

CHARACTERIZATION OF ORDER TYPES OF POINTWISE LINEARLY ORDERED FAMILIES OF BAIRE CLASS 1 FUNCTIONS

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Abstract

In the 1970s M. Laczkovich posed the following problem: Let $\mathcal{B}_1(X)$ denote the set of Baire class 1 functions defined on a Polish space X equipped with the pointwise ordering.

Characterize the order types of the linearly ordered subsets of $\mathcal{B}_1(X)$.

The main result of the present paper is a complete solution to this problem.

We prove that a linear order is isomorphic to a linearly ordered family of Baire class 1 functions iff it is isomorphic to a subset of the following linear order that we call $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$, where $[0, 1]_{\searrow 0}^{<\omega_1}$ is the set of strictly decreasing transfinite sequences of reals in $[0, 1]$ with last element 0, and $<_{altlex}$, the so called *alternating lexicographical ordering*, is defined as follows: if $(x_\alpha)_{\alpha \leq \xi}, (x'_\alpha)_{\alpha \leq \xi'} \in [0, 1]_{\searrow 0}^{<\omega_1}$ are distinct, and δ is the minimal ordinal where the two sequences differ then we say that

$$(x_\alpha)_{\alpha \leq \xi} <_{altlex} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Using this characterization we easily reprove all the known results and answer all the known open questions of the topic.

1. INTRODUCTION

Let $\mathcal{F}(X)$ be a class of real valued functions defined on a Polish space X , e.g. $C(X)$, the set of continuous functions. The natural partial ordering on this space is the pointwise ordering $<_p$, that is, we say that $f <_p g$ if for every $x \in X$ we have $f(x) \leq g(x)$ and there exists at least one x such that $f(x) < g(x)$. If we would like to understand the structure of this partially ordered set (poset), the first step is to describe its linearly ordered subsets.

For example, if $X = [0, 1]$ and $\mathcal{F}(X) = \mathcal{C}([0, 1])$ then it is a well known result that the possible order types of the linearly ordered subsets of $\mathcal{C}([0, 1])$ are the real order types (that is, the order types of the subsets of the reals). Indeed, a real order type is clearly representable by constant functions, and if $\mathcal{L} \subset \mathcal{C}([0, 1])$ is a linearly ordered family of continuous functions then (by continuity) $f \mapsto \int_0^1 f$ is a *strictly* monotone map of \mathcal{L} into the reals.

The next natural class to look at is the class of Lebesgue measurable functions. However, it is not hard to check that the assumption of measurability is rather meaningless here. Indeed, if \mathcal{L} is a linearly ordered family of *arbitrary* real functions and $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a map that maps the Cantor set onto \mathbb{R} and is zero outside of the Cantor set then $f \mapsto f \circ \varphi$ is a strictly monotone map of \mathcal{L} into the class of Lebesgue measurable functions.

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Therefore it is more natural to consider the class of Borel measurable functions. However, P. Komjáth [9] proved that it is already independent of *ZFC* (the usual axioms of set theory) whether the class of Borel measurable functions contains a strictly increasing transfinite sequence of length ω_2 .

The next step is therefore to look at subclasses of the Borel measurable functions, namely the Baire hierarchy. A function is of *Baire class 1* if it is the pointwise limit of continuous functions. The set of (real valued) Baire class 1 functions defined on a space X will be denoted by $\mathcal{B}_1(X)$. A function is of *Baire class 2* if it is the pointwise limit of Baire class 1 functions. Komjáth actually also proved that in his above mentioned result the set of Borel measurable function can be replaced by the set of Baire class 2 functions. This explains why the Baire class 1 case seem to be the most interesting one.

Back in the 1970s M. Laczkovich [11] posed the following problem:

Problem 1.1. *Characterize the order types of the linearly ordered subsets of $(\mathcal{B}_1(X), <_p)$.*

We will use the following notation:

Definition 1.2. Let $(P, <_P)$ and $(Q, <_Q)$ be two posets. We say that P is *embeddable into Q* , in symbols $(P, <_P) \hookrightarrow (Q, <_Q)$ if there exists a map $\Phi : P \rightarrow Q$ such that for every $p, q \in P$ if $p <_P q$ then $\Phi(p) <_Q \Phi(q)$. (Note that an embedding may not be 1-to-1 in general. However, an embedding of a *linearly* ordered set is 1-to-1.) If $(L, <_L)$ is a linear ordering and $(L, <_L) \hookrightarrow (Q, <_Q)$ then we also say that L is *representable in Q* .

Whenever the ordering of a poset $(P, <_P)$ is clear from the context we will use the notation $P = (P, <_P)$. Moreover, when Q is not specified, the term “representable” will refer to representability in $\mathcal{B}_1(X)$.

The earliest result that is relevant to Laczkovich’s problem is due to Kuratowski. He showed that for any Polish space X we have $\omega_1, \omega_1^* \not\hookrightarrow \mathcal{B}_1(X)$, or in other words, there is no ω_1 -long strictly increasing or decreasing sequence of Baire class 1 functions (see [10, §24. III.2.]).

It seems conceivable at first sight that this is the only obstruction, that is, every linearly ordered set that does not contain ω_1 -long strictly increasing or decreasing sequences is representable in $\mathcal{B}_1(\mathbb{R})$. First, answering a question of Gerlits and Petruska, this conjecture was consistently refuted by P. Komjáth [9] who showed that no Suslin line (ccc linearly ordered set that is not separable) is representable in $\mathcal{B}_1(\mathbb{R})$. Komjáth’s short and elegant proof uses the very difficult set-theoretical technique of forcing. Laczkovich [12] asked if a forcing-free proof exists.

Elekes and Steprāns [5] continued this line of research. On the one hand they proved that consistently Kuratowski’s result is a characterization for order types of cardinality $< \mathfrak{c}$. On the other hand they strengthened Komjáth’s result by constructing in *ZFC* a linearly ordered set L not containing Suslin lines or ω_1 -long strictly increasing or decreasing sequences such that L is not representable in $\mathcal{B}_1(X)$.

Among other results, M. Elekes [2] proved that if X and Y are both uncountable σ -compact or both non- σ -compact Polish spaces then for a linearly ordered set L we have $L \hookrightarrow \mathcal{B}_1(X) \iff L \hookrightarrow \mathcal{B}_1(Y)$. Then he asked if the same holds if X is an uncountable σ -compact and Y is a non- σ -compact Polish space. Moreover, he also asked whether the same linearly ordered sets can be embedded into the set of *characteristic* functions in $\mathcal{B}_1(X)$ as into $\mathcal{B}_1(X)$. Notice that a characteristic function χ_A is of Baire class 1 if and only if A is simultaneously F_σ and G_δ (denoted by $A \in \Delta_2^0(X)$, see the Preliminaries section below). Moreover, $\chi_A <_p \chi_B \iff A \subsetneq_{\overline{\neq}} B$, hence the above question is equivalent to whether $L \hookrightarrow (\mathcal{B}_1(X), <_p)$ implies $L \hookrightarrow (\Delta_2^0(X), \subsetneq_{\overline{\neq}})$. He also asked if *duplications* and

completions of representable orders are themselves representable, where the duplication of L is $L \times \{0, 1\}$ ordered lexicographically.

Our main aim in this paper is to solve Problem 1.1 and consequently answer the above mentioned questions. The solution proceeds by constructing a *universal* linearly ordered set for $\mathcal{B}_1(X)$, that is, a linear order that is representable in $\mathcal{B}_1(X)$ such that every representable linearly ordered set is embeddable into it. Of course such a linear order only provides a useful characterization if it is sufficiently simple combinatorially to work with. We demonstrate this by providing new, simpler proofs of the known theorems (including a forcing-free proof of Komjáth's theorem), and also by answering the above mentioned open questions.

The universal linear ordering can be defined as follows.

Definition 1.3. Let $[0, 1]_{\searrow 0}^{<\omega_1}$ be the set of strictly decreasing well-ordered transfinite sequences in $[0, 1]$ with last element zero. Let $\bar{x} = (x_\alpha)_{\alpha \leq \xi}, \bar{x}' = (x'_\alpha)_{\alpha \leq \xi'} \in [0, 1]_{\searrow 0}^{<\omega_1}$ be distinct and let δ be the minimal ordinal such that $x_\delta \neq x'_\delta$. We say that

$$(x_\alpha)_{\alpha \leq \xi} <_{\text{attlex}} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Now we can formulate our main result.

Theorem 1.4. (*Main Theorem*) *Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L, <)$:*

- (1) $(L, <) \hookrightarrow (\mathcal{B}_1(X), <_p)$,
- (2) $(L, <) \hookrightarrow ([0, 1]_{\searrow 0}^{<\omega_1}, <_{\text{attlex}})$.

In fact, $(\mathcal{B}_1(X), <_p)$ and $([0, 1]_{\searrow 0}^{<\omega_1}, <_{\text{attlex}})$ are embeddable into each other.

Using this theorem one can reduce every question concerning the linearly ordered subsets of $\mathcal{B}_1(X)$ to a purely combinatorial problem. We were able to answer all of the known such questions and we reproved easily the known theorems as well. The most important results are:

- Answering a question of Laczkovich [12], we give a new, forcing free proof of Komjáth's theorem. (Theorem 4.2)
- The class of ordered sets representable in $\mathcal{B}_1(X)$ does not depend on the uncountable Polish space X . (Corollary 3.15)
- There exists an embedding $(\mathcal{B}_1(X), <_p) \hookrightarrow (\Delta_2^0(X), \subsetneq)$, hence a linear ordering is representable by Baire class 1 functions iff it is representable by Baire class 1 *characteristic* functions. (Corollary 3.14)
- The duplication of a representable linearly ordered set is representable. More generally, countable lexicographical products of representable sets are representable. (Corollary 5.5 and Theorem 5.2)
- There exists a linearly ordered set that is representable in $\mathcal{B}_1(X)$ but none of its completions are representable. (Theorem 5.12)

The paper is organized as follows. In Section 3 we first prove that there exists an embedding $\mathcal{B}_1(X) \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$, then that $[0, 1]_{\searrow 0}^{<\omega_1} \hookrightarrow \mathcal{B}_1(X)$. The former result heavily builds on a theorem of Kechris and Louveau. Unfortunately for us, they only consider the case of compact Polish spaces, while it is of crucial importance in our proof to use their theorem for arbitrary Polish spaces. Moreover, their proof seems to contain a slight error. Hence it was unavoidable to reprove their result, which is the content of Section 6. Section 4 contains the new proofs of the known results, while in Section 5 we answer the above open questions. Finally, in Section 7 we formulate some new open problems.

2. PRELIMINARIES

Our terminology will mostly follow [7] and [13].

Let X be a *Polish space*, that is, a complete, separable and metrizable topological space. $\mathcal{B}_1(X)$ denotes the set of the pointwise limits of continuous functions defined on X , this is called the class of *Baire class 1* functions.

$USC(X)$ stands for the set of *upper semicontinuous* functions, that is, the set of functions f for which for every $r \in \mathbb{R}$ the set $f^{-1}((-\infty, r))$ is open in X . It is easy to see that the infimum of USC functions is also USC.

If $\mathcal{F}(X)$ is a class of real valued functions then we will denote by $b\mathcal{F}(X)$ and $\mathcal{F}^+(X)$ the set of bounded and nonnegative functions in $\mathcal{F}(X)$, respectively.

$\mathcal{K}(X)$ will stand for the set of the nonempty compact subsets of X endowed with the Hausdorff metric. It is well known (see [7, Section 4.F]) that if X is Polish then so is $\mathcal{K}(X)$. Moreover, the compactness of X is equivalent to the compactness of $\mathcal{K}(X)$.

As usual, we denote the ξ th *additive and multiplicative Borel classes* of a Polish space X by $\Sigma_\xi^0(X)$ and $\Pi_\xi^0(X)$, respectively. We will also use the notation $\Delta_\xi^0(X) = \Sigma_\xi^0(X) \cap \Pi_\xi^0(X)$. We call a set A *ambiguous*, if $A \in \Delta_2^0(X)$. Sometimes the following equivalent definition is also used for the first Baire class: $f \in \mathcal{B}_1(X) \iff$ the preimage of every open set under f is in $\Sigma_2^0(X)$ (see [7, 24.10]). This easily implies that a characteristic function χ_A is of Baire class 1 if and only if $A \in \Delta_2^0(X)$. The above equivalent definition also implies that USC functions are of Baire class 1.

For a function $f : X \rightarrow \mathbb{R}$ the *subgraph* of f is the set $sgr(f) = \{(x, r) \in X \times \mathbb{R} : r \leq f(x)\}$. Notice that a function is USC if and only if its subgraph is closed.

Let $(P, <_p)$ be a poset. Let us introduce the following notation for the set of well-ordered sequences in P :

$$\sigma P = \{F : \alpha \rightarrow P \mid \alpha \text{ is an ordinal, } F \text{ is strictly increasing}\}.$$

We will use the notation σ^*P for the reverse well-ordered sequences, that is,

$$\sigma^*P = \{F : \alpha \rightarrow P \mid \alpha \text{ is an ordinal, } F \text{ is strictly decreasing}\}.$$

Then $\sigma^*[0, 1]$ is the set of strictly decreasing well-ordered transfinite sequences of reals in $[0, 1]$.

For a poset P , if $\bar{p} \in \sigma^*P$ and the domain of \bar{p} is ξ then we will write \bar{p} as $(p_\alpha)_{\alpha < \xi}$, where $p_\alpha = \bar{p}(\alpha)$. We will call the ordinal ξ the length of \bar{p} , in symbols $l(\bar{p})$.

Let H and H' be two subsets of the linearly ordered set $(L, <_L)$. We will say that $H \leq_L H'$ or $H <_L H'$ if for every $h \in H$ and $h' \in H'$ we have $h \leq_L h'$ or $h <_L h'$, respectively.

Now if $\bar{p}, \bar{p}' \in \sigma^*P$ and $\bar{p} \not\leq \bar{p}'$, $\bar{p}' \not\leq \bar{p}$ then there exists a minimal ordinal δ such that $p_\delta \neq p'_\delta$. This ordinal is denoted by $\delta(\bar{p}, \bar{p}')$.

Let α be a successor ordinal, then $\alpha - 1$ will stand for its predecessor. Now, since every ordinal α can be uniquely written in the form $\alpha = \gamma + n$ where γ is limit and n is finite, we let $(-1)^\alpha = (-1)^n$ and refer to the parity of n as the parity of α .

A poset $(T, <_T)$ is called a *tree* if for every $t \in T$ the ordering $<_T$ restricted to the set $\{s : s <_T t\}$ is a well-ordering. We denote by $Lev_\alpha(T)$ the α th level of T , that is, the set $\{t \in T : <_T \upharpoonright_{\{s : s <_T t\}} \text{ has order type } \alpha\}$. An α -*chain* C is a subset of a tree such that $<_T \upharpoonright_C$ is a well-ordering in type α , whereas an *antichain* is a set that consists of \leq_T -incomparable elements. A set $D \subset T$ is called *dense* if for every $t \in T$ there exists a $p \in D$ such that $t \leq_T p$. A set is called *open* if for every $p \in D$ we have $\{t \in T : t \geq_T p\} \subset D$.

A tree $(T, <_T)$ of cardinality \aleph_1 is called an Aronszajn tree, if for every $\alpha < \omega_1$ we have $|Lev_\alpha(T)| \leq \aleph_0$ and T contains no ω_1 -chains. An Aronszajn tree is called a Suslin tree if it contains no uncountable antichains.

A Suslin line is a linearly ordered set that is ccc (it contains no uncountable pairwise disjoint collection of nonempty open intervals) but not separable.

We will call a poset $(P, <_P)$ \mathbb{R} -special (\mathbb{Q} -special) if there exists an embedding $P \hookrightarrow \mathbb{R}$ ($P \hookrightarrow \mathbb{Q}$).

Every ordinal is identified with the set of its predecessors, in particular, $2 = \{0, 1\}$.

