

All Maximum 2-Part Sperner Families

PÉTER L. ERDÖS

*Institute of Mathematics and Computer Science, Karl Marx University of Economics,
Budapest, Pf. 482, H-1828, Hungary*

AND

G. O. H. KATONA

*Mathematical Institute of the Hungarian Academy of Sciences,
Budapest, Pf. 127, H-1364, Hungary*

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Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ be a partition of an n -element set. Suppose that the family \mathcal{F} of some subsets of X satisfy the following condition: if there is an inclusion $F_1 \subsetneq F_2$ ($F_1, F_2 \in \mathcal{F}$) in \mathcal{F} , the difference $F_2 - F_1$ cannot be a subset of X_1 or X_2 . Kleitman (*Math. Z.* **90** (1965), 251–259) and Katona (*Studia Sci. Math. Hungar.* **1** (1966), 59–63) proved 20 years ago that $|\mathcal{F}|$ is at most n choose $\lfloor n/2 \rfloor$. We determine all families giving equality in this theorem. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let us start with a classic theorem of Sperner [9]:

If $\mathcal{F} \subseteq 2^X$ is a family of distinct subsets of an n -element set X such that $F_1 \subsetneq F_2$ holds for all $F_1, F_2 \in \mathcal{F}$ then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

Kleitman [6] and Katona [5] independently discovered that the condition of this theorem can be weakened while its statement remains true:

Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ be a partition of X ($|X| = n$). Suppose that the family $\mathcal{F} \subseteq 2^X$ satisfies the following condition:

$$F_1 \subsetneq F_2, F_1, F_2 \in \mathcal{F} \text{ imply } F_2 - F_1 \not\subset X_1 \text{ and } F_2 - F_1 \not\subset X_2. \quad (1)$$

Then

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}. \quad (2)$$

The families satisfying (1) are called *2-part Sperner families*. The main aim of the present paper is to determine all maximum 2-part Sperner families, that is, the ones with equality in (2). It is worth mentioning that all of them have the following *homogeneity* property: if $F \in \mathcal{F}$ then $|F \cap X_1| = |G \cap X_1|$, $|F \cap X_2| = |G \cap X_2|$ imply $G \in \mathcal{F}$. This is not true for more than 2 parts. (See [4] for analogous questions.)

The proof is based on a theorem of [2]. We state it to make the paper self-contained.

Let \mathcal{F} be a 2-part Sperner family, and let p_{ij} denote the number of members $F \in \mathcal{F}$ such that $|F \cap X_1| = i$, $|F \cap X_2| = j$ ($0 \leq i \leq n_1 = |X_1|$, $0 \leq j \leq n_2 = |X_2|$). The *profile-matrix* $P(\mathcal{F})$ is defined by the entries p_{ij} . It can be considered as a point of the $(n_1 + 1)(n_2 + 1)$ -dimensional space. Consider the set μ of all such points. The *extreme points* of μ are the ones which cannot be expressed as convex linear combinations of other points of μ . The next statement determines all extreme points of μ .

THEOREM A Particular case of Theorem 2.1 of [2]). *The extreme points of the set of profile-matrices of all 2-part Sperner families are the $(n_1 + 1) \times (n_2 + 1)$ matrices having either 0 or $\binom{n_1}{i} \binom{n_2}{j}$ as the ij th entry but having at most one non-zero entry in each row or column.*

For interested readers we also suggest the recent survey paper [3] on more-part Sperner theorems.

2. DETAILS

$|\mathcal{F}| = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij}$ is a linear function of the variables p_{ij} . It follows that $|\mathcal{F}|$ will be maximum for some extreme points described in Theorem A and may be for some convex linear combinations of these maximum extreme points.

At first we determine the extreme points maximizing $|\mathcal{F}| = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij}$. The non-zero entries of the extreme points (matrices) are in different rows and in different columns. The *partial transversals* are defined accordingly: $I \subset \{0, \dots, n_1\} \times \{0, \dots, n_2\}$ is a partial transversal iff $(i_1, j_1), (i_2, j_2) \in I$, $(i_1, j_1) \neq (i_2, j_2)$ imply $i_1 \neq i_2, j_1 \neq j_2$. So we have to maximize

$$\sum_{i=0}^{n_1} \sum_{j=0}^{n_2} p_{ij} = \sum_{(i,j) \in I} \binom{n_1}{i} \binom{n_2}{j} \quad (3)$$

for partial transversals I .

