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POLYTOPES DETERMINED BY COMPLEMENTFREE SPERNER FAMILIES

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The profile of a family of subsets of an n-element set is a vector $f=(f_0,\ldots,f_n)$, where f_k denotes the number of k-element sets in the family. Using a new method the extreme points of the convex hull of the profiles of all complementfree Sperner families over an n-element set are determined.

Let $N=\{1,\ldots,n\}$ and $\binom{N}{i}=\{X\subseteq N: |X|=i\}$. If $\mathcal{F}$ is a family of subsets of N, then let $\mathcal{F}_i=\{X\in\mathcal{F}: |X|=i\}$ and $f_i=|\mathcal{F}_i|$, $i=0,\ldots,n$. The vector $f=(f_0,\ldots,f_n)$ is called the *profile* of $\mathcal{F}$.

If $\mathbb{A}$ is a class of families, let $\mu(\mathbb{A})$ be the set of profiles of the families belonging to $\mathbb{A}$, $\langle\mu(\mathbb{A})\rangle$ be its convex hull in the space $\mathbb{R}^{n+1}$ and $\varepsilon(\mathbb{A})$ be the set of all extreme points (i.e. vertices) of the polytope $\langle\mu(\mathbb{A})\rangle$. This subject was first studied by P.L. Erdős, P. Frankl and G.O.H. Katona who determined $\varepsilon(\mathbb{A})$ for some special classes [4]. The determination of $\varepsilon(\mathbb{A})$ is motivated by the following fact: If w is any (weight-)function from $\{0,\ldots,n\}$ into $\mathbb{R}$, then

$$\max_{\mathcal{F}\in\mathbb{A}}\sum_{X\in\mathcal{F}} w(|X|)=\max_{f\in\varepsilon(\mathbb{A})}\sum_{i=0}^{n} w(i)f_i \tag{1}$$

(see [4] or [3, pp. 172 ff]).

In this paper we will present a new method of proof that a given set of vectors is the set $\varepsilon(\mathbb{A})$ of all extreme points arising from the considered class $\mathbb{A}$. The idea is simple: To prove that a vector f is contained in some bounded polytope P it is enough to show that f is contained in some smaller bounded polytope (e.g. a simplex) all of whose extreme points belong to P. Thereby the consideration of canonical families is useful.

Let $\mathbb{A}_1$ be the class of all Sperner families ($\mathcal{F}$ is a Sperner family if $X, Y\in\mathcal{F}$, $X\subseteq Y$ imply $X=Y$), $\mathbb{A}_2$ be the class of all complementfree Sperner families ($\mathcal{F}$ is *complementfree* if $X\in\mathcal{F}$ implies $N\setminus X\notin\mathcal{F}$) and, if $n=2k$, let $\mathbb{A}_3$ be the class of all Sperner families for which $f_k\le\frac{1}{2}\binom{n}{k}$ holds. It is easy to see that $\mathbb{A}_2\subseteq\mathbb{A}_1$ and, for even n, $\mathbb{A}_2 \subseteq \mathbb{A}_3$, hence

$$\langle \mu(\mathbb{A}_2) \rangle \subseteq \langle \mu(\mathbb{A}_1) \rangle, \tag{2}$$

$$\langle \mu(\mathbb{A}_2) \rangle \subseteq \langle \mu(\mathbb{A}_3) \rangle \quad (n = 2k). \tag{3}$$

Let

$$\boldsymbol{O} = (\underset{\hat{0}}{0}, \ldots, \underset{\hat{n}}{0}) \quad \text{and} \quad \boldsymbol{u}_i = \left(\underset{\hat{0}}{0}, \ldots, 0, \underset{\hat{i}}{\binom{n}{i}}, 0, \ldots, \underset{\hat{n}}{0}\right), \quad i = 0, \ldots, n.$$

Further let for $n = 2k$

$$\boldsymbol{v}_i = \left(\underset{\hat{0}}{0}, \ldots, 0, \underset{\hat{i}}{\binom{n-1}{i-1}}, 0, \ldots, 0, \underset{\hat{k}}{\tfrac{1}{2}\binom{n}{k}}, 0, \ldots, \underset{\hat{n}}{0}\right), \quad i = 0, \ldots, k-1$$