3. THE MAIN RESULT

3.1. $\mathcal{B}_1(X) \hookrightarrow ([0, 1]_{\searrow 0}^{<\omega_1}, <_{attlex})$. Recall that

$$[0, 1]_{\searrow 0}^{<\omega_1} = \{\bar{x} \in \sigma^*[0, 1] : \min \bar{x} = 0\}$$

and also that for $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$, $\bar{x}' = (x'_\alpha)_{\alpha \leq \xi'}$ $\in [0, 1]_{\searrow 0}^{<\omega_1}$ distinct and $\delta = \delta(\bar{x}, \bar{x}')$ we say that

$$(x_\alpha)_{\alpha \leq \xi} <_{attlex} (x'_\alpha)_{\alpha \leq \xi'} \iff (\delta \text{ is even and } x_\delta < x'_\delta) \text{ or } (\delta \text{ is odd and } x_\delta > x'_\delta).$$

Theorem 3.1. *Let X be a Polish space. Then $\mathcal{B}_1(X) \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$.*

In order to prove the theorem we have to make some preparation. We will use results of Kechris and Louveau [8]. They developed a method to decompose a Baire class 1 function into a sum of a transfinite alternating series, which is analogous to the well known Hausdorff-Kuratowski analysis of Δ_2^0 sets.

First we define the generalized sums.

Definition 3.2. ([8]) Suppose that $(f_\beta)_{\beta < \alpha}$ is a pointwise decreasing sequence of non-negative bounded USC functions for an ordinal $\alpha < \omega_1$. Let us define the *generalized alternating sum* $\sum_{\beta < \alpha}^* (-1)^\beta f_\beta$ by induction on α as follows:

$$\sum_{\beta < 0}^* (-1)^\beta f_\beta = 0$$

and

$$\sum_{\beta < \alpha}^* (-1)^\beta f_\beta = \sum_{\beta < \alpha-1}^* (-1)^\beta f_\beta + (-1)^{\alpha-1} f_{\alpha-1}$$

if α is a successor and

$$\sum_{\beta < \alpha}^* (-1)^\beta f_\beta = \sup\{\sum_{\gamma < \beta}^* (-1)^\gamma f_\gamma : \beta < \alpha, \beta \text{ even}\}$$

if $\alpha > 0$ is a limit.

Every nonnegative bounded Baire class 1 function can be canonically decomposed into such a sum. For this we need the notion of upper regularization.

Definition 3.3. ([8]) Let $f : X \rightarrow \mathbb{R}$ be a nonnegative bounded function. The *upper regularization* of f is defined as

$$\hat{f} = \inf\{g : f \leq_p g, g \in \text{USC}(X)\}.$$

Note that \hat{f} is USC, since the infimum of USC functions is USC. Also, clearly $\hat{f} = f$ if f is USC.

Definition 3.4. ([8]) Let

$$g_0 = f, f_0 = \hat{g}_0,$$

if α is a successor then let

$$g_\alpha = f_{\alpha-1} - g_{\alpha-1}, f_\alpha = \hat{g}_\alpha,$$

if $\alpha > 0$ is a limit then let

$$g_\alpha = \inf_{\substack{\beta < \alpha \\ \beta \text{ even}}} g_\beta \text{ and } f_\alpha = \widehat{g}_\alpha.$$

Now if there exists a minimal ξ such that $f_\xi \equiv f_{\xi+1}$ then let $\Phi(f) = (f_\alpha)_{\alpha \leq \xi}$.

Note that we need some results of Kechris and Louveau for arbitrary Polish spaces, however in [8] the authors proved the theorems only in the compact Polish case, although the proofs still work for the general case as well. Unfortunately, in our proof the non- σ -compact statement plays a significant role, hence we must check the validity of their results on such spaces. The results used are summarized in Proposition 3.5 and the proof can be found in Section 6. Notice that the original proof seems to contain a small error, but it can be corrected with the same ideas.

Proposition 3.5. ([8]) *Let X be a Polish space and $f \in b\mathcal{B}_1^+(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^*bUSC^+$ and we have*

- (1) $f = \sum_{\beta < \alpha}^* (-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ for every $\alpha \leq \xi$,
- (2) $f_\xi \equiv 0$,
- (3) $f = \sum_{\alpha < \xi}^* (-1)^\alpha f_\alpha$.

Proof. See Section 6. □

Proposition 3.6. *Let X be a Polish space and $f_0, f_1 \in b\mathcal{B}_1^+(X)$. Suppose that $f_0 <_p f_1$ and let $\Phi(f_0) = (f_\alpha^0)_{\alpha \leq \xi_0}$ and $\Phi(f_1) = (f_\alpha^1)_{\alpha \leq \xi_1}$. Then $\Phi(f_0) \neq \Phi(f_1)$ and if $\delta = \delta(\Phi(f_0), \Phi(f_1))$ then $f_\delta^0 <_p f_\delta^1$ if δ is even and $f_\delta^0 >_p f_\delta^1$ if δ is odd.*

Proof. First notice that if $f_0 \neq f_1$ then by (3) of Proposition 3.5 we have that $\Phi(f_0) \neq \Phi(f_1)$.

Let $(g_\beta^0)_{\beta \leq \xi_0}$ and $(g_\beta^1)_{\beta \leq \xi_1}$ be the appropriate sequences (used in Definition 3.4 with $\widehat{g}_\beta^i = f_\beta^i$).

We show by induction on β that for every even ordinal $\beta \leq \delta$ we have $g_\beta^0 \leq_p g_\beta^1$ and for every odd ordinal $\beta \leq \delta$ we have $g_\beta^0 \geq_p g_\beta^1$.

For $\beta = 0$ by definition $g_0^0 = f_0$ and $g_0^1 = f_1$, so $g_0^0 \leq_p g_0^1$.

Suppose that we are done for every $\gamma < \beta$.

- for limit β we have that

$$g_\beta^0 = \inf_{\substack{\gamma < \beta \\ \gamma \text{ even}}} g_\gamma^0$$

so by the inductive hypothesis obviously $g_\beta^0 \leq_p g_\beta^1$.

- if β is an odd ordinal, since $\beta - 1 < \delta$ we have $f_{\beta-1}^0 = f_{\beta-1}^1$ so

$$g_\beta^0 = f_{\beta-1}^0 - g_{\beta-1}^0 \geq_p f_{\beta-1}^0 - g_{\beta-1}^1 = f_{\beta-1}^1 - g_{\beta-1}^1 = g_\beta^1$$

by $\beta - 1$ being even and using the inductive hypothesis.

- if β is an even successor, the calculation is similar, using that $g_{\beta-1}^0 \geq_p g_{\beta-1}^1$ we obtain

$$g_\beta^0 = f_{\beta-1}^0 - g_{\beta-1}^0 \leq_p f_{\beta-1}^0 - g_{\beta-1}^1 = f_{\beta-1}^1 - g_{\beta-1}^1 = g_\beta^1.$$

Consequently, the induction shows that $g_\delta^0 \leq_p g_\delta^1$ if δ is even and $g_\delta^0 \geq_p g_\delta^1$ if δ is odd.

Therefore, since $\widehat{g}_\delta^i = f_\delta^i$ we have that $f_\delta^0 \leq_p f_\delta^1$ if δ is even and $f_\delta^0 \geq_p f_\delta^1$ if δ is odd. But by the definition of δ it is clear that $f_\delta^0 \neq f_\delta^1$, hence $f_\delta^0 <_p f_\delta^1$ if δ is even and $f_\delta^0 >_p f_\delta^1$ if δ is odd. This finishes the proof of Proposition 3.6. □

Now to finish the proof of Theorem 3.1 we need the following folklore lemma.

Lemma 3.7. *There exists an order preserving embedding $\Psi_0 : USC^+(X) \hookrightarrow [0, 1]$ where the image of the function $f \equiv 0$ is 0. In particular, there is no uncountable strictly monotone transfinite sequence in $USC^+(X)$.*

Proof. Fix a countable basis $\{B_n : n \in \omega\}$ of $X \times [0, \infty)$. Assign to each $f \in USC^+$ the real

$$r_f = 1 - \sum_{B_n \cap sgr(f) = \emptyset} 2^{-n-1}.$$

If $f <_p g$ then $sgr(f) \subsetneq sgr(g)$ so, as the subgraph of an USC function is a closed set, there exists an $n \in \omega$ such that B_n is an open neighborhood of a point in $sgr(g) \setminus sgr(f)$. Thus, $\{n : B_n \cap sgr(f) = \emptyset\} \supsetneq \{n : B_n \cap sgr(g) = \emptyset\}$. Consequently, $r_f < r_g$. \square

Proof of Theorem 3.1. Let $\Psi : \sigma^*USC^+(X) \rightarrow \sigma^*[0, 1]$ be the map that applies the above Ψ_0 to every coordinate of the sequences in $\sigma^*USC^+(X)$. Thus, Ψ is order preserving coordinate-wise.

Clearly, $h(x) = \frac{1}{\pi} \arctan(x) + 1$ is an order preserving homeomorphism from \mathbb{R} to $(0, 1)$ and for $f \in \mathcal{B}_1(X)$ let $H(f) = h \circ f$. Composing the functions in $\mathcal{B}_1(X)$ with h we still have Baire class 1 functions and this does not effect the pointwise ordering. Thus, H is an order preserving map from $\mathcal{B}_1(X)$ into $b\mathcal{B}_1^+(X)$.

Let $\Theta = \Psi \circ \Phi \circ H$. Notice that as $H : \mathcal{B}_1(X) \rightarrow b\mathcal{B}_1^+(X)$, $\Phi : b\mathcal{B}_1^+(X) \rightarrow \sigma^*bUSC^+(X)$ and $\Psi : \sigma^*USC^+(X) \rightarrow \sigma^*[0, 1]$, the map Θ is well defined.

Now, by Lemma 3.7 we have that Ψ_0 maps the constant zero function to zero and by (2) of Proposition 3.5 we have that for every function f its Φ image ends with the constant zero function. Thus, the Θ image of every function f ends with zero. Therefore, Θ maps into $[0, 1]_{\searrow 0}^{<\omega_1}$.

If $f_0 <_p f_1$ are Baire class 1 functions then clearly $H(f_0) <_p H(f_1)$ hence by Proposition 3.6 we have that if $\delta = \delta(\Phi(H(f_0)), \Phi(H(f_1)))$, then $\Phi(H(f_0))(\delta) <_p \Phi(H(f_1))(\delta)$ if δ is even and $\Phi(H(f_0))(\delta) >_p \Phi(H(f_1))(\delta)$ if δ is odd. Since Ψ is order preserving coordinate-wise, we obtain that Θ is an order preserving embedding of $\mathcal{B}_1(X)$ into $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$, which finishes the proof of the theorem. \square

3.2. $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex}) \hookrightarrow \mathcal{B}_1(X)$.

Theorem 3.8. *The linearly ordered set $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$ can be represented by Δ_2^0 subsets of $\mathcal{K}([0, 1]^2)$ ordered by inclusion.*

Proof. First we define a map $\Psi : [0, 1]_{\searrow 0}^{<\omega_1} \rightarrow \mathcal{K}([0, 1]^2)$, basically assigning to each sequence its closure (as a subset of the interval). However, such a map cannot distinguish between continuous sequences and sequences omitting a limit point. To remedy this we place a line segment on each limit point contained in the sequence.

Let $\bar{x} \in [0, 1]_{\searrow 0}^{<\omega_1}$, with $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$. Now let

$$\Psi(\bar{x}) = \overline{\{(x_\alpha, 0) : \alpha \leq \xi\}} \cup$$

$$\bigcup \{ \{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\} \}.$$

Lemma 3.9. $\Psi(\bar{x})$ is a compact set for every $\bar{x} \in [0, 1]_{\searrow 0}^{<\omega_1}$.

Proof. Clearly, it is enough to show that if $(p_n, q_n) \rightarrow (p, q)$ is a convergent sequence such that for every n we have

$$(3.1) \quad (p_n, q_n) \in \bigcup \{ \{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] : \text{if } 0 < \alpha < \xi \text{ and } x_\alpha = \inf\{x_\beta : \beta < \alpha\} \}$$

then $(p, q) \in \Psi(\bar{x})$.

Obviously, $p_n = x_{\alpha_n}$ for some ordinals α_n . First, if the sequence x_{α_n} is eventually constant, then there exists an α such that $p = x_\alpha$ and except for finitely many n 's by (3.1) we have $q_n \in [0, x_\alpha - x_{\alpha+1}]$. So $(p, q) \in \{x_\alpha\} \times [0, x_\alpha - x_{\alpha+1}] \subset \Psi(\bar{x})$.

Now if the sequence $(x_{\alpha_n})_{n \in \omega}$ is not eventually constant, since the sequence $(x_\alpha)_{\alpha \leq \xi}$ is strictly decreasing and well-ordered then (passing to a subsequence of $(x_{\alpha_n})_{n \in \omega}$ if necessary) we can suppose that $(x_{\alpha_n})_{n \in \omega}$ is a strictly decreasing sequence.

Using the fact that $(x_{\alpha_n})_{n \in \omega}$ is a strictly decreasing subset of $(x_\alpha)_{\alpha \leq \xi}$ we obtain that $x_{\alpha_n} - x_{\alpha_{n+1}} \leq x_{\alpha_n} - p$. Hence from (3.1) we obtain

$$0 \leq q_n \leq x_{\alpha_n} - x_{\alpha_{n+1}} \leq x_{\alpha_n} - p \rightarrow 0$$

so $q_n = 0$. Therefore,

$$(p, q) = \left(\lim_{n \rightarrow \infty} x_{\alpha_n}, 0 \right) \in \overline{\{(x_\alpha, 0) : \alpha \leq \xi\}} \subset \Psi(\bar{x}).$$

□

Now we define a decreasing sequence of subsets of $\mathcal{K}([0, 1]^2)$ for each $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$ and $\alpha \leq \xi$ as follows:

$$(3.2) \quad \mathcal{H}_\alpha^{\bar{x}} = \{ \Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha \}.$$

We will use the following notations for an even ordinal $\alpha \leq \xi$:

$$(3.3) \quad \mathcal{K}_\alpha^{\bar{x}} = \overline{\mathcal{H}_\alpha^{\bar{x}}} (= \overline{\{ \Psi(\bar{z}) : \bar{z}|_\alpha = \bar{x}|_\alpha, z_\alpha \leq x_\alpha \}}),$$

and if $\alpha + 1 \leq \xi$ then

$$(3.4) \quad \mathcal{L}_\alpha^{\bar{x}} = \overline{\mathcal{H}_{\alpha+1}^{\bar{x}}} (= \overline{\{ \Psi(\bar{z}) : \bar{z}|_{\alpha+1} = \bar{x}|_{\alpha+1}, z_{\alpha+1} \leq x_{\alpha+1} \}}).$$

Finally, if $\alpha = \xi$ then let $\mathcal{L}_\alpha^{\bar{x}} = \emptyset$. So $\mathcal{K}_\alpha^{\bar{x}}$ and $\mathcal{L}_\alpha^{\bar{x}}$ is defined for every even $\alpha \leq \xi$.

Notice that the sequence $(\overline{\mathcal{H}_\alpha^{\bar{x}}})_{\alpha \leq \xi}$ is a decreasing sequence of closed sets.

To each $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$ let us assign

$$\mathcal{A}^{\bar{x}} = \bigcup_{\alpha \leq \xi, \alpha \text{ even}} (\mathcal{K}_\alpha^{\bar{x}} \setminus \mathcal{L}_\alpha^{\bar{x}}).$$

By [7, 22.27], since $\mathcal{A}^{\bar{x}}$ is a transfinite difference of a decreasing sequence of closed sets, we have $\mathcal{A}^{\bar{x}} \in \Delta_2^0(\mathcal{K}([0, 1]^2))$.

To overcome some technical difficulties we prove the following lemma.