It is intuitively clear that (3) is maximum if the great numbers (n_i) are paired with great (n_j)'s and the little ones with little ones. Some easy lemmas leading in this direction are:

LEMMA 1. Let a_1, \dots, a_u and b_1, \dots, b_v be integers and I a partial transversal. Suppose that $(i_1, j_2), (i_2, j_1) \in I$ and $a_{i_1} > a_{i_2}, b_{j_1} > b_{j_2}$ hold and define $I' = (I - \{(i_1, j_2), (i_2, j_1)\}) \cup \{(i_1, j_1), (i_2, j_2)\}$. Then

$$\sum_{(i,j) \in I'} a_i b_j < \sum_{(i,j) \in I} a_i b_j. \quad (4)$$

Proof. We have $\sum_{(i,j) \in I'} a_i b_j - \sum_{(i,j) \in I} a_i b_j = a_{i_1} b_{j_1} + a_{i_2} b_{j_2} - a_{i_1} b_{j_2} - a_{i_2} b_{j_1} = (a_{i_1} - a_{i_2})(b_{j_1} - b_{j_2}) > 0$, proving (4). ■

LEMMA 2. Let $a_1 > a_2 \geq a_3 \geq \dots \geq a_u > 0$ and $b_1 > b_2 \geq b_3 \geq \dots \geq b_v > 0$ be integers. If I is a partial transversal maximizing

$$\sum_{(i,j) \in I} a_i b_j \quad (5)$$

then $(1, 1) \in I$.

Proof. Suppose, on the contrary, that $(1, 1) \notin I$. We will find contradictions distinguishing several cases. If there is no other pair with component 1 in I then any element (i, j) can be replaced by $(1, 1)$ and this is a contradiction by $a_i b_j < a_1 b_1$ and the maximality of I .

If $(i, 1) \in I$ ($i \neq 1$) but $(1, j) \notin I$ for any j then $(i, 1)$ can be replaced by $(1, 1)$. This is a contradiction by $a_i b_j < a_1 b_1$. The case when $(1, j) \in I$ ($j \neq 1$) but $(i, 1) \in I$ holds for no i can be settled in the same way.

Finally suppose that $(i, 1) \in I$ and $(1, j) \in I$ ($i \neq 1 \neq j$). Then $(i, j) \notin I$, because I is a partial transversal. Replacing $(i, 1)$ and $(1, j)$ by $(1, 1)$ and (i, j) , Lemma 1 gives the contradiction. ■

LEMMA 3. Let $a_1 > a_2 \geq a_3 \geq \dots \geq a_u > 0$ and $b_1 = b_2 > b_3 \geq \dots \geq b_v > 0$ be integers. If I is a partial transversal maximizing (5) then either $(1, 1) \in I$ or $(1, 2) \in I$ holds.

Proof. Suppose, on the contrary, that none of them holds. The proof of Lemma 2 can be repeated, since it does not lead here to a contradiction only if $(1, 2)$ is involved. However, it is not in I by the indirect assumption. ■

LEMMA 4. Let $a_1 = a_2 > a_3 \geq \dots \geq a_u > 0$ and $b_1 = b_2 > b_3 \geq \dots \geq b_v > 0$ be integers. If I is a partial transversal maximizing (5) then either $(1, 1), (2, 2) \in I$ or $(1, 2), (2, 1) \in I$ hold.

Proof. Suppose that none of $(1, 1)$, $(2, 2)$, $(1, 2)$, $(2, 1)$ is in I . Then the proof of Lemma 1 leads to a contradiction, since $(2, 2)$, $(1, 2)$, and $(2, 1)$ are not involved in the changes. Hence at least one of $(1, 1)$, $(2, 2)$, $(1, 2)$, and $(2, 1)$, say (i, j) , is in I . Delete a_i and b_j from the numbers. The remaining numbers satisfy the conditions of Lemma 2, thus $(3-i, 3-j) \in I$. ■

Now we are able to determine all partial transversals I maximizing (3); however, we have to distinguish cases according to the parity of n_1 and n_2 .

