On splitting infinite-fold covers

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December 27, 2007

Abstract

Let X be a set, κ be a cardinal number and let $\mathcal{H}$ be a family of subsets of X which covers each $x \in X$ at least κ times. What assumptions can ensure that $\mathcal{H}$ can be decomposed into κ many disjoint subcovers?

We examine this problem under various assumptions on the set X and on the cover $\mathcal{H}$: among other situations, we consider covers of topological spaces by closed sets, interval covers of linearly ordered sets and covers of $\mathbb{R}^n$ by polyhedra and by arbitrary convex sets. We focus on these problems mainly for infinite κ. Besides numerous positive and negative results, many questions turn out to be independent of the usual axioms of set theory.

1 Introduction

Let X be a set, κ and λ be cardinal numbers and let $\mathcal{H}$ be a family of subsets of X which covers each $x \in X$ at least κ times. What assumptions on $\mathcal{H}$ can ensure that $\mathcal{H}$ can be decomposed into λ many disjoint subcovers? That is, which κ-fold cover can be split into λ many subcovers?

Depending on personal taste, every mathematician can readily formulate the "most relevant" context for the splitting problem; therefore splitting covers has a long-standing tradition. Unarguably, the most studied version of the problem is when X is a topological space and $\mathcal{H}$ is an open cover with special combinatorial properties. We do not attempt to summarize the vast amount of results in this direction, the interested reader is referred to [17] and the references therein. Nevertheless, we note that the literature on combinatorial properties of open covers is mainly concerned with how the combinatorics of open covers is related to topological properties of the underlying space. Therefore a strong topological motivation for considering the given special classes of open covers is always present, and splitting is concerned as far as one is looking for "nice" disjoint subcovers of a "not so nice" open cover. Moreover, for most of the problems discussed in these papers the open covers are automatically countable. In the present paper we do not work on open covers, and as we will see, we treat the problem of splitting covers from a more set theoretic point of view.

Another well-understood variant of the splitting problem deals with covers of finite structures. The most interesting questions in this area ask for splitting the edge covers of (hyper)graphs, and almost optimal solutions of the relevant problems have already been found long ago. But none of the available results concern infinite graphs or infinite-fold covers. In Section 4 we give a full solution to the splitting problem of infinite-fold edge covers of graphs; we will recall the related finite combinatorial results there.

The situation turns out to be less clear if we are interested in the splitting of finite-fold covers of infinite sets, even in the seemingly simple case of covers of the plane by such familiar objects as circles, triangles or rectangles. To start with positive results, J. Pach and G. Tóth [14] showed that for every

MSC codes: Primary 03E05, 03E15 Secondary 03C25, 03E04, 03E35, 03E40, 03E50, 03E65, 05C15, 06A05, 52A20, 52B11
Key Words: splitting infinite cover, coloring, convex set, linearly ordered set, interval, closed, compact, cardinal, Cohen model, Continuum Hypothesis, Martin's Axiom

centrally symmetric open convex polygonal region R in the plane there is a constant $c(R)$ such that every $c(R)k^2$-fold cover of the plane with translates of R can be decomposed into k disjoint covers; while G. Tardos and G. Tóth [18] obtained that any 43-fold cover of the plane with translates of an open triangle can be decomposed into two disjoint covers. In contrast with these results, J. Pach, G. Tardos and G. Tóth [13] constructed for every $1 < k < \omega$ a k-fold cover of the plane (1) by open strips, (2) by axis-parallel open rectangles, (3) by the homothets of an arbitrary open concave quadrilateral which cannot be decomposed into two disjoint covers.

However, the problem whether for a given convex subset R of the plane there is a k such that any k-fold cover of the plane with translates/homothets of R can be decomposed into two disjoint subcovers is far from being solved. E.g. we do not know the answer when R is an open or closed disk; but we remark that a positive answer may be hidden in a more than 100 page-long manuscript of J. Mani-Levitska and J. Pach. We also note that the situation in the 3-space can turn out to be completely different: in another unpublished work of J. Mani-Levitska and J. Pach, for every $k < \omega$ a k-fold cover of $\mathbb{R}^3$ with open unit balls is constructed which cannot be decomposed into two disjoint covers.

Our investigations were initiated by the question of J. Pach whether any infinite-fold cover of the plane by axis-parallel rectangles can be decomposed into two disjoint subcovers (see also [1, Concluding remarks pp. 12]). After answering this question in the negative for ω-fold covers, we started a systematic study of splitting infinite-fold covers in the spirit of J. Pach et al.; in the present paper we would like to publish our first results and state numerous open problems.