Lemma 3.10. *Let $\bar{z} \in [0, 1]_{\searrow 0}^{< \omega_1}$ and β be an ordinal such that $\beta + 1 \leq l(\bar{z})$.*

- (1) *If $K \in \overline{\mathcal{H}_{\beta+1}^{\bar{z}}}$, β is a limit ordinal, $\inf\{z_\gamma : \gamma < \beta\} = z_\beta$ and $l(\bar{z}) > \beta + 1$ then $(z_\beta, z_\beta - z_{\beta+1}) \in K$.*
- (2) *If $K \in \overline{\mathcal{H}_\beta^{\bar{z}}}$ and β is a successor then $(z_{\beta-1}, 0) \in K$.*
- (3) *If $K \in \overline{\mathcal{H}_\beta^{\bar{z}}}$, β is a limit ordinal and $\inf\{z_\gamma : \gamma < \beta\} > z_\beta$ OR β is a successor then*

$$K \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0, 1]) = \emptyset$$

(notice that if β is a successor then $\inf\{z_\gamma : \gamma < \beta\} = z_{\beta-1}$).

Proof. For (2) and (1) just notice that by equation (3.2) whenever $\Psi(\bar{w}) \in \mathcal{H}_\beta^{\bar{z}}$ ($\mathcal{H}_{\beta+1}^{\bar{z}}$, respectively) then $\Psi(\bar{w})$ contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$). Consequently, every compact set which is in the closure of $\mathcal{H}_\beta^{\bar{z}}$ (or $\mathcal{H}_{\beta+1}^{\bar{z}}$) contains the point $(z_{\beta-1}, 0)$ (the point $(z_\beta, z_\beta - z_{\beta+1})$).

(3) can be proved similarly: by the definition of $\mathcal{H}_\beta^{\bar{z}}$ for every \bar{w} such that $\Psi(\bar{w}) \in \mathcal{H}_\beta^{\bar{z}}$ we have

$$\Psi(\bar{w}) \cap ((z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0, 1]) = \emptyset.$$

Now since the set $U = (z_\beta, \inf\{z_\gamma : \gamma < \beta\}) \times [0, 1]$ is relatively open in $[0, 1]^2$, the set $\{K \in \mathcal{K}([0, 1]^2) : K \cap U = \emptyset\}$ is closed. Hence $\mathcal{H}_\beta^{\bar{z}} \subset \{K \in \mathcal{K}([0, 1]^2) : K \cap U = \emptyset\}$ implies that every $K \in \overline{\mathcal{H}_\beta^{\bar{z}}}$ is disjoint from U . So we proved the lemma. \square

In order to show that $\bar{x} \mapsto \mathcal{A}^{\bar{x}}$ is an embedding it is enough to prove the following claim.

Main Claim. If $\bar{x} <_{\text{altlex}} \bar{y}$ then $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$.

To verify this we have to distinguish two cases.

Case 1. $\delta = \delta(\bar{x}, \bar{y})$ is even. Then $x_\delta < y_\delta$ and $\delta + 1 < l(\bar{y})$. We will show the following lemma.

Lemma 3.11. $\mathcal{K}_\delta^{\bar{x}} \subsetneq \mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}$.

Proof of Lemma 3.11. From $x_\delta < y_\delta$ we have

$$\{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq x_\delta\} \subset \{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq y_\delta\}$$

so $\mathcal{K}_\delta^{\bar{x}} \subset \mathcal{K}_\delta^{\bar{y}}$.

First, we prove that

$$(3.5) \quad \mathcal{K}_\delta^{\bar{x}} \subset \mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}.$$

Here we have to separate two subcases.

SUBCASE 1. δ is a limit ordinal and $y_\delta = \inf\{y_\alpha : \alpha < \delta\}$.

On the one hand, using (1) of Lemma 3.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}_\delta^{\bar{y}} (= \overline{\mathcal{H}_{\delta+1}^{\bar{y}}})$ we have $(y_\delta, y_\delta - y_{\delta+1}) \in K$.

On the other hand, from (3) of Lemma 3.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta$) we have that for every $K \in \mathcal{K}_\delta^{\bar{x}} (= \overline{\mathcal{H}_\delta^{\bar{x}}})$ we have $K \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0, 1]) = \emptyset$. In particular, as $y_\delta \in (x_\delta, \inf\{x_\alpha : \alpha < \delta\})$, we have $(y_\delta, y_\delta - y_{\delta+1}) \notin K$. So we obtain $\mathcal{K}_\delta^{\bar{x}} \cap \mathcal{L}_\delta^{\bar{y}} = \emptyset$, hence by $\mathcal{K}_\delta^{\bar{x}} \subset \mathcal{K}_\delta^{\bar{y}}$ we have $\mathcal{K}_\delta^{\bar{x}} \subset \mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}$.

SUBCASE 2. δ is a limit and $y_\delta < \inf\{y_{\delta'} : \delta' < \delta\}$ or δ is a successor.

Using (2) of Lemma 3.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta + 1$) we obtain that every $K \in \mathcal{L}_\delta^{\bar{y}} (= \overline{\mathcal{H}_{\delta+1}^{\bar{y}}})$ contains the point $(y_\delta, 0)$. From (3) of Lemma 3.10 (with $\bar{z} = \bar{x}$, $\beta = \delta$) we have that for every $K \in \mathcal{K}_\delta^{\bar{x}} (= \overline{\mathcal{H}_\delta^{\bar{x}}})$ the set $K \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0, 1])$ is empty. But $y_\delta \in (x_\delta, \inf\{x_\alpha : \alpha < \delta\})$ so $\mathcal{K}_\delta^{\bar{x}} \cap \mathcal{L}_\delta^{\bar{y}} = \emptyset$. This finishes the proof of equation (3.5).

Second, in order to prove $\mathcal{K}_\delta^{\bar{x}} \neq \mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}$ let \bar{w} be such that $\bar{w}|_\delta = \bar{x}|_\delta$, $x_\delta, y_{\delta+1} < w_\delta < y_\delta$ and $w_{\delta+1} = 0$. Clearly, $\Psi(\bar{w}) \in \mathcal{K}_\delta^{\bar{y}}$.

By (3) of Lemma 3.10 (used for $\bar{z} = \bar{x}$ and $\beta = \delta$) we have that $\Psi(\bar{w}) \in \mathcal{K}_\delta^{\bar{x}} (= \overline{\mathcal{H}_\delta^{\bar{x}}})$ would imply $\Psi(\bar{w}) \cap ((x_\delta, \inf\{x_\alpha : \alpha < \delta\}) \times [0, 1]) = \emptyset$, but $(w_\delta, 0) \in (x_\delta, y_\delta) \times [0, 1]$ and $\inf\{x_\alpha : \alpha < \delta\} = \inf\{y_\alpha : \alpha < \delta\} \geq y_\delta$ which is a contradiction. Hence $\Psi(\bar{w}) \notin \mathcal{K}_\delta^{\bar{x}}$.

Now we prove $\Psi(\bar{w}) \notin \mathcal{L}_\delta^{\bar{y}}$. Suppose the contrary, then using (3) of Lemma 3.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta + 1$) one can obtain that for every $K \in \mathcal{L}_\delta^{\bar{y}} (= \overline{\mathcal{H}_{\delta+1}^{\bar{y}}})$ the set $K \cap ((y_{\delta+1}, y_\delta) \times [0, 1])$ is empty. But clearly $(w_\delta, 0) \in \Psi(\bar{w}) \cap ((y_{\delta+1}, y_\delta) \times [0, 1])$, a contradiction. So $\Psi(\bar{w}) \notin \mathcal{L}_\delta^{\bar{y}}$.

Thus, it follows that $\Psi(\bar{w}) \in (\mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}) \setminus \mathcal{K}_\delta^{\bar{x}}$. From this and from equation (3.5) we can conclude Lemma 3.11. \square

Now we prove the Main Claim in Case 1. If δ' is even and $\delta' < \delta$, the definitions (3.3) and (3.4) of $\mathcal{K}_{\delta'}^{\bar{y}}$ and $\mathcal{L}_{\delta'}^{\bar{y}}$ depend only on $(x_\alpha)_{\alpha \leq \delta'+1}$ so

$$(3.6) \quad \mathcal{K}_{\delta'}^{\bar{x}} = \mathcal{K}_{\delta'}^{\bar{y}}$$

and

$$(3.7) \quad \mathcal{L}_{\delta'}^{\bar{x}} = \mathcal{L}_{\delta'}^{\bar{y}}.$$

Now, from Lemma 3.11 we have $\mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}$, since for every $K \in \mathcal{A}^{\bar{x}}$ we have either $K \in \mathcal{K}_{\delta'}^{\bar{x}} \setminus \mathcal{L}_{\delta'}^{\bar{x}} = \mathcal{K}_{\delta'}^{\bar{y}} \setminus \mathcal{L}_{\delta'}^{\bar{y}}$ for some $\delta' < \delta$ or $K \in \mathcal{K}_\delta^{\bar{x}}$.

Moreover, we claim that using Lemma 3.11 one can prove that $\mathcal{A}^{\bar{x}} \subsetneq \mathcal{A}^{\bar{y}}$. From the definition of $\mathcal{A}^{\bar{x}}$, from the fact that the sequence $(\mathcal{H}_\alpha^{\bar{x}})_{\alpha \leq \xi} = (\mathcal{K}_0^{\bar{x}}, \mathcal{L}_0^{\bar{x}}, \mathcal{K}_1^{\bar{x}}, \mathcal{L}_1^{\bar{x}}, \dots)$ is decreasing and from equations (3.6) and (3.7) follows that

$$(\mathcal{K}_\delta^{\bar{x}})^c \cap \mathcal{A}^{\bar{x}} = \bigcup_{\delta' < \delta, \delta' \text{ even}} \mathcal{K}_{\delta'}^{\bar{x}} \setminus \mathcal{L}_{\delta'}^{\bar{x}} = \bigcup_{\delta' < \delta, \delta' \text{ even}} \mathcal{K}_{\delta'}^{\bar{y}} \setminus \mathcal{L}_{\delta'}^{\bar{y}} = (\mathcal{K}_\delta^{\bar{y}})^c \cap \mathcal{A}^{\bar{y}}$$

So $\mathcal{A}^{\bar{x}} \subset (\mathcal{K}_\delta^{\bar{y}})^c \cup \mathcal{K}_\delta^{\bar{x}}$. Hence, if $K \in (\mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}}) \setminus \mathcal{K}_\delta^{\bar{x}}$ then

$$K \in \mathcal{K}_\delta^{\bar{y}} \setminus \mathcal{L}_\delta^{\bar{y}} \subset \mathcal{A}^{\bar{y}}$$

and

$$K \notin (\mathcal{K}_\delta^{\bar{y}})^c \cup \mathcal{K}_\delta^{\bar{x}} \supset \mathcal{A}^{\bar{x}}$$

so indeed, we obtain that the containment is strict, hence we are done with Case 1.

Case 2. $\delta = \delta(\bar{x}, \bar{y})$ is odd.

Then $x_\delta > y_\delta$ and $\delta + 1 < l(\bar{x})$. Notice that as the length of \bar{x} is larger than $\delta + 1$, the sets $\mathcal{K}_{\delta+1}^{\bar{x}}$ and $\mathcal{L}_{\delta+1}^{\bar{x}}$ are defined.

Now for every even $\delta' \leq \delta - 1$ the definition of $\mathcal{K}_{\delta'}^{\bar{x}}$ and $\mathcal{K}_{\delta'}^{\bar{y}}$ depend only on $(x_\alpha)_{\alpha \leq \delta'}$ and $(y_\alpha)_{\alpha \leq \delta'}$. Thus for every even $\delta' \leq \delta - 1$

$$(3.8) \quad \mathcal{K}_{\delta'}^{\bar{x}} = \mathcal{K}_{\delta'}^{\bar{y}}$$

and also for every even $\delta' < \delta - 1$

$$(3.9) \quad \mathcal{L}_{\delta'}^{\bar{x}} = \mathcal{L}_{\delta'}^{\bar{y}}.$$

We will show the following:

Lemma 3.12. (1) $\mathcal{K}_{\delta-1}^{\bar{x}} \setminus \mathcal{L}_{\delta-1}^{\bar{x}} \subset \mathcal{K}_{\delta-1}^{\bar{y}} \setminus \mathcal{L}_{\delta-1}^{\bar{y}}$
(2) $\mathcal{K}_{\delta+1}^{\bar{x}} \subset \mathcal{K}_{\delta-1}^{\bar{y}} \setminus \mathcal{L}_{\delta-1}^{\bar{y}}$.

Proof of Lemma 3.12. It is easy to prove (1): from equation (3.8) we get $\mathcal{K}_{\delta-1}^{\bar{x}} = \mathcal{K}_{\delta-1}^{\bar{y}}$. Moreover, $\mathcal{L}_{\delta-1}^{\bar{x}} \supset \mathcal{L}_{\delta-1}^{\bar{y}}$, since

$$\mathcal{L}_{\delta-1}^{\bar{x}} = \{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{x}|_\delta, z_\delta \leq x_\delta\} \supset \{\Psi(\bar{z}) : \bar{z}|_\delta = \bar{y}|_\delta, z_\delta \leq y_\delta\} = \mathcal{L}_{\delta-1}^{\bar{y}}$$

holds by $x_\delta > y_\delta$.

Now we show (2). First, $\mathcal{K}_{\delta+1}^{\bar{x}} \subset \mathcal{K}_{\delta-1}^{\bar{x}} = \mathcal{K}_{\delta-1}^{\bar{y}}$, using that the sequence $(\mathcal{K}_\alpha^{\bar{x}})_{\alpha \leq \delta+1}$ is decreasing.

So it suffices to show that $\mathcal{K}_{\delta+1}^{\bar{x}} \cap \mathcal{L}_{\delta-1}^{\bar{y}} = \emptyset$. Using (3) of Lemma 3.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we obtain that for every $K \in \mathcal{L}_{\delta-1}^{\bar{y}} (= \overline{\mathcal{H}_{\delta}^{\bar{y}}})$, we have $K \cap ((y_{\delta}, y_{\delta-1}) \times [0, 1]) = \emptyset$.

However, by (2) of Lemma 3.10 (used with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) if $K \in \mathcal{K}_{\delta+1}^{\bar{x}} (= \overline{\mathcal{H}_{\delta+1}^{\bar{x}}})$ then $(x_{\delta}, 0) \in K$. Therefore, $x_{\delta} \in (y_{\delta}, y_{\delta-1})$ implies that the intersection $\mathcal{K}_{\delta+1}^{\bar{x}} \cap \mathcal{L}_{\delta-1}^{\bar{y}}$ must be empty. So we are done with the lemma. \square

Now we prove the Main Claim in Case 2. By definition of $\mathcal{A}^{\bar{x}}$ and by the fact that the sequence $(\mathcal{H}_{\alpha}^{\bar{x}})_{\alpha \leq \xi} = (\mathcal{K}_{0}^{\bar{x}}, \mathcal{L}_{0}^{\bar{x}}, \mathcal{K}_{1}^{\bar{x}}, \mathcal{L}_{1}^{\bar{x}}, \dots)$ is decreasing we have that if $K \in \mathcal{A}^{\bar{x}}$ then either $K \in \mathcal{K}_{\delta'}^{\bar{x}} \setminus \mathcal{L}_{\delta'}^{\bar{x}} = \mathcal{K}_{\delta'}^{\bar{y}} \setminus \mathcal{L}_{\delta'}^{\bar{y}}$ for some even $\delta' < \delta - 1$ or $K \in \mathcal{K}_{\delta-1}^{\bar{x}} \setminus \mathcal{L}_{\delta-1}^{\bar{x}}$ or $K \in \mathcal{K}_{\delta+1}^{\bar{x}}$. Hence using equations (3.8) and (3.9) and Lemma 3.12 we obtain

$$(3.10) \quad \mathcal{A}^{\bar{x}} \subset \mathcal{A}^{\bar{y}}.$$

In order to show that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$ it is enough to find a \bar{w} such that

$$(3.11) \quad \Psi(\bar{w}) \in \mathcal{K}_{\delta-1}^{\bar{y}} \setminus \mathcal{L}_{\delta-1}^{\bar{y}} \subset \mathcal{A}^{\bar{y}}$$

and

$$(3.12) \quad \Psi(\bar{w}) \notin \mathcal{K}_{\delta+1}^{\bar{x}} \cup (\mathcal{L}_{\delta-1}^{\bar{x}})^c \supset \mathcal{A}^{\bar{x}}.$$

Take $\bar{w}|_{\delta} = \bar{y}|_{\delta}$ and w_{δ} such that $x_{\delta+1}, y_{\delta} < w_{\delta} < x_{\delta}$ and $w_{\delta+1} = 0$.