LEMMA 5. If n_1 and n_2 are both even then I is a partial transversal (3) iff

$$\left(\frac{n_1}{2}, \frac{n_2}{2}\right) \in I \quad (6)$$

and exactly one of the following two rows hold for each $k = 1, 2, \dots, \min \left\{\frac{n_1}{2}, \frac{n_2}{2}\right\}$:

$$\left(\frac{n_1}{2} - k, \frac{n_2}{2} - k\right) \in I, \quad \left(\frac{n_1}{2} + k, \frac{n_2}{2} + k\right) \in I, \quad (7)$$

$$\left(\frac{n_1}{2} - k, \frac{n_2}{2} + k\right) \in I, \quad \left(\frac{n_1}{2} + k, \frac{n_2}{2} - k\right) \in I. \quad (8)$$

Proof. We use first Lemma 2 with the numbers $\binom{n_1}{0}, \dots, \binom{n_1}{n_1}$ and $\binom{n_2}{0}, \dots, \binom{n_2}{n_2}$ ordered decreasingly, respectively. This proves (6). Delete $\binom{n_1}{n_1/2}$ and $\binom{n_2}{n_2/2}$ from the numbers. The remaining numbers satisfy the conditions of Lemma 4, therefore either (7) or (8) holds with $k = 1$. The proof of the necessity of (6)–(8) can be completed by induction.

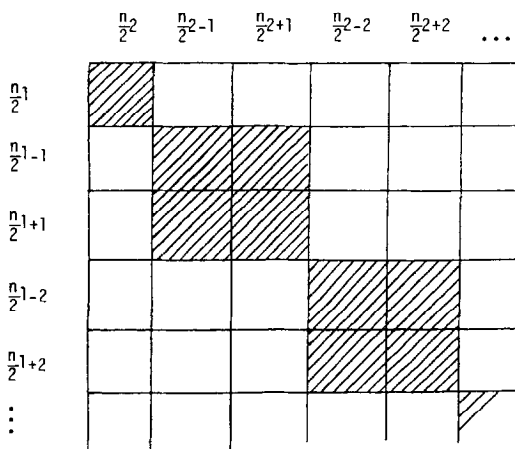


FIGURE 1

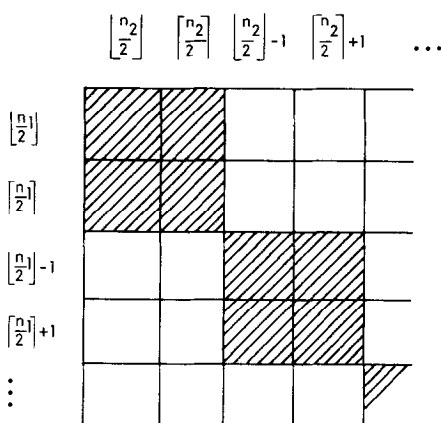


FIGURE 2

On the other hand, it is easy to see that all such I 's give the same value for (3), maximizing it. ■

This result can be better visualized if the rows and the columns of the matrix are ordered according to the decreasing order of the binomial coefficients (Fig. 1). I has to contain two opposite corners of each 2×2 shaded block and the 1×1 shaded one.

The proof of the next lemma is analogous.

LEMMA 6. *If n_1 and n_2 are both odd then I is a partial transversal minimizing (3) iff exactly one of the following two rows holds for each $k = 0, 1, \dots, \min\{\lfloor n_1/2 \rfloor, \lfloor n_2/2 \rfloor\}$ (Fig. 2):*

$$\begin{aligned} \left(\left\lfloor \frac{n_1}{2} \right\rfloor - k, \left\lfloor \frac{n_2}{2} \right\rfloor - k \right) \in I, & \quad \left(\left\lceil \frac{n_1}{2} \right\rceil + k, \left\lceil \frac{n_2}{2} \right\rceil + k \right) \in I \\ \left(\left\lceil \frac{n_1}{2} \right\rceil + k, \left\lfloor \frac{n_2}{2} \right\rfloor - k \right) \in I, & \quad \left(\left\lfloor \frac{n_1}{2} \right\rfloor - k, \left\lceil \frac{n_2}{2} \right\rceil + k \right) \in I. \end{aligned}$$

The proof of the remaining case, when the parities are different, is again analogous. However the formulation of the statement is less convenient.

