NO FINITE-INFINITE ANTICHAIN DUALITY IN THE HOMOMORPHISM POSET $\mathbb{D}$ OF DIRECTED GRAPHS

PÉTER L. ERDŐS AND LAJOS SOUKUP

Abstract. $\mathbb{D}$ denotes the homomorphism poset of finite directed graphs. An *antichain duality* is a pair $\langle \mathcal{F}, \mathcal{D} \rangle$ of antichains of $\mathbb{D}$ such that $(\mathcal{F}\rightarrow) \cup (\rightarrow\mathcal{D}) = \mathbb{D}$ is a partition.

A *generalized duality pair in* $\mathbb{D}$ is an antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ with finite $\mathcal{F}$ and $\mathcal{D}$. We give a simplified proof of the Foniok - Nešetřil - Tardif theorem for the special case $\mathbb{D}$, which gave full description of the generalized duality pairs in $\mathbb{D}$.

Although there are many antichain dualities $\langle \mathcal{F}, \mathcal{D} \rangle$ with infinite $\mathcal{D}$ and $\mathcal{F}$, we can show that there is no antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ with finite $\mathcal{F}$ and infinite $\mathcal{D}$.

1. Introduction

If G and H are finite directed graphs, write $G \leq H$ or $G \rightarrow H$ iff there is a *homomorphism* from G to H, that is, a map $f : V(G) \rightarrow V(H)$ such that $\langle f(x), f(y) \rangle$ is a directed edge $\in E(H)$ whenever $\langle x, y \rangle \in E(G)$. The relation $\leq$ is a quasi-order and so it induces an equivalence relation: G and H are *homomorphism-equivalent* if and only if $G \leq H$ and $H \leq G$. The *homomorphism poset* $\mathbb{D}$ is the partially ordered set of all equivalence classes of finite directed graphs, ordered by $\leq$.

In this paper we continue the study of *antichain dualities* in $\mathbb{D}$ (what was started, in this form, in [2]). An *antichain duality* is a pair $\langle \mathcal{F}, \mathcal{D} \rangle$ of disjoint antichains of $\mathbb{D}$ such that $\mathbb{D} = (\mathcal{F}\rightarrow) \cup (\rightarrow\mathcal{D})$ is a partition of $\mathbb{D}$, where the set $\mathcal{F}\rightarrow := \{G : F \leq G \text{ for some } F \in \mathcal{F}\}$, and $\rightarrow\mathcal{D} := \{G : G \leq D \text{ for some } D \in \mathcal{D}\}$.

Special cases of antichain dualities were introduced and studied earlier in many papers. A *duality pair* is an antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ with

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$\mathcal{F} = \{F\}$ and $\mathcal{D} = \{D\}$. This particularly useful instance of the antichain duality was introduced by Nešetřil and Pultr in [16]. Clearly, $\langle F, D\rangle \in \mathbb{D} \times \mathbb{D}$ is a duality pair if

$$F\!\to\; = \;\not\to\! D, \tag{1}$$

(where, recall, $F\!\to\; := \{G : F \to G\}$ and $\not\to\! D := \{G : G \not\to D\}$). In [18], Nešetřil and Tardif gave full description of the single duality pairs in $\mathbb{D}$ (see Theorems 2.2 and 3.3 below).

Foniok, Nešetřil and Tardif studied a generalization of the notion of single duality pairs, which is another special case of antichain dualities ([8, 9, 10]): a *generalized duality pair* is an antichain duality $\langle \mathcal{F}, \mathcal{D}\rangle$ such that both $\mathcal{F}$ and $\mathcal{D}$ are finite. The generalized duality pairs (in the much more general context of *homomorphism poset of relational structures*) are characterized in [9]. The specialization of this general result of Foniok, Nešetřil and Tardif to the homomorphism poset of directed graphs $\mathbb{D}$ states that every maximal finite antichain (with the two exceptional cases of 1-element maximal antichains $\{P_1\}$ and $\{P_2\}$) has exactly one partition $\langle \mathcal{F}, \mathcal{D}\rangle$ which is generalized duality pair.

