Chapter 1

ERDŐS-KO-RADO THEOREMS OF HIGHER ORDER

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Abstract We survey conjectured and proven Ahlswede-type higher-order generalizations of the Erdős-Ko-Rado theorem.

Keywords: poset, Erdős-Ko-Rado theorem, partition lattice, subspace lattice

This paper is dedicated to the 60^{th} birthday of Professor Rudolf Ahlswede.

1. INTRODUCTION

Rudolf Ahlswede's seminal work in extremal combinatorics includes:
• the Ahlswede-Daykin (or *Four Function*) inequality [4, 5] which provides for a common generalization of many correlation inequalities;
• the Ahlswede-Zhang identity, which unexpectedly turns the familiar LYM inequality into an identity [13];
• the complete solution (in joint work with L. Khachatrian [6, 7]) for maximizing the number of t-intersecting k-element sets—a problem dating back to the 30's [22];
• breakthrough results in Erdős type number theory (using the shifting technique in joint works [9, 10, 11] with L. Khachatrian) on problems like what is the maximum number of positive integers up to n such that no two of them are relatively primes, and related results.

The present survey paper focuses on *higher order extremal problems* in the sense of Ahlswede [3, 14]. The traditional questions about set systems sound like "how many sets can one have under certain restrictions" while the new higher order questions ask "how many *families of sets* can one have under certain restrictions". R. Ahlswede *et al.* have started this research, with strong motivation from information theory [3, 14]. They propose that any problem about set systems may give rise to four higher-order problems. For illustration, the classic Erdős-Ko-Rado theorem [22] sets an upper bound, on how many pairwise intersecting k-element subsets of an n-element set can one find. The four higher-order problems each ask how many *pairwise disjoint* families of k-element subsets of an n-element set can have such that for any two families:

(1) there exists an element of the first family which intersects all elements of the second family;
(2) there exists an element of the first family and an element of the second family that intersect;
(3) for all elements of the first family there exists an element of the second family, which intersects it;
(4) all elements of the first family intersect all elements of the second family.

One may not expect, of course, that all new problems generated in this way make sense and are interesting. But some of them yield elegant generalizations of known results. Ahlswede conjectured a bound $\binom{n-1}{k-1}$ for the problem (1), which would have given a higher-level generalization of the classic Erdős-Ko-Rado theorem. (For an intersecting family of k-sets $\{A_i : i \in I\}$ one makes the family of *singleton* families $\{\{A_i\} : i \in I\}$. If an upper bound holds for the second family, then it holds for the first family.) However, it was shown in [1] that although the conjecture holds for $k = 2, 3$, it is false for $k \geq 8$. The proof of the counterexample uses the *probabilistic method*. In this paper we restrict our interest to higher order generalizations of the Erdős-Ko-Rado theorem. The higher order generalizations of Sperner's theorem [8, 14, 15] will not be considered here.

In this paper we do not take narrowly the definition of Ahlswede-type higher-order extremal problems, since we rather do not insist on the pairwise disjointness of the families, but require that the sets in the same family have a certain additional structural property (make classes of a partition or be comparable for inclusion, etc.).

It is instructive to compare the concept of higher order generalization to other generalizing principles in combinatorics. Gian-Carlo Rota taught us to look for analogues of theorems valid on the power set lattice on the subspace lattice and the partition lattice. In the setting of Erdős-Ko-Rado theorems, Miklós Simonovits and Vera Sós initiated the study of "structured intersection theorems" [34, 35]: they look for the largest number of "structures" (graphs, arithmetic progressions, etc.) that pairwise intersect in a required type of "substructre".

If we understand higher order generalization in a broader sense, where we want to bound the number of families instead of the number of sets, it turns out that these three directions for generalization frequently overlap.

Excellent references on Erdős-Ko-Rado type theorems for set systems are [20, 28, 30].

2. INTERSECTING CHAINS IN POSETS

This section reviews results on intersecting chains in posets. A *k-chain* in a poset is a set of k distinct poset elements, such that any two elements are comparable in the poset. We say that two chains in a poset *intersect*, if they share at least one poset element. P. L. Erdős, Faigle, and Kern [24] pointed out that certain frequently studied problems well belong to this line. For example, let $M_1, M_2, \ldots, M_n$ be n pairwise disjoint sets of the same cardinality q. The associated *generalized Boolean algebra* (or *sequence space*) consists of the family

$$\mathcal{B}(n,q) = \{C \subseteq M_1 \cup \ldots \cup M_n : |C \cap M_i| \leq 1,\ i = 1, \ldots, n\}$$

ordered by inclusion. Observe that $\mathcal{B}(n,q)$ may be viewed as the collection of chains of an order $P = P(n,q)$ on $M_1 \cup \ldots \cup M_n$ with order relation

$$x < y \qquad if \qquad i < j$$

for all $x \in M_i, y \in M_j$. Frankl and Füredi [29] and Deza and Frankl [20] proved that, for $q \geq 2$ and $k = 1, \ldots, n$, there are at most $\binom{n-1}{k-1}q^{k-1}$ pairwise intersecting k-chains. Their method did not apply for the case $q = 1$. This is, however, the "classical" power set case, and therefore the original Erdős-Ko-Rado theorem also fits into this framework by solving the case $q = 1$.