LEMMA 7. *If n_1 is even and n_2 is odd, then I is a partial transversal maximizing (3) iff I contains exactly one element of the following sets for each k :*

$$\begin{aligned}
& \left\{ \left(\frac{n_1}{2}, \left\lfloor \frac{n_2}{2} \right\rfloor \right), \left(\frac{n_1}{2}, \left\lceil \frac{n_2}{2} \right\rceil \right) \right\}, \\
& \left\{ \left(\frac{n_1}{2} - k, \left\lfloor \frac{n_2}{2} \right\rfloor - k + 1 \right), \left(\frac{n_1}{2} - k, \left\lceil \frac{n_2}{2} \right\rceil + k - 1 \right), \right. \\
& \quad \left. \left(\frac{n_1}{2} + k, \left\lfloor \frac{n_2}{2} \right\rfloor - k + 1 \right), \left(\frac{n_1}{2} + k, \left\lceil \frac{n_2}{2} \right\rceil + k - 1 \right) \right\}, \\
& \quad k = 1, 2, \dots, \min \left\{ \frac{n_1}{2}, \left\lceil \frac{n_2}{2} \right\rceil \right\}, \\
& \left\{ \left(\frac{n_1}{2} - k, \left\lfloor \frac{n_2}{2} \right\rfloor - k \right), \left(\frac{n_1}{2} - k, \left\lceil \frac{n_2}{2} \right\rceil + k \right), \right. \\
& \quad \left. \left(\frac{n_1}{2} + k, \left\lfloor \frac{n_2}{2} \right\rfloor - k \right), \left(\frac{n_1}{2} + k, \left\lceil \frac{n_2}{2} \right\rceil + k \right) \right\} \\
& \quad k = 1, 2, \dots, \min \left\{ \frac{n_1}{2}, \left\lfloor \frac{n_2}{2} \right\rfloor \right\}.
\end{aligned}$$

The statement is visualized in Fig. 3. I has to contain exactly one element of each shaded block (1×2 or 2×2). And it has to be a partial transversal, of course. A typical example is shown in Fig. 4. The case when n_1 is odd and n_2 is even can be formulated and proved analogously.

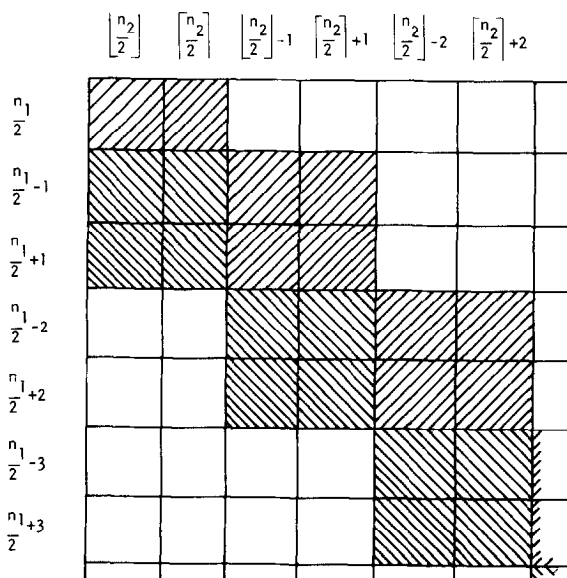


FIGURE 3

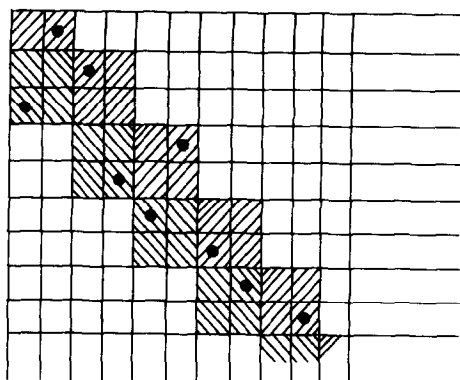


FIGURE 4

By this we have finished the first part of our work; the extreme points maximizing $|\mathcal{F}|$ are determined. In the rest of the paper we show that there are no other 2-part Sperner families with equality in (2).

