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HOW TO SPLIT ANTICHAINS IN INFINITE POSETS*

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A maximal antichain A of poset P *splits* if and only if there is a set $B \subset A$ such that for each $p \in P$ either $b \le p$ for some $b \in B$ or $p \le c$ for some $c \in A \setminus B$. The poset P is *cut-free* if and only if there are no $x < y < z$ in P such that $[x,z]_P = [x,y]_P \cup [y,z]_P$. By [1] every maximal antichain in a finite cut-free poset splits. Although this statement for infinite posets fails (see [2])) we prove here that if a maximal antichain in a cut-free poset "resembles" to a finite set then it splits. We also show that a version of this theorem is just equivalent to Axiom of Choice.

We also investigate possible strengthening of the statements that "A does not split" and we could find a maximal strengthening.

1. Introduction

Given a poset $\mathcal{P} = (P, <)$ and subset $A \subset P$ we define the *upset* $A^\uparrow$ and the *downset* $A^\downarrow$ of A as follows:

$$A^\uparrow = \{p \in P : \exists a \in A \ a \le_P p\}$$

and

$$A^\downarrow = \{p \in P : \exists a \in A \ p \le_P a\}.$$

An *antichain* in P is a set of pairwise incomparable elements. If A is a maximal antichain in P then clearly $P = A^\uparrow \cup A^\downarrow$. We say that A *splits* if

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there is $B \subset A$ such that $P = B^{\uparrow} \cup (A \setminus B)^{\downarrow}$. Some maximal antichain may split in a trivial way: e.g. $P = A^{\uparrow}$. Some antichains can not split for the following trivial reason: there are $x, y, z \in P$ such that $x <_P y <_P z$ and y is the only element in the antichain which is comparable to x or z.

Let us remark that the splitting property can be considered as a generalization of property-B, for an explanation see [7].

You can not expect an "easy" characterization of the maximal antichains in finite posets which splits because this question is NP-complete, see [1]. However in the same paper it was also shown that if a finite poset P has a property which is just a bit stronger than the lack of above type obstacle points y then every maximal antichain of P splits. To recall that result we should introduce some new notions.

An element $y \in P$ is called *cutting point* if and only if there are $x, z \in P$ such that $x <_P y <_P z$ and $[x,z] = [x,y] \cup [y,z]$. (The interval $[x,z] = \{y \in P : x \le y \le z\}$.) We say that P is *cut-free* if there is no cutting point in it. (This property was called *dense*, see e.g. [1], but the current wording seems to be more adequate.)

Theorem 1.1 **([1, Theorem 3.1]).** *Let $\mathcal{P}$ be a finite cut-free poset. Then every maximal antichain A splits.*

This result yields immediately following question: what about infinite posets?

Ahlswede and Khachatrian showed ([2]) that the plain generalization of Theorem 1.1 for infinite posets fails: the finite-subset-lattice $\langle [\omega]^{<\omega}, \subset \rangle$, which is cut-free, contains an infinite antichain which does not split.

In Section 2 we prove Theorem 2.7 saying that if a maximal antichain of an infinite poset satisfies some extra assumptions than it splits. This result yield that if a maximal antichain of a cut-free poset "resembles" a finite antichain then it splits (see Theorem 2.10).

On the other hand, in Section 3 we show that the non-splitting behavior of the poset $\langle [\omega]^{<\omega}, \subset \rangle$ is not exceptional: if an infinite poset is rich enough in elements then it should contain non-splitting antichains, see Theorem 3.6. Let us recall that Ahlswede and Khachatrian use number theory in [2] to construct a non-splitting antichain; our proof is purely combinatorial. Besides this result in Section 3 we also investigate possible strengthening of the statements that "A does not split". To formulate these results we introduce the following notation. If P is a poset and $A \subset P$ is a maximal antichain put

$$\mathfrak{S}(A) = \{\langle B, C \rangle : B \subset A, C \subset A, P = B^{\uparrow} \cup C^{\downarrow}\}.$$

Clearly A splits if and only if there is $\langle B,C\rangle \in \mathfrak{S}(A)$ with $B\cap C=\emptyset$. The maximal strengthening of the above mention result of Ahlswede and Khachatrian would be a cut-free poset P and a maximal antichain $A\subset P$ with $\mathfrak{S}(P)=\{\langle A,A\rangle\}$, but Corollary 3.3 says that this is not possible. In Theorem 3.8 we show that Theorem 3.6 is the maximal possible strengthening.

Quite surprisingly, the technique we developed to construct non-splitting antichain can be used to build splitting antichains as well, see Theorem 3.9.

Our notation is standard. Put $A^{\updownarrow}=A^{\downarrow}\cup A^{\uparrow}$. If $x\in P$ write $x^{\uparrow}$ for $\{x\}^{\uparrow}$, $x^{\downarrow}$ for $\{x\}^{\downarrow}$ and $x^{\updownarrow}$ for $\{x\}^{\updownarrow}$. If $A\subset P$ and $\mathcal{P}$ is not clear form the context we write $A^{\uparrow_{\mathcal{P}}}$ for $A^{\uparrow}$, and $A^{\downarrow_{\mathcal{P}}}$ for $A^{\downarrow}$. On the poset $\mathcal{P}$ we always think the poset $\mathcal{P}=(P,<)$.

2. Positive theorems

Definition 2.1. Let $\mathcal{P}$ be a poset and $A\subset P$ be a maximal antichain. An element $x\in A^{\downarrow}\setminus A$ is *high* if and only if there is no $y\in x^{\uparrow}\cap(A^{\downarrow}\setminus A)$ with $y^{\uparrow}\cap A\subsetneqq x^{\uparrow}\cap A$. An element $z\in A^{\uparrow}\setminus A$ is *low* if and only if there is no $v\in z^{\downarrow}\cap(A^{\uparrow}\setminus A)$ with $v^{\downarrow}\cap A\subsetneqq z^{\downarrow}\cap A$.