It is worth pointing out, that these results can be strengthened to Bollobás type inequalities (see [24] or Engel [21]).

P. L. Erdős, Faigle, and Kern [24], among other results on intersecting chains, posed the problem of finding the largest number of pairwise intersecting k-chains in B_n^c, where B_n^c denotes the poset of sets $\{X \subseteq \{1, 2, ..., n\} :\ c \leq |X| \leq n - c\}$ for inclusion. Füredi solved this problem first, using the kernel method, for $c = 0, 1$ and $n > 6k \log k$ (personal communication). Ahlswede and Cai [2] solved the problem for $c = 0$. For an arbitrary value of c it was solved by Ákos Seress and the authors ([25, 26]). More precisely:

Definition. For $c \leq m \leq n - c$, let $\mathcal{T}_{n,k}^c(m)$ denote the set of those k-chains in B_n^c, which contain as element the initial segment $\{1, \ldots, m\}$. Clearly $|\mathcal{T}_{n,k}^c(m)|$ is also the cardinality of the set of those k-chains in $B_{n,k}^c$ which contain a specified (but otherwise arbitrary) subchain of length 1 with specified size m.

Theorem 1 ([26]) *Let $c \geq 1$ and let $\mathcal{F}$ be a family of intersecting k-chains in B_n^c. Then $|\mathcal{F}| \leq |\mathcal{T}_{n,k}^c(m)|$, and there is an injection $\phi : \mathcal{F} \to \mathcal{T}_{n,k}^c(c)$ such that every chain $\mathcal{L} = (L_1, L_2, \ldots, L_k) \in \mathcal{F}$ and its image $\phi(\mathcal{L}) = \mathcal{H} = (H_1, H_2, \ldots, H_k) \in \mathcal{T}_{n,k}^c(c)$ satisfy*

$$|L_k| \geq |H_k|.$$

The proof is based on a version of the shifting technique and uses mathematical induction. It is interesting to remark, that the same technique could apply for t-intersecting k-chains, if we had an easy base case for the induction, that we do not have. In lack of a good base case, the corresponding result in [26] uses the *kernel method*, and therefore does not give all n's for which the theorem holds. Finding the exact threshold for t-intersecting problem seems to be a very challenging problem.

The following problem fits the scheme of "structured intersection theorems" [34, 35] of Simonovits and Vera Sós: given a graph G, what is the maximum number of pairwise intersecting complete k-subgraphs? The maximum number of pairwise intersecting k-chains in a poset is exactly this problem if G is the comparability graph of the poset.

Whenever the poset elements are sets, the maximum number of pairwise intersecting k-chains in a poset fits the description of higher-order problems. Rota type analogues also came into play. Czabarka [16, 18] made the q-analogue of the shifting technique and gave new proofs to Hsieh's theorem [33]—the q-analogue of the classic Erdős-Ko-Rado theorem—in this way. Czabarka [17] also obtained a q-analogue of the shifting proof of the theorem of Seress and the authors on intersecting k-chains in B_n^c to intersecting k-chains for subspaces in an n-dimensional linear space over $GF(q)$, although for $c = 0, 1$ only.

Here we cite two other, general theorems of Seress and the authors [26] on intersecting chains in posets. These are the basis to prove result on t-intersecting chains in B_n^c. The first is an Erdős-Ko-Rado type result, the second is a Hilton-Milner type result.

Let us be given a fixed k and a sequence of posets P_n. For a given t-chain $\mathcal{L}$, let $\mathcal{T}_{n,k}(\mathcal{L})$ denote the set of k-chains in P_n which contain $\mathcal{L}$ as a subset. Define $T_{n,k}(\mathcal{L}) = |\mathcal{T}_{n,k}(\mathcal{L})|$. Also define $r_t(n) = \max T_{n,k}(\mathcal{L})$, where the maximum is taken for t-chains $\mathcal{L}$ in P_n.

Theorem 2 *For fixed $1 \leq t < k$, and a sequence of posets P_n, let us be given a family $\mathcal{F}_n$ of t-intersecting k-chains in P_n. Assume that*

$$\lim_{n\to\infty} r_{t+1}(n)/r_t(n) = 0.$$

Then, for n sufficiently large, $|\mathcal{F}_n| \leq r_t(n)$, and equality implies that the elements of $\mathcal{F}_n$ share a t-subchain.