Meanwhile, independently from the above mentioned investigations, R. Ahlswede, P.L. Erdős and N. Graham [1] introduced the notion of "splitting" of maximal antichains. If P is a poset and A is a maximal antichain in P, then a *split* of A is a partition $\langle B, C\rangle$ of A such that $P = (B\!\to) \cup (\to\! C)$. (In the papers dealing with the splitting property usually the notations $B^{\uparrow}$ and $C^{\downarrow}$ are used instead of $B\!\to$ and $\to\! C$.) The poset P has the *splitting property* iff all of its maximal antichains have a split. They obtained sufficient conditions for the splitting property in finite posets, from which they proved, in particular, that all finite Boolean lattices possess it. The splitting property was also studied for infinite posets; see [5, 6].

A split of a maximal antichain in $\mathbb{D}$ is just an antichain duality.

As we already mentioned, the "finite-finite" antichain dualities, i.e. antichain dualities $\langle \mathcal{F}, \mathcal{D}\rangle$ with finite $\mathcal{F}$ and $\mathcal{D}$, were described in [9]. It is quite natural to ask, what about the other possible (by cardinality type) antichain dualities?

Using techniques from [6], Duffus, Erdős, Nešetřil and Soukup studied (in [2]) the existence of "infinite-infinite" antichain dualities in $\mathbb{D}$, i.e. antichain dualities $\langle \mathcal{F}, \mathcal{D}\rangle$ with infinite $\mathcal{F}$ and $\mathcal{D}$ and got the following results: Although there exist infinite-infinite antichain dualities in $\mathbb{D}$, it is not true that every infinite antichain has exactly one split: some of them have infinitely many splits, some others do not have splits at all, and there are maximal infinite antichains with exactly one split as well. At the very moment there is no reasonable descriptions of "infinite-infinite" antichains dualities, the structure of these antichain dualities is unknown. It seems to be a promising research subject.

However, there are two more, theoretically possible, classes of antichain dualities: the "finite-infinite" and the "infinite-finite" ones. In this paper we study the first one, and we show that somewhat surprising fact, that there is no antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ in $\mathbb{D}$ where $\mathcal{F}$ is finite while $\mathcal{D}$ is infinite (see Theorem 4.1).

We also include a short proof of the (specialized) Foniok - Nešetřil - Tardif's theorem ([9]) on generalized duality pairs (see Section 3). In Section 2 we give the technical prerequisites for the proofs. Finally in Section 5 we give some "background" information concerning the problem of the existence of infinite - finite antichain dualities.

In addition to the selected individual papers, we refer the reader to the book [13] by Hell and Nešetřil that is devoted to graph homomorphisms. Chapter 3 of it gives a thorough introduction and many of the key results on maximal antichains and dualities in posets of (undirected or directed) finite graphs.

A preliminary, extended abstract version of this paper was published in the proceedings of the ROGICS'08 ([7]).

2. Prerequisites

Given a poset $\mathcal{P} = (P, \leq)$ and $A \subset P$ denote

$A{\to} = \{p \in P : \exists a \in A\ a \leq_P p\}; \qquad {\to}A = \{p \in P : \exists a \in A\ p \leq_P a\},$

furthermore let

$$A^{\Uparrow} = (A{\to}) \setminus A \quad \text{and} \quad A^{\Downarrow} = ({\to}A) \setminus A,$$

so for example $A^{\Uparrow}$ contains everything which is really "above" A.

Let A be a maximal antichain in P. A partition $\langle B, C \rangle$ of A is a *split of* A iff $P = (B{\to}) \cup ({\to}C)$. We say that A *splits* iff A has a split.

$\mathbb{D}$ denotes the homomorphism poset of the finite directed graphs. For each $F \in \mathbb{D}$ (which is an equivalence class of the finite directed graphs) let the *core of* F be a digraph C from the equivalence class F with minimal number of vertices.

Claim 2.1. *If $C' \in F$, C is a core and $f : C \to C'$ is a homomorphism, then f is injective. Hence, the core of F is unique.*

Indeed, since there is a homomorphism $g : C' \to C$, we have that C and $f(C)$ are homomorphism equivalent. Since C has the minimal number of vertices in F, we have that $f(C)$ and C have the same number of vertices, i.e. f is injective. □

We will use the following convention: if φ is a property of digraphs, then we say that $G \in \mathbb{D}$ *has property* φ iff the core of G has property φ. Similarly, if $G \in \mathbb{D}$, then $|G|$ is the cardinality of the core of G.

