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ANY FOUR INDEPENDENT EDGES OF A 4-CONNECTED GRAPH ARE CONTAINED IN A CIRCUIT

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L. Lovász [2] raised the following problem.

Conjecture. Suppose G is a k-connected graph ($k \geqq 2$), $e_1, e_2, \ldots, e_k \in E(G)$ are independent edges, and if k is odd then $G-\{e_1, e_2, \ldots, e_k\}$ is connected. Then G contains a circuit using all the edges $e_1, e_2, \ldots, e_k$.

This conjecture is proved for $k=3$ by Lovász [3; 6. § 67]. In general, R. Häggkvist and C. Thomassen [1] proved a slightly weaker statement that the same conclusion follows if G is $(k+1)$-connected.

Now we prove that the conjecture of Lovász holds for $k=4$.

THEOREM. *In a* 4-*connected graph, any four independent edges are contained in a circuit.*

This result effects on a conjecture of Erdős and Gallai. Using this theorem, L. Pyber [4] proved that every graph of n vertices can be covered by $1{,}5n$ circuits or edges. (Without this result, a greater constant could be proved by the method of Pyber.)

PROOF OF THE THEOREM. Let us fix the 4-connected graph G and the independent edges $x_1y_1, x_2y_2, x_3y_3, x_4y_4 \in E(G)$. By 4-connectivity (using Menger's theorem), there exist four vertex-disjoint paths from the vertices x_1, y_1, x_3, y_3 to the vertices x_2, y_2, x_4, y_4. These paths P_1, P_2, P_3, P_4 with the edges $x_1y_1, x_2y_2, x_3y_3, x_4y_4$ constitute one or two circuits. In the first case it is a desired circuit, so without loss of generality, we may suppose that the paths P_1, P_2, P_3 and P_4 lead from x_1, y_1, x_3 and y_3 to x_2, y_2, x_4 and y_4, respectively; P_1, P_2 and the edges x_1, y_1, x_2, y_2 constitute the circuit C_1, P_3, P_4 and the edges x_3y_3, x_4y_4 constitute the circuit C_2.

Now again by 4-connectivity and Menger's theorem, there exist four vertex-disjoint paths Q_1, Q_2, Q_3, Q_4 from C_1 to C_2. The circuits C_1, C_2 and the paths Q_1, Q_2, Q_3, Q_4 constitute a subgraph H. In what follows, we deal with this subgraph H.

We introduce some notation. The paths are denoted by the sequence of labelled vertices in them. For a path P from x to y, $[xy]$, $[xy)$, $(xy]$, (xy) denote the vertex-sets $V(P)$, $V(P)-\{y\}$, $V(P)-\{x\}$, $V(P)-\{x, y\}$, respectively. The subpaths of P_1, P_2, P_3 and P_4 are called arcs.

First make a very simple observation which however is used several times.

Fact 1. If two vertex-disjoint paths Q_i connect the same pair of paths P_j then deleting the inner points and the edges of the arcs between the endpoints of these paths we get a desired circuit.

So we may suppose that the paths Q_1, Q_2, Q_3, Q_4 lead from the arcs P_1, P_1, P_2, P_2 to the arcs P_4, P_3, P_3, P_4, respectively. Let $w_1, w_2, z_1, z_2 \in V(C_1)$, $z_4, w_3, w_4, z_3 \in V(C_2)$ be the endpoints of the paths Q_1, Q_2, Q_3, Q_4, respectively. If the vertices w_3 and z_4 do not separate the vertices w_4 and z_3 in C_2 then the disjoint subpaths w_3z_4 and w_4z_3 of C_2 with Q_1, Q_2, Q_3 and Q_4 constitute two paths such that both paths can replace one arc of C_1 and this new circuit is a desired one.

So we may suppose that the vertices w_3 and z_4 separate the vertices w_4 and z_3 in C_2. Now without loss of generality, we may suppose that we have the subgraph H in Figure 1. (We drew the subgraph H so that the figure should show the large symmetry of the situation.) Of course, it may occur that $w_1=x_1$, $w_2=x_2$, $w_3=x_3$, $w_4=x_4$, $z_1=y_1$, $z_2=y_2$, $z_2=y_3$ or $z_4=y_4$.