($\binom{n-1}{0-1}$ is defined to be zero) and

$$\boldsymbol{w}_j = \left(\underset{\hat{0}}{0}, \ldots, 0, \underset{\hat{k}}{\tfrac{1}{2}\binom{n}{k}}, 0, \ldots, 0, \underset{\hat{j}}{\binom{n-1}{j}}, 0, \ldots, \underset{\hat{n}}{0}\right), \quad j = k+1, \ldots, n.$$

It is easy to prove that

$$\varepsilon(\mathbb{A}_1) = \{\boldsymbol{O}, \boldsymbol{u}_0, \ldots, \boldsymbol{u}_n\},$$

which is essentially the same as the well-known LYM inequality (see [4] or [3, p. 174]). If n is odd, then obviously $\varepsilon(\mathbb{A}_1) \subseteq \mu(\mathbb{A}_2)$ because the families $\emptyset$, $\binom{N}{0}, \ldots, \binom{N}{n}$ are complementfree Sperner families. It follows directly $\langle \mu(\mathbb{A}_1) \rangle \subseteq \langle \mu(\mathbb{A}_2) \rangle$ and because of (2) $\langle \mu(\mathbb{A}_1) \rangle = \langle \mu(A_2) \rangle$, hence

$$\varepsilon(\mathbb{A}_1) = \varepsilon(\mathbb{A}_2) \quad \text{if } n \text{ is odd.}$$

From now on let $n = 2k$,

$$V = \{\boldsymbol{O}, \boldsymbol{u}_0, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n, \boldsymbol{v}_0, \ldots, \boldsymbol{v}_{k-1}, \boldsymbol{w}_{k+1}, \ldots, \boldsymbol{w}_n\}$$

and $\langle V \rangle$ be the convex hull of V. We use the above mentioned method to prove the following central result of the paper.

Theorem. $\varepsilon(\mathbb{A}_2) = \varepsilon(\mathbb{A}_3) = V$.

Proof. First we note that the vectors of V are really profiles of families in $\mathbb{A}_2$ and $\mathbb{A}_3$: The families $\emptyset$, $\binom{N}{i}$ $(i = 0, \ldots, k-1,\ k+1, \ldots, n)$, $\{X \subseteq \binom{N}{k} : n \notin X\} \cup \{X \subseteq \binom{N}{i} : n \in X\}$ $(i = 0, \ldots, k-1)$, $\{X \subseteq \binom{N}{k} : n \in X\} \cup \{X \subseteq \binom{N}{j} : n \notin X\}$ $(j = k+1, \ldots, n)$ belong to $\mathbb{A}_2$ and $\mathbb{A}_3$ and have the corresponding profiles. Consequently,

$$\langle V \rangle \subseteq \langle \mu(\mathbb{A}_2) \rangle \subseteq \langle \mu(\mathbb{A}_3) \rangle \tag{4}$$

(note (3)). In the main step (Lemma 1, 2, 3) we show that every profile of a family of $\mathbb{A}_3$ can be written as a convex combination of vectors of V. This implies $\langle \mu(\mathbb{A}_3) \rangle \subseteq \langle V \rangle$ and because of (4) it follows $\langle \mu(\mathbb{A}_2) \rangle = \langle \mu(\mathbb{A}_3) \rangle = \langle V \rangle$. Since no vector of V can be written as a convex combination of the other vectors of V it follows finally $\varepsilon(\mathbb{A}_2) = \varepsilon(\mathbb{A}_3) = V$. □

Lemma 1. *If $f \in \mu(\mathbb{A}_3)$ and $f_k = 0$, then $f \in \langle V \rangle$.*

Proof. Because f is the profile of a Sperner family the LYM inequality (see e.g. [3, p. 9]) yields that

$$f = \left(1 - \sum_{\substack{i=0\\ i\neq k}}^{n} \frac{f_i}{\binom{n}{i}}\right)\mathbf{0} + \sum_{\substack{i=0\\ i\neq k}}^{n} \frac{f_i}{\binom{n}{i}} \boldsymbol{u}_i$$

is a convex combination of $\mathbf{0}, \boldsymbol{u}_0, \ldots, \boldsymbol{u}_{k-1}, \boldsymbol{u}_{k+1}, \ldots, \boldsymbol{u}_n$. □

Lemma 2. *If $f \in \mu(\mathbb{A}_3)$ and $f_{k+1} = \cdots = f_n = 0$,* then $f \in \langle V \rangle$.