The following lemma is a part of the folklore, a sharpening of the so-called LYM-inequality ([7, 8, 10]) (which was proved independently by Bollobás [1] in a more general form).

LEMMA 8. *Let \mathcal{H} be a Sperner family on an n -element set. The number of i -element members is p_i . Then*

$$\sum_{i=0}^n \frac{p_i}{\binom{n}{i}} \leq 1$$

with equality only when $p_j = \binom{n}{j}$ for some $0 \leq j \leq n$.

The next lemma is a similar statement for 2-part Sperner families.

LEMMA 9. *Let \mathcal{F} be a 2-part Sperner family on $X = X_1 \cup X_2$ ($X_1 \cap X_2 = \emptyset$, $|X_1| = n_1$, $n_2 = |X_2|$), and let p_{ij} denote the number of its members F such that $|F \cap X_1| = i$, $|F \cap X_2| = j$. Suppose that the following conditions hold for some indices u, v ($0 \leq u \leq n_1$, $0 \leq v \leq n_2$):*

$$p_{uv} > 0, \tag{9}$$

$$\sum_{j=0}^{n_2} \frac{p_{uj}}{\binom{n_1}{u} \binom{n_2}{j}} = 1, \tag{10}$$

$$\sum_{i=0}^{n_1} \frac{p_{iv}}{\binom{n_1}{i} \binom{n_2}{v}} = 1. \tag{11}$$

Then

$$p_{uv} = \binom{n_1}{u} \binom{n_2}{v}. \quad (12)$$

Proof. Introduce the following notations:

$$\mathcal{F}_1(A) = \{F: F \subset X_2, A \cup F \in \mathcal{F}\} \quad (A \subset X_1),$$

$$\mathcal{F}_2(B) = \{F: F \subset X_1, F \cup B \in \mathcal{F}\} \quad (B \subset X_2),$$

$$p_j(A) = |\{F: |F| = j, F \in \mathcal{F}_1(A)\}|,$$

$$q_i(B) = |\{F: |F| = i, F \in \mathcal{F}_2(B)\}|.$$

Observe that the above families are Sperner families for each $A \subset X_1$, $B \subset X_2$. Therefore

$$\sum_{j=0}^{n_2} \frac{p_j(A)}{\binom{n_2}{j}} \leq 1 \quad (13)$$

holds for any $A \subset X_1$. Summing up for all sets $A \subset X_1$ with $|A| = u$ we obtain

$$\sum_{\substack{A \subset X_1 \\ |A| = u}} \sum_{j=0}^{n_2} \frac{p_j(A)}{\binom{n_2}{j}} \leq \binom{n_1}{u}. \quad (14)$$

As $\sum_{A \subset X_1, |A| = u} p_j(A) = p_{uj}$, (14) is equivalent to

$$\sum_{j=0}^{n_2} \frac{p_{uj}}{\binom{n_1}{u} \binom{n_2}{j}} \leq 1.$$

By (10) we have equality here and in (14), consequently (13) hold with equality for all $A \subset X_1$, $|A| = u$. By Lemma 8, one of the numbers $p_j(A)$, say $p_{j(A)}(A)$, is equal to $\binom{n_2}{j(A)}$, the other ones are zero. $\sum_{A \subset X_1, |A| = u} p_v(A) = p_{uv}$ and (9) imply the existence of an $A^* \subset X_1$, $|A^*| = u$ such that $p_v(A^*) > 0$. This means that $j(A^*) = v$ for this A^* : $p_v(A^*) = \binom{n_2}{v}$.