Now, in order to see (3.11) clearly $\Psi(\bar{w}) \in \mathcal{K}_{\delta-1}^{\bar{y}}$. On the other hand if $K \in \mathcal{L}_{\delta-1}^{\bar{y}} (= \overline{\mathcal{H}_{\delta}^{\bar{y}}})$ by (3) of Lemma 3.10 (with $\bar{z} = \bar{y}$ and $\beta = \delta$) we have $K \cap ((y_{\delta}, y_{\delta-1}) \times [0, 1]) = \emptyset$. But $y_{\delta} < w_{\delta} < x_{\delta} < x_{\delta-1} = y_{\delta-1}$, so $(w_{\delta}, 0) \in \Psi(\bar{w}) \cap ((y_{\delta}, y_{\delta-1}) \times [0, 1])$. Therefore, $\Psi(\bar{w}) \notin \mathcal{L}_{\delta-1}^{\bar{y}}$.

In order to prove (3.12) it is obvious that $\Psi(\bar{w}) \in \mathcal{L}_{\delta-1}^{\bar{x}}$. Now using again (3) of Lemma 3.10 (with $\bar{z} = \bar{x}$ and $\beta = \delta + 1$) we obtain that whenever $K \in \mathcal{K}_{\delta+1}^{\bar{x}} (= \overline{\mathcal{H}_{\delta+1}^{\bar{x}}})$ then $K \cap ((x_{\delta+1}, x_{\delta}) \times [0, 1]) = \emptyset$. However, $w_{\delta} \in (x_{\delta+1}, x_{\delta})$ hence $(w_{\delta}, 0) \in \Psi(\bar{w}) \cap ((x_{\delta+1}, x_{\delta}) \times [0, 1])$, so $\Psi(\bar{w}) \notin \mathcal{K}_{\delta+1}^{\bar{x}}$.

So we can conclude that $\mathcal{A}^{\bar{x}} \neq \mathcal{A}^{\bar{y}}$. Thus, using equation (3.10) we can finish the proof of the Main Claim in Case 2 and hence we obtain Theorem 3.8 as well. \square

3.3. The main theorem.

Theorem 3.13. (Main Theorem) *Let X be an uncountable Polish space. Then the following are equivalent for a linear ordering $(L, <)$:*

- (1) $(L, <) \leftrightarrow (\mathcal{B}_1(X), <_p)$
- (2) $(L, <) \leftrightarrow ([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$
- (3) $(L, <) \leftrightarrow (\Delta_2^0(X), \underline{\subset})$

In fact, $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$, $(\Delta_2^0(X), \underline{\subset})$ and $(\mathcal{B}_1(X), <_p)$ are embeddable into each other.

Proof. $(\mathcal{B}_1(X), <_p) \leftrightarrow ([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$: Theorem 3.1.

$([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex}) \leftrightarrow (\Delta_2^0(X), \underline{\subset})$: we proved in Theorem 3.8 that $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex}) \leftrightarrow (\Delta_2^0(\mathcal{K}([0, 1]^2)), \underline{\subset})$. Now, [2, Theorem 1.2] states that the class of linear orderings representable in Δ_2^0 coincide for all uncountable σ -compact Polish spaces. Hence, if C is the Cantor space, then $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex}) \leftrightarrow (\Delta_2^0(C), \underline{\subset})$. If X is an uncountable Polish space then there exists a continuous injection $h : C \rightarrow X$. Now, since $h(C)$ is a closed set in X we have that $A \mapsto h(A)$ is an inclusion-preserving embedding $(\Delta_2^0(C), \underline{\subset}) \hookrightarrow (\Delta_2^0(X), \underline{\subset})$. Consequently, $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex}) \leftrightarrow (\Delta_2^0(X), \underline{\subset})$.

$(\Delta_2^0(X), \subseteq) \hookrightarrow (\mathcal{B}_1(X), <_p)$: if A is a Δ_2^0 set then χ_A is a Baire class 1 function and $A \mapsto \chi_A$ is an order preserving $(\Delta_2^0(X), \subseteq) \hookrightarrow (\mathcal{B}_1(X), <_p)$ map. \square

We immediately obtain the answers to Questions 5.2 and 5.3 from [5].

Corollary 3.14. *There exists an embedding $\mathcal{B}_1(X) \hookrightarrow \Delta_2^0(X)$, hence a linear ordering is representable by Baire class 1 functions iff it is representable by Baire class 1 characteristic functions.*

The equivalence of (1) and (2), implies that the embeddability of a linearly ordered set into the set of Baire class 1 functions does not depend on the underlying Polish space (provided of course that the Polish space is uncountable). This result answers Question 1.5 from [2] affirmatively.

Corollary 3.15. *If X and Y are uncountable Polish spaces and $L \hookrightarrow \mathcal{B}_1(X)$ then $L \hookrightarrow \mathcal{B}_1(Y)$.*

From now on we will simply use the notation $\mathcal{B}_1(X) = \mathcal{B}_1$.

4. NEW PROOFS OF KNOWN THEOREMS

In this section we would like to demonstrate the strength and applicability of our characterization by providing new, simpler proofs of the theorems of Kuratowski, Komjáth, Elekes and Steprāns. In case of Komjáth's result our proof does not use the technique of forcing, which is an answer to a question of Laczko [12].

We would like to remark here that the above authors mainly investigated $\mathcal{B}_1(\mathbb{R})$ and $\mathcal{B}_1(\omega^\omega)$, but as we saw in Corollary 3.15 the statements do not depend on the underlying Polish space, so we will state them slightly more generally.

4.1. Kuratowski's theorem.

Theorem 4.1. *(Kuratowski, [10, §24. III.2.]) ω_1 and ω_1^* are not representable in \mathcal{B}_1 .*

Proof. By the Main Theorem it is enough to prove that $\omega_1 \not\hookrightarrow [0, 1]_{<0}^{<\omega_1}$ and $\omega_1^* \not\hookrightarrow [0, 1]_{<0}^{<\omega_1}$. We will prove the former statement, the proof of the latter is the same.

Suppose that $(f_\alpha)_{\alpha < \omega_1}$ is a strictly increasing sequence in $[0, 1]_{<0}^{<\omega_1}$. Now we define a sequence $\{s_\alpha : \alpha < \omega_1\} \subset \sigma^*[0, 1]$ that is strictly increasing with respect to containment. Notice that this will yield a contradiction, since $\cup_{\alpha < \omega_1} s_\alpha$ would be an ω_1 -long strictly decreasing sequence of reals.

We define the sequence s_α by induction on α with the following properties:

$$(4.1) \quad l(s_\alpha) = \alpha \text{ and } \{\gamma : s_\alpha \subset f_\gamma\} \text{ contains an end segment of } \omega_1.$$

First, $s_0 = \emptyset$ clearly works. Now suppose that we are done for every $\beta < \alpha$.

If α is a limit let $s_\alpha = \cup_{\beta < \alpha} s_\beta$. Then

$$\{\gamma : s_\alpha \subset f_\gamma\} = \bigcap_{\beta < \alpha} \{\gamma : s_\beta \subset f_\gamma\}$$

so the set $\{\gamma : s_\alpha \subset f_\gamma\}$ is the intersection of countably many sets that contain end segments, hence it contains an end segment. Therefore, (4.1) holds.

Let α be a successor. Let $S = \{\gamma : s_{\alpha-1} \subset f_\gamma\}$. If $\gamma, \gamma' \in S$ with $\gamma < \gamma'$ then clearly $f_\gamma <_{\text{attlex}} f_{\gamma'}$. By $s_{\alpha-1} \subset f_\gamma$, $s_{\alpha-1} \subset f_{\gamma'}$ and $l(s_{\alpha-1}) = \alpha - 1$ we obtain that $\delta(f_\gamma, f_{\gamma'}) \geq \alpha - 1$. So either $f_\gamma(\alpha - 1) = f_{\gamma'}(\alpha - 1)$ or $f_\gamma(\alpha - 1) < f_{\gamma'}(\alpha - 1)$ if $\alpha - 1$ is even and $f_\gamma(\alpha -$

1) $> f_{\gamma'}(\alpha - 1)$ if $\alpha - 1$ is odd. Therefore, $f_\gamma(\alpha - 1) \leq f_{\gamma'}(\alpha - 1)$ if $\alpha - 1$ is even and $f_\gamma(\alpha - 1) \geq f_{\gamma'}(\alpha - 1)$ if $\alpha - 1$ is odd. Consequently, the map $\gamma \mapsto f_\gamma(\alpha - 1)$ is order preserving from S to the unit interval if $\alpha - 1$ is even and order reversing if $\alpha - 1$ is odd. But S contains an end segment by induction, and $[0, 1]$ contains no subset of type ω_1 or ω_1^* , hence this map attains a constant value, say r on an end segment. Thus, $s_\alpha = s_{\alpha-1} \hat{\wedge} r$ satisfies (4.1). \square

4.2. Komjáth's theorem. Komjáth [9] has shown using forcing that a Suslin line is not representable in $\mathcal{B}_1(\mathbb{R})$. Laczko [12] asked if a forcing-free proof exists. Now we provide such a proof.

Theorem 4.2. (Komjáth, [9]) *A Suslin line is not representable in \mathcal{B}_1 .*

NOTATION. Let $(T, <_T)$ be a tree. We denote by $T|_{succ}$ the set $\{t \in T : t \in Lev_\alpha(T), \alpha \text{ is a successor}\}$ ordered by the restriction of $<_T$. Notice that $T|_{succ}$ is also a tree, but it is not a subtree of T . If $t \in \sigma^*[0, 1]$ we will use the notation I_t for the set $\{\bar{x} \in [0, 1] \stackrel{<_{\omega_1}}{\underset{0}{<}} : t \subset \bar{x}\}$.

Lemma 4.3. *Suppose that $\mathcal{S} \subset [0, 1] \stackrel{<_{\omega_1}}{\underset{0}{<}}$ is a nowhere separable Suslin line. Then $\sigma^*[0, 1]|_{succ}$ contains a Suslin tree.*

Proof. Let

$$(4.2) \quad T = \{t \in \sigma^*[0, 1] : |\mathcal{S} \cap I_t| \geq 2\}.$$

We claim that (T, \subseteq) is a Suslin tree.

First, T is clearly a subtree of $(\sigma^*[0, 1], \subseteq)$ and $\sigma^*[0, 1]$ does not contain uncountable chains hence this is true for T as well.

Second, let $A \subset T$ be an antichain. Notice that for every pair of incomparable nodes $t, t' \in T$ the sets I_t and $I_{t'}$ are disjoint intervals of $([0, 1] \stackrel{<_{\omega_1}}{\underset{0}{<}}, <_{attlex})$, hence $I_t \cap \mathcal{S}$ and $I_{t'} \cap \mathcal{S}$ are also disjoint intervals in \mathcal{S} . By (4.2) these intervals are non-degenerate. Since $A \subset T$ is an antichain the set $\{I_t \cap \mathcal{S} : t \in A\}$ is a collection of pairwise disjoint nonempty intervals in \mathcal{S} . Using that \mathcal{S} is nowhere separable for every t we can select a $J_t \subset I_t$ such that $\mathcal{S} \cap J_t$ is a nonempty open interval. By definition \mathcal{S} is ccc so the set $\{J_t \cap \mathcal{S} : t \in A\}$ is countable. Hence A is countable, showing that T does not contain uncountable antichains.

Third, it is left to show that T is uncountable. Suppose the contrary. Notice first that for every $t \in T$ the set $\{r \in [0, 1] : \mathcal{S} \cap I_t \cap_r \neq \emptyset\}$ is countable, otherwise, choosing points $\bar{p}_r \in \mathcal{S} \cap I_t \cap_r$ the map $r \mapsto \bar{p}_r$ would give an uncountable real subtype of \mathcal{S} , which is impossible (see [13, Proposition 3.5]). Hence, as T is also countable, we can select a countable subset D of \mathcal{S} with the following property: for every $t \in T$ and $r \in [0, 1]$ such that $\mathcal{S} \cap I_t \cap_r \neq \emptyset$ there exists a point $\bar{p} \in D$ such that $\bar{p} \in I_t \cap_r$.

We claim that D is dense in \mathcal{S} which will contradict the non-separability of \mathcal{S} . In order to see this let $J \subset \mathcal{S}$ be a nonempty open interval. By passing to a subinterval of J (using that \mathcal{S} is nowhere separable) we can assume that J is of the form $[\bar{x}, \bar{y}] \cap \mathcal{S}$ with $\bar{x} \neq \bar{y}$. Let $\bar{z} \in (\bar{x}, \bar{y}) \cap \mathcal{S}$ (such a \bar{z} exists by the fact that \mathcal{S} is nowhere separable). Clearly $\bar{x} <_{attlex} \bar{z} <_{attlex} \bar{y}$. Let $\delta_{\bar{x}} = \delta(\bar{x}, \bar{z})$ and $\delta_{\bar{y}} = \delta(\bar{y}, \bar{z})$. Then $l(\bar{z}) \geq \max\{\delta_{\bar{x}}, \delta_{\bar{y}}\} + 1$ and

$$(4.3) \quad \bar{x}(\delta_{\bar{x}}) < \bar{z}(\delta_{\bar{x}}) \iff \delta_{\bar{x}} \text{ even and } \bar{z}(\delta_{\bar{y}}) < \bar{y}(\delta_{\bar{y}}) \iff \delta_{\bar{y}} \text{ even.}$$

Suppose that $\delta_{\bar{x}} \geq \delta_{\bar{y}}$, the proof of the other case is the same. If $t = \bar{x} \cap \bar{z}$, then $\{\bar{x}, \bar{z}\} \subset I_t$, so by (4.2) we have $t \in T$. Clearly,

$$\bar{z} \in \mathcal{S} \cap I_{\bar{z}|\delta_{\bar{x}}+1} = \mathcal{S} \cap I_t \cap_{\bar{z}(\delta_{\bar{x}})}$$

hence, by the definition of D we obtain that there exists a $\bar{p} \in D \cap I_{t \frown \bar{z}(\delta_{\bar{x}})}$. We have $\bar{p}|_{\delta_{\bar{x}+1}} = \bar{z}|_{\delta_{\bar{x}+1}}$ so from $\delta_{\bar{x}} \geq \delta_{\bar{y}}$ we get

$$\delta(\bar{x}, \bar{p}) = \delta_{\bar{x}} \text{ and } \delta(\bar{y}, \bar{p}) = \delta_{\bar{y}},$$

moreover

$$\bar{p}(\delta_{\bar{x}}) = \bar{z}(\delta_{\bar{x}}) \text{ and } \bar{p}(\delta_{\bar{y}}) = \bar{z}(\delta_{\bar{y}}).$$

Therefore, using (4.3) we obtain that $\bar{x} <_{\text{altlex}} \bar{p} <_{\text{altlex}} \bar{y}$, so $\bar{p} \in D \cap (\bar{x}, \bar{y}) \subset D \cap J$. So D is a countable dense subset of \mathcal{S} , a contradiction.

This yields that T is uncountable, hence it is indeed a Suslin tree.