Lemma 2.2. *If $\mathcal{P}$ is a poset, $A\subset P$ is a maximal antichain which does not contain cutting points, $x\in A^{\downarrow}\setminus A$ is high and $z\in A^{\uparrow}\setminus A$ is low then $|[x,z]\cap A|\neq 1$.*

Proof. Assume on the contrary that $[x,z]\cap A=\{y\}$. Since y is not a cutting point there is $u\in[x,z]$ such that y and u are incomparable. By the indirect assumption we have $u\notin A$. If $u\in A^{\uparrow}$ then $u^{\downarrow}\cap A\subset(z^{\downarrow}\cap A)\setminus\{y\}$, i.e. z is not low. Hence $u\in A^{\downarrow}$. But then $u^{\uparrow}\cap A\subset(x^{\uparrow}\cap A)\setminus\{y\}$, i.e. x is not high. Contradiction. ∎

Definition 2.3. Given a family $\mathcal{A}\subset\mathcal{P}(X)$ a well-ordering $\prec$ of X is called *maximizing well-ordering for* $\mathcal{A}$ if and only if $\max_{\prec}A$ exists for each $A\in\mathcal{A}$. The family $\mathcal{A}$ is said to be *maximizing* if and only if there is a maximizing well-ordering for $\mathcal{A}$.

For example, the family $[X]^{<\omega}$ is clearly maximizing because any well-ordering of X is maximizing for this family.

If $\mathcal{A}\subset\mathcal{P}(X)$ and $\prec$ is a well-ordering of X let $\mathrm{MIN}(\mathcal{A},\prec)=\{\min_{\prec}A : A\in\mathcal{A}\}$ and $\mathrm{MAX}(\mathcal{A},\prec)=\{\max_{\prec}A : A\in\mathcal{A}$ and $\max_{\prec}A$ exists$\}$.

In [9] Klimó gave a characterization of maximizing families. Although he used a different terminology we can formulate his result as follows:

Theorem 2.4 **([9, Theorem 7]).** *$\mathcal{A}\subset\mathcal{P}(X)$ is a maximizing family if and only if there is a function $f:\mathcal{A}\to X$ such that $f(A)\in A$ for each $A\in\mathcal{A}$ and there is no sequence $\langle A_i : i<\omega\rangle$ in $\mathcal{A}$ such that $f(A_i)\neq f(A_{i+1})\in A_i$ for each $i<\omega$ and the set $\{A_i : i<\omega\}$ is infinite.*

Definition 2.5. Given a family $\mathcal{A}\subset\mathcal{P}(X)$ a set $Y\subset X$ is called a *point cover* if and only if $A\cap Y\neq\emptyset$ for each $A\in A$. Y is a *minimal point cover* if and only if it is a point cover but no proper subset of Y is a point cover.

The following lemma gives us a method to construct splits of certain antichains in certain posets.

Lemma 2.6. *Let $\mathcal{P}$ be a poset and $A\subset P$ be a maximal antichain. Assume that there are two functions $\underline{\mathrm{B}}$ and $\overline{\mathrm{B}}$ such that*

(i) *$\underline{\mathrm{B}}: A^{\uparrow}\setminus A\to\mathcal{P}(A)$ and $\emptyset\neq\underline{\mathrm{B}}(y)\subset A\cap y^{\downarrow}$ for each $y\in A^{\uparrow}\setminus A$,*
(ii) *$\overline{\mathrm{B}}: A^{\downarrow}\setminus A\to\mathcal{P}(A)$ and $\emptyset\neq\overline{\mathrm{B}}(x)\subset A\cap x^{\uparrow}$ for each $x\in A^{\downarrow}\setminus A$,*
(iii) *$|\underline{\mathrm{B}}(y)\cap\overline{\mathrm{B}}(x)|\neq 1$ for each $x\in A^{\downarrow}\setminus A$ and $y\in A^{\uparrow}\setminus A$*

Write $\overline{\mathcal{B}}=\{\overline{\mathrm{B}}(x):x\in A^{\downarrow}\setminus A\}$ and $\underline{\mathcal{B}}=\{\underline{\mathrm{B}}(x):x\in A^{\uparrow}\setminus A\}$.

(1) *If $\prec$ is a maximizing well-ordering of $\overline{\mathcal{B}}$ then $\mathrm{MIN}(\underline{\mathcal{B}},\prec)\cap\mathrm{MAX}(\overline{\mathcal{B}},\prec)=\emptyset$, and so $\mathcal{A}$ splits.*
(2) *If $C\subset A$ is a minimal point cover of $\overline{\mathcal{B}}$ then $\langle A\setminus C, C\rangle\in\mathfrak{S}(A)$ and so A splits.*

Proof. (1) Indeed, $\max_{\prec}\overline{\mathrm{B}}(x)=\min_{\prec}\underline{\mathrm{B}}(y)$ would imply that $\overline{\mathrm{B}}(x)\cap\underline{\mathrm{B}}(y)=\{\max_{\prec}\overline{\mathrm{B}}(x)\}$ which contradicts to property (iii) in the choice of $\underline{\mathrm{B}}$ and $\overline{\mathrm{B}}$.

Since clearly $A^{\downarrow}\setminus A\subset\mathrm{MIN}(\overline{\mathcal{B}},\prec)^{\downarrow}$ and $A^{\uparrow}\setminus A\subset\mathrm{MAX}(\underline{\mathcal{B}},\prec)^{\uparrow}$ we have that A splits.

(2) Since C is a point cover we have $A^{\downarrow}\setminus A\subset C^{\downarrow}$. To prove the other property assume on the contrary that $A^{\uparrow}\setminus A\not\subset(A\setminus C)^{\uparrow}$, i.e. there is $y\in A^{\uparrow}\setminus A$ such that $\underline{\mathrm{B}}(y)\subset C$. Pick an arbitrary $z\in\underline{\mathrm{B}}(y)$. Since $C\setminus\{z\}$ is not a point cover of $\overline{\mathcal{B}}$ there is $x\in A^{\downarrow}\setminus A$ such that $\overline{\mathrm{B}}(x)\cap C=\{z\}$. But then $\{z\}\subset\overline{\mathrm{B}}(x)\cap\underline{\mathrm{B}}(y)\subset\overline{\mathrm{B}}(x)\cap C=\{z\}$ which contradicts (iii). ∎

Theorem 2.7. *Let $\mathcal{P}$ be a poset and $A\subset P$ be a maximal antichain which does not contain cutting points. Assume that*

(i) *for each $y\in A^{\uparrow}\setminus A$ there is a low $z\in A^{\uparrow}\setminus A$ with $z\leq y$,*
(ii) *for each $x\in A^{\downarrow}\setminus A$ there is a high $t\in A^{\downarrow}\setminus A$ such that $x\leq t$,*

If either

(1) *the family $\{x^{\uparrow}\cap A : x \text{ is high }\}$ is maximizing or*
(2) *the family $\{x^{\uparrow}\cap A : x \text{ is high }\}$ has a minimal point cover*

then A splits.