Proof. Let $\mathcal{F}$ be a canonical Sperner family with profile f which exists by a theorem of Daykin, Godfrey and Hilton [1]. (The construction of a *canonical Sperner family* is as follows: First define a total (lexicographic) order $<_L$ on $\binom{N}{i}$ in the following way; $x <_L Y$ iff $max\{x : x \in X \setminus Y\} > \max\{x : x \in Y \setminus X\}$. If h is the largest index such that $f_h \neq 0$ take the first f_h sets of $\binom{N}{h}$ in the order $<_L$. Then take the first f_{h-1} sets in $\binom{N}{h-1}$ which are not contained in already chosen sets, further take the first f_{h-2} sets in $\binom{N}{h-2}$ which are not contained in sets taken up to now, and so on.) Then $f_k \leq \frac{1}{2}\binom{n}{k}$ and $f_{k+1} = \cdots = f_n = 0$ imply $n \notin X$ for all $X \in \mathcal{F}_k$. Let s be the smallest number with the property $n \notin X$ for all $X \in \mathcal{F}_{s+1}$ $(0 \leq s \leq k-1)$. Further define

$$\mathcal{F}_s' = \{X \in \mathcal{F}_s : n \notin X\}, \qquad \mathcal{F}_s'' = \{X \in \mathcal{F}_s : n \in X\},$$
$$f_s' = |\mathcal{F}_s'|, \qquad f_s'' = |\mathcal{F}_s''|,$$

and put

$$\mathcal{F}' = \mathcal{F}_s' \cup \mathcal{F}_{s+1} \cup \cdots \cup \mathcal{F}_k,$$
$$\mathcal{F}'' = \mathcal{F}_s'' \cup \mathcal{F}_{s-1} \cup \cdots \cup \mathcal{F}_0.$$

Obviously, $\mathcal{F}'$ is a Sperner family of subsets of $\{1, \ldots, n-1\}$ and, since $\mathcal{F}$ is canonical, $\mathcal{F}''$ is a Sperner family of subsets of $\{1, \ldots, n\}$ containing the element n. Thus the LYM inequality implies

$$\frac{f_s'}{\binom{n-1}{s}} + \sum_{i=s+1}^{k} \frac{f_i}{\binom{n-1}{i}} \leq 1 \tag{5}$$

$$\frac{f_s''}{\binom{n-1}{s-1}} + \sum_{i=1}^{s-1} \frac{f_i}{\binom{n-1}{i-1}} \leq 1. \tag{6}$$

Now fix $f_0, \ldots, f_{s-1}, f_s''$ and consider the simplex S in the space $\mathbb{R}^{n+1}$ which is given by

$$\begin{aligned}
&x_i = f_i \quad (i = 0, \ldots, s-1),\\
&x_s \geqslant f_s'',\\
&x_i \geqslant 0 \quad (i = s+1, \ldots, k),\\
&x_i = 0 \quad (i = k+1, \ldots, n),\\
&\frac{x_s - f_s''}{\binom{n-1}{s}} + \sum_{i=s+1}^{k} \frac{x_i}{\binom{n-1}{i}} \leqslant 1.
\end{aligned}$$

Because of (5) and $f_s = f_s' + f_s''$ the vector $\boldsymbol{f}$ is contained in S. To prove $\boldsymbol{f} \in \langle V \rangle$ it is enough to show that all extreme points of S are contained in $\langle V \rangle$. The extreme points of S are

$$\boldsymbol{a} = (\underset{\hat{0}}{f_0}, \ldots, f_{s-1}, \underset{\hat{s}}{f_s''}, 0, \ldots, \underset{\hat{n}}{0}),$$

$$\boldsymbol{a}_s = \left(\underset{\hat{0}}{f_0}, \ldots, f_{s-1}, \underset{\hat{s}}{f_s'' + \binom{n-1}{s}}, 0, \ldots, \underset{\hat{n}}{0}\right),$$

$$\boldsymbol{a}_i = \left(\underset{\hat{0}}{f_0}, \ldots, f_{s-1}, f_s'', 0, \ldots, 0, \underset{\hat{i}}{\binom{n-1}{i}}, 0, \ldots, \underset{\hat{n}}{0}\right), \quad i = s+1, \ldots, k.$$

