Antichains and duality pairs in the digraph-poset extended abstract *

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Abstract

Let $\mathbb{D}$ denote the partially ordered sets of homomorphism classes of finite directed graphs, ordered by the homomorphism relation. Order theoretic properties of this poset have been studied extensively, and have interesting connections to familiar graph properties and parameters. This paper studies the generalized duality pairs in $\mathbb{D}$.

1 Introduction

Let $\vec{G}$ and $\vec{H}$ be two directed graphs and write $\vec{G} \leq \vec{H}$ or $\vec{G} \to \vec{H}$ provided that there is a homomorphism from $\vec{G}$ to $\vec{H}$, that is, a map $f : V(\vec{G}) \to V(\vec{H})$ such that for all $\langle x, y\rangle \in E(\vec{G})$, $\langle f(x), f(y)\rangle \in E(\vec{H})$. (Since in this paper we deal mainly with directed graphs - or *digraphs* for short - therefore, with some abuse of notation, we do not use the arrow notation when we can do it without danger of misunderstanding.) When we have $G \to H$ then we say that G admits an H*-coloration*. (The origin of this notion is the fact that an undirected graph G has a homomorphism into the ℓ-vertices (undirected) complete graph iff G is ℓ-colorable in the usual sense.)

The relation $\leq$ is a quasi-order and so it induces an equivalence relation: we say that G and H are *homomorphism-equivalent* or *hom-equivalent* and write $G \sim H$ if and only if $G \leq H$ and $H \leq G$. The *homomorphism poset* $\mathbb{D}$ is the partially ordered set of all equivalence classes of finite directed graphs, ordered by the $\leq$. We will often abuse notation by replacing the classes that comprise $\mathbb{D}$ with their members.

This partially ordered set is of significant intrinsic interest and are useful tool in the study of directed graph properties. For instance, it is easily seen that it is a countable distributive lattice: the supremum, or *join*, of any pair is their disjoint sum, and the infimum, or *meet*, is their categorical or relational product. Poset $\mathbb{D}$ is "predominantly" dense – it is shown by Nešetřil and Tardif [16]. Furthermore it embeds all countable partially ordered sets – see [19] for a presentation. The latter statement holds for the posets of all directed trees or paths, respectively, see [12] and [13].

The maximal chains and antichains of an ordered set are well known objects of interest. In this case, maximal antichains are particularly relevant because of their relationship to the notion of a *homomorphism duality*, introduced by Nešetřil and Pultr [14]: say that an ordered pair $\langle F, D\rangle$ of directed graphs, is a *duality pair* if

$$F \to \, = \, \not\to D \tag{1}$$

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where $F\!\rightarrow\; = \; \{G : F \to G\}$ and $\not\rightarrow\! D \; = \; \{G : G \not\rightarrow D\}$. Equivalently, the set of all structures is partitioned by the *upset* [or *final segment*] $F\!\rightarrow$ and the *downset* [or *initial segment*] $\rightarrow\! D$. [Here we also use the other common notation $F^{\uparrow}$ and $D^{\downarrow}$ for upsets and downsets, respectively.]

In the lattice $\mathbb{D}$ of directed graphs, each tree can play the role of F in (1). In fact, in [16], Nešetřil and Tardif obtain a correspondence between duality pairs and *gap*s in the homomorphism order for general relational structures. They note, among others, in [17] that the 2-element maximal antichains in $\mathbb{D}$ are exactly the duality pairs $\langle F, D\rangle$ where F is a tree and D is its *dual*, where the dual can be directly constructed.