Proof. Let $L=\{y\in A^{\uparrow}\setminus A : y \text{ is low}\}$ and $H=\{x\in A^{\downarrow}\setminus A : x \text{ is high}\}$. Let $M=A\cup H\cup L$ and let Q be the subordering of P with the underlining set M. Since $[x,y]\subset M$ for each $\{x,y\}\in[M]^2$, (i.e. M is "convex" in P) the antichain A does not contain cutting points in Q.

Since A is clearly a maximal antichain in Q, every element of H is high in Q and every element of L is low. Thus, by Lemma 2.2, we have

$$(1)\qquad |[x,y]\cap A|\neq 1 \text{ for each } x\in H \text{ and } y\in L.$$

Let $\overline{\mathrm{B}}(x)=x^{\uparrow}\cap A$ and $\underline{\mathrm{B}}(y)=y^{\downarrow}\cap A$. We want to apply Lemma 2.6. Properties (i)–(ii) are clear. Since $\overline{\mathrm{B}}(x)\cap\underline{\mathrm{B}}(y)=[x,y]\cap A$, property (1) implies that the functions $\underline{\mathrm{B}}$ and $\overline{\mathrm{B}}$ satisfies Lemma 2.6.(iii).

Since (1) implies Lemma 2.6.(1), and (2) implies Lemma 2.6.(2) hence we have that A splits in Q: there is $B\subset A$ such that $B^{\uparrow}=L$ and $(A\setminus B)^{\downarrow}=H$ in Q. Since $L^{\uparrow}=A^{\uparrow}\setminus A$ in P and $H^{\downarrow}=L^{\downarrow}\setminus A$ in P we have that $B^{\uparrow}=A^{\uparrow}\setminus A$ and $(A\setminus B)^{\downarrow}=A^{\downarrow}\setminus A$ in P. Thus B witnesses that A splits. ∎

Let us remark the nontrivial fact that condition (1) is stronger than (2): as Klimó proved in [9] a maximizing family $\mathcal{A}$ has a minimal point cover. However we included the statement with proof here because you can get two different splits for A when $\{x^{\uparrow}\cap A : x \text{ is high }\}$ is maximizing: one applying Lemma 2.6.(1) directly and the other by finding a minimal point cover for $\{x^{\uparrow}\cap A : x \text{ is high }\}$ and then applying Lemma 2.6.(2).

A poset $\mathcal{P}=\langle P,<\rangle$ is called *well-founded* (or satisfies the *Descending Chain Condition*), if there exists no infinite descending chain: if $x_1\geq x_2\geq\cdots\geq x_n\geq\ldots$ then there exists an integer i such that $x_i=x_j$ for all $j>i$.

Theorem 2.8. *Let $\mathcal{P}$ be a well-founded poset and let A be a maximal, cutting point free antichain, such that for every $p\in A^{\downarrow}\setminus A$ there exists element $x(p)\in A^{\downarrow}\setminus A$ with $p\leq a(p)$ such that $a(p)^{\uparrow}\cap A$ is finite. Then A splits.*

Proof. We want to apply Theorem 2.7. Property (ii) holds by assumptions. Moreover $x^{\uparrow}\cap A$ is finite for each high elements and so Property (1) holds. The minimal elements of $A^{\uparrow}\setminus A$ are all low, hence Property (i) also holds. ∎

The next observation provides a very useful tool to manipulate the antichain pairs in $\mathfrak{S}(A)$ of maximal antichains in cut-free posets.

Lemma 2.9. *Assume that $\mathcal{P}$ is a poset, $A\subset P$ is a maximal antichain, and $\langle B,C\rangle\in\mathfrak{S}(A)$. Then for each $y\in B\cap C$ if y is not a cutting point then either $\langle B\setminus\{y\},C\rangle\in\mathfrak{S}(A)$ or $\langle B,C\setminus\{y\}\rangle\in\mathfrak{S}(A)$.*

Proof. Assume on the contrary that this is not true, so there are $x,z\in P$ such that $x<y<z$, $x\notin (C\setminus\{y\})^{\downarrow}$ and $z\notin (B\setminus\{y\})^{\uparrow}$. Since y is not a cutting point, there is $t\in[x,z]$ such that y and t are incomparable. Then $t\in(B\setminus\{y\})^{\uparrow}\cup(C\setminus\{y\})^{\downarrow}$. If $y'<t$ for some $y'\in B\setminus\{y\}$ then $y'<z$, contradiction. If $t<y'$ for some $y'\in C\setminus\{y\}$ then $x<y'$, contradiction. ∎

Theorem 2.10. *Let A be a maximal antichain in the poset $\mathcal{P}$ such that A does not contain cutting points and*

$$|(x^{\updownarrow})\cap A|<\omega \text{ for all } x\in P,$$

then A splits.

This result is a direct generalization of Theorem 1.1 ([1]). We give here two different proofs. However it is not clear yet the complexity of these methods to find a splitting (at least of the second one) in the case of finite cut-free posets. It is also a question whether all possible splitting arise along the second method.

First proof. Consider the poset $Q(P)=\langle\mathfrak{S}(A),\prec\rangle$ where $\langle B,C\rangle\prec\langle B',C'\rangle$ if and only if $B\supset B'$ and $C\supset C'$.