Finally, notice that T is a subtree of $\sigma^*[0, 1]$ so $T|_{\text{succ}} \subset \sigma^*[0, 1]|_{\text{succ}}$. Let $T' = T|_{\text{succ}}$. Clearly, T' is a subset of T and by definition the ordering of T' is the restriction of the ordering of T , so T' does not contain uncountable chains or antichains. In order to see that T' is uncountable first notice that the lengths of the elements in T are unbounded in ω_1 , therefore the lengths of the elements on the successor levels are also unbounded. Hence T' is uncountable so T' is also a Suslin tree, which completes the proof of the lemma. \square

For the sake of completeness we will prove the following classical facts about Suslin trees.

Lemma 4.4. *If D is a dense open subset of the Suslin tree T then $T \setminus D$ is countable.*

Proof. Let A be a maximal antichain in D . Clearly, A is countable. Let α be such that $\alpha > \sup\{l(s) : s \in A\}$. Now, if $\beta \geq \alpha$ arbitrary and $t \in \text{Lev}_\beta(T)$ then by the density of D there exists an $s_0 \in D$ such that $t \leq_T s_0$. From the facts that A is maximal and $\beta \geq \alpha$ we obtain that for some $s_1 \in A$ we have $s_1 \leq_T s_0$ and hence $s_1 \leq_T t$. But then, as D is open and $A \subset D$ we obtain that $t \in D$. This finishes the proof of the lemma. \square

Lemma 4.5. *A Suslin tree is not \mathbb{R} -special.*

Proof. Suppose the contrary. Let T be a Suslin tree and $f : T \rightarrow \mathbb{R}$ be an order preserving map. We can suppose that $f(T)$ is a subset of $[0, 1]$.

Let $n \in \omega$ and

$$D_n = \{t \in T : (\forall s \geq_T t)(f(s) \leq f(t) + \frac{1}{n+1})\}.$$

Clearly, D_n is open. We will show that it is also dense in T . In order to see this let $t_0 \in T$ be arbitrary. Then either $t_0 \in D_n$ or there exists an $t_1 \geq_T t_0$ such that $f(t_1) > f(t_0) + \frac{1}{n+1}$. Repeating this argument for t_1 we obtain either that $t_1 \in D_n$ or a $t_2 \geq_T t_1$ such that $f(t_2) > f(t_1) + \frac{1}{n+1} > f(t_0) + \frac{2}{n+1}$, etc. $f(T) \subset [0, 1]$ implies that this procedure stops after at most $n+2$ steps, hence we obtain an $s \geq_T t_0$ such that $s \in D_n$. Therefore, the sets D_n are dense open subsets of T . By Lemma 4.4 the complement of $\bigcap_{n \in \omega} D_n$ is countable, hence there exists $s <_T t$ such that $s, t \in \bigcap_{n \in \omega} D_n$. But then clearly $f(t) = f(s)$, a contradiction. \square

Now we are ready to prove the main result of this subsection.

Proof of Theorem 4.2. Suppose the contrary and let \mathcal{S}' be a subset of \mathcal{B}_1 order isomorphic to a Suslin line. By the Main Theorem there exists an embedding $\Phi_0 : \mathcal{S}' \hookrightarrow [0, 1]_{\leq \omega_0}^{\omega_1}$. For $p, q \in \mathcal{S}'$ let $p \sim q$ if the interval $[p, q]$ is separable. Then \sim is an equivalence relation and $\mathcal{S} = \mathcal{S}' / \sim$ is a nowhere separable Suslin line (for the details see [13, Section 3.]). For every \sim equivalence class $[\cdot]$ fix a representative $p \in \mathcal{S}'$. It is easy to see that every equivalence class is an interval, so the map $\Phi([p]) = \Phi_0(p)$ is an order preserving embedding of \mathcal{S} into $[0, 1]_{\leq \omega_0}^{\omega_1}$.

Now we can use Lemma 4.3 for $\Phi(\mathcal{S})$. This yields that there exists a Suslin tree $T \subset \sigma^*[0, 1]_{succ}$. Assign to each $t \in T$ the last element of t , namely, let $f(t) = t(l(t) - 1)$.

Let $s, t \in T$ such that $s <_T t$. Then, as $s \neq t$, the sequences s and t are strictly decreasing and (using that $s <_T t \iff s \subsetneq t$) t is an end extension of s we obtain that $f(t) < f(s)$. Therefore, the map $1 - f$ is a strictly monotone map from the Suslin tree T to \mathbb{R} . This contradicts Lemma 4.5. \square

4.3. Linearly ordered sets of cardinality $< \mathfrak{c}$ and Martin's Axiom. In this subsection we reprove the results of Elekes and Steprāns from [5]. To formulate the statements, we need some preparation.

Suppose that $(L, <_L)$ is a linearly ordered set. A partition tree T_L of L is defined as follows: the elements of T_L are certain nonempty open intervals of L ordered by reverse inclusion. T_L is constructed by induction. Let $Lev_0(T_L) = \{L\}$.

Suppose that for an ordinal α we have defined $Lev_\beta(T_L)$ for all $\beta < \alpha$. If α is a successor, for every $I \in Lev_{\alpha-1}(T_L)$ fix nonempty intervals I_0 and I_1 such that $I_0 \cup I_1 = I$ and $I_0 \cap I_1 = \emptyset$ if such I_0, I_1 exist. Let

$$Lev_\alpha(T_L) = \bigcup \{I_0, I_1 : I \in Lev_{\alpha-1}(T_L)\}.$$

Now if α is a limit ordinal let

$$Lev_\alpha(T_L) = \left\{ \bigcap_{\beta < \alpha} I_\beta : I_\beta \in Lev_\beta(T_L), \bigcap_{\beta < \alpha} I_\beta \neq \emptyset \right\}.$$

Somewhat ambiguously if $t \in T_L$ we will denote the corresponding interval of L by N_t .

We first verify the next proposition, which is interesting in its own right.

Proposition 4.6. *Let L be a linear ordering such that T_L , a partition tree of L , is \mathbb{R} -special. Then $L \hookrightarrow \mathcal{B}_1$.*

Proof. Without loss of generality we can suppose that we have a strictly decreasing map $\Phi : T_L \rightarrow (0, 1)$.

Lemma 4.7. *There exists a map $\Psi_0 : T_L \rightarrow \sigma^*[0, 1]$ with the following properties for every $t, s \in T_L$:*

- (1) if $s \leq_{T_L} t$ then $\Psi_0(s) \subset \Psi_0(t)$,
- (2) if $N_s <_L N_t$ then $\Psi_0(s) <_{attlex} \Psi_0(t)$,
- (3) $\inf \Psi_0(t) \geq \Phi(t)$.

Proof. We define Ψ_0 inductively on the levels of T_L . Suppose that we are done for every $\beta < \alpha$.

If α is a limit ordinal and $t \in Lev_\alpha(T_L)$, let

$$(4.4) \quad \Psi_0(t) = \bigcup_{t' <_{T_L} t} \Psi_0(t').$$

Now let α be a successor ordinal. First notice that for every $t \in Lev_\alpha(T_L)$ by the fact that Φ is strictly decreasing and the inductive hypothesis for $t|_\alpha$ we have

$$(4.5) \quad \Phi(t) < \Phi(t|_\alpha) \leq \inf \Psi_0(t|_\alpha).$$

Let

$$A = \{t \in Lev_\alpha(T_L) : (\exists s \in Lev_\alpha(T_L))(s \neq t \wedge t|_\alpha = s|_\alpha)\}.$$

Now, if $t \notin A$ then using (4.5) there exists an $r \in [0, 1]$ such that

$$(4.6) \quad \Phi(t) < r < \inf \Psi_0(t|_\alpha).$$

So let

$$(4.7) \quad \Psi_0(t) = \Psi_0(t|_\alpha) \frown r.$$

Notice that if $t \in A$ then there exists exactly one $s \neq t$ such that $s \in Lev_\alpha(T_L)$ and $t|_\alpha = s|_\alpha$. Hence A is the union of pairs $\{s, t\}$ such that $s, t \in Lev_\alpha(T_L)$ and $t \neq s$ and $t|_\alpha = s|_\alpha$. We will define $\Psi_0(s)$ and $\Psi_0(t)$ simultaneously for such pairs. Since s and t are incomparable, the intervals N_s and N_t are disjoint, so either $N_s <_L N_t$ or $N_s >_L N_t$. Using (4.5) and $s|_\alpha = t|_\alpha$ we obtain

$$\Phi(t), \Phi(s) < \Phi(t|_\alpha) \leq \inf \Psi_0(t|_\alpha).$$

From this it follows that we can choose $r, q \in (0, 1)$ such that

$$(4.8) \quad \Phi(t), \Phi(s) < r, q < \inf \Psi_0(t|_\alpha)$$

and

$$(4.9) \quad N_s <_L N_t \iff \Psi_0(t|_\alpha) \frown q <_{attlex} \Psi_0(t|_\alpha) \frown r,$$

so let

$$(4.10) \quad \Psi_0(t) = \Psi_0(t|_\alpha) \frown r \text{ and } \Psi_0(s) = \Psi_0(t|_\alpha) \frown q = \Psi_0(s|_\alpha) \frown q.$$

Thus, we have defined Ψ_0 on $Lev_\alpha(T_L)$ (first on the complement of A then on A as well). We claim that Ψ_0 satisfies properties (1)-(3).

We check (1). Let $s <_{T_L} t$ and $t \in Lev_\alpha(T_L)$. If α is a limit ordinal then by (4.4) clearly $\Psi_0(s) \subset \Psi_0(t)$. If α is a successor then $s \leq_{T_L} t|_\alpha$, hence from the inductive hypothesis and from equations (4.6) and (4.10) we obtain (1).

In order to prove (2) let s and t be given with $N_s <_L N_t$. If $s|_\alpha = t|_\alpha$ then $s, t \in Lev_\alpha(T_L)$ and α is a successor. Then by equations (4.9) and (4.10) clearly (2) holds. If $s|_\alpha \neq t|_\alpha$ then there exists an ordinal $\beta < \alpha$, $s' \subset s$ and $t' \subset t$ such that $s', t' \in Lev_\beta(T_L)$ and $N_{s'} < N_{t'}$. Hence from the inductive hypothesis $\Psi_0(s') <_{attlex} \Psi_0(t')$ so from property (1) we have $\Psi_0(s) <_{attlex} \Psi_0(t)$.

Finally, in order to see (3) if α is a limit just notice that $\Phi(t) \leq \Phi(t')$ whenever $t' \leq_{T_L} t$ so by the inductive hypothesis we have

$$\Phi(t) \leq \inf_{t' <_{T_L} t} \Phi(t') \leq \inf_{t' <_{T_L} t} (\inf \Psi_0(t')) = \inf \Psi_0(t).$$

If α is a successor then for $t \notin A$ by (4.6) and (4.7), while for $t \in A$ by (4.8) and (4.10) we get (3).

Thus the induction works, so we have proved that such a Ψ_0 exists. \square

Now we define the embedding $L \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$. For $x \in L$ let

$$\Psi(x) = \left(\bigcup_{t \in T_L, x \in N_t} \Psi_0(t) \right) \frown 0.$$

By the definition of a partition tree, if for s and t we have $x \in N_t \cap N_s$ then s and t are \leq_{T_L} -comparable. Hence by property (1) of Ψ_0 for every $x \in L$ we have $\Psi_0(x) \in \sigma^*[0, 1]$. Moreover, by $\text{ran}(\Phi) \subset (0, 1)$ and by property (3) we have that concatenating $\bigcup_{t \in T_L, x \in N_t} \Psi_0(t)$ with zero will give an element in $[0, 1]_{\searrow 0}^{<\omega_1}$.

We claim that the map Ψ is order preserving between $(L, <_L)$ and $([0, 1]_{\searrow 0}^{<\omega_1}, <_{attlex})$. Let $x, y \in L$ with $x <_L y$. Then there exist $s, t \in T_L$ such that $x \in N_s$ and $y \in N_t$ and $N_s <_L N_t$. Then by property (2) of Ψ_0 we have $\Psi_0(s) <_{attlex} \Psi_0(t)$. Therefore, $\Psi_0(s) \subset \Psi(x)$ and $\Psi_0(t) \subset \Psi(y)$ implies $\Psi(x) <_{attlex} \Psi(y)$. \square

Theorem 4.8. (MA) *If L is a linearly ordered set of cardinality $< \mathfrak{c}$ then L is representable in \mathcal{B}_1 iff L does not contain ω_1 or ω_1^* .*

Proof. Let T_L be a partition tree of L . We claim that T_L does not contain uncountable chains. Suppose the contrary, let $\{t_\alpha : \alpha < \omega_1\} \subset T_L$ be a chain. Then N_{t_α} (denoted by N_α later on) is a strictly decreasing sequence of intervals in L . Therefore, for every α there exists an $x_\alpha \in N_\alpha \setminus N_{\alpha+1}$ such that either $N_{\alpha+1} <_L \{x_\alpha\}$ or $N_{\alpha+1} >_L \{x_\alpha\}$. Without loss of generality we can suppose that the set $R = \{\alpha : (\exists x_\alpha \in N_\alpha \setminus N_{\alpha+1})(N_{\alpha+1} <_L \{x_\alpha\})\}$ is uncountable. But then the sequence $(x_\alpha)_{\alpha \in R}$ is strictly decreasing in L and R is unbounded in ω_1 so $(x_\alpha)_{\alpha \in R}$ is order isomorphic to ω_1^* .

Notice that as every level of T_L contains pairwise disjoint nonempty intervals of L , from $|L| < \mathfrak{c}$ it follows that the cardinality of every level is strictly less than \mathfrak{c} . Moreover, since T_L does not contain uncountable chains, using that under Martin's Axiom \mathfrak{c} is a regular cardinal we obtain that $|T_L| < \mathfrak{c}$.

Now it is easy to prove the theorem using a result of Baumgartner, Malitz and Reinhardt (see [1]) which states that assuming Martin's Axiom every tree with cardinality $< \mathfrak{c}$ that does not contain ω_1 -chains is \mathbb{Q} -special. We have seen that T_L does not contain uncountable chains and $|T_L| < \mathfrak{c}$, hence it is \mathbb{Q} -special (in particular \mathbb{R} -special), so by Proposition 4.6 we have $L \hookrightarrow [0, 1]_{\searrow 0}^{< \omega_1}$. By the Main Theorem this implies $L \hookrightarrow \mathcal{B}_1$. \square

5. NEW RESULTS

5.1. Countable products and gluing. In this section we will answer Questions 2.2, 2.5 and 3.10 from [2]. Concerning the last question we would like to point out that in fact it has been already solved in [5].

Elekes [2] investigated several operations on collections of linearly ordered sets, and asked whether the closure of a simple collection of orderings under these operations coincide with the linearly ordered subsets of \mathcal{B}_1 . We will first prove that the set of linearly ordered subsets of \mathcal{B}_1 is closed under the application of these operations.

Definition 5.1. Let L be a linearly ordered set and for every $p \in L$ fix a linearly ordered set L_p . Then the set $\{(p, q) : p \in L, q \in L_p\}$ ordered lexicographically (that is, $(p, q) <_g (p', q')$ if and only if $p <_L p'$ or $p = p'$ and $q <_{L_p} q'$) is called the *gluing of the L_p 's along L* .

Theorem 5.2. (1) Let $\{L_\beta : \beta < \alpha\}$ be a countable collection of linearly ordered sets that are representable in \mathcal{B}_1 . Then the set $\prod_{\beta < \alpha} L_\beta$ ordered lexicographically is also representable.

(2) Suppose that L and every $(L_p)_{p \in L}$ is representable in \mathcal{B}_1 . Then the gluing of L_p 's along L is also representable in \mathcal{B}_1 .

NOTATION. Throughout this section if $\bar{x} = (x_\alpha)_{\alpha \leq \xi}$ is a transfinite sequence of reals and $a, b \in \mathbb{R}$ we will abbreviate the sequence $(ax_\alpha + b)_{\alpha \leq \xi}$ by $a\bar{x} + b$.