Foniok, Nešetřil and Tardif [8, 9] are concerned with the most general circumstance. Let $\mathcal{F}$ and $\mathcal{D}$ both be finite subsets in the poset $\mathbb{D}$. Call $\langle \mathcal{F}, \mathcal{D}\rangle$ a *generalized finite duality* if

$$\bigcup_{F\in\mathcal{F}} F\!\rightarrow \; = \bigcap_{D\in\mathcal{D}} \not\rightarrow\! D. \tag{2}$$

Here one can assume - without extra cost - that both classes are antichains. Indeed, since $\mathcal{F}$ is finite therefore the set $\mathcal{F}'$ of minimum elements is an antichain, and $\mathcal{F}^{\uparrow} = (\mathcal{F}')^{\uparrow}$. Similarly, the maximal elements of $\mathcal{D}$ form the antichain $\mathcal{D}'$ for which $\mathcal{D}^{\downarrow} = (\mathcal{D}')^{\downarrow}$. Furthermore an easy consequence of (2) is that any element of $\mathcal{F}$ is incomparable with each element of $\mathcal{D}$.

Another consequence of this definition is the following formula:

$$\mathbb{D} = \left(\bigcup_{F\in\mathcal{F}} F\!\rightarrow\right) \; \cup \; \left(\bigcup_{D\in\mathcal{D}} \rightarrow\! D\right). \tag{3}$$

(Here, of course, the incomparability of the elements of $\mathcal{F}$ and $\mathcal{D}$ is not a requirement.) The generalized finite dualities are characterized in [8, 9]. Foniok, Nešetřil and Tardif also show that all maximal finite antichains A (with three exceptional cases) yields the generalized finite duality $\langle \mathcal{F}, A \setminus \mathcal{F}\rangle$.

It is quite natural to ask, in more general circumstances, if maximal antichains possess these sorts of partitions.

Indeed, Ahlswede, Erdős and Graham [1] introduced the notion of "splitting" a maximal antichain. Say that a maximal antichain A of a poset P *splits* if A can be partitioned into two subsets B and C such that $P = B^{\uparrow} \cup C^{\downarrow}$; say that P has the *splitting property* if all of its maximal antichains split. They obtained sufficient conditions for the splitting property in finite posets, from which they proved, in particular, that all finite Boolean lattices possess it. It is also a natural notion for infinite posets; see [5, 6].

On one hand side the correspondence between generalized dualities and maximal antichains obtained in [8, 9] and the partition in (3) demonstrate that for $\mathbb{D}$ essentially all finite maximal antichains split. On the other side: dropping the finiteness condition in the definition (2) one may have the notion of *generalized duality*. Using techniques from [6], Duffus, P.L. Erdős, Nešetřil and Soukup studied this further generalized notion. They proved, that under rather mild conditions, any finite, non-maximal antichain of $\mathbb{D}$ can be extended, on one hand side, into a generalized duality pair of antichains or, on the other side, into a maximal infinite antichain what does not split.

In case of general duality one has to take special care for the antichain properties since the procedure described after formula (2) can not be done in case of infinite classes.

In this paper we mainly study the remaining cases, where at least one class of $\langle \mathcal{F}, \mathcal{D}\rangle$ is finite. In Section 3 we give a new and transparent proof of the Foniok - Nešetřil - Tardif's theorem on finite duality pairs. In Section 2 we give the technical prerequisites for the proofs. Finally in Section 4 we will discuss those (possible) generalized duality pairs where both classes form antichains, and one of them is finite while the other one is infinite.

In addition to the selected individual papers, we refer the reader to the book [12] by Hell and Nešetřil that is devoted to graph homomorphisms. Chapter 3 of it gives a thorough introduction and many of the key results on maximal antichains and dualities in posets of (undirected or directed) finite graphs.

2 Prerequisites

Given a poset $\mathcal{P} = (P, <)$ and $A \subset P$ let

$$A^{\uparrow} = \{p \in P : \exists a \in A \; a \leq_P p\}; \qquad A^{\downarrow} = \{p \in P : \exists a \in A \; p \leq_P a\},$$

and

$$\mathbf{ABOVE}(A) = A^{\uparrow} \setminus A \quad \text{and} \quad \mathbf{BELOW}(A) = A^{\downarrow} \setminus A.$$

Let A be a maximal antichain in P. A partition (B, C) of A is a *split of* A iff $P = B^{\uparrow} \cup C^{\downarrow}$. We say that A *splits* if A has a split.

