use Lemma 4.1 with binomial coefficients, again. The only difference is that $\binom{n_{1}}{0}$ is missing now. If $n_{1}$ is odd, $\binom{n_{1}}{0}$
is the last one in the above ordering. Its coefficient is

$$
\left(\left\lceil\frac{n_{1}}{2}\right\rceil+\left\lfloor\frac{n_{2}}{2}\right\rfloor\right)
$$

The sum is smaller by this amount. On the other hand, if $n_{1}$ is even then $\binom{n_{1}}{n_{1}}$ is the last number in the ordering, it leaps forward to replace $\binom{n_{1}}{0}$. Therefore the coefficient of $\binom{n_{1}}{n_{1}}$, that is,

$$
\binom{n_{2}}{\left\lceil\frac{n_{1}}{2}\right\rceil+\left\lfloor\frac{n_{1}}{2}\right\rfloor-n_{1}}
$$

will be deleted. In both cases the missing term is $\binom{n_{2}}{\left[\begin{array}{c}n \\ 2\end{array}\right]}$.
Remark 4.4. Actually, we proved a sharper theorem than Lih's one. We used only the 2-part Sperner condition rather than the usual Sperner condition. On the other hand, we did not determine here the optimal families as Griggs [10] did. It could be done with the present method, but it is rather inconvenient. (See the analogous result on all 2-part Sperner families in [5].)

In some further investigations the following inequality will be needed. We think it is interesting in its own right, too.

Lemma 4.5. If $I \subset\left\{0, \ldots, n_{1}\right\} \times\left\{0, \ldots, n_{2}\right\}$ is an antichain then

$$
\begin{equation*}
\sum_{(i, j, j \in I} \frac{\binom{n_{1}}{i}\binom{n_{2}}{j}}{\binom{n_{1}+n_{2}}{i+j}} \leq 1 \tag{13}
\end{equation*}
$$

holds.
Proof. Let $I$ be an antichain. Suppose that $(i, j) \in I$ and that $(I-\{i, j\}) \cup\{(i+1, j)\}$ is also an antichain. An easy calculation shows that

$$
\frac{\binom{n_{1}}{i}\binom{n_{2}}{j}}{\binom{n_{1}+n_{2}}{i+j}}<\frac{\binom{n_{1}}{i+1}\binom{n_{2}}{j}}{\binom{n_{1}+n_{2}}{i+1+j}}
$$

holds iff $\frac{n_{2}}{n_{1}+1}<\frac{j}{i+1}$. In this case the sum (13) is increased. We say briefly that the change

$$
\begin{equation*}
(i, j) \rightarrow(i+1, j) \text { is increasing iff } \frac{n_{2}}{n_{1}+1}<\frac{j}{i+1} \tag{14}
\end{equation*}
$$

One can see in the same way that

$$
\begin{gather*}
(i, j) \rightarrow(i-1, j) \text { is increasing iff } \frac{n_{2}}{n_{1}+1}>\frac{j}{i},  \tag{15}\\
(i, j) \rightarrow(i, j+1) \text { is increasing iff } \frac{n_{1}}{n_{2}+1}<\frac{i}{j+1}, \tag{16}
\end{gather*}
$$

and

$$
\begin{equation*}
(i, j) \rightarrow(i, j-1) \text { is increasing iff } \frac{n_{1}}{n_{2}+1}>\frac{i}{j} \tag{17}
\end{equation*}
$$

Suppose that $I$ is an antichain maximizing the left hand side of (13). We will show that $I$ consists of all pairs with a constant $i+j$.

Order the elements of $I$ according to their first components. It is easy to see that the second components are then ordered backwards. Let $(i, j)$ and $(i+k, j-l)$, $k \geq 1, l \geq 1$, be two neighbouring elements of $I$ in this order. Suppose first that $k \geq 2$. The change $(i, j) \rightarrow(i+1, j)$ cannot be increasing by the optimality of $I$. Therefore $\frac{n_{2}}{n_{1}+1} \geq \frac{j}{i+1}$ must hold by (14). Hence $\frac{n_{2}}{n_{1}+1}>\frac{j-l}{i+k}$ follows, consequently $(i+k, j-l)(i+k-1, j-l)$ is an increasing change by (15). This contradiction proves $k=1$.

Suppose now $l \geq 2$. The change $(i, j) \rightarrow(i, j-1)$ cannot be increasing by the optimality of $I$. Therefore $\frac{n_{1}}{n_{2}+1} \leq \frac{i}{j}$ must hold by (17). Hence $\frac{n_{1}}{n_{2}+1}<\frac{i+1}{j-l+1}$ follows, consequently $(i+1, j-l) \rightarrow(i+1, j-l+1)$ is an increasing change by (16). This contradiction proves $l=1$.

We have proved that the neighbouring elements of $I$ are of the form $(i, j)$ and $(i+1, j-1)$. Consequently, there is a constant $c$ such that the optimal $I$ consists of all elements $(i, j)$ satisfying $i+j=c$. For such an $I(13)$ holds with equality, whence it holds for any antichain.

It will be easy to prove the next theorem after this lemma.
Theorem 4.6. (Griggs [10]) $\mathscr{F} \in \mathbb{S}(l)$ implies the inequality

$$
\begin{equation*}
\sum_{F \in \mathscr{F}} \frac{1}{\binom{n-l}{|F|-l}} \frac{\binom{\left|F \cap X_{1}\right|}{l}}{\binom{\left|X_{1}\right|}{l}} \leq 1 \tag{18}
\end{equation*}
$$

Proof. If $\mathscr{F} \in \mathbb{S}(l)$ then its profile-matrix $P(\mathscr{F})=\left(p_{i j}\right)$ has only zeros in the 0th, 1st, $\ldots,(l-1)$ st rows. (18) can be rewritten into the form

$$
\begin{equation*}
\sum_{i=l}^{n_{1}} \sum_{j=0}^{n_{2}} p_{i j} \frac{1}{\binom{n-l}{i+j-l}} \frac{\binom{i}{l}}{\binom{n_{1}}{l}} \leq 1 \tag{19}
\end{equation*}
$$

The left hand side of $(19)$ is a linear combination of the entries of $P(\mathscr{F})$. It attains its maximum for one of the extreme points determined by Theorem 3.4. Therefore, we have to prove (19) only for these extreme points:

$$
\sum_{(i, j) \in I}\binom{n_{1}}{i}\binom{n_{2}}{j} \frac{1}{\binom{n-l}{i+j-l}} \frac{\binom{i}{l}}{\binom{n_{1}}{l}} \leq 1
$$

should be proved for any antichain (not containing elements with a first component $<l$ ). It is easy to check that this is equivalent to

$$
\sum_{(i, j) \in I} \frac{\binom{n_{1}-l}{i-l}\binom{n_{2}}{j}}{\binom{n_{1}+n_{2}-l}{i+j-l}} \leq 1
$$

This latter inequality is a consequence of (13).
Of course, the original proof in [10] is easier and shorter than ours, which however shows some new connections.

We are indebted to the referees for their suggestions.

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Received: August 9, 1985
Revised: December 11, 1985