We want to apply the Zorn lemma to find a maximal elements of $Q(P)$. So let $\langle\langle B_\xi,C_\xi\rangle:\xi<\eta\rangle$ be an increasing chain in $Q(P)$. Put $B=\cap\{B_\xi:\xi<\eta\}$ and $C=\cap\{C_\xi:\xi<\eta\}$. Let $x\in P$ be arbitrary. Since $(x^{\updownarrow})\cap A$ is finite there is $\zeta<\eta$ such that $(x^{\updownarrow})\cap B=(x^{\updownarrow})\cap B_\zeta$ and $(x^{\updownarrow})\cap C=(x^{\updownarrow})\cap C_\zeta$. Since $x\in B_\zeta^{\uparrow}\cup C_\zeta^{\downarrow}$ we have $x\in B^{\uparrow}\cup C^{\downarrow}$. Since x was arbitrary we have $\langle B,C\rangle\in\mathfrak{S}(A)$, and so $\langle B,C\rangle$ is the required upper bound of $\langle\langle B_\xi,C_\xi\rangle:\xi<\eta\rangle$.

Thus the Zorn lemma implies that $Q(P)$ has a maximal element $\langle B,C\rangle$. But then $B\cap C=\emptyset$ by Lemma 2.9. ∎

Second proof. Apply Theorem 2.7. Since (1) and (2) clearly holds we can apply that result to get that A splits. ∎

Finally we give one more application of Theorem 2.7: we prove a theorem on the subset lattice of the natural numbers.

Let A be a maximal antichain in $\mathcal{P}(\omega)$ and let $x\in(A^{\downarrow}\setminus A)$. Denote $\mathrm{Card}(A)$ the set of the cardinalities present in A, and denote $\mathrm{Card}_x(A)$ the set of cardinalities of those elements in A which are comparable to x. We say that this x *behaves well* if $|\mathrm{Card}_x(A)|=\omega$ then $\omega\in\mathrm{Card}_x(A)$ as well. If, for example, $|\mathrm{Card}(A)|$ is finite, then every element behaves well.

Theorem 2.11. *Let A be a maximal antichain in $\mathcal{P}(\omega)$. Assume that*

$$\forall y\in(A^{\uparrow}\setminus A)\quad:\quad[y^{\downarrow}\cap A\cap[\omega]^{<\omega}]\neq\emptyset, \tag{2}$$

furthermore every element $x\in A^{\downarrow}\setminus A$ behaves well. Then A splits.

Proof. Let $I = A\cap[\omega]^{\omega}$, $F = A\cap[\omega]^{<\omega}$ and $Q=\mathcal{P}(\omega)\setminus I^{\downarrow}$. Clearly F is a maximal antichain in Q. Next we show that:

Claim 2.12. *For each $c\in(F^{\downarrow}\setminus F)\cap Q$ there is a high $h\in Q$ with $c\subset h$.*

$\mathrm{Card}_c(F)$ is finite, because c behaves well. Write $n=\max\mathrm{Card}_c(F)$. Fix $f\in F\cap c^{\uparrow}\cap[\omega]^n$ and pick $h\in[\omega]^{n-1}$ with $c\subset h\subset f$. Then h is high in Q because it is maximal in $(F^{\downarrow}\setminus F)\cap Q$.

Claim 2.13. *For each $b\in F^{\uparrow}\setminus F$ there is a low $\ell\in Q$ such that $\ell\subset b$ and $\ell^{\downarrow}\cap F$ is finite.*

Indeed, let $j=\min\{|f|:f\in F\cap b^{\downarrow}\}$, pick $f\in b^{\downarrow}\cap F\cap[\omega]^j$ and let $\ell\in[\omega]^{j+1}$ with $f\subset\ell\subset b$. Then ℓ is minimal in $F^{\downarrow}\setminus F$ hence it is low in Q. Moreover $\ell^{\downarrow}\cap F$ is clearly finite.

Hence we can apply Theorem 2.7 for Q^{-1} (the dual of poset Q) and F to yield that F splits in Q: there is $G\subset F$ such that $G^{\uparrow}\setminus G=F^{\uparrow}\setminus G$ and $F\setminus G^{\downarrow}=F^{\downarrow}$.

Then G shows that A splits in P. Indeed, $F^{\uparrow}=A^{\uparrow}$ because of assumption (2). Hence $G^{\uparrow}=A^{\uparrow}$ in P. On the other hand, if $c\in A^{\downarrow}$ then either $c\in Q$ and so $c\in(F\setminus G)^{\downarrow}$, or $c^{\uparrow}\cap A\cap[\omega]^{\omega}\neq\emptyset$ and so $c\in(A\setminus F)^{\downarrow}\subset(A\setminus G)^{\downarrow}$. ∎

3. Negative theorems

In this Section we study maximal antichains of countable posets, together the possible structures of non-splitting maximal antichains.

To start we give some consequences of Lemma 2.9. At first we have:

Corollary 3.1. *If a maximal antichain A does not split in a cut-free poset $\mathcal{P}$ then $|B\cap C|=\omega$ for each $\langle B,C\rangle\in\mathfrak{S}(A)$.*

Which in turns gives a direct generalization of Theorem 1.1:

Corollary 3.2. *Every finite maximal antichain splits in every cut-free poset.*

We think that in the future Lemma 2.9 will provide the standard proof of Theorem 1.1. Lemma 2.9 also shows that in cut-free posets there are no maximal antichains A with maximally degenerated $\mathfrak{S}(A)$:

Corollary 3.3. *There exits no cut-free poset $\mathcal{P}$ such that $\mathfrak{S}(A)=\{\langle A,A\rangle\}$ for some maximal antichain $A\subset P$.*

On the other hand, in Theorem 3.6 below we show that the structure of $\mathfrak{S}(A)$ can be quite degenerated: it might happen that every pair in $\mathfrak{S}(A)$ contains A itself. To formulate this result we need one more definition.

Definition 3.4. A poset $\mathcal{P}$ is *loose* if and only if for each $x \in P$ and $F \in [P]^{<\omega}$ if $x \notin F^{\uparrow}$ then there is $y \in x^{\uparrow} \setminus \{x\}$ such that $y \notin F^{\downarrow} \cup F^{\uparrow}$.

Assume that $\mathcal{P}$ is loose and $p \in P$. Let $F = \emptyset$. Then $p \notin F^{\uparrow}$ hence by looseness there is $y \in P$ with $y \in x^{\uparrow} \setminus \{x\}$, i.e. $y > x$. Thus we have:

Remark. A loose poset does not have maximal elements. Especially, it is infinite.