All sets F satisfying $F \cap X_1 = A^*$, $|F \cap X_2| = v$ are in \mathcal{F} . Choose one of them, its intersection with X_2 will be denoted by B^* . Therefore $B^* \subset X_2$, $|B^*| = v$, $A^* \cup B^* \in \mathcal{F}$, $A^* \in \mathcal{F}_2(B^*)$, and

$$q_u(B^*) > 0 \quad (15)$$

all hold. $\mathcal{F}_2(B)$ is a Spencer family; it satisfies

$$\sum_{i=0}^{n_1} \frac{q_i(B)}{\binom{n_1}{i}} \leq 1. \quad (16)$$

The sum of these inequalities for all $B \subset X_2$, $|B| = v$ leads to

$$\sum_{i=0}^{n_1} \frac{p_{iv}}{\binom{n_1}{i} \binom{n_2}{j}} \leq 1$$

because $\sum_{B \subset X_2, |B|=v} q_i(B) = p_{iv}$. The equality in (11) implies that we must have equality in (16) for all $B \subset X_2$, $|B| = v$, including B^* . By Lemma 8, exactly one of $q_i(B^*)$ is non-zero, and by (15) this is $q_u(B^*) = \binom{n_1}{u}$. Therefore all sets $A \subset X_1$, $|A| = u$ are in $\mathcal{F}_2(B^*)$, that is, $A \cup B^* \in \mathcal{F}$ holds for them. But this holds for all $B^* \subset X_2$, $|B^*| = v$, therefore \mathcal{F} includes all sets $A \cup B$, where $A \subset X_1$, $B \subset X_2$, $|A| = u$, $|B| = v$. Hence $p_{uv} = \binom{n_1}{u} \binom{n_2}{v}$. ■

THEOREM. *Let $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$, $|X_1| = n_1$, $|X_2| = n_2$. The maximally sized 2-part Sperner families are of the form*

$$\mathcal{F} = \{F: |F \cap X_1| = i, |F \cap X_2| = j, (i, j) \in I\}$$

where I is a partial transversal described in Lemmas 5–7 (Figs. 1–4.).

Proof. Lemmas 5–7 determined the extreme points maximizing $|\mathcal{F}|$ for the 2-part Sperner families. To prove the theorem we only have to show that no proper convex linear combination of these maximum extreme points can be the profile matrix of a 2-part Sperner family.

Suppose that M is the profile matrix of a 2-part Sperner family and M is a convex linear combination of extreme points described in Lemmas 5–7:

$$M(m_{ij}) = \sum_{k=1}^m \lambda_k S(I_k) \quad \left(\lambda_1, \dots, \lambda_m \geq 0, \sum_{k=1}^m \lambda_k = 1 \right), \quad (17)$$

where $S(I_k)$ is the extreme point determined by the partial transversal I_k :

$$S(I_k) = (s_{ij}^k)_{1 \leq i \leq n_1, 1 \leq j \leq n_2}$$

and

$$s_{ij}^k = \begin{cases} \binom{n_1}{i} \binom{n_2}{j} & \text{if } (i, j) \in I_k, \\ 0 & \text{otherwise.} \end{cases}$$

Consider first the case when n_1 and n_2 are of equal parity. By symmetry we may suppose that $n_1 \leq n_2$. It is obvious from Lemmas 5 and 6 that all of these $S(I_k)$'s contain exactly one non-zero entry in each row and each column with index j such that $(n_2 - n_1)/2 \leq j \leq (n_2 + n_1)/2$ (the first $n_1 + 1$ columns in the ordering of the figures). Hence we have

$$\sum_{j=0}^{n_2} \frac{s_{ij}^k}{\binom{n_1}{i} \binom{n_2}{j}} = 1, \quad (0 \leq i \leq n_1) \quad (18)$$

and

$$\sum_{i=0}^{n_1} \frac{s_{ij}^k}{\binom{n_1}{i} \binom{n_2}{j}} = 1, \quad \left(\frac{n_2 - n_1}{2} \leq j \leq \frac{n_2 + n_1}{2} \right). \quad (19)$$