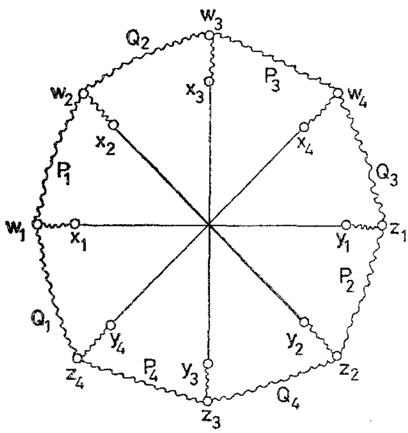

Fig. 1

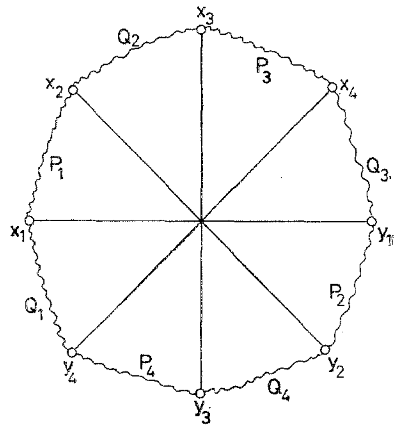

Fig. 2

Suppose that H is a subgraph as in Figure 1 such that the sum of the lengths of the arcs w_1x_1, w_2x_2, w_3x_3, w_4x_4, z_1y_1, z_2y_2, z_3y_3, z_4y_4 is minimum. Suppose that e.g. $w_1 \neq x_1$. Then the path $w_1x_1y_1z_1$ contains inner vertices and there is a path in $G-\{w_1, z_1\}$ from $(w_1x_1y_1z_1)$ to the remaining part of $H-\{w_1, z_1\}$ by 4-connectivity. By symmetry, we may assume that this path leads from a vertex $u \in (w_1x_1]$. If this path leads to a vertex $v \in [w_1w_2] \cup [w_1z_4]$ then adding this path to H and deleting the inner vertices and the edges of the arc vw_1 we obtain a subgraph like in Figure 1 such that the path ux_1 is shorter than w_1x_1, a contradiction. If this path leads to a vertex in $(w_3x_3]$, $(w_3w_4]$ or $[w_4x_4]$ ($[y_4z_4)$, $(z_4z_3]$ or $[y_3z_3]$, resp.) then this path and Q_2 (Q_1 resp.) are two vertex-disjoint paths from the arc P_1 to the arc P_3 (P_4 resp.) and we are done by Fact 1. If this path leads to a vertex v in (w_4z_1) ((z_3z_2), resp.) then the path uvw_4 (uvz_3, resp.) and Q_2 (Q_1, resp.) are two vertex-disjoint paths from P_1 to P_3 (P_4, resp.) and we are ready by Fact 1 again. The other possibilities can be settled by the axial symmetry of Figure 1 with the axis $w_1x_1y_1z_1$.

So we may assume that $x_1=w_1$, $x_2=w_2$, $x_3=w_3$, $x_4=w_4$, $y_1=z_1$, $y_2=z_2$, $y_3=z_3$, $y_4=z_4$, like in Figure 2.

Now by 4-connectivity, there is a path P in $G-\{x_2, y_1, y_4\}$ from $(x_2 x_1 y_4)$ to the remaining part of $H-\{x_2, y_1, y_4\}$. By symmetry, we may assume that P leads from $[x_1 y_4)$. We distinguish two cases.

Case 1. P starts at x_1.

If P leads to a vertex $v\in(x_3x_4]\cup(x_4y_1)$ then P or P together with the path vx_4 and Q_2 are two vertex-disjoint paths from P_1 to P_3 and we are done by Fact 1. If P leads to $(y_3y_2]\cup(y_2y_1)$ then we are done by axial symmetry. So in this case P leads to a neighbouring subpath of the circuit $x_1x_2x_3x_4y_1y_2y_3y_4x_1$, i.e. to $(x_2x_3]$ or $(y_4y_3]$.

Case 2. P leads from a vertex $u\in(x_1y_4)$.

If P leads to a vertex v in (x_4y_1) ($[x_4x_3)$, resp.) the paths x_1u, P_1vx_4 (x_1u, P, resp.) constitute a further path from P_1 to P_3 and we are done by Fact 1. If P leads to a vertex $v\in(x_2x_3]$ then the paths and edges P, vx_3, x_3y_3, y_3y_2, y_2x_2, x_2x_1, x_1y_1, y_1x_4, x_4y_4, y_4u constitute a desired circuit. If P leads to $[y_1y_2)$ or $[y_2y_3)$ then we are done by symmetry.

So in both cases, we obtained

Fact 2. Only neighbouring segments of the circuit $x_1x_2x_3x_4y_1y_2y_3y_4x_1$ are connected by any path openly disjoint to H.

Without loss of generality, we may assume that there is such a path P from $[x_1y_4)$ to $(y_4y_3]$. Let $u\in[x_1y_4)$ and $v\in(y_4y_3]$ be the points nearest to x_1 and y_3 in the path x_1y_4 and y_4y_3, respectively, which occur as the endpoint of such a path. (It may happen that u and v belong to different paths.) Choose H (with the constraints $x_1=w_1, ..., y_4=z_4$) so that the sum of the lengths of paths ux_1 and vy_3 should be the minimum.

By 4-connectivity, there is a path R in $G-\{u, v, x_4\}$ from (uy_4v) to the remaining part of H. R does not lead from (y_4v) to $[x_1u)$ by the definition of u. If R leads from $x\in(uy_4]$ to $y\in[x_1u)$ then replacing the path xy of H by R we obtain a subgraph H_0 such that there is a path from y to $(y_4y_3]$ (via u), a contradiction to the choice of H. Similarly, R does not lead from (uy_4v) to $(vy_3]$. But according to Fact 2, only neighbouring segments can be connected by R. Without loss of generality, we may assume that R leads from $x\in(uy_n]$ to $y\in(x_1x_2]$. Now let R_1 be a path from $(y_4y_3]$ to u. (There exists such an R_1 by the definition of u.) If R and R_1 are vertex-disjoint then y_4x with R and R_1 with ux_1 are vertex-disjoint paths from P_4 to P_1 and we are finished by Fact 1. And if R and R_1 have a vertex in common then $R_1\cup R_2$ contains a path from $(x_1x_2]$ to $(y_4y_3]$, a contradiction to Fact 2.

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