Claim 3.5. $\langle [\omega]^{<\omega}, \subset \rangle$ *is loose.*

Proof. Indeed, if $x \in [\omega]^{<\omega}$ and F is a finite subset of $[\omega]^{<\omega}$ with $x \notin F^{\uparrow}$ then let n be a natural number not belonging to x or any set in F, and put $y = x \cup \{n\}$. Let $f \in F$. Then $\emptyset \neq f \setminus x = f \setminus y$ hence $y \notin F^{\uparrow}$. Moreover, $n \in y \setminus f$ and so $y \notin F^{\downarrow}$. ∎

Theorem 3.6. *Assume that $\mathcal{P} = \langle P, \leq \rangle$ is a countable, loose poset. Then $\mathcal{P}$ contains a maximal antichain A such that*

(i) *if $\langle B, C \rangle \in \mathfrak{S}(A)$ then $B = A$,*
(ii) *if A is finite then $\cap \{C : \langle B, C \rangle \in \mathfrak{S}(A)\} \neq \emptyset$,*
(iii) *if A is infinite then so is C for each $\langle B, C \rangle \in \mathfrak{S}(A)$,*
(iv) *if $\mathcal{P}$ is cut-free then A is infinite.*

Proof. Let $\langle p_n : n < \omega \rangle$ be an enumeration of the elements of P. By induction on $n \in \omega$ we choose elements $x_n, y_n, z_n \in P$ with $x_n < y_n < z_n$ as follows.

Let $m_n = \min\{m : p_m \notin \{y_i : i < n\}^{\uparrow} \cup \{y_i : i < n\}^{\downarrow}\}$. If m_n is not defined then the we stop the construction. Assume that m_n is defined. Since $y_i < z_i$ we have $p_{m_n} \notin \{y_i, z_i : i < n\}^{\uparrow}$. Furthermore since $\mathcal{P}$ is loose there is $x_n \in P$ with $p_{m_n} < x_n$ such that

$$x_n \notin \{y_i, z_i : i < n\}^{\uparrow} \cup \{y_i, z_i : i < n\}^{\downarrow}.$$

Applying the looseness of $\mathcal{P}$ once more there is $y_n \in P$ with $x_n < y_n$ such that

$$y_n \notin \{y_i, z_i : i < n\}^{\uparrow} \cup \{y_i, z_i : i < n\}^{\downarrow}. \tag{3}$$

Applying the looseness of $\mathcal{P}$ a third time there is $z_n \in P$ with $y_n < z_n$ such that

$$z_n \notin \{y_i, z_i : i < n\}^{\uparrow} \cup \{y_i, z_i : i < n\}^{\downarrow}. \tag{4}$$

We claim that $A=\{y_n : n<\omega, y_n \text{ is defined}\}$ has the required properties. First observe, that A is an antichain by Property (3).

By induction on n we can see that $m_n \ge n$ and so $p_n \in \{y_i : i<n\}^{\uparrow} \cup \{y_i : i<n\}^{\downarrow} \cup {y_n}^{\downarrow}$, hence the antichain A is maximal.

Assume that $\langle B, C\rangle \in \mathfrak{S}(A)$. Let n be arbitrary such that m_n is defined. By Property (4) we have $z_n \notin \{y_i : i<n\}^{\uparrow} \cup \{y_i : i<n\}^{\downarrow}$. By Property (3) we have $z_n \notin \{y_i : i>n\}^{\uparrow} \cup \{y_i : i>n\}^{\downarrow}$. Since $y_n < z_n$ we have $z_n \notin A^{\downarrow} \cup (A\setminus\{y_n\})^{\uparrow}$. Thus $z_n \in B^{\uparrow} \cup C^{\downarrow}$ implies then $y_n \in B$. Hence $B=A$. (That is (i) holds.)

Since $p_{m_n} < y_n$ we have $p_{m_n} \notin A^{\uparrow}$. By the choice of m_n we have $p_{m_n} \notin \{y_i : i<n\}^{\uparrow} \cup \{y_i : i<n\}^{\downarrow}$. Thus $p_{n_m} \in \{y_k \in C : k \ge n\}^{\downarrow}$. Hence $\{m : x_m \in C\}$ is cofinal in $\{m : x_m \text{ is defined}\}$. Therefore $y_n \in C$ provided that A is finite and $n=\max\{n' : m_{n'} \text{ is defined}\}$ and so (ii) holds, and C is infinite provided that A is infinite. (That is (iii) holds.) Let's remark that one can prove (iii) by observing that if C would be finite then Lemma 2.9 and Property (i) together would prove that $\langle B, \emptyset\rangle \in \mathfrak{S}(A)$, a clear contradiction.

Properties (ii) and (iii) imply that A does not split. Since, according to Corollary 3.2, finite antichains split in a cut-free posets we have that A is infinite provided that P is cut-free. (That is (iv) holds.) ∎

Since $\langle [\omega]^{<\omega}, \subset\rangle$ is loose and cut-free, we can apply Theorem 3.6 to get the following corollary.

Corollary 3.7. *$\langle [\omega]^{<\omega}, \subset\rangle$ contains a maximal antichain A such that if $\langle B, C\rangle \in \mathfrak{S}(A)$ then $A=B$ and C is infinite, and so A does not split.*

This result is a farfetched generalization of the construction given by Ahlswede and Khachatrian in [2].

The following result shows that even more can be said about maximal antichains A in cut-free posets, where every pair in $\mathfrak{S}(A)$ contains A itself, showing also that Theorem 3.6 is sharp in a certain sense.

Theorem 3.8. *Assume that $\mathcal{P}=\langle P, \le\rangle$ is a countable, cut-free poset, $A\subset P$ is a maximal antichain such that $A=B$ for each $\langle B, C\rangle \in \mathfrak{S}(A)$. Then there is $\langle A, C\rangle \in \mathfrak{S}(A)$ with $|A\setminus C|=\omega$.*

Proof. Since $\langle A\setminus\{a\}, A\rangle \notin \mathfrak{S}(A)$ we can pick $z_a \in P$ such that $a<z_a$ and $z_a \notin (A\setminus\{a\})^{\uparrow}$ for each $a\in A$.