The poset $\mathbb{D}$ is a countable distributive lattice: the supremum, or *join*, of any pair is their disjoint sum, and the infimum, or *meet*, is their *categorical product*. (In this latter the vertices are ordered pairs from the vertex sets, and $\langle xy, vz\rangle$ is a directed edge iff $\langle x, v\rangle$ and $\langle y, z\rangle$ are directed edges.)

Poset $\mathbb{D}$ is "predominantly" dense – it is shown by Nešetřil and Tardif [18]. Furthermore it embeds all countable partially ordered sets – see [20]. The latter statement holds for the posets of all directed trees or even paths, respectively, see [13] and [14].

The equivalence classes of finite directed forests and directed trees in $\mathbb{D}$ will be denoted by $\mathbb{F}$ and by $\mathbb{T}$, respectively. The core of the equivalence class of a directed forest is a directed forest as well, while the core of any connected graph is connected. Given a directed graph D denote by $\mathrm{Comp}(D)$ the set of connected components of D.

For a given oriented *walk* (walk of directed edges where the edges are not necessarily directed consecutively) its *net-length* is the (absolute) difference between the numbers of edges oriented in one direction and in the other direction. Given $D \in \mathbb{D}$ let the net-length $\ell(D)$ of D be the supremum of the net-length of the oriented walks in D. An oriented cycle is *balanced* iff the same number of oriented edges are going forward and backward. Otherwise it is *unbalanced*. Clearly $\ell(D) = \infty$ iff D contains unbalanced cycle(s).

Write $\mathbb{B} = \{D \in \mathbb{D} : \ell(D) < \infty\}$ and $\mathbb{U} = \{D \in \mathbb{D} : \ell(D) = \infty\}$. The graphs in $\mathbb{B}$ are the *balanced* ones. (It is easy to see that if $\ell(D) < \infty$ then $\ell(D) \le |D|$ as well.) Clearly $\mathbb{T} \subsetneqq \mathbb{F} \subsetneqq \mathbb{B}$.

We need a fundamental result of J. Nešetřil and C. Tardif ([18]) which shows that each directed tree has a dual:

Theorem 2.2. *For each tree* $T \in \mathbb{T} \setminus \left\{\vec{P}_0, \vec{P}_1, \vec{P}_2\right\}$, *there is a* $D_T \in \mathbb{D}$ *such that* $\langle T, D_T\rangle$ *is a duality pair. The graph* D_T *is called the* the dual *of* T.

We will use the following special case of the general Directed Sparse Incomparability Lemma of Nešetřil from [15]. A proof of the following version can be found in [2].

Theorem 2.3 (Directed Sparse Incomparability Lemma)**.** *For each directed graph* $H \in \mathbb{D} \setminus \mathbb{F}$ *and for all integers* $m, k \in \mathbb{N}$ *there is a directed graph* H' *such that*

(1) $k < \mathrm{girth}(H') < \infty$, (*this is the* girth *of the underlying undirected graph, the girth of a tree is* ∞),
(2) *for each directed graph* G *with* $|V(G)| < m$ *we have* $H' \to G$ *if and only if* $H \to G$.
(3) $H \not\to H'$.

3. The Foniok - Nešetřil - Tardif Theorem

In this Section we give a short proof for the Foniok - Nešetřil - Tardif Theorem (in the special case of $\mathbb{D}$) using only the Nešetřil -Tardif Theorem and the Directed Sparse Incomparability Lemma (Theorems 2.2 and 2.3).

We start with two auxiliary results: The connected components of any core are pairwise incomparable digraphs in $\mathbb{D}$. Hence

Fact 3.1. *If* $G \in \mathbb{D}$ *and* $C \to \vec{P}_2$ *for some connected component* C *of the core of* G *then* $G \to \vec{P}_2$.

Lemma 3.2. *If* $\mathcal{B} \subset \mathbb{D}$ *is finite and* $X \in \mathcal{B} \setminus \mathbb{F}$ *with* $X \notin \big(\mathcal{B} \setminus \{X\}\big)^{\Downarrow}$ *then* $\{X\}^{\Downarrow} \not\subset \big(\mathcal{B} \setminus \{X\}\big)^{\Downarrow}$.