First we need a technical lemma.

Lemma 5.3. Suppose that L is a linearly ordered set and there exists an embedding $\Psi : L \hookrightarrow [0, 1]_{\searrow 0}^{< \omega_1}$. Then there exists an embedding $\Psi' : L \hookrightarrow [0, 1]_{\searrow 0}^{< \omega_1}$ such that for every $p \in L$ the length $l(\Psi'(p))$ is an even ordinal.

Proof. It is easy to see that

$$\Psi'(p) = \begin{cases} (\frac{1}{2}\Psi(p) + \frac{1}{2}) \frown 0 & \text{if } l(\Psi(p)) \text{ is odd} \\ (\frac{1}{2}\Psi(p) + \frac{1}{2}) \frown \frac{1}{4} \frown 0 & \text{if } l(\Psi(p)) \text{ is even} \end{cases}$$

is also order preserving and takes every point $p \in L$ to a sequence with even length. \square

Proof of Theorem 5.2. First we prove (1). The representability of L_β for every $\beta < \alpha$ by the Main Theorem imply that there exist embeddings $\Psi_\beta : L_\beta \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$. Using Lemma 5.3 we can suppose that for every $\beta < \alpha$ and $p \in L_\beta$ the length of $\Psi_\beta(p)$ is even.

Fix now a sequence $(y_\beta)_{\beta < \alpha} \in \sigma^*[\frac{1}{2}, 1]$. For $\bar{p} = (p_\beta)_{\beta < \alpha} \in \prod_{\beta < \alpha} L_\beta$ let

$$\Psi(\bar{p}) = (\bigwedge_{\beta < \alpha} (\frac{y_\beta - y_{\beta+1}}{2} \Psi_\beta(p_\beta) + y_{\beta+1})) \wedge 0,$$

where $\bigwedge_{\beta < \alpha}$ denotes concatenation of the sequences in type α .

We claim that Ψ is an embedding of $(\prod_{\beta < \alpha} L_\beta, <_{lex})$ into $([0, 1]_{\searrow 0}^{<\omega_1}, <_{altlex})$. It is easy to see that for every $\bar{p} \in \prod_{\beta < \alpha} L_\beta$ we have $\Psi(\bar{p}) \in [0, 1]_{\searrow 0}^{<\omega_1}$.

Now we prove that Ψ is order preserving. Let $\bar{p} <_{lex} \bar{q}$ with $\bar{p} = (p_\beta)_{\beta < \alpha}$, $\bar{q} = (q_\beta)_{\beta < \alpha}$ and let $\delta = \delta(\bar{p}, \bar{q})$, then $p_\delta <_{L_\delta} q_\delta$. It is easy to see that

$$\delta(\Psi(\bar{p}), \Psi(\bar{q})) = \sum_{\beta < \delta} l(\Psi_\beta(p_\beta)) + \delta(\Psi_\delta(p_\delta), \Psi_\delta(q_\delta)).$$

In particular, since every length in the previous equation is even we get that the $\delta(\Psi(\bar{p}), \Psi(\bar{q}))$ and $\delta(\Psi_\delta(p_\delta), \Psi_\delta(q_\delta))$ are of the same parity. Using this, $p_\delta <_{L_\delta} q_\delta$ and the fact that Ψ_δ is order preserving, we obtain that $\Psi(\bar{p}) <_{altlex} \Psi(\bar{q})$, which finishes the proof of (1).

(2) can be proved similarly. Fix an order preserving embedding $\Psi_0 : L \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$ such that for every $p \in L$ we have that $l(\Psi_0(p))$ is even. For every $p \in L$ let us also fix embeddings $\Psi_p : L_p \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$. Then

$$\Psi(p, q) = (\frac{1}{2}(\Psi_0(p)) + \frac{1}{2}) \wedge (\frac{1}{8}(\Psi_p(q)) + \frac{1}{4}) \wedge 0$$

works. □

Definition 5.4. Let L be a linearly ordered set. The set $L \times 2$ ordered lexicographically is called the *duplication of L* .

Corollary 5.5. *A linearly ordered set is representable in \mathcal{B}_1 then its duplication is also representable.*

The first part of Theorem 5.2 answers Question 2.5, while Corollary 5.5 answers Question 2.2 from [2] affirmatively.

Now let us define the above mentioned operations on collections of linearly ordered sets. Suppose that \mathcal{H} is an arbitrary set of ordered sets.

Definition 5.6. Let $\alpha < \omega_1$ be an ordinal, then

$$\mathcal{H}^\alpha = \{L_1 \subset L^\alpha : L \in \mathcal{H}\},$$

where L^α is ordered lexicographically. Let us denote by \mathcal{H}^* the closure of \mathcal{H} under the operation $\mathcal{H} \mapsto \mathcal{H}^\alpha$ for every $\alpha < \omega_1$.

Definition 5.7. $\mathcal{S}(\mathcal{H})$ denotes the closure of \mathcal{H} under gluing.

It can be shown that such \mathcal{H}^* and $\mathcal{S}(\mathcal{H})$ exist.

Suppose that every element of \mathcal{H} is representable in \mathcal{B}_1 . The first part of Theorem 5.2 clearly implies that every element of \mathcal{H}^* , while the second part yields that every element of $\mathcal{S}(\mathcal{H})$ is representable in \mathcal{B}_1 . So it is natural to ask the following:

Question 5.8. (Elekes, [2, Question 3.10.]) *Does $\mathcal{S}(\{[0, 1]^\alpha : \alpha < \omega_1\})^\omega$ or $\mathcal{S}(\{[0, 1]^\alpha : \alpha < \omega_1\})^*$ equal to the linearly ordered sets representable in \mathcal{B}_1 ?*

To answer this question we need a property that is invariant under the above operations.

Definition 5.9. We will say that a linearly ordered set L has property $(*)$ if every uncountable subset of L contains an uncountable subset order-isomorphic to a subset of \mathbb{R} .

Proposition 5.10. *Suppose that every $L \in \mathcal{H}$ has property $(*)$. Then $(*)$ holds for every element of \mathcal{H}^* and $\mathcal{S}(\mathcal{H})$ as well.*

Proof. In order to prove that every element of \mathcal{H}^* has the required property it is enough to prove that if $\alpha < \omega_1$ and L has property $(*)$ then so does L^α .

We prove this by induction on α . Suppose that we are done for every $\beta < \alpha$ and let $L_1 \subset L^\alpha$ be uncountable.

Observe that if there exists an ordinal $\beta < \alpha$ such that $L_2 = \{\bar{p} \in L^\beta : (\exists \bar{q})(\bar{p} \wedge \bar{q} \in L_1)\}$ is uncountable then using that $L_2 \subset L^\beta$ and the inductive hypothesis we obtain that L_2 contains an uncountable real order type R_2 . Thus, there exists an $R_1 \subset L_1$ such that for every $\bar{p} \in R_2$ there exists a unique \bar{q} such that $\bar{p} \wedge \bar{q} \in R_1$. It is easy to see that since L^α is ordered lexicographically we have that R_1 is an uncountable real order type in L_1 (in fact it is isomorphic to R_2).

So we can suppose that there is no such a β .

If α is a successor then using the above observation for $\beta = \alpha - 1$ we obtain that the set $\{\bar{p} \in L^{\alpha-1} : (\exists q \in L)(\bar{p} \wedge q \in L_1)\}$ is countable. By the uncountability of L_1 there exists a $\bar{p} \in L^{\alpha-1}$ such that the set $\{q : \bar{p} \wedge q \in L_1\}$ is uncountable. But this is a subset of L , so by the assumption on L there exists an uncountable real order type $R \subset \{q : \bar{p} \wedge q \in L_1\}$. Then $\{\bar{p} \wedge q : q \in R\}$ is an uncountable real order type in L_1 .

Suppose now that α is a limit ordinal. By the above observation for every $\beta < \alpha$ the set $\{\bar{p} \in L^\beta : (\exists \bar{q})(\bar{p} \wedge \bar{q} \in L_1)\}$ is countable. So there exist countable sets $D_\beta \subset L_1$ with the following property: whenever for a point $\bar{p} \in L^\beta$ there exists a \bar{q} such that $\bar{p} \wedge \bar{q} \in L_1$ then there exists a \bar{q}' such that $\bar{p} \wedge \bar{q}' \in D_\beta$. Let $D = \bigcup_{\beta < \alpha} D_\beta$, then D is a countable set.

We claim that D is dense in L_1 (equipped with the order topology). In order to prove this let $\bar{x}, \bar{y} \in L_1$ such that $(\bar{x}, \bar{y}) \cap L_1$ is nonempty. Choose a $\bar{z} \in (\bar{x}, \bar{y}) \cap L_1$. Since α is a limit there exists a $\beta < \alpha$ such that $\beta > \max\{\delta(\bar{x}, \bar{z}), \delta(\bar{y}, \bar{z})\}$. Then there exists a $\bar{w} \in D_\beta \subset D$ such that $\bar{w}|_\beta = \bar{z}|_\beta$. But then clearly $\bar{w} \in (\bar{x}, \bar{y}) \cap L_1 \cap D$. So D is indeed dense. Consequently, L_1 contains an uncountable real order type (see [13, 3.2. Corollary]). This proves that L^α has property $(*)$, so it is true for every element of \mathcal{H}^* .

In order to prove that every element of $\mathcal{S}(\mathcal{H})$ has property $(*)$ one can use similar ideas: just use the above observation and the same argument as in the case of successor α . \square

Now we are ready to answer Question 5.8. An *Aronszajn line* is an uncountable linearly ordered set that does not contain ω_1 , ω_1^* and uncountable sets isomorphic to a subset of \mathbb{R} . An Aronszajn line is called *special* if it has an \mathbb{R} -special partition tree. Special Aronszajn lines exist, see [13, Theorem 5.1, 5.2]. Notice that Proposition 4.6 immediately gives the following important corollary:

Corollary 5.11. *If A is a special Aronszajn line then $A \leftrightarrow \mathcal{B}_1$.*

This corollary was proved by Elekes and Steprāns. Although it is not mentioned explicitly in the Elekes-Steprāns paper, the embeddability of the Aronszajn line answers the questions of Elekes negatively: on the one hand an Aronszajn line does not contain uncountable real order types. On the other hand by Proposition 5.10 every element of

every collection of linear orderings obtainable from $\{[0, 1]\}$ by the operations $\mathcal{H} \mapsto \mathcal{H}^*$ or $\mathcal{H} \mapsto \mathcal{S}(\mathcal{H})$ has property (*).

5.2. Completion. Now we will answer Question 2.7 from [2] negatively.

Theorem 5.12. *There exists a linearly ordered set such that it is representable in \mathcal{B}_1 , but none of its completions are representable.*

Proof. Let $L \supset [0, 1]_{\searrow 0}^{<\omega_1}$ be a completion of $[0, 1]_{\searrow 0}^{<\omega_1}$, that is, a complete linear order containing $[0, 1]_{\searrow 0}^{<\omega_1}$ as a dense subset. If it was representable then by Corollary 5.5 there would be an order preserving embedding $\Psi : L \times 2 \hookrightarrow [0, 1]_{\searrow 0}^{<\omega_1}$. We will denote the lexicographical ordering on $L \times 2$ by $<_{L \times 2}$ and somewhat ambiguously the lexicographical ordering on $[0, 1]_{\searrow 0}^{<\omega_1} \times 2$ by $<_{altlex \times 2}$. Notice that $<_{altlex \times 2}$ is the restriction of $<_{L \times 2}$ to $[0, 1]_{\searrow 0}^{<\omega_1} \times 2$.

NOTATION. For each $s \in \sigma^*[0, 1]$ let J_s be the basic interval in $[0, 1]_{\searrow 0}^{<\omega_1} \times 2$ assigned to s , that is, the set $\{\bar{x} \in [0, 1]_{\searrow 0}^{<\omega_1} : s \subset \bar{x}\} \times 2$. We will use the notation

$$(5.1) \quad I(s) = \Psi(\inf(J_s)) \text{ and } S(s) = \Psi(\sup(J_s)).$$

Notice that if L is complete then the set $L \times 2$ ordered lexicographically is also a complete linearly ordered set, hence $I(s)$ and $S(s)$ exist for every $s \in \sigma^*[0, 1]$.

Let us define a map $\Phi : \sigma^*[0, 1] \rightarrow [0, 1]$ as follows:

Definition 5.13. For $s \in \sigma^*[0, 1]$ let

$$\delta_s = \delta(I(s), S(s))$$

and

$$\Phi(s) = \max\{I(s)(\delta_s), S(s)(\delta_s)\}.$$

Let us also use the notation

$$\phi(s) = \min\{I(s)(\delta_s), S(s)(\delta_s)\}.$$

Notice that Φ and ϕ are well defined, since for every $s \in \sigma^*[0, 1]$ the interval J_s contains at least two elements (one with last element 0 and another with 1), so $I(s)$ and $S(s)$ must differ. From this we have for all s that

$$(5.2) \quad 0 \leq \phi(s) < \Phi(s).$$

In the following lemma we collect the easy observations that will be needed in the proof of the theorem.

Lemma 5.14. *Let $s, t, u \in \sigma^*[0, 1]$ with $s \subset t$. Then*

- (1) $\delta_s \leq \delta_t$,
- (2) (a) $\Phi(s) \geq \Phi(t)$,
(b) $\max\{I(t)(\delta_s), S(t)(\delta_s)\} \leq \Phi(s)$,
- (3) if $\delta \leq \delta_t$ then $\Phi(t) \leq \max\{I(t)(\delta), S(t)(\delta)\}$,
- (4) if $\Phi(s) = \Phi(t)$ then $\delta_s = \delta_t$,
- (5) if $r, q \in [0, 1]$ such that $t \hat{\ } r \leq_{altlex} t \hat{\ } q$ then
 - (a) $I(t \hat{\ } r)|_{\delta_t} = S(t \hat{\ } r)|_{\delta_t} = I(t \hat{\ } q)|_{\delta_t} = S(t \hat{\ } q)|_{\delta_t}$,
 - (b) $I(t \hat{\ } r)(\delta_t) \leq S(t \hat{\ } r)(\delta_t) \leq I(t \hat{\ } q)(\delta_t) \leq S(t \hat{\ } q)(\delta_t)$ if δ_t is even,
 - (c) $I(t \hat{\ } r)(\delta_t) \geq S(t \hat{\ } r)(\delta_t) \geq I(t \hat{\ } q)(\delta_t) \geq S(t \hat{\ } q)(\delta_t)$ if δ_t is odd,
- (6) if $t \leq_{altlex} u$ and δ is an even ordinal such that $I(t)|_{\delta} = S(t)|_{\delta} = I(u)|_{\delta}$ then

$$I(t)(\delta) \leq S(t)(\delta) \leq I(u)(\delta).$$

Proof. $J_s \supset J_t$, so by the fact that Ψ is order preserving we get

$$I(s) \leq_{altlex} I(t) \leq_{altlex} S(t) \leq_{altlex} S(s).$$

Therefore, by the definition of $<_{altlex}$ it is clear that $\delta_s \leq \delta_t$, so we have (1).

Now we show part (b) of (2). It is easy to see from the definition of $<_{altlex}$ that for every $\bar{x} \in [\inf(J_s), \sup(J_s)]$ we have $\Psi(\bar{x})(\delta_s) \in [\phi(s), \Phi(s)]$. In particular, as $[\inf(J_t), \sup(J_t)] \subset [\inf(J_s), \sup(J_s)]$ we obtain

$$(5.3) \quad \max\{I(t)(\delta_s), S(t)(\delta_s)\} \in [\phi(s), \Phi(s)],$$

which gives part (b). Since $I(t)$ and $S(t)$ are strictly decreasing sequences, using (1) we have

$$I(t)(\delta_t) \leq I(t)(\delta_s) \text{ and } S(t)(\delta_t) \leq S(t)(\delta_s).$$

Hence, (5.3) yields that $\Phi(t) \leq \Phi(s)$. Thus we have verified (2).