These inequalities imply

$$\begin{aligned} \sum_{j=0}^{n_2} \frac{m_{ij}}{\binom{n_1}{i} \binom{n_2}{j}} &= \sum_{j=0}^{n_2} \sum_{k=1}^m \lambda_k \frac{s_{ij}^k}{\binom{n_1}{i} \binom{n_2}{j}} \\ &= \sum_{k=1}^m \lambda_k \sum_{j=0}^{n_2} \frac{s_{ij}^k}{\binom{n_1}{i} \binom{n_2}{j}} \\ &= \sum_{k=1}^m \lambda_k = 1, \quad (0 \leq i \leq n_1) \end{aligned}$$

and

$$\sum_{i=0}^{n_1} \frac{m_{ij}}{\binom{n_1}{i} \binom{n_2}{j}} = 1 \quad \left(\frac{n_2 - n_1}{2} \leq j \leq \frac{n_2 + n_1}{2} \right). \quad (21)$$

On the other hand, all entries m_{ij} with $j < (n_2 - n_1)/2$ or $(n_2 + n_1)/2 < j$ are 0. Therefore, for any u ($0 \leq u \leq n_1$) there is a v ($(n_2 - n_1)/2 \leq v \leq (n_2 + n_1)/2$) satisfying $m_{uv} > 0$. The entries m_{ij} satisfy conditions (9)–(11) of Lemma 9 by (20) and (21). We obtain $m_{uv} = \binom{n_1}{u} \binom{n_2}{v}$. So, in each row i of M there is an entry such that $m_{i, v(i)} = \binom{n_1}{i} \binom{n_2}{v(i)}$. By (20) $v(i)$ are distinct, that is, M is equal to $S(I)$ for some partial transversal I , having exactly one non-zero value in each row and each column between $(n_2 - n_1)/2$ and $(n_2 + n_1)/2$. So $M = S(I_k)$ for some k ($1 \leq k \leq m$).

The situation is somewhat different if n_1 and n_2 have different parities. Suppose first that n_1 is even, n_2 is odd, and $n_1 < n_2$. The other cases can be treated analogously.

In this case (as it is easy to see by Lemma 7) $S(I_k)$'s again contain exactly one non-zero entry in each row. It is also true for the columns j such that $(n_2 - n_1 + 1)/2 \leq j \leq (n_2 + n_1 - 1)/2$. However, columns $(n_2 - n_1 - 1)/2$ and $(n_2 + n_1 + 1)/2$ are exceptional. Exactly one of them contains a non-zero entry of $S(I_k)$. Therefore (18) remains valid, but (19) holds only from $(n_2 - n_1 + 1)/2$ to $(n_2 + n_1 - 1)/2$. The same can be said about (20) and (21).

For any $(0 \leq u \leq n_1)$ there is a $v = v(u)$ satisfying $m_{uv} > 0$. If $1 \leq u \leq n_1 - 1$ then $(n_2 - n_1 + 1)/2 \leq v(u) \leq (n_2 + n_1 - 1)/2$ must hold because no $S(I_k)$ has a non-zero entry with indices $1 \leq u \leq n_1 - 1$ and $v < (n_2 - n_1 + 1)/2$ or $(n_2 + n_1 - 1)/2 < v$, by Lemma 7. Lemma 9 can be applied for $m_{u,v(u)}$ if $1 \leq u \leq n_1 - 1$:

$$m_{u,v(u)} = \binom{n_1}{u} \binom{n_2}{v(u)}.$$

A particular case of (20) is the following equality:

$$\sum_{j=0}^{n_2} \frac{m_{0j}}{\binom{n_2}{j}} = 1. \quad (22)$$

Here m_{0j} is the number of members $F \in \mathcal{F}$ such that $F \cap X_1 = \emptyset$, $|F \cap X_2| = j$. Using the notations of the proof of Lemma 9, $m_{0j} = p_j(\emptyset)$. Since $\mathcal{F}_1(\emptyset)$ is a Sperner family, (22) and Lemma 8 lead to $m_{0j} = \binom{n_1}{0} \binom{n_2}{j}$ for some $j = v(0)$. The existence of a $v(n_1)$ such that $m_{n_1,v(n_1)} = \binom{n_1}{n_1} \binom{n_2}{v(n_1)}$ can be proved similarly. (21) implies that $v(0), v(1), \dots, v(n_1)$ are all distinct. Therefore $M = S(I)$ for some partial transversal I . It must be one of the I_k 's. ■

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