The equivalence classes of finite directed forests and directed trees will be denoted by $DFOREST$ and by $DTREE$, respectively. Given a directed graph D denote by $\mathrm{Comp}(D)$ the set of connected components of D.

For a given oriented path (directed path where the edges are not necessarily are directed consecutively) its *net-length* is the (absolute) difference between the numbers of edges oriented in one way or the other. Given $D \in \mathbb{D}$ let $\ell(D)$ be the net-length of D, i.e. the supremum of the net-length of the paths in D. Clearly $\ell(D) = \infty$ iff D contains an unbalanced circle.

Write $\mathbb{B} = \{D \in \mathbb{D} : \ell(D) < \infty\}$ and $\mathbb{U} = \{D \in \mathbb{D} : \ell(D) = \infty\}$. Clearly $DFOREST \cap \mathbb{B} = \emptyset$. The graphs in $\mathbb{B}$ are the *balanced* ones.

We need a result of J. Nešetřil and C. Tardif ([**?**]) which shows that each directed tree has a unique dual:

Theorem 2.1. *For each $T \in DTREE \setminus \{\vec{P}_0, \; \vec{P}_1, \vec{P}_2, \}$ there is a unique $D_T \in \mathbb{D}$ such that $\langle T, D_T \rangle$ is a duality, i.e. T and D_T are incomparable and $T \rightarrow = \not\rightarrow D_T$.*

We will use the Directed Sparse Incomparability Lemma in the following form (see [2]):

Theorem 2.2 (Directed Sparse Incomparability Lemma). *For each directed graph $H \in \mathbb{D} \setminus DFOREST$ and for all integers $m, k \in \mathbb{N}$ there is a directed graph H' such that*

(1) $k < \mathrm{girth}(H') < \infty$, *(this is the* girth *of the underlying undirected graph, the girth of a tree is ∞),*

(2) *for each directed graph G with $|V(G)| < m$ we have $H' \rightarrow G$ if and only if $H \rightarrow G$.*

(3) $H \not\rightarrow H'$.

3 The Foniok-Nešetřil-Tardif theorem

In this Section we give a short and transparent proof for the FNT Theorem using only the Directed Sparse Incomparability Lemma (Theorem 2.2) and the Nesetril-Tardif Theorem (Theorem 2.1).

Lemma 3.1. *Let $\mathcal{F}$ be antichain of balanced kernels that no kernel is a directed path of length at most 2. Then no kernel contains component compatible with directed path of length at most 2.*

Proof. Due to the condition every kernel contains component of net-length at least 3. Such a component can embed those paths. $\square$

Theorem 3.2 (Foniak, Nesetril Tardif, [7]).
(A) *If $\mathcal{F} \subset DFOREST$ is a finite antichain of forests, $\mathcal{F} \neq \{\vec{P}_0\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$, then it has a finite dual $D_{\mathcal{F}}$, and so $\mathcal{F} \cup D_{\mathcal{F}}$ is a splitting finite maximal antichain in $\mathbb{D}$.*
(B) *Let $\mathcal{A} \subset \mathbb{D}$ be a finite maximal antichain in $\mathbb{D}$, $\mathcal{A} \neq \{\vec{P}_0\}$, $\{\vec{P}_1\}$, $\{\vec{P}_2\}$. Then $\mathcal{A}$ splits and*

$$\langle \mathcal{A} \cap DFOREST, \mathcal{A} \cap \mathbb{U} \rangle$$

is the unique split of $\mathcal{A}$. Especially $\mathcal{A} = (\mathcal{A} \cap DFOREST) \cup D_{\mathcal{A} \cap DFOREST}$. Moreover each balanced graph in $\mathcal{A}$ is directed forest.