In order to see (3), use again that the sequences $I(t)$ and $S(t)$ are decreasing. Hence from $\delta \leq \delta_t$ and the definition of δ_t we have (3):

$$\Phi(t) = \max\{I(t)(\delta_t), S(t)(\delta_t)\} \leq \max\{I(t)(\delta), S(t)(\delta)\}.$$

In order to prove (4) using (1) it is enough to show that $\delta_s < \delta_t$ implies $\Phi(t) < \Phi(s)$. If $\delta_s < \delta_t$ then by the definition of δ_t , the fact that the sequences $I(t)$ and $S(t)$ are strictly decreasing and (5.3), we obtain

$$\Phi(t) = \max\{I(t)(\delta_t), S(t)(\delta_t)\} < \max\{I(t)(\delta_s), S(t)(\delta_s)\} \leq \Phi(s),$$

which proves (4).

Now we prove (5). Notice that $t \frown r \leq_{altlex} t \frown q$ implies that $J_{t \frown r} \leq_{altlex \times 2} J_{t \frown q}$. Thus,

$$\inf(J_{t \frown r}) \leq_{L \times 2} \sup(J_{t \frown r}) \leq_{L \times 2} \inf(J_{t \frown q}) \leq_{L \times 2} \sup(J_{t \frown q}).$$

Consequently, by the fact that Ψ is order preserving, we get

$$(5.4) \quad I(t \frown r) \leq_{altlex} S(t \frown r) \leq_{altlex} I(t \frown q) \leq_{altlex} S(t \frown q).$$

From $J_{t \frown r}, J_{t \frown q} \subset J_t$ it is clear that

$$\begin{aligned} I(t) &\leq_{altlex} I(t \frown r) \leq_{altlex} S(t \frown r) \\ &\leq_{altlex} I(t \frown q) \leq_{altlex} S(t \frown q) \leq_{altlex} S(t). \end{aligned}$$

Thus, from the definition of δ_t we have

$$I(t)|_{\delta_t} = I(t \frown r)|_{\delta_t} = S(t \frown r)|_{\delta_t} = I(t \frown q)|_{\delta_t} = S(t \frown q)|_{\delta_t} = S(t)|_{\delta_t},$$

so this shows that (a) holds. Now using (a), the definition of $<_{altlex}$ and (5.4) we obtain (b) and (c) of (5) as well.

The proof of (6) is similar to the previous argument: $t \leq_{altlex} u$ implies $J_t \leq_{L \times 2} J_u$, consequently $I(t) \leq_{altlex} S(t) \leq_{altlex} I(u)$. Since by assumption δ is even and $I(t)|_{\delta} = S(t)|_{\delta} = I(u)|_{\delta}$, the definition of $<_{altlex}$ implies

$$I(t)(\delta) \leq S(t)(\delta) \leq I(u)(\delta).$$

□

The following lemma is the essence of our proof.

Lemma 5.15. *There exists a $\not\leq$ -increasing sequence $\{s_\alpha\}_{\alpha < \omega_1}$ such that $s_\alpha \in \sigma^*[0, 1]$, $l(s_\alpha) = \alpha$ and*

$$(*) \quad (\forall r \in s_\alpha)(\Phi(s_\alpha) < r).$$

Proof. We define s_α by induction on α .

Suppose that we have defined s_β for $\beta < \alpha$. Then by the inductive hypothesis for every $\beta < \alpha$ we have

$$(5.5) \quad (\forall r \in s_\beta)(\Phi(s_\beta) < r).$$

Now we define s_α for limit and successor α 's separately.

α IS A LIMIT. Let $s_\alpha = \bigcup_{\beta < \alpha} s_\beta$. If $r \in s_\alpha$ is arbitrary then $r \in s_\beta$ for some $\beta < \alpha$. Notice that part (a) of (2) of Lemma 5.14 and (5.5) imply

$$(s_\beta \subset s_\alpha \text{ and } r \in s_\beta) \Rightarrow \Phi(s_\alpha) \leq \Phi(s_\beta) < r.$$

Hence, using $s_\beta \subset s_\alpha$ we obtain $\Phi(s_\alpha) < r$ so s_α satisfies requirement (*).

α IS A SUCCESSOR. Let $\alpha = \beta + 1$.

Our aim is to find a real x such that

$$(5.6) \quad s_\beta \wedge x \in \sigma^*[0, 1] \text{ and } \Phi(s_\beta \wedge x) < x.$$

Clearly, this ensures that $s_\alpha = s_\beta \wedge x$ satisfies (*).

Now notice that (5.5) yields

$$(5.7) \quad s_\beta \wedge \Phi(s_\beta) \in \sigma^*[0, 1].$$

Now we have to separate two cases.

First, suppose that

$$\Phi(s_\beta \wedge \Phi(s_\beta)) < \Phi(s_\beta).$$

Let $x = \Phi(s_\beta)$. It is clear that x satisfies (5.6) by induction, so $s_\alpha = s_\beta \wedge x$ is a suitable choice for (*).

Second, suppose that $\Phi(s_\beta \wedge \Phi(s_\beta)) \geq \Phi(s_\beta)$. Since $s_\beta \subset s_\beta \wedge \Phi(s_\beta)$, by part (a) of (2) of Lemma 5.14 we have $\Phi(s_\beta \wedge \Phi(s_\beta)) \leq \Phi(s_\beta)$, so in fact

$$(5.8) \quad \Phi(s_\beta \wedge \Phi(s_\beta)) = \Phi(s_\beta).$$

Moreover, by (4) of Lemma 5.14 we obtain that (5.8) implies

$$(5.9) \quad \delta_{s_\beta \wedge \Phi(s_\beta)} = \delta_{s_\beta}.$$

In order to find an x that satisfies (5.6) we will distinguish 3 cases according to the parity of β and δ_{s_β} .

Case 1. β and δ_{s_β} have the same parity.

By (5.2) we can choose an

$$(5.10) \quad x \in (\phi(s_\beta \wedge \Phi(s_\beta)), \Phi(s_\beta \wedge \Phi(s_\beta))) = (\phi(s_\beta \wedge \Phi(s_\beta)), \Phi(s_\beta))$$

where the equality holds because of (5.8).

We claim that x has property (5.6). Clearly, $x < \Phi(s_\beta)$ and therefore by (5.5) we have $s_\beta \wedge x \in \sigma^*[0, 1]$, hence the first part of (5.6) holds. Now we can use (5) of Lemma 5.14 (part (b) with $t = s_\beta$, $r = x$, $q = \Phi(s_\beta)$ if δ_{s_β} and β are even and part (c) with $t = s_\beta$, $r = \Phi(s_\beta)$, $q = x$ if they are odd) and we obtain

$$(5.11) \quad \max\{I(s_\beta \wedge x)(\delta_{s_\beta}), S(s_\beta \wedge x)(\delta_{s_\beta})\} \leq \\ \min\{I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}), S(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta})\} = \phi(s_\beta \wedge \Phi(s_\beta)) < x,$$

where the equality follows from the definition of ϕ and (5.9) and the last inequality follows from (5.10).

By (1) of Lemma 5.14 we have $\delta_{s_\beta} \leq \delta_{s_\beta \wedge x}$ and (3) of Lemma 5.14 implies

$$\Phi(s_\beta \wedge x) \leq \max\{I(s_\beta \wedge x)(\delta_{s_\beta}), S(s_\beta \wedge x)(\delta_{s_\beta})\}.$$

Combining this inequality with (5.11) we obtain that the second part of (5.6) holds for x . So $s_\alpha = s_\beta \wedge x$ satisfies (*), hence we are done with the first case.

Case 2. β is even and δ_{s_β} is odd.

Then clearly, by (5.8), (5.9) and the odd parity of δ_{s_β}

$$\begin{aligned} \Phi(s_\beta) &= \Phi(s_\beta \wedge \Phi(s_\beta)) = \\ &= \max\{I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta \wedge \Phi(s_\beta)}), S(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta \wedge \Phi(s_\beta)})\} = \\ &= \max\{I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}), S(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta})\} = I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}). \end{aligned}$$

Thus,

$$(5.12) \quad \Phi(s_\beta) = I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}).$$

Let $z < \Phi(s_\beta)$ be arbitrary. Clearly, by the parity of β we get $s_\beta \wedge z <_{\text{altlex}} s_\beta \wedge \Phi(s_\beta)$.

Hence, using part (c) of (5) of Lemma 5.14 with $t = s_\beta$, $r = z$ and $q = \Phi(s_\beta)$ we obtain

$$(5.13) \quad I(s_\beta \wedge z)(\delta_{s_\beta}) \geq S(s_\beta \wedge z)(\delta_{s_\beta}) \geq I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}) \geq S(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}).$$

Now, part (b) of (2) of Lemma 5.14 applied to s_β and $s_\beta \wedge z$ yields

$$(5.14) \quad \max\{I(s_\beta \wedge z)(\delta_{s_\beta}), S(s_\beta \wedge z)(\delta_{s_\beta})\} \leq \Phi(s_\beta).$$

Comparing this inequality with (5.13) and (5.12) we have

$$(5.15) \quad I(s_\beta \wedge z)(\delta_{s_\beta}) = S(s_\beta \wedge z)(\delta_{s_\beta}) = I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}).$$

Therefore, as by (1) of Lemma 5.14 $\delta_{s_\beta \wedge z} \geq \delta_{s_\beta}$, we obtain that

$$(5.16) \quad \text{for every } z < \Phi(s_\beta) \text{ we have } \delta_{s_\beta \wedge z} \geq \delta_{s_\beta} + 1.$$

Notice that (a) of (5) of Lemma 5.14 applied to $s_\beta \wedge z$ and $s_\beta \wedge \Phi(s_\beta)$ and (5.9) imply that

$$(5.17) \quad I(s_\beta \wedge z)|_{\delta_{s_\beta}} = S(s_\beta \wedge z)|_{\delta_{s_\beta}} = I(s_\beta \wedge \Phi(s_\beta))|_{\delta_{s_\beta}} = S(s_\beta \wedge \Phi(s_\beta))|_{\delta_{s_\beta}}.$$

Now the even parity of $\delta_{s_\beta} + 1$, $s_\beta \wedge z <_{\text{altlex}} s_\beta \wedge \Phi(s_\beta)$, (5.15) and (5.17) show that (6) of Lemma 5.14 can be applied for $t = s_\beta \wedge z$ and $u = s_\beta \wedge \Phi(s_\beta)$ and $\delta = \delta_{s_\beta} + 1$. This yields for every $z < \Phi(s_\beta)$ that

$$(5.18) \quad \begin{aligned} &\max\{I(s_\beta \wedge z)(\delta_{s_\beta} + 1), S(s_\beta \wedge z)(\delta_{s_\beta} + 1)\} \leq \\ &\leq I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta} + 1) < I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}) = \Phi(s_\beta), \end{aligned}$$

where the last inequality follows from the fact that $I(s_\beta \wedge \Phi(s_\beta))$ is strictly decreasing and the equality comes from (5.12).

So by equations (5.16), (5.18) and (3) of Lemma 5.14 for an $x \in (I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta} + 1), \Phi(s_\beta))$ we obtain

$$\begin{aligned} \Phi(s_\beta \wedge x) &\leq \max\{I(s_\beta \wedge x)(\delta_{s_\beta} + 1), S(s_\beta \wedge x)(\delta_{s_\beta} + 1)\} \\ &\leq I(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta} + 1) < x. \end{aligned}$$

Thus, the second part of (5.6) holds for x . The first part is clear from $x < \Phi(s_\beta)$ and (5.5), hence $s_\alpha = s_\beta \wedge x$ is an appropriate choice for (*).

Case 3. β is odd and δ_{s_β} is even.

Then s_β has a least element $\min s_\beta$, and by induction and (5.8) $\min s_\beta > \Phi(s_\beta) = \Phi(s_\beta \wedge \Phi(s_\beta))$. Now let $x \in (\Phi(s_\beta), \min s_\beta)$. Then we have $s_\beta \wedge x \in \sigma^*[0, 1]$, so the first part of (5.6) holds. Since β is odd, we have $s_\beta \wedge x <_{\text{altlex}} s_\beta \wedge \Phi(s_\beta)$. Therefore, from the fact that δ_{s_β} is even using part (b) of (5) of Lemma 5.14 it follows that

$$(5.19) \quad \begin{aligned} I(s_\beta \wedge x)(\delta_{s_\beta}) &\leq S(s_\beta \wedge x)(\delta_{s_\beta}) \leq S(s_\beta \wedge \Phi(s_\beta))(\delta_{s_\beta}) \\ &\leq \Phi(s_\beta \wedge \Phi(s_\beta)) = \Phi(s_\beta) < x, \end{aligned}$$

where the last \leq uses (5.9) while the equality comes from (5.8). Hence, using (1) of Lemma 5.14 we get $\delta_{s_\beta \wedge x} \geq \delta_{s_\beta}$, so by (3) of Lemma 5.14 and (5.19) we obtain

$$\Phi(s_\beta \wedge x) \leq \max\{I(s_\beta \wedge x)(\delta_{s_\beta}), S(s_\beta \wedge x)(\delta_{s_\beta})\} < x,$$

thus, again x satisfies the second part of (5.6) so $s_\alpha = s_\beta \wedge x$ is a good choice for (*).

Thus, in any case we can carry out the induction. \square

In order to prove the theorem just notice that Lemma 5.15 gives an ω_1 -long $\overset{\subsetneq}{\neq}$ -increasing sequence of elements in $\sigma^*[0, 1]$. But then $\bigcup_{\alpha < \omega_1} s_\alpha$ would be an ω_1 -long decreasing sequence of reals, which is a contradiction. Therefore no completion of $([0, 1]_{\searrow 0}^{< \omega_1}, <_{altlex})$ can be embedded into itself and this finishes the proof of the theorem. \square

Remark 5.16. Let C be the following set:

$$\{\bar{x} \wedge x_\xi \wedge 0 : \bar{x} \in \sigma^*[0, 1], \xi \text{ is even, } l(\bar{x}) = \xi + 1, x_\xi \neq 0\}.$$

The ordering $<_{altlex}$ extends to the set $C \cup [0, 1]_{\searrow 0}^{< \omega_1}$ naturally and it is not hard to show that this ordering is complete. By Theorem 5.12 this is not representable in \mathcal{B}_1 . However, one can show that this ordering does not contain ω_1 , ω_1^* and Suslin lines. Thus, we obtain another proof of [5, Theorem 4.1].

6. PROOF OF PROPOSITION 3.5

Proposition 3.5. ([8]) Let X be a Polish space and $f \in b\mathcal{B}_1^+(X)$. Then $\Phi(f)$ is defined, $\Phi(f) \in \sigma^*bUSC^+$ and we have

- (1) $f = \sum_{\beta < \alpha}^* (-1)^\beta f_\beta + (-1)^\alpha g_\alpha$ for every $\alpha \leq \xi$,
- (2) $f_\xi \equiv 0$,
- (3) $f = \sum_{\alpha < \xi}^* (-1)^\alpha f_\alpha$.

Proof. First we show that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^*bUSC^+$. In order to prove this, we will show the following lemma.

Lemma 6.1. *The functions g_α and f_α (assigned to f in Definition 3.4) are bounded nonnegative and the sequence (f_α) is decreasing.*

Proof. It follows trivially from the definition of the upper regularization that if g is an arbitrary function then

$$(6.1) \quad g \text{ is bounded} \Rightarrow \widehat{g} \text{ exists, bounded and } \widehat{g} \geq_p g.$$

Now we prove the statement of the lemma by induction on α . If $\alpha = 0$ then $g_0 = f$ and $f_0 = \widehat{f}$, hence from $f \in b\mathcal{B}_1^+(X)$ and (6.1) clearly follows that g_0 and f_0 are bounded nonnegative functions.