Proof. Let $n = \max\{|Q| : Q \in \mathcal{B}\}$. Since $X \notin \mathbb{F}$ we can apply the Directed Sparse Incomparability Lemma for X and $k = m = n + 1$ to obtain a graph Y satisfying conditions 2.3.(1)–(3). Then $Y \to Q$ implies $X \to Q$ for $Q \in \mathcal{B}$ because $|Q| < m$. Thus $Y \in \{X\}^{\Downarrow} \setminus (\mathcal{B} \setminus \{X\})^{\Downarrow}$. □

Now we are in the position to present our proof. From now on, for sake of brevity, we will say that an antichain is *trivial* if it is one of $\left\{\vec{P}_0\right\}$, $\left\{\vec{P}_1\right\}$, $\left\{\vec{P}_2\right\}$.

Theorem 3.3 (Foniok - Nešetřil - Tardif [9])**.**
(i) *If* $\mathcal{F} \subset \mathbb{F}$ *is a non-trivial finite antichain, then there is a finite antichain* $\mathcal{D}_{\mathcal{F}}$ *in* $\mathbb{D}$ *such that* $(\mathcal{F}, \mathcal{D}_{\mathcal{F}})$ *is an antichain duality.*
(ii) *If* $\langle \mathcal{F}, \mathcal{D} \rangle$ *is a generalized duality pair in* $\mathbb{D}$, $\mathcal{F}$ *is a non-trivial antichain, then* $\mathcal{F} \cup \mathcal{D}$ *is also an antichain.*
(iii) *If* $\mathcal{A}$ *is a non-trivial, finite, maximal antichain in* $\mathbb{D}$, *then* $\mathcal{A}$ *possesses exactly one split, namely* $\langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U} \rangle$. *Hence* $\mathcal{A} \cap \mathbb{U} = \mathcal{D}_{\mathcal{A} \cap \mathbb{F}}$, *and so* $\mathcal{A} \cap (\mathbb{B} \setminus \mathbb{F}) = \emptyset$ (*that is,* $\mathcal{A}$ *does not contain any balanced graph which is not a forest*).

Proof of Theorem 3.3. (i) First observe that $\mathcal{F} \neq \emptyset$ because $\mathbb{D}$ does not have maximal elements. So we can write $\mathcal{F} = \{F^i : i < n\}$ and

$\mathrm{Comp}(F_i) = \{F^i_j : j < k_i\}$ for $i < n$. Since $F^i_j \in \mathbb{T}$, by Theorem 2.2 we have $\mathbb{D} \setminus (F^i_j \rightarrow) = \rightarrow D_{F^i_j}$ for some digraph $D_{F^i_j} \in D$. Then

$$\mathbb{D} \setminus \bigcup_{i<n} (F^i \rightarrow) = \bigcap_{i<n} (\mathbb{D} \setminus (F^i \rightarrow)) = \bigcap_{i<n} (\mathbb{D} \setminus \bigcap_{j<k_i} (F^i_j \rightarrow)) =$$
$$\bigcap_{i<n} (\bigcup_{j<k_i} (\mathbb{D} \setminus (F^i_j \rightarrow))) = \bigcap_{i<n} \bigcup_{j<k_i} (\rightarrow D_{F^i_j}) = \bigcup_{f \in \prod_{i<n} k_i} \bigcap_{i<n} (\rightarrow D_{F^i_{f(i)}}) =$$
$$\bigcup_{f \in \prod_{i<n} k_i} (\rightarrow \prod_{i<n} D_{F^i_{f(i)}}).$$

Let $\mathcal{D}_{\mathcal{F}}$ be just the $\mathbb{D}$-maximal elements of $\left\{\prod_{i<n} D_{F^i_{f(i)}} : f \in \prod_{i<n} k_i\right\}$. Then, by the equation above, we have $\mathbb{D} \setminus (\mathcal{F}\rightarrow) = \rightarrow(\mathcal{D}_{\mathcal{F}})$, which proves (i).

(ii) Let $\mathcal{B} = \mathcal{F} \cup \mathcal{D}$. If $X \in \mathcal{F}$ then $X \notin \mathcal{B} \setminus \{X\}^{\Downarrow}$. So, by Lemma 3.2, if $X \notin \mathbb{F}$ then there is $Y \in \mathbb{D}$ such that $D \notin \mathcal{F}^{\Uparrow} \cup \mathcal{D}^{\Downarrow}$, which is a contradiction. Therefore $\mathcal{F} \subset \mathbb{F}$.