Proof of Theorem 3.2. (A) Write $\mathcal{F} = \{F^i : i < n\}$ and $\mathrm{Comp}(F_i) = \{F^i_j : j < k_i\}$ for $i < n$. (In this paper the indices run from 0 to upper-bound−1.) By Theorem 2.1 we have $\mathbb{D} \setminus (F^i_j \to) = \to D_{F^i_j}$. Then

$$\mathbb{D} \setminus \bigcup_{i<n} (F^i \to) = \bigcap_{i<n} (\mathbb{D} \setminus (F^i \to)) = \bigcap_{i<n} (\mathbb{D} \setminus \bigcap_{j<k_i} (F^i_j \to)) =$$

$$\bigcap_{i<n} (\bigcup_{j<k_i} (\mathbb{D} \setminus (F^i_j \to))) = \bigcap_{i<n} \bigcup_{j<k_i} (\to D_{F^i_j}) = \bigcup_{f \in \prod_{i<n} k_i} \bigcap_{i<n} (\to D_{F^i_{f(i)}}) =$$

$$\bigcup_{f \in \prod_{i<n} k_i} (\to \prod_{i<n} D_{F^i_{f(i)}}).$$

So $\mathcal{D}_{\mathcal{F}}$ is just the $\mathbb{D}$-maximal elements of $\left\{ \prod_{i<n} D_{F^i_{f(i)}} : f \in \prod_{i<n} k_i \right\}$, which proves (A).

Next we give a proof of (B). Let $\mathcal{A} \subset \mathbb{D}$ be a maximal finite antichain in $\mathbb{D}$, $\mathcal{A} \neq \{\vec{P}_0\}, \{\vec{P}_1\}, \{\vec{P}_2\}$.

Let $\vec{P}(2,n)$ be the following oriented path: $\vec{P}_2$ then n zigzag steps, then $\vec{P}_2$ again, then n zigzag steps, etc, up to n recursion, that is

$$\vec{P}(2,n) = 1\Big((10)^n 1\Big)^n 1.$$

Clearly $\ell(\vec{P}(2,n)) = n+1$. [See Figure 1.]

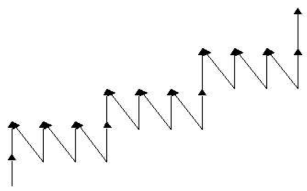

Figure 1: Graph $\vec{P}(2,3)$

Let $\ell = \max\{\ell(Q) : Q \in \mathcal{A} \cap \mathbb{B}\}$ and $n = \max\{|Q| : Q \in \mathcal{A}\}$.

Lemma 3.3. $\mathbf{BELOW}(\mathcal{A}) = \mathbf{BELOW}(\mathcal{A} \cap \mathbb{U})$.

Proof of the Lemma. Assume on the contrary that there exists $X \in \mathbf{BELOW}(\mathcal{A}) \setminus \mathbf{BELOW}(\mathcal{A} \cap \mathbb{U})$. Let $Y = X + \vec{P}(2, \ell+1)$.

For $Q \in \mathcal{A}$ if $Q \to Y$ then some component C of Q maps into $\vec{P}(2,\ell)$ and so $C \to \vec{P}_2$ because ℓ is large enough. But this is excluded by Lemma 3.1, a contradiction.

Moreover $Y \not\to Q$ for $Q \in \mathcal{A} \cap \mathbb{B}$ because $\ell(Y) > \ell(Q)$. For $Q \in \mathcal{A} \cap \mathbb{U}$ we have $Y \not\to Q$ because $X \to Y$ and $X \not\to Q$.

Hence Y is incomparable to any element of the maximal antichain $\mathcal{A}$. Contradiction. □

Lemma 3.4. $\mathbf{BELOW}(\{X\}) \not\subset \mathbf{BELOW}(\mathcal{A} \setminus \{X\})$ *for* $X \in \mathcal{A} \setminus \mathit{DFOREST}$.