For a t-chain $\mathcal{X} \subset P_n$ and $y \notin \mathcal{X}$, let $T(\mathcal{X}, y)$ denote the number of k-chains which contain X and y. For a t-chain $\mathcal{X}$ and a k-chain $\mathcal{L}$ in P_n, such that $|\mathcal{X} \cup \mathcal{L}| = k+1$, let $y^*_{\mathcal{L}} \in \mathcal{L} \setminus \mathcal{X}$ such that $T(\mathcal{X}, y^*_{\mathcal{L}})$ minimize $T(\mathcal{X}, y_{\mathcal{L}})$ for the elements $y \in \mathcal{L} \setminus \mathcal{X}$, and set

$$\tau(\mathcal{X}, \mathcal{L}) = \sum_{y \in \mathcal{L} \setminus \mathcal{X},\ y \neq y^*_{\mathcal{L}}} T(\mathcal{X}, y).$$

Also define

$$M_\tau(n) = \max_{\mathcal{X}, \mathcal{L}} \ \tau(\mathcal{X}, \mathcal{L}),$$

and

$$M^*_\tau(n) = \max_{\substack{\mathcal{X}, \mathcal{L}: \\ \tau(\mathcal{X}, \mathcal{L}) = M_\tau(n)}} T(\mathcal{X}, y^*_{\mathcal{L}}).$$

Now the following Hilton-Milner type theorem [32] holds:

Theorem 3 *For fixed $1 \leq t < k$, and a sequence of posets P_n, let us be given a maximum sized family $\mathcal{F}_n$ of non-trivially t-intersecting k-chains in P_n. Assume further that*

$$\lim_{n \to \infty} r_{t+2}(n)/M^*_\tau(n) = 0.$$

then, for n sufficiently large, $\mathcal{F}_n$ has one of the following two descriptions:

(i) there exists a t-chain $\mathcal{X}$ and a $(k+1-t)$-chain $\mathcal{Y}$, such that $\mathcal{X} \cap \mathcal{Y} = \emptyset$; and $\mathcal{F}_n$ is the following set of k-chains:

$$\{\mathcal{L} : \ \mathcal{X} \subseteq \mathcal{L} \text{ and } \mathcal{L} \cap \mathcal{Y} \neq \emptyset\} \cup \{\mathcal{L} : \ \mathcal{Y} \subseteq \mathcal{L} \text{ and } |\mathcal{L} \cap \mathcal{X}| = t-1\},$$

where the second set of chains is non-empty;

(ii) there exists a $(t+2)$-chain $\mathcal{Z}$, and $\mathcal{F}_n$ is the following set of k-chains:

$$\{\mathcal{L} : \ |\mathcal{L} \cap \mathcal{Z}| \geq t+1\},$$

and $|\bigcap_{\mathcal{L} \in \mathcal{F}_n} \mathcal{L} \cap \mathcal{Z}| \leq t-1$.

These theorems provide for a common generalization of the classic Erdős-Ko-Rado theorem and the theorem on intersecting chains in B^c_n. The proofs depend on the kernel method and may allow for generalization to other hereditary families than chains.

3. INTERSECTING PARTITIONS

This section poses some new problems on intersecting set partitions. A *partition* is a collection of disjoint non-empty sets whose union is the universe. We are going to consider different definitions for intersecting partitions. All of them are related to the type (2) higher-order problem. First, we say that two partitions of n elements *intersect in a class* if the two partitions share a class. It is natural to conjecture, that the largest number of k-partitions of an n-set that pairwise intersect in a class can be obtained by taking a fixed singleton and all $(k-1)$-partitions of the remaining $n-1$ elements.

Second, we say that that two partitions of an n-element set *intersect in a pair* if there exist respective classes C_1, C_2 of the two partitions such that $|C_1 \cap C_2| \geq 2$. This is the Rota type analogue of the intersection property to the partition lattice: two partitions intersect if their meet is above an atom. (We think about the partition lattice such that 0 is the finest partition and 1 is the coarsest partition.) This problem fits well the scheme of Simonovits and Vera Sós: consider those graphs on n vertices, which are vertex-disjoint unions of cliques. Give the largest number of those graphs which pairwise share at least one edge.

Conjecture 1 *If $n \leq 2k-1$, then the largest number of k-partitions of an n-set that pairwise intersect in a pair is $S(n-1,k)$. This bound can be attained by taking a fixed pair and all k-partitions of the n elements that have this pair in one class.*

Note that if $n = 2k$, then we can freely add to the above construction any partition which has a single class of size $k+1$ and $k-1$ singletons. Therefore, for $n = 2k$, the construction in the conjecture is no longer optimal.