If α is a successor then by definition $g_\alpha = \widehat{g_{\alpha-1}} - g_{\alpha-1}$ so by the second part of (6.1) we have $g_\alpha \geq_p 0$. Moreover, since $g_{\alpha-1}$ is bounded $\widehat{g_{\alpha-1}}$ is also bounded. Thus, g_α is the difference of two bounded functions, therefore it is also bounded. Therefore, by (6.1) f_α exists (notice that we have defined the upper regularization only for bounded functions) and also bounded and nonnegative.

Now we show that the sequence (f_α) is decreasing. By the nonnegativity of $g_{\alpha-1}$ we have $f_{\alpha-1} - g_{\alpha-1} \leq_p f_{\alpha-1}$, so

$$f_\alpha = f_{\alpha-1} - \widehat{g_{\alpha-1}} \leq_p \widehat{f_{\alpha-1}} = f_{\alpha-1}.$$

For limit α we have

$$(6.2) \quad g_\alpha = \inf\{g_\beta : \beta < \alpha \text{ and } \beta \text{ is even}\},$$

so clearly $g_\alpha \geq_p 0$ and g_α is bounded. Hence using again (6.1) we obtain that f_α is bounded and nonnegative.

Now for every β we have $g_\beta \leq_p f_\beta$. Therefore, if β is an even ordinal and $\beta < \alpha$ then by (6.2) we have

$$g_\alpha \leq_p g_\beta \leq_p f_\beta,$$

so $f_\alpha = \widehat{g_\alpha} \leq_p \widehat{f_\beta} = f_\beta$. But if β is odd, then $\beta + 1$ is even and $\beta + 1 < \alpha$. Using (6.2) we obtain $g_\alpha \leq_p g_{\beta+1}$ hence by the definition of f_α and $f_{\beta+1}$ and the inductive hypothesis we have $f_\alpha \leq_p f_{\beta+1} \leq_p f_\beta$. This finishes the proof of the lemma. \square

Clearly, by the definition of upper regularization, the functions f_α are upper semicontinuous. Therefore, by Lemma 6.1 we obtain that (f_α) is a decreasing sequence of nonnegative USC functions, so it must stabilize for some countable ordinal ξ ([10] or Lemma 3.7). Therefore, for every function in $f \in b\mathcal{B}_1^+(X)$ we have that $\Phi(f)$ is defined and $\Phi(f) \in \sigma^*bUSC^+(X)$.

Now we need the following lemma.

Lemma 6.2. *Let $(f_\alpha)_{\alpha < \xi} \in \sigma^*USC^+$. Then $\sum_{\alpha < \xi}^* (-1)^\alpha f_\alpha$ is a Baire class 1 function.*

Proof. We prove the lemma by induction on ξ .

First, if ξ is a successor just use that Baire class 1 functions are closed under addition and subtraction.

Second, if ξ is a limit, by definition of the alternating sums we have that

$$\sum_{\alpha < \xi}^* (-1)^\alpha f_\alpha = \sup\{\sum_{\beta < \alpha}^* (-1)^\beta f_\beta : \alpha < \xi, \alpha \text{ even}\}.$$

For even $\alpha < \xi$ we have

$$(*) \quad \sum_{\beta < \alpha}^* (-1)^\beta f_\beta = \sum_{\beta < \alpha+1}^* (-1)^\beta f_\beta - f_\alpha.$$

Again, for even α

$$\sum_{\beta < \alpha}^* (-1)^\beta f_\beta + f_\alpha - f_{\alpha+1} = \sum_{\beta < \alpha+2}^* (-1)^\beta f_\beta$$

so since the sequence $(f_\alpha)_{\alpha < \xi}$ is decreasing the sequence $(\sum_{\beta < \alpha}^* (-1)^\beta f_\beta)_{\alpha \text{ even}}$ is increasing. Similarly, the sequence $(\sum_{\beta < \alpha+1}^* (-1)^\beta f_\beta)_{\alpha \text{ even}}$ is decreasing. Notice that if $(r_\beta)_{\beta < \alpha}$ and $(t_\beta)_{\beta < \alpha}$ are decreasing transfinite sequences of nonnegative reals such that $r_\beta - t_\beta$ is increasing, then

$$\sup\{r_\beta - t_\beta : \beta < \alpha\} = \inf\{r_\beta : \beta < \alpha\} - \inf\{t_\beta : \beta < \alpha\}.$$

Therefore, applying (*) and these facts we have

$$\begin{aligned} & \sup\{\sum_{\beta < \alpha}^* (-1)^\beta f_\beta : \alpha < \xi \text{ even}\} = \\ & \inf\{\sum_{\beta < \alpha+1}^* (-1)^\beta f_\beta : \alpha < \xi \text{ even}\} - \inf\{f_\alpha : \alpha < \xi \text{ even}\}. \end{aligned}$$

The infimum of USC functions is also USC, hence the right-hand side of the equation is the difference of the infimum of a countable family of Baire class 1 functions and a USC function. Therefore, $\sup\{\sum_{\beta < \alpha}^* (-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is the infimum of a countable family of Baire class 1 functions. Moreover, by the inductive hypothesis, this function is also the supremum of a countable family of Baire class 1 functions. Now, using the fact that a function is Baire class 1 if and only if the preimage of every open set is $\Sigma_2^0(X)$ it is easy to see that if a function h is the infimum of a countable family of Baire class 1 functions then for every $a \in \mathbb{R}$ we have that $h^{-1}((-\infty, a))$ is in $\Sigma_2^0(X)$. Similarly, if h is

the supremum of a countable family of Baire class 1 functions then the sets $h^{-1}((a, \infty))$ are also in $\Sigma_2^0(X)$. But this implies that a function that is both an infimum and a supremum of countable families of Baire class 1 functions is also Baire class 1.

So, as an infimum and supremum of countable families of Baire class 1 functions, the function $\sup\{\sum_{\beta < \alpha}^* (-1)^\beta f_\beta : \alpha < \xi \text{ even}\}$ is also a Baire class 1 function, which completes the inductive proof. \square

Now we prove (1) of the Proposition by induction on α .

For $\alpha = 0$ this is clear. If α is a successor, then $g_{\alpha-1} = f_{\alpha-1} - g_\alpha$, so

$$\begin{aligned} f &= \sum_{\beta < \alpha-1}^* (-1)^\beta f_\beta + (-1)^{\alpha-1} g_{\alpha-1} = \\ &= \sum_{\beta < \alpha-1}^* (-1)^\beta f_\beta + (-1)^{\alpha-1} (f_{\alpha-1} - g_\alpha) = \sum_{\beta < \alpha}^* (-1)^\beta f_\beta + (-1)^\alpha g_\alpha. \end{aligned}$$

For limit α notice that we have by induction for every even $\beta < \alpha$

$$f = \sum_{\gamma < \beta}^* (-1)^\gamma f_\gamma + g_\beta.$$

Then, using that the sequence $(f_\beta)_{\beta < \alpha}$ is decreasing, the sequence $(\sum_{\gamma < \beta}^* (-1)^\gamma f_\gamma)_{\beta \text{ even}}$ is increasing, so $(g_\beta)_{\beta \text{ even}}$ is decreasing as their sum is constant f .

Notice that if $(r_\beta)_{\beta < \alpha}$ is an increasing and $(t_\beta)_{\beta < \alpha}$ is a decreasing transfinite sequence of nonnegative reals such that $r_\beta + t_\beta = c$ is constant, then

$$c = \sup\{r_\beta + t_\beta : \beta < \alpha\} = \sup\{r_\beta : \beta < \alpha\} + \inf\{t_\beta : \beta < \alpha\}.$$

So

$$\begin{aligned} f &= \sup_{\beta \text{ even}, \beta < \alpha} \left(\sum_{\gamma < \beta}^* (-1)^\gamma f_\gamma + g_\beta \right) = \\ &= \sup_{\beta \text{ even}, \beta < \alpha} \sum_{\gamma < \beta}^* (-1)^\gamma f_\gamma + \inf_{\beta \text{ even}, \beta < \alpha} g_\beta = \sum_{\beta < \alpha}^* (-1)^\beta f_\beta + g_\alpha, \end{aligned}$$

where the last equality follows from the definition of $\sum_{\beta < \alpha}^* (-1)^\beta f_\beta$ and g_α .

This proves the induction hypothesis, so we have (1).

After rearranging the equality in (1) we have that

$$(-1)^{\alpha+1} g_\alpha = \sum_{\beta < \alpha}^* (-1)^\beta f_\beta - f.$$

By Lemma 6.2 we have that the sum on the right-hand side of the equation is a Baire class 1 function, therefore g_α is also Baire class 1. We have that $f_{\xi+1} \equiv f_\xi$, so by Definition 3.4 we have $\widehat{g_\xi} - g_\xi = \widehat{g_\xi}$. Hence in order to prove (2) it is enough to show the following claim.

Claim. *If g is a nonnegative, bounded Baire class 1 function such that $\widehat{g} = \widehat{g} - g$ then $g \equiv 0$.*

Proof of the Claim. Suppose the contrary. Then there exists an $\varepsilon > 0$ such that $\{x : g(x) > \varepsilon\} \neq \emptyset$. Let $K = \overline{\{x : g(x) > \varepsilon\}}$. Since g is a Baire class 1 function we have that there exists an open set V such that

$$\varepsilon > \text{osc}(g, K \cap V) \quad (= \sup_{x, y \in K \cap V} |g(x) - g(y)|)$$

and $K \cap V$ is not empty (see [7, 24.15]).

The function $\limsup_{y \rightarrow x} g(y)$ (here in the \limsup we do not exclude those sequences which contain x) is USC. Therefore, by definition $\widehat{g} \leq_p \limsup g$. Hence letting $h = \widehat{g} - g$ we have that

$$(6.3) \quad h \leq_p \limsup(g) - g.$$

Now, we claim that

$$(6.4) \quad (\limsup(g) - g)|_{V \cap K} \leq \varepsilon.$$

Suppose the contrary. Then there exists an $x \in V \cap K$ such that $(\limsup_{y \rightarrow x} g(x)) - g(x) > \varepsilon$. Consequently, there exists a sequence $y_n \rightarrow x$, such that $\lim_{n \rightarrow \infty} g(y_n) > g(x) + \varepsilon$. Using the nonnegativity of g and the fact that $g|_{K^c} \leq \varepsilon$ we get that $y_n \in K \cap V$ except for finitely many n 's. But then $\text{osc}(g, K \cap V) > \varepsilon$, a contradiction. So we have (6.4) and using (6.3) we obtain

$$(6.5) \quad h|_{V \cap K} \leq \varepsilon.$$

Observe now that if for a bounded function f and an open set U we have that $f|_U \leq \varepsilon$, then $\widehat{f}|_U \leq \varepsilon$ (clearly, if $|f| < K$ then the function $K \cdot \chi_{U^c} + \varepsilon \cdot \chi_U$ is an USC upper bound of f).

By the above observation used for g on K^c we have that $\widehat{g}|_{K^c} \leq \varepsilon$, in particular from $h = \widehat{g} - g \leq_p \widehat{g}$ we obtain that $h|_{K^c} \leq \varepsilon$. Then from (6.5) we get $h|_V \leq \varepsilon$. So finally, using the above observation for h and V we obtain $\widehat{h}|_V \leq \varepsilon$.

The set $\{x : g(x) > \varepsilon\}$ is dense in K , hence there exists an $x_0 \in V \cap \{x : g(x) > \varepsilon\}$. On the one hand $\widehat{g}(x_0) \geq g(x_0) > \varepsilon$, on the other by $x \in V$ we get $\widehat{h}(x_0) \leq \varepsilon$. This contradicts the assumption that $\widehat{g} = \widehat{h}$. \square

So we have proved (2) of Proposition 3.5.

(3) easily follows from Lemma 6.1, (1), (2) since $0 \leq g_\xi \leq f_\xi \equiv 0$. This finishes the proof of the proposition. \square

7. OPEN PROBLEMS

Probably the most natural and intriguing problem is the following. Recall that the α th level of the Baire hierarchy in a space X is denoted by $\mathcal{B}_\alpha(X)$. Unless stated otherwise, X is an uncountable Polish space.

Problem 7.1. *Let $2 \leq \alpha < \omega_1$. Characterize the order types of the linearly ordered subsets of $\mathcal{B}_\alpha(X)$. For instance, does there exist a (simple) universal linearly ordered set for $\mathcal{B}_\alpha(X)$? And how about the class of Borel measurable functions $\cup_{\alpha < \omega_1} \mathcal{B}_\alpha(X)$?*

We remark here that Komjáth [9] proved that under the Continuum Hypothesis every ordered set of cardinality at most \mathfrak{c} can be represented in $\mathcal{B}_2(X)$ (hence in $\mathcal{B}_\alpha(X)$ for any $\alpha \geq 2$ as well). Nevertheless, a ZFC result would be very interesting and in light of our solution to Laczkovich's problem now it seems conceivable that one can construct relatively simple universal linearly ordered sets in these cases as well. As a first step in this direction it would be interesting to see if the result of Kechris and Louveau can be generalized to $\mathcal{B}_\alpha(X)$. Actually, closely related results from this paper have already been generalised from the Baire class 1 case to the Baire class α case in [3].

Let $(L_n)_{n \in \omega}$ and L be linearly ordered sets. We say that L is a *blend* of $(L_n)_{n \in \omega}$ if L can be partitioned to pairwise disjoint subsets $(L'_n)_{n \in \omega}$ such that L_n is order isomorphic to L'_n for every n . Elekes [2] proved that if the duplication and completion of every representable ordering was representable then countable blends of representable orderings would also be representable. As we have seen (Theorem 5.12), the second condition of this theorem fails, hence it is quite natural to ask the following.

Problem 7.2. *Suppose that the linearly ordered sets L_n are representable in $\mathcal{B}_1(X)$ and L is a blend of $(L_n)_{n \in \omega}$. Does it follow that L is also representable in $\mathcal{B}_1(X)$?*

The authors would expect a negative answer using similar ideas and techniques as in the proof of Theorem 5.12.

Elekes and Kunen [4] investigated Problem 1.1 in general, for non-Polish X . This raises the next question:

Problem 7.3. *Let X be a topological space (e. g. a separable metric space). Characterize the order types of the linearly ordered subsets of $\mathcal{B}_1(X)$. For instance, does there exist a (simple) universal linearly ordered set for $\mathcal{B}_1(X)$?*

We believe that an affirmative answer might be useful in answering Question 7.1 using topology refinements.

The next problem concerns characterizing all the subposets of our function spaces instead of only the linearly ordered ones. For example, it is not hard to check that $\mathcal{F}(X) = \mathcal{C}([0, 1])$ contains an isomorphic copy of a poset P iff $(\mathcal{P}(\omega), \subseteq)$ does.

Problem 7.4. *Characterize, up to poset-isomorphism, the subsets of $\mathcal{B}_1(X)$. Does there exist a simple, informative universal poset? For instance, is $\Delta_2^0(X)$ or $USC_{\searrow 0}^{<\omega_1}(X)$ universal?*

Here $USC_{\searrow 0}^{<\omega_1}$ is defined analogously to $[0, 1]_{\searrow 0}^{<\omega_1}$ and is ordered by the natural modification of $<_{\text{altlex}}$. Notice that our method of proving that $(\mathcal{B}_1(X), <_p) \hookrightarrow (\Delta_2^0(X), \subseteq)$ does not give a poset isomorphism between $\mathcal{B}_1(X)$ and its image. In fact, the image is linearly ordered. Unfortunately, it can be easily seen that even the Kechris-Louveau-type embedding $\mathcal{B}_1(X) \rightarrow bUSC_{\searrow 0}^{<\omega_1}$, that is, assigning to every Baire class 1 function its canonical resolution as a sum is not a poset isomorphism.

At first sight Laczko's problem seems to be closely related to the theory of Rosenthal compacta [6].

Problem 7.5. *Explore the connection between the topic of our paper and the theory of Rosenthal compacta.*

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