Proof of the Lemma. Since $X \notin \mathit{DFOREST}$ we can apply the Directed Sparse Incomparability Lemma for X and $k = m = n+1$ to obtain a graph Y. Then $Y \to Q$ implies $X \to Q$ for $Q \in \mathcal{A}$ because $|Q| < m$. Thus $Y \in \mathbf{BELOW}(\{X\}) \setminus \mathbf{BELOW}(\mathcal{A} \setminus \{X\})$. □

Lemmas 3.3 and 3.4 together yield

$$\mathcal{A} \cap \mathit{DFOREST} = \mathcal{A} \cap \mathbb{B}, \tag{4}$$

that is every balanced element in $\mathcal{A}$ is a directed forest.

Lemma 3.5. $\mathbf{ABOVE}(\mathcal{A} \cap \mathit{DFOREST}) = \mathbf{ABOVE}(\mathcal{A})$.

Proof. Assume on the contrary that

$$X \in \mathbf{ABOVE}(\mathcal{A}) \setminus \mathbf{ABOVE}(\mathcal{A} \cap DFOREST). \tag{5}$$

Then $Q \to X$ for some $Q \in \mathcal{A} \cap \mathbb{U}$ and so $X \in \mathbb{U}$. Hence we can apply the Directed Sparse Incomparability Lemma for X and $k = m = \max\{n+1, |X|+1\}$ to obtain a graph Y.

Assume first that $Q' \to Y$ for some $Q' \in \mathcal{A}$. Then the image of Q' is a forest in Y because $girth(Y) > |Q'|$. Hence $Q' \in \mathbb{B}$. Thus $Q' \in DFOREST$ by (4). But $Q' \to Y \to X$ which is not possible by our assumption (5).

Hence $Y \to Q$ for some $Q \in \mathcal{A}$. But then $X \to Q$ because $|Q| < n+1$. Contradiction because $X \in \mathbf{ABOVE}(\mathcal{A})$. □

Lemma 3.6. $\mathbf{ABOVE}(\{F\}) \not\subset \mathbf{ABOVE}(\mathcal{A} \setminus \{F\})$ *for* $F \in \mathcal{A} \cap DFOREST$*.*

Proof. Let $F \in \mathcal{A} \cap DFOREST$. Let $Y = F + \vec{P}(2, \max(\ell, n)+1)$. Then $F \to Y$ but $\ell(Y) > \ell(F)$ so $Y \not\to F$. So $Y \in \mathbf{ABOVE}(F)$.

Assume now that there exists a $Q \in \mathcal{A} \setminus \{F\}$ with $Q \to Y$. Then every connected component C of Q can be mapped either into F or into $\vec{P}(2,\ell)$. In the latter, however, $C \to \vec{P}_2$ because ℓ is large enough. But this contradicts to Lemma 3.1. □

By equation (4) $(\mathcal{A} \cap DFOREST, \mathcal{A} \cap \mathbb{U})$ is a partition of $\mathcal{A}$. Hence Lemmas 3.3 and 3.5 imply that $\langle \mathcal{A} \cap DFOREST, \mathcal{A} \cap \mathbb{U} \rangle$ is a split in $\mathbb{D}$.

If $\langle \mathcal{B}, \mathcal{C} \rangle$ is a split of $\mathcal{A}$ then $\mathcal{C} \supset \mathcal{A} \cap \mathbb{U}$ by Lemma 3.4, and $\mathcal{B} \supset \mathcal{A} \cap DFOREST$ by Lemma 3.6. Hence $\langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{A} \cap DFOREST, \mathcal{A} \cap \mathbb{U} \rangle$. This proves Theorem 3.2. □

4 Duality pairs with both finite and infinite classes

The Foniok - Nešetřil - Tardif result (Theorem 3.2) characterized completely the finite duality pairs in $\mathbb{D}$.

Similarly Duffus, P.L. Erdős, Nešetřil and Soukup studied in details those generalized duality pairs where both classes form infinite antichains ([2]). More preciously they showed, that every non-maximal finite antichain A in $\mathbb{D}$ can be extended, in one hand side, into a splitting maximal antichain and it can also be extended into an another, a non-splitting one as well. These latter results depend on the fact, that $\mathbb{D}$ has a *loose kernel*, and so any non-maximal finite antichain A which belongs to this kernel can be extended into the still not maximal antichain A' with one more elements. These results give another measure of the richness of poset $\mathbb{D}$.