Third, we say that that two partitions of n elements *intersect in a co-pair* if there exist a two-partition $\{C_1, C_2\}$ of $\{1, 2, ..., n\}$ such that both partitions refine $\{C_1, C_2\}$. This is also a Rota type analogue of the intersection property on the partition lattice: two partitions intersect if their join is under a co-atom.

Conjecture 2 *If $n \geq 2k-1$, then the largest number of k-partitions of an n-set that pairwise intersect in a co-pair is $S(n-1,k-1)$. This bound can be attained by taking a fixed singleton and all $(k-1)$-partitions of the remaining $n-1$ elements.*

Note that if $n = 2k-2$, then we can freely add to the above construction any partition which has a single class of size $k-1$ and $k-1$ singletons. Such k-partitions intersect in a co-pair every other k-partitions, otherwise the other partition would have a class whose size exceeds k, which is impossible. Therefore, for $n = 2k-2$, the construction in the conjecture is no longer optimal. The threshold in this conjecture is somewhat bold, the conjecture might require a larger value of n.

Theorem 4 *For fixed $k > t \geq 1$ and $n > n_0(k)$, the largest number of k-partitions of an n-set that pairwise intersect in at least t classes is $S(n-t, k-t)$. This bound can be attained by taking t singletons fixed and all $(k-t)$-partitions of the remaining $n-t$ elements.*

For the proof of the theorem we review facts about sunflowers that we use in the kernel method. A set system $\{A_1, A_2, \ldots, A_m\}$ is called a *sunflower* or *delta-system*, if $A_i \cap A_j = \bigcap_{l=1}^{m} A_l$ for all $1 \leq i < j \leq m$. The sets A_i are called the *petals* and $\bigcap_{l=1}^{m} A_l$ is called the *kernel* of the sunflower.

We say that a set system is of *rank* k, if $|H| \leq k$ for all $H \in \mathcal{H}$; and $\mathcal{H}$ is *t-intersecting*, if $|H_1 \cap H_2| \geq t$ for all $H_1, H_2 \in \mathcal{H}$. For $t \geq 1$, we say that $\mathcal{H}$ is *non-trivially* t-intersecting, if it is t-intersecting, and $|\bigcap \mathcal{H}| < t$. We say that $\mathcal{H}$ is *critically* t-intersecting, if it is t-intersecting, and deleting any $x \in H$ from any $H \in \mathcal{H}$, the resulting set system $\mathcal{H} \setminus \{H\} \cup \{H \setminus \{x\}\}$ is not t-intersecting.

Estimates in the kernel method are usually based on the following simple observation.

Lemma 1 *Let $\mathcal{H}$ be a critically t-intersecting system ($t \geq 1$) of rank k. Then $\mathcal{H}$ does not contain a sunflower with $k+1$ petals.*

Proof. Indeed, if $\{H_1, H_2, \ldots, H_{k+1}\}$ is a sunflower in $\mathcal{H}$, then any $H \in \mathcal{H}$ must intersect the kernel K of the sunflower in at least t elements, since a $\leq k$-element set cannot intersect each of the $k+1$ disjoint sets $H_1 \setminus K$, $H_2 \setminus K$,$\ldots$, $H_{k+1} \setminus K$. Hence the deletion of $H_1 \setminus K$ from H_1 (if $H_1 \neq K$) results a t-intersecting set system, contradicting the minimality of $\mathcal{H}$. □

We will also need the Erdős-Rado theorem [23]:

Lemma 2 *For every i and l, there exists a number $f(i,l)$, such that any family of $f(i,l)$ sets of size i each, contains a sunflower with l petals.* □

Now we return to the proof of the theorem. Identify a partition $\mathcal{P}$ with the k-element set of its classes. Throw out classes of partitions until we obtain a critically intersecting family $\mathcal{H}$. Let $\mathcal{H}_i$ denote the set of i-element collections in $\mathcal{H}$. If $\mathcal{H}_t \neq \emptyset$, then we have t identical classes present in all partitions, and the theorem follows by the monotonicity of $S(n,k)$, the Stirling number of the second kind, in n.

If $\mathcal{H}_t = \emptyset$, then from the Lemmas we have $|\mathcal{H}_i| \le f(i, k+1)$. Any element of $\mathcal{H}_i$ can be extended in at most $S(n-i, k-i)$ ways toward a partition $\mathcal{P}$. Hence the total number of partitions in this case is at most

$$\sum_{i=t+1}^{k} f(i, k+1) S(n-i, k-i).$$

Using the fact that for fixed k the asymptotic formula

$$S(n,k) \sim \frac{k^n}{k!}$$

holds ([19] p. 293), it follows that the number of partitions is $o(S(n-t, k-t))$.

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