Hence, by (i), the non-trivial finite antichain $\mathcal{F} \subset \mathbb{D}$ has the finite dual $D_{\mathcal{F}}$ such that $(\mathcal{F}\rightarrow) \cup (\rightarrow D_{\mathcal{F}})$ is a partition of $\mathbb{D}$. Hence $\rightarrow D_{\mathcal{F}} = \rightarrow \mathcal{D}$. Since $\mathcal{D}$ and $D_{\mathcal{F}}$ are antichains we have $\mathcal{D} = D_{\mathcal{F}}$. Thus $\mathcal{F} \cup \mathcal{D} = \mathcal{F} \cup D_{\mathcal{F}}$ is an antichain which was to be proved.

(iii) Let $\mathcal{A}$ be a non-trivial, finite, maximal antichain in $\mathbb{D}$, we are going to show that it possesses exactly one split.

To start with identify the oriented paths of length k and the 0-1 sequences of length k in a natural way: 1 means a forward, 0 a backward edge.

Let $R(n)$ be the oriented path described by the sequence $\big((10)^n 1\big)^n$, where the superscript n means the concatenation of n copies of the base. Then $R(n)$ is the concatenation of n "blocks", $R(n) = Q_1 {}^\frown \cdots {}^\frown Q_i {}^\frown \cdots {}^\frown Q_n$, where a block Q_i is described by $(10)^n 1$, i.e. it is n zigzag followed by a "forward" edges.

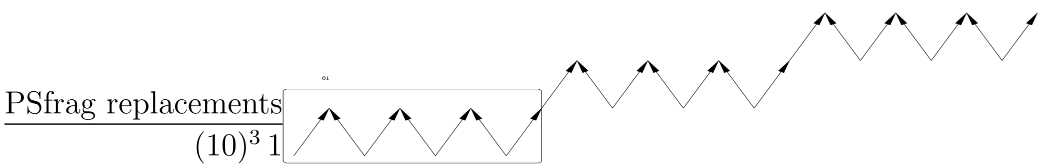

FIGURE 1. Path $R(3)$ assigned to $\big((10)^3 1\big)^3$

Clearly $\ell(R(n)) = n$, and

(2) if $C \in \mathbb{D}$ is connected, $|C| \le n$ and $C \to R(n)$, then $C \to P_2$.

Indeed, if $f : C \to R(n)$ then the image of C is inside two consecutive "blocks", $f : C \to Q_i {}^\frown Q_{i+1}$, and clearly $Q_i {}^\frown Q_{i+1} \to P_2$.

Let $\ell = \max\{\ell(Q) : Q \in \mathcal{A} \cap \mathbb{B}\}$ and $n = \max\{|Q| : Q \in \mathcal{A}\}$.

Lemma 3.4. $\mathcal{A}^{\Downarrow} = (\mathcal{A} \cap \mathbb{U})^{\Downarrow}$.

Proof of the lemma. Assume that $X \in \mathcal{A}^{\Downarrow}$. Let Y be the disjoint union of the graphs X and $R(n+1)$. Then Y and some $Q \in \mathcal{A}$ are comparable.

If $Q \to Y$ then $Q \not\to X$ implies that $C \to R(n+1)$ for some connected component C of the core of Q. Since $|C| \le |Q| < n+1$ we have $C \to \vec{P}_2$ by (2). Thus $Q \to \vec{P}_2$ by the Fact 3.1, and so $\mathcal{A}$ is one of the trivial antichains, a contradiction.

Therefore $Y \to Q$. Thus $R(n+1) \to Q$. Since $\ell(R(n+1)) = n+1 > n \ge \ell = \max\{\ell(Q) : Q \in \mathcal{A} \cap \mathbb{B}\}$, we have $Q \notin \mathbb{B}$, i.e $Q \in \mathbb{U}$, which proves the lemma. $\square$

Since $\mathcal{A}$ is an antichain, applying Lemma 3.2 for $\mathcal{B} = \mathcal{A}$ we obtain

Lemma 3.5. $\big(\{X\}^{\Downarrow}\big) \not\subset \big(\mathcal{A} \setminus \{X\}\big)^{\Downarrow}$ *for* $X \in \mathcal{A} \setminus \mathbb{F}$.

Lemmas 3.4 and 3.5 together yield

$$\mathcal{A} \cap \mathbb{F} = \mathcal{A} \cap \mathbb{B}, \tag{3}$$

that is, every balanced element in $\mathcal{A}$ is equivalent to a directed forest.

Lemma 3.6. $(\mathcal{A} \cap \mathbb{F})^{\Uparrow} = \mathcal{A}^{\Uparrow}$.