In this last Section we will mention one theorem and some further remarks on generalized duality pairs $\langle \mathcal{F}, \mathcal{D} \rangle$ in $\mathbb{D}$ where one class is finite while the other one is infinite.

Let start the remarks with a quite surprising result:

Theorem 4.1. *There is no generalized duality pair* $\langle \mathcal{F}, \mathcal{D} \rangle$*, forming an antichain, where* $|\mathcal{F}| < \omega$ *and* $|\mathcal{D}| = \omega$*,*

The proof of this fact will be published elsewhere. When all elements of $\mathcal{F}$ are forests, then the statement is almost trivial. However in the case of infinite $\mathcal{D}$ the proof of Theorem 3.2 does not give this property.

Let's discuss now the last open case: the existence of infinite-finite duality pairs in $\mathbb{D}$. This case has a huge theoretical importance due to its close connection to the *constrain satisfaction problems*.

More specifically let $\langle \mathcal{F}, \mathcal{D} \rangle$ be a generalized duality pair where $\mathcal{F} \subset DTREE$ but it can be infinite, while $\mathcal{D}$ consists of preciously one element D. Then we say that D has the *tree duality*. The following theorem is a seminal result in the theory of constrain satisfaction problems:

Theorem 4.2 (Hell - Nešetřil - Zhu [11]). *If digraph* D *has the tree duality then the* D*-colorability of any directed graph* G *can be decided in polynomial time.*

(Here the tree-duality can be strengthened to *bounded treewidth duality.*) The basic tool to prove this statement is the so-called *consistency check procedure*. This procedure is always finite and it succeeds if and only if $G \to D$. Furthermore

Theorem 4.3 (Hell - Nešetřil - Zhu [11]). *The* "tree duality of H" *is equivalent to the following property:* $\forall G : G \to H$ *if and only if the consistency check for* G *with respect to* H *succeeds.*

As far as these authors are aware the only known applicable instances of Theorems 4.2 and 4.3 are generalized duality pairs where the classes $\mathcal{F}$ do not form antichains. It is still an open problem whether there exists duality pair $\langle \mathcal{F}, \mathcal{D} \rangle$ with infinite antichain $\mathcal{F} \subset DTREE$ and $|\mathcal{D}| = 1$.

Here we give some easy observations, without proofs, which may be useful.

Remark 4.4. *Let* $\langle \mathcal{F}, \mathcal{D} \rangle$ *be a generalized duality pair where* $|\mathcal{D}|$ *is finite. Then* $\mathcal{F} \subset DFOREST$.

Remark 4.5. *Assume that* $\mathcal{F} \subset DTREE$. *Then we have* $|\mathcal{D}| = 1$.

Probably finding an infinite tree duality - where the trees form an antichain - is not easy. Therefore here is a less ambitious problem for that end:

Let $\langle \mathcal{F}, \{D\} \rangle$ be an infinite duality pair where $\mathcal{F} \subset DTREE$ and the second class consists of preciously one digraph. Then

1. $\forall F \in \mathcal{F}$ we have $F \not\to D$, but
2. for each tree $F' \in \mathbf{BELOW}(\mathcal{F})$ we have $F' \to D$.

Let say a digraph D is a *weak dual* of the infinite antichain $\mathcal{F}$ of directed trees, if they satisfy these two conditions. (The word weak refers for the fact that $\mathbb{D} \setminus \mathcal{F}^{\uparrow}$ is much-much bigger, than $\mathbf{BELOW}(\mathcal{F})$, but if $\{D\}$ is the (usual) dual of $\mathcal{F}$ then (2) holds for all elements of $\mathbb{D} \setminus \mathcal{F}^{\uparrow}$.)

Problem Construct an infinite antichain $\mathcal{F}$ of oriented trees in $\mathbb{D}$ with a week dual D.

5 Acknoledgement

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