We claim that $x^{\uparrow}\cap A$ is infinite for each $x\in A^{\downarrow}\setminus A$ and this statement finishes the proof: Indeed, in this case there is $C\in[A]^{\omega}$ such that $|(x^{\uparrow}\cap A)\cap C| = |(x^{\uparrow}\cap A)\setminus C| = \omega$ for each $x\in A^{\downarrow}\setminus A$, and so $\langle A, C\rangle \in \mathfrak{S}(A)$ with $|A\setminus C|=\omega$. (This is the well-known Bernstein's Lemma [3].)

To prove our claim assume on the contrary that $B=x^{\uparrow}\cap A$ is finite for some $x\in A^{\downarrow}\setminus A$. Choose x such that $|B|$ is minimal. Clearly $|B|>0$. Let

$y \in B$ be arbitrary. Then $x < y < z_y$ and $\mathcal{P}$ is cut-free so there is $t \in [x, z_y]$ which is incomparable with y.

Now $t \in A^{\downarrow}$ because $a \leq t$ would imply $a < z_y$ and so $a = y$ for any $a \in A$, but t and y were incomparable. Moreover $t^{\uparrow} \cap A \subset (x^{\uparrow} \cap A) \setminus \{y\}$, which contradicts the minimality of the cardinality of $x^{\uparrow} \cap A$. ∎

Till now we used the looseness to show that certain antichain can not split, or to restrict the structure of $\mathfrak{S}(A)$. The next theorem shows that the looseness can be used even in the other direction: to guarantee the existence of splitting antichains.

Theorem 3.9. *Assume that $\mathcal{P} = \langle P, \leq \rangle$ is a countable poset such that $\mathcal{P}$ and $\mathcal{P}^{-1}$ are loose. Then $\mathcal{P}$ contains a maximal antichain A which splits.*

Proof. Write $\mathcal{P} = \{p_n : n < \omega\}$. By induction on n we will construct finite disjoint subsets B_n and C_n of P such that

(i) $B_n \cup C_n$ is an antichain,
(ii) $B_{n-1} \subset B_n$ and $C_{n-1} \subset C_n$,
(iii) $p_{n-1} \in B_n^{\uparrow} \cup C_n^{\downarrow}$.

It is enough to show that we can carry out the induction because taking $B = \cup\{B_n : n \in \omega\}$ and $C = \cup\{C_n : n \in \omega\}$ we have that $A := B \cup C$ is a maximal antichain having the splitting $\langle B, C \rangle$.

Let $B_0 = C_0 = \emptyset$. Assume that B_{n-1} and C_{n-1} are constructed. Write $p = p_{n-1}$. If $p \in B_{n-1}^{\uparrow} \cup C_{n-1}^{\downarrow}$ then let $C_n = C_{n-1}$ and $B_n = B_{n-1}$. So we can assume that $p \notin B_{n-1}^{\uparrow} \cup C_{n-1}^{\downarrow}$.

Case 1. $p \notin C_{n-1}^{\uparrow}$.

Then $p \notin (C_{n-1} \cup B_{n-1})^{\uparrow}$. Since $\mathcal{P}$ is loose there is $p \leq q$ such that $q \notin (C_{n-1} \cup B_{n-1})^{\uparrow} \cup (C_{n-1} \cup B_{n-1})^{\downarrow}$, i.e. $B_{n-1} \cup C_{n-1} \cup \{q\}$ is an antichain. Let $C_n = C_{n-1} \cup \{q\}$ and $B_n = B_{n-1}$. Then $p \in q^{\downarrow} \subset C_n^{\downarrow}$, B_n and C_n are disjoint and $B_n \cup C_n$ is an antichain.

Case 2. $p \notin B_{n-1}^{\downarrow}$.

Then $p \notin (B_{n-1} \cup C_{n-1})^{\downarrow}$. Since $\mathcal{P}^{-1}$ is loose there is $q \leq p$ such that $q \notin (B_{n-1} \cup C_{n-1})^{\downarrow} \cup (B_{n-1} \cup C_{n-1})^{\uparrow}$, i.e. $C_{n-1} \cup B_{n-1} \cup \{q\}$ is an antichain. Let $B_n = B_{n-1} \cup \{q\}$ and $C_n = C_{n-1}$. Then $p \in q^{\uparrow} \subset B_n^{\uparrow}$, C_n and B_n are disjoint and $C_n \cup B_n$ is an antichain.

Case 3. $p \in B_{n-1}^{\downarrow} \cap C_{n-1}^{\uparrow}$.

Then there is $b \in B_{n-1}$ and $c \in C_{n-1}$ such that $c \leq p \leq b$, i.e. $B_{n-1} \cup C_{n-1}$ is not an antichain. Contradiction, this case is not possible which finishes the proof. ∎

The maximal antichains in the poset $\mathbb{Z}$ of the integer are the singletons and they clear don't split.

Problem 3.10. Is there a countable cut-free poset P which does not contain splitting maximal antichains?

Consider the following countable, well-founded, cut-free poset. Let the underlying set of $\mathcal{P}$ be $\omega \times \omega$. Put $\langle n, m \rangle <_P \langle n', m' \rangle$ if and only if $n < n'$. Then the antichains in $\mathcal{P}$ are the sets $\{n\} \times \omega$ for $n < \omega$, and $\{n\} \times \omega$ splits because

$$P = \{\langle n, i \rangle\}^{\downarrow} \cup \{\langle n, j \rangle\}^{\uparrow}$$

whenever $i \neq j$. We do not have any characterization of posets having only splitting maximal antichains.

Till now we were interested the existence of splitting of maximal antichains. One can ask, however, how many different splits can be found.

Problem 3.11. Fix a cardinal κ. Is there a countable cut-free poset P having a maximal antichain A such that

$$\kappa_P \stackrel{\text{def}}{=} |\{B : \langle B, A \setminus B \rangle \in \mathfrak{S}(A)\}| = \kappa?$$

In general, we do not know the answer. Since $|A|$ is countable 2^A can be considered as a topological space homeomorphic to the ω^{th} power of the two element discrete topological space $\mathbf{2} = \{0, 1\}$, i.e. to the Cantor set. Hence we have the Borel hierarchy on 2^A. Since $\mathfrak{S}(A)$ is a G_δ-subset of $2^A \times 2^A$ hence either $\mathfrak{S}(A)$ is at most countable or has cardinality 2^ω by [8, Theorem 11.18(iii)]. The case $\kappa = 2^\omega$ is trivial. The case $\kappa = 1$ is also trivial: let P be well-founded and A be the minimal points of P. However the κ_P can be 1 in a less trivial way.