The vector $\boldsymbol{a}$ is the profile of the Sperner family $\mathcal{F}''$ and $\boldsymbol{a}_i$ is the profile of the Sperner family $\mathcal{F}'' \cup \{X \in \binom{N}{i} : n \notin X\}$ $(i = s, \ldots, k)$. By Lemma 1, $\boldsymbol{a}, \boldsymbol{a}_s, \ldots, \boldsymbol{a}_{k-1} \in \langle V \rangle$. Finally, $\boldsymbol{a}_k \in \langle V \rangle$ because

$$\boldsymbol{a}_k = \left(1 - \frac{f_s''}{\binom{n-1}{s-1}} - \sum_{i=1}^{s-1} \frac{f_i}{\binom{n-1}{i-1}}\right)\boldsymbol{v}_0 + \sum_{i=1}^{s-1} \frac{f_i}{\binom{n-1}{i-1}} \boldsymbol{v}_i + \frac{f_s''}{\binom{n-1}{s-1}} \boldsymbol{v}_s$$

is a convex combination of $\boldsymbol{v}_0, \ldots, \boldsymbol{v}_s$ (note (6)). □

Lemma 3. *If $\boldsymbol{f} \in \mu(\mathbb{A}_3)$, then $\boldsymbol{f} \in \langle V \rangle$.*

Proof. By a theorem of Daykin, Godfrey and Hilton [1] (see also [3, p. 122]) there exists a "reflexed" Sperner family $\mathcal{F}'$ with profile

$$\boldsymbol{f}' = (\underset{\hat{0}}{f_0 + f_n}, f_1 + f_{n-1}, \ldots, f_{k-1} + f_{k+1}, \underset{\hat{k}}{f_k}, 0, \ldots, \underset{\hat{n}}{0}).$$

Because $\boldsymbol{f}' \in \mu(\mathbb{A}_3)$ we can write $\boldsymbol{f}'$ by Lemma 2 in the form

$$\boldsymbol{f}' = \lambda \boldsymbol{O} + \sum_{i=0}^{k-1} \lambda_i \boldsymbol{u}_i + \mu_i \boldsymbol{v}_i,$$

where λ, λ_i, $\mu_i \geqslant 0$ $(i = 0, \ldots, k-1)$ and $\lambda + \sum_{i=0}^{k-1} \lambda_i + \mu_i = 1$. Now it is easy to check that

$$\begin{aligned} \boldsymbol{f} = \lambda \boldsymbol{O} &+ \sum_{\substack{i=0 \\ f_i + f_{n-i} \neq 0}}^{k-1} \frac{f_i}{f_i + f_{n-i}} (\lambda_i \boldsymbol{u}_i + \mu_i \boldsymbol{v}_i) \\ &+ \sum_{\substack{j=k+1 \\ f_j + f_{n-j} \neq 0}}^{n} \frac{f_j}{f_j + f_{n-j}} (\lambda_{n-j} \boldsymbol{u}_j + \mu_{n-j} \boldsymbol{w}_j) \end{aligned}$$

is a convex combination of the vectors of V. Hence $\boldsymbol{f} \in \langle V \rangle$. $\square$

Finally, let us mention that the equality $\varepsilon(\mathbb{A}_2) = V$ was also proved among other things in [2], but with a completely different method, namely the general theory which was developed by P.L. Erdős, P. Frankl and G.O.H. Katona in [4] and [5]. Using the relation (1) with the weight function $w \equiv 1$ and the description of the polytope $\langle \mu(\mathbb{A}_2) \rangle$ one can derive immediately an old result of Purdy [6] (for another proof see [7]) which states that the maximum size of a complementfree Sperner family equals $\binom{n}{\lfloor n/2 \rfloor + 1}$.

References

[1] D.E. Daykin, J. Godfrey and A.J.W. Hilton, Existence theorems for Sperner families, J. Combin. Theory Ser. A 17 (1974) 245–251.

[2] K. Engel and P.L. Erdős, Sperner families satisfying additional conditions and their convex hulls, Graphs and Combinatorics 5 (1989) 47–56.

[3] K. Engel and H.-D.O.F. Gronau, Sperner theory in partially ordered sets (BSB B.G. Teubner Verlagsgesellschaft, Leipzig, 1985).

[4] P.L. Erdős, P. Frankl and G.O.H. Katona, Intersecting Sperner families and their convex hulls, Combinatorica 4 (1984) 21–34.

[5] P.L. Erdős, P. Frankl and G.O.H. Katona, Extremal hypergraph problems and convex hulls, Combinatorica 5 (1985) 11–26.

[6] G. Purdy, A result on Sperner collections, Utilitas Math. 12 (1977) 95–99.

[7] C. Greene and A.J.W. Hilton, Some results on Sperner families, J. Combin. Theory Ser. A 26 (1979) 202–209.