Proof. Assume that $X \in (\mathcal{A} \setminus \mathbb{F})^{\Uparrow}$. Then $Q \to X$ for some $Q \in \mathcal{A} \cap \mathbb{U}$ and so $X \in \mathbb{U}$. Hence we can apply the Directed Sparse Incomparability Lemma for X and $k = m = n + 1$ to obtain a graph Y satisfying conditions 2.3.(1) – (3).

Then Y and some $Q' \in \mathcal{A}$ are comparable.

If $Y \to Q'$, then $X \to Q'$ because $|Q'| < n + 1 = m$. Thus $Q \to X \to Q'$ implies $Q = X = Q' \in \mathcal{A}$, which contradicts $X \in \mathcal{A}^{\Uparrow}$.

Therefore $Q' \to Y$. The image of Q' is a forest in Y because $girth(Y) > |Q'|$. Hence $Q' \in \mathbb{B}$, therefore $Q' \in \mathbb{F}$ by (3). So $Q' \to Y \to X$ implies $X \in (\mathcal{A} \cap \mathbb{F})^{\Uparrow}$ which was to be proved. $\square$

Lemma 3.7. $\{F\}^{\Uparrow} \not\subset \big(\mathcal{A} \setminus \{F\}\big)^{\Uparrow}$ *for each* $F \in \mathcal{A} \cap \mathbb{F}$.

Proof. Let $F \in \mathcal{A} \cap \mathbb{F}$. Put $Y = F + R(n+1)$. Then $F \to Y$ but $\ell(Y) > \ell(F)$ so $Y \not\to F$. Hence $Y \in F^{\Uparrow}$.

Let $Q \in \mathcal{A}$ with $Q \to Y$. Then every connected component C of Q can be mapped either into F or into $R(n+1)$. If $C \to R(n+1)$, then $C \to \vec{P}_2$ by (2). Thus $Q \to \vec{P}_2$ by the Fact 3.1, and so $\mathcal{A}$ is one of the trivial antichains, a contradiction. Therefore $C \to F$, and so $Q \to F$, which implies $Q = F$. Thus $Y \in \{F\}^{\Uparrow} \setminus (\mathcal{A} \setminus \{F\})^{\Uparrow}$. □

By equation (3), $\big(\mathcal{A} \cap \mathbb{F}\big) \cup \big(\mathcal{A} \cap \mathbb{U}\big)$ is a partition of $\mathcal{A}$. Hence Lemmas 3.4 and 3.6 imply that $\langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U} \rangle$ is a split in $\mathbb{D}$.

If $\langle \mathcal{B}, \mathcal{C} \rangle$ is a split of $\mathcal{A}$ then $\mathcal{C} \supset \mathcal{A} \cap \mathbb{U}$ by Lemma 3.5, and $\mathcal{B} \supset \mathcal{A} \cap \mathbb{F}$ by Lemma 3.7. Hence $\langle \mathcal{B}, \mathcal{C} \rangle = \langle \mathcal{A} \cap \mathbb{F}, \mathcal{A} \cap \mathbb{U} \rangle$. This proves Theorem (iii). □

4. There is no finite-infinite antichain duality

In the remaining part of this paper we will study those antichain dualities, where one class is finite while the other one is infinite. In this Section we will show that there exists no antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ in $\mathbb{D}$ such that $\mathcal{F}$ is finite while $\mathcal{D}$ is infinite.

An ordered pair $\langle D, E \rangle$ of finite digraphs is a *gap in* $\mathbb{D}$ iff D is strictly below E and there is no H strictly between D and E. In the coming proof we need the following week version of a theorem of Nešetřil and Tardif:

Proposition 4.1 ([18])**.** *If $C \notin \mathbb{T}$ and C is connected, then $\langle G, G + C \rangle$ can not be a gap for any $G \in \mathbb{D}$.*

Proof. Assume that G is strictly below $G + C$. Apply the Directed Sparse Incomparability Lemma 2.3 for C with parameters $k = m = \max\{|C|, |G|\} + 1$ to get a digraph H satisfying Theorem 2.3 (1) – (3). Then $G + H$ is strictly between G and $G + C$. □

The main result of this paper is the following:

Theorem 4.2. *There is no antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ in $\mathbb{D}$ with $|\mathcal{F}| < \omega$ and $|\mathcal{D}| = \omega$.*

Proof. Assume on the contrary that $\langle \mathcal{F}, \mathcal{D} \rangle$ is such a pair.