Claim 3.12. *There is a countable, cut-free poset P and an infinite maximal antichain $A \subset P$ such that*

(i) $\forall a \in A \; \exists x, y \in P \; x < a < y$,
(ii) $|\{B \subset A : \langle B, A \setminus B \rangle \in \mathfrak{S}(A)\}| = 1$.

Proof. Consider the poset Q on Figure 1. The poset Q is cut-free. The set $A = \{b, c\}$ is a maximal antichain in Q and $\{B \subset A : \langle B, A \setminus B \rangle \in \mathfrak{S}(A)\} = \{\{b\}\}$. Let P be the disjoint union of countable many copies of Q. ∎

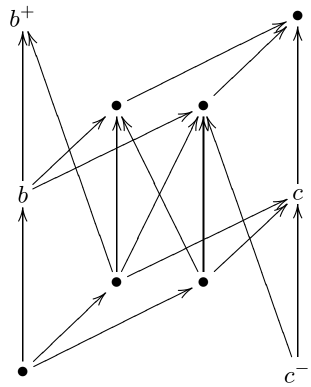


Figure 1. Poset Q

4. Some set-theory

In this section we will use the standard set-theoretical notation throughout, see e.g. [8].

The answers to the questions which we investigated in connection with countable posets in Section 3 does not depend on the actual set-theoretical universe in which we work. The reason is that all the statements can be formulated as a $\Sigma^1_2(a)$ or $\Pi^1_2(a)$ formula with some parameter $a \in \omega^\omega$, and so they are absolute by Schoenfield's absoluteness theorem, [8, Theorem 25.20]. For example, given a countable poset $\mathcal{P}$ and maximal antichain $A \subset \mathcal{P}$ statements like "A splits", or "no maximal antichains of $\mathcal{P}$ splits", or "every maximal antichain of $\mathcal{P}$ splits" are all absolute: their truth value depends on only $\mathcal{P}$ and A and independent of the set-theoretical universe. Same argument gives that although we do not know the answer to the problem 3.10 we can expect a yes or no answer in ZFC.

The situation changes dramatically if we consider uncountable partially ordered sets. We will give an example after Proposition 4.3 that given a poset $\mathcal{P}$ of size ω_1 and maximal antichain $A \subset \mathcal{P}$ the statement "A splits" can depend on the set-theoretical universe in which we live. We will also show that axiom $\clubsuit$ can be reformulated as a statement on splitting property of certain antichains in certain posets, see proposition 4.3.

Definition 4.1. Let $\mathcal{L}$ be the set of the countable limit ordinals. We say that $\langle T_\alpha : \alpha \in \mathcal{L} \rangle$ is a $\clubsuit$-*sequence* if and only if $T_\alpha \subset \alpha$ is cofinal for each $\alpha \in \mathcal{L}$ and for each $X \in [\omega_1]^{\omega_1}$ there is $\alpha \in \mathcal{L}$ with $T_\alpha \subset X$. Axiom $\clubsuit$ holds if and only if there is a $\clubsuit$-sequence.

It is well-known that axiom $\clubsuit$ is independent from ZFC: there is a $\clubsuit$ sequence in the constructible universe L of Gödel but Martin's Axiom excludes the existence of such a sequence.

Definition 4.2. Given a sequence $\mathcal{T}=\{T_\beta:\beta\in\mathcal{L}\}$, where $T_\alpha\subset\alpha$ is cofinal, we define the poset $Q(\mathcal{T})$ as follows. The underlying set of $Q(\mathcal{T})$ is $(\{2\}\times\mathcal{L})\cup(2\times\omega_1)$. Let $\langle 0,\eta\rangle\prec\langle 1,\xi\rangle$ if and only if $\eta<\xi$. Let $\langle 1,\zeta\rangle\prec\langle 2,\beta\rangle$ if and only if $\zeta\in T_\beta$. Let $\le_{Q(\mathcal{T})}$ be the partial ordering generated by $\prec$.

The poset $Q(\mathcal{T})$ is clearly cut-free.

Proposition 4.3. *Let $\mathcal{T}=\{T_\beta:\beta\in\mathcal{L}\}$, where $T_\alpha\subset\alpha$ is cofinal. The maximal antichain $A=\{1\}\times\omega_1$ splits in $Q(\mathcal{T})$ if and only if $\mathcal{T}$ is not a $\clubsuit$-sequence.*

Proof. If $\mathcal{T}$ is not a $\clubsuit$ sequence then there is $X\in[\omega_1]^{\omega_1}$ such that $T_\alpha\setminus X\neq\emptyset$ for each $\alpha\in\mathcal{L}$. Let $B=\{1\}\times(\omega_1\setminus X)$ and $C=\{1\}\times X$. Then for each $\alpha\in\mathcal{L}$ there is $\xi\in\omega_1\setminus X$ with $\xi\in T_\alpha$ and so $\langle 1,\xi\rangle\prec\langle 2,\alpha\rangle$, i.e. $B^\uparrow\supset\{2\}\times\mathcal{L}$. Moreover for each $\eta<\omega_1$ there is $\xi\in X$ with $\eta<\xi$ and so $\langle 0,\xi\prec\langle 1,\eta\rangle\rangle$. Thus $\{0\}\times\omega_1\subset C^\downarrow$. Hence $B^\uparrow\cup C^\downarrow=Q(\mathcal{T})$.