Let $\mathcal{T} = \bigcup\{\mathrm{Comp}(F) : F \in \mathcal{F}\} \cap \mathbb{T}$ and $\mathcal{C} = \bigcup\{\mathrm{Comp}(F) : F \in \mathcal{F}\} \setminus \mathbb{T}$. The sets $\mathcal{T}$ and $\mathcal{C}$ are finite and since $\vec{P}_0, \vec{P}_1, \vec{P}_2 \notin \mathcal{T}$ by Fact 3.1, we can put $\mathcal{A} = \{D_T : T \in \mathcal{T}\}$, where D_T is the dual of the directed tree T. Let

$$\mathcal{A}^* = \Big\{ \prod \mathcal{B}' : \mathcal{B}' \subset \mathcal{A} \Big\}$$

and $\mathcal{D}_1 = \mathcal{D} \cap \mathcal{A}^*$.

Since $\mathcal{D}$ is infinite, we can fix $E \in \mathcal{D} \setminus \mathcal{D}_1$. Put $\mathcal{B} = \{D_T : T \in \mathcal{T} \text{ and } T \not\to E\}$.

Let $G_0 = \prod \mathcal{B}$ provided $\mathcal{B} \neq \emptyset$, while in case of $\mathcal{B} = \emptyset$ let G_0 be an arbitrary graph which is strictly bigger than E (that is $E \to G_0 \not\to E$).

Then $E \to G_0$, but they are not homomorphism equivalent: This is clear if $\mathcal{B} = \emptyset$. If $\mathcal{B} \neq \emptyset$, then $E \to D_T$ for each $D_T \in \mathcal{B}$. Thus $E \to G_0$. But $G_0 \in \mathcal{A}^*$ and $E \notin \mathcal{A}^*$, so G_0 and E are not equivalent.

Enumerate $\mathcal{C}$ as $\{C_i : i < m\}$. By finite induction on $i \leq m$ define graphs $G_0, G_1, .., G_m$ such that

(1) $E \to G_m \to \cdots \to G_1 \to G_0$,
(2) if $C_i \not\to E$ then $C_i \not\to G_{i+1}$.

The digraph G_0 satisfies (1). Assume that we have already constructed $G_0, ..., G_i$. If $C_i \not\to E$, but $C_i \to G_i$ then consider the ordered pair $(E, E + C_i)$. Since C_i is not a tree, this pair is not a gap by Proposition 4.1. Let G_{i+1} be any element which is strictly between E and $E + C_i$.

Since $\langle \mathcal{F}, \mathcal{D} \rangle$ is a antichain duality and $G_m \in \{E\}^{\Uparrow} \subseteq \mathcal{D}^{\Uparrow}$ we have $F \to G_m$ for some $F \in \mathcal{F}$. Since $F \not\to E$, there is a $C \in \text{Comp}(F)$ such that $C \not\to E$.

If $C \in \mathcal{T}$, then $D_C \in \mathcal{B}$, and so $G_m \to G_0 = \prod \mathcal{B} \to D_C$. Thus $C \not\to G_m$ and so $F \not\to G_m$ as well.

Assume now that $C \in \mathcal{C}$. Then $C = C_i$ for some $i < m$ and so $C \not\to E$ implies $C \not\to G_{i+1}$. Thus $C \not\to G_m$, and so $F \not\to G_m$ as well.

Contradiction, Theorem 4.2 is proved. □

5. Do there exist infinite-finite antichain dualities?!

In this Section we discuss briefly the last open case: the possible existence of infinite-finite antichain dualities in $\mathbb{D}$. This case has some theoretical interest due to its close connection to the constrain satisfaction problems.

More specifically let $\langle \mathcal{F}, \mathcal{D} \rangle$ be an antichain duality where $\mathcal{F} \subset \mathbb{T}$ but it can be infinite, while $\mathcal{D}$ consists of precisely one element D. After Hell, Nešetřil and Zhu, we say that D has a *tree duality.* The following theorem is a seminal result in the constrain satisfaction problem:

Theorem 5.1 (Hell - Nešetřil - Zhu [12]). *If digraph D has the tree duality then the D-colorability of each directed graph G (that is the existence of a $G \to D$ homomorphism) can be decided in polynomial time.*

The basic tool to prove this statement is the so-called *consistency check procedure*. This procedure is always finite and it succeeds if and only if $G \to D$. An even stronger result applies as well:

Theorem 5.2 (Hell - Nešetřil - Zhu [12])**.** *The* "tree duality of H" *is equivalent to the following property:* $\forall G : (G \to H$ *if and only if the consistency check for* G *with respect to* H *succeeds*$)$.