Assume now that $\mathcal{T}$ is a $\clubsuit$-sequence and let $\langle B,C\rangle\in\mathfrak{S}(A)$. We show that $A\setminus B$ is countable and C is uncountable. If $C\subset A$ is countable then $\langle 0,\sup\{\alpha:\langle 1,\alpha\rangle\in C\}+1\rangle\notin C^\downarrow$. Assume on the contrary that e.g. $A\setminus B$ is uncountable. Then $X=\{\xi:\langle 1,\xi\rangle\notin B\}\in[\omega_1]^{\omega_1}$ and so there is $\alpha\in\mathcal{L}$ with $T_\alpha\subset X$. Let $x=\langle 2,\alpha\rangle$. Then $A\cap x^\downarrow=\{1\}\times T_\alpha$ and so $B\cap x^\downarrow=\emptyset$, i.e. $x\notin B^\uparrow$. Since $C^\downarrow$ is disjoint to $\{2\}\times\mathcal{L}$ we obtain that $x\notin B^\uparrow\cup C^\downarrow$, a contradiction. Hence the set $A\setminus B$ is countable. ∎

Example. Fix a $\clubsuit$-sequence $\mathcal{T}=\langle T_\beta:\beta\in\mathcal{L}\rangle$ in L. Then, by proposition 4.3, the antichain $A=\{1\}\times\omega_1$ does not split in $Q(\mathcal{T})$. It is well-known that there is a c.c.c generic extension of L in which Martin's Axiom holds, and so axiom $\clubsuit$ fails, especially $\mathcal{T}$ is not a $\clubsuit$-sequence. Hence, applying proposition 4.3 again we obtain that A splits in this generic extension. Hence the statement "A splits" is not absolute.

As we have seen splitting property can be used to formulate an equivalent of axiom $\clubsuit$. The next proposition shows that even the Axiom of Choice can be reformulated in a similar way.

Theorem 4.4. *(ZF) The Axiom of Choice is equivalent to the statement of Theorem 2.8.*

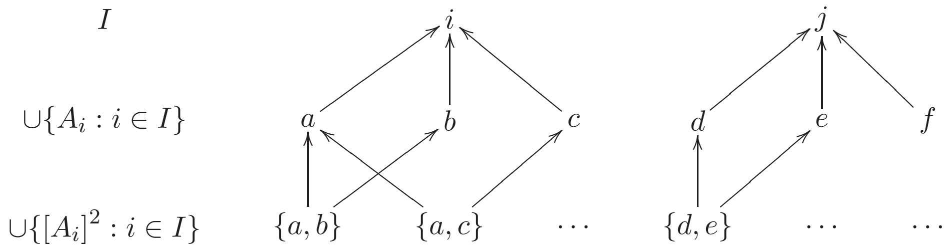
Proof. Assume that the statement of Theorem 2.8 holds. Let $\mathcal{A}=\{A_i:i\in I\}$ be a family of pairwise disjoint nonempty sets. Without loss of generality

$|A_i| \neq 1$ for each $i \in I$. We need to show that there is a choice function on $\mathcal{A}$. To do so define the poset $R(\mathcal{A}) = \langle R, \leq_R \rangle$ as follows:

$$R = I \cup \left(\cup\left\{A_i : i \in I\right\}\right) \cup \left(\cup\left\{[A_i]^2 : i \in I\right\}\right), \tag{5}$$

$$\prec = \left\{\langle a, i\rangle : i \in I, a \in A_i\right\} \cup \left\{\langle \{a,b\}, a\rangle : a \in A_i, b \in A_i \setminus \{a\}, i \in I\right\}, \tag{6}$$

and let $\leq_R$ be the partial order generated by $\prec$.



The poset $R(\mathcal{A})$ is well-founded and cut-free. The set $A = \cup\{A_i : i \in I\}$ is a maximal antichain in it and $|x^{\uparrow} \cap A| = 2$ for each $x \in A^{\downarrow} \setminus A = \cup\{[A_i]^2 : i \in I\}$. Hence A splits by theorem 2.8, $R = B^{\uparrow} \cup (A \setminus B)^{\downarrow}$ for some $B \subset A$. Since $I \subset B^{\uparrow}$ we have $B \cap A_i \neq \emptyset$ for each $i \in I$. On the other hand $|A_i \cap B| \leq 1$. Indeed $\{b,c\} \in [B]^2 \cap [A_i]^2$ would imply that $\{b,c\} \notin (A \setminus B)^{\downarrow}$. Hence $|B \cap A_i| = 1$ for each $i \in I$ and so we have a choice function f on $\mathcal{A}$: let $f(i) = \cup(A_i \cap B)$ for $i \in I$. ∎

Let us conclude this Section with a generalization of property "loose" to bigger cardinals. The proofs of the results are very similar to those in the first part of Section 3, therefore we leave them to the diligent reader.

Definition 4.5. Given a cardinal κ, a poset $\mathcal{P}$ is *κ-loose* if and only if for each $x \in P$ and $F \in [P]^{<\kappa}$ if $x \notin F^{\uparrow}$ then there is $y \in x^{\uparrow} \setminus \{x\}$ such that $y \notin F^{\downarrow} \cup F^{\uparrow}$.

Claim 4.6. *If κ and λ cardinal such that $\lambda < \kappa$ or $\lambda = \kappa = \mathrm{cf}(\kappa)$ then $\langle [\kappa]^{<\lambda}, \subseteq \rangle$ is κ-loose.*

Theorem 4.7. *Assume that $\mathcal{P} = \langle P, \leq \rangle$ is a κ-loose poset of cardinality κ. Then $\mathcal{P}$ contains a maximal antichain A such that*

(i) *if $\langle B, C \rangle \in \mathfrak{S}(A)$ then $B = A$,*
(ii) *$\mathrm{cf}(|C|) = \mathrm{cf}(|A|)$ for each $\langle B, C \rangle \in \mathfrak{S}(A)$.*
(iii) *if $\mathcal{P}$ is cut-free then A is infinite.*

Corollary 4.8. *If $\kappa^{<\lambda}=\lambda=\kappa$ then $\langle[\kappa]^{<\lambda},\subseteq\rangle$ contains a maximal antichain which does not split. In particular,*

(i) *for each infinite cardinal κ the poset $\langle[\kappa]^{<\omega},\subseteq\rangle$ contains maximal antichain which does not split,*
(ii) *if the continuum hypothesis holds then $\langle[\omega_1]^{\omega},\subseteq\rangle$ contains maximal antichain which does not split.*

Proof. Since $|[\kappa]^{<\lambda}|=\kappa^{<\lambda}=\kappa$, and $\langle[\kappa]^{<\lambda},\subseteq\rangle$ is cut-free and κ-loose we can apply Theorem 4.7 to get the required maximal antichain. ∎

Corollary 4.9. *If $2^{\omega}=\omega_1$ then the cut-free poset $\mathcal{P}(\omega)/[\omega]^{<\omega}$ contains an antichain which does not split.*

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