As far as these authors are aware, it is still an open problem whether there exists antichain duality $\langle \mathcal{F}, \mathcal{D} \rangle$ with infinite $\mathcal{F} \subset \mathbb{T}$ and $|\mathcal{D}| = 1$.

We want to add that one of the (anonymous) referees was kind enough to point out, there is known example of infinite - singleton antichain duality in case of a more general relational structure with three relations. However it is not known whether this example can be mapped into the graph homomorphism poset $\mathbb{D}$.

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References

[1] R. Ahlswede - P.L. Erdős - N. Graham, A splitting property of maximal antichains, *Combinatorica* **15** (1995) 475–480

[2] D. Duffus - P.L. Erdős - J. Nešetřil - L. Soukup: Antichains in the homomorphism order of graphs, *Comment Math Univ Carolinae* **48** (4) (2007), 571–583.

[3] P. Erdős: Graph theory and probability. *Canad. J. Math.* **11** (1959) 34–38

[4] P. Erdős - A. Hajnal: On chromatic number of graphs and set systems, *Acta Math. Hungar.* **17** (1966), 61–99

[5] P. L. Erdős: Splitting property in infinite posets, *Discrete Math.* **163** (1997) 251–256

[6] P. L. Erdős - L. Soukup: How to split antichains in infinite posets, *Combinatorica* **27** (2) (2007), 147–161.

[7] P.L. Erdős - L. Soukup: Antichains and duality pairs in the digraph-poset – extended abstract; *Proc. Int. Conf. on Relations, Orders and Graphs: Interaction with Computer Science, ROGICS '08* May 12-17, Mahdia-Tunisia (Ed. Y. Boudabbous, N. Zaguia) ISBN: 978-0-9809498-0-3 (2008), 327–332.

[8] J. Foniok - J. Nešetřil - C. Tardif: Generalised dualities and finite maximal antichains, in *Graph-Theoretic Concepts in Computer Science'06)* (F.V. Fomin (ed.)) **LNCS 4271** (2006), 27-36.

[9] J. Foniok - J. Nešetřil - C. Tardif: Generalized dualities and maximal finite antichains in the homomorphism order of relational structures, *Europ. J. Comb* **29** (2008), 881-899.

[10] J. Foniok - J. Nešetřil - C. Tardif: On Finite Maximal Antichains in the Homomorphism Order, *Electronic Notes in Discrete Mathematics* **29** (2007), 389-396.

[11] P. Hell - J.Nešetřil: *Graphs and Homomorphisms*, Oxford Univ. Press (2004)

[12] P. Hell - J. Nešetřil - X. Zhu: Duality and polynomial testing of tree homomorphisms, *Trans. Amer. Math. Soc.* **348** (1996), 1281–1297.

[13] J. Hubička - J. Nešetřil: Universal partial order represented by means of oriented trees and other simple graphs, *Eur. J. Combinatorics* **26** (2005), 765-778.

[14] J. Hubička - J. Nešetřil: Finite Paths are Universal, *Order* **22** (2005), 21-40.

[15] J. Nešetřil, Sparse incomparability for relational structures, in preparation

[16] J. Nešetřil - A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, *Discrete Math.* **22** (1978) 287–300

[17] J. Nešetřil - V. Rödl: Chromatically optimal rigid graphs, *J. Comb. Theory Ser. (B)* **46** (1989) 133–141

[18] J. Nešetřil - C. Tardif: Duality theorems for finite structures (characterising gaps and good characterisations) *J. Comb. Theory Ser. (B)* **80** (2000) 80–97

[19] J. Nešetřil - Xuding Zhu: On sparse graphs with given colorings and homomorphisms, *J. Comb. Theory Ser. (B)* **90** (2004), 161–172

[20] A. Pultr - V. Trnková: *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North-Holland, Amsterdam (1980)

(Péter L. Erdős and Lajos Soukup) Alfréd Rényi Institute of Mathematics, Budapest, P.O. Box 127, H-1364 Hungary

E-mail address: `elp@renyi.hu, soukup@renyi.hu`