We have organized the paper to add structure as we go along. In Section 3, for any pair of cardinals κ and λ, we study the splitting of covers of κ by sets in $[\kappa]^{\leq\lambda}$. In Section 4, we discuss the splitting of edge-covers of finite or infinite graphs. In the remaining sections of the paper we study covers by convex sets. In Section 5, we show that a cover of a linearly ordered set by convex sets is "maximally" decomposable. After finishing this work it turned out that similar results were obtained much earlier by R. Aharoni, A. Hajnal, E. C. Milner [1]. Since our proofs are significantly simpler and yield slightly stronger results we decided not to leave them out.

In Section 6, as a preliminary study to covers by convex sets on the plane, we show that the splitting problem for covers by closed sets is independent of ZFC. Roughly speaking, under Martin's Axiom an indecomposable cover of $\mathbb{R}$ can be obtained even by the translates of one compact set; while in a Cohen extension of a model with GCH, every uncountable-fold cover by closed sets is "maximally" decomposable. From these results, in Section 7 we easily get that the splitting problem for covers of $\mathbb{R}^n$ by convex sets is independent of ZFC. This independence is accompanied by two ZFC results. We show that for very general classes of sets, including e.g. polyhedra, balls or arbitrary affine varieties, an uncountable-fold cover by such sets is "maximally" decomposable. On the other hand, we construct an ω-fold cover of the plane by closed axis-parallel rectangles which cannot be decomposed into two disjoint subcovers. We close the paper with a collection of open problems.

2 Terminology

In this section we fix the notation which will be used in all of the forthcoming sections. We denote by On and Card the class of ordinals and the class of cardinals, respectively. If κ is an ordinal, $\mathrm{Lim}(\kappa)$ denotes the set of limit ordinals below κ.

When we consider covers of a set X, we do not want to exclude to use a set $H \subseteq X$ multiple times. This approach is motivated both by theoretical and by practical reasons. First, the classical results for splitting finite-fold covers of finite graphs allow graphs with multiple edges, so it is reasonable to keep this generality while extending these results for infinite graphs and infinite-fold covers. Second, the natural operation of restricting a cover of X to a subspace of X can easily result in a cover where some of the covering sets are used multiple times. Moreover, this generality does not cause any additional complication. The following definition makes our notion *cover* precise.

Definition 2.1 Let X be an arbitrary set, let $\mathcal{H} \subseteq 2^X$ be an arbitrary family of subsets of X and let $m\colon \mathcal{H} \to \mathrm{On} \setminus \{0\}$ be an arbitrary function. Then the *cover of X by $\mathcal{H}$ with multiplicity m* is $\mathbf{H} = \{H \times m(H)\colon H \in \mathcal{H}\}$. For $x \in X$, let $\mathbf{H}(x) = \{\langle H, \alpha\rangle\colon x \in H \in \mathcal{H},\ \alpha < m(H)\}$. A cover is *simple* if $m(H) = 1$ $(H \in \mathcal{H})$; for simple covers we identify $\mathbf{H}$ with $\mathcal{H}$.

Let $Y \subseteq X$ and let κ be a cardinal number. Then $\mathbf{H}$ is a *κ-fold cover of Y* if $|\mathbf{H}(x)| \geq \kappa$ for every $x \in Y$; $\mathbf{H}$ is a *κ-fold cover* if it is a κ-fold cover of $\cup\mathcal{H}$.

In the sequel $\mathcal{H}$, m and $\mathbf{H}$ will always be as in Definition 2.1. To ease notation the decomposition of a cover will be realized by coloring the covering sets.

Definition 2.2 Let $\mathbf{H}$ be a cover of X, let $Y \subseteq X$ and let $\kappa \in \mathrm{Card}$. A partial function $c : \mathbf{H} \to \kappa$ is a *good κ-coloring of $\mathbf{H}$ over Y*, or simply a *good coloring over Y*, if for every $x \in Y$ and every $\alpha < \kappa$ there exists $H \in \mathcal{H}$ and $\chi < m(H)$ such that $x \in H$ and $c(\langle H, \chi\rangle) = \alpha$. Similarly, $c : \mathbf{H} \to \kappa$ is a *good κ-coloring of $\mathbf{H}$*, or simply a *good coloring*, if it is a good coloring over $\cup\mathcal{H}$.

Clearly, a cover (of Y) has a good κ-coloring (over Y) if and only if it can be partitioned into κ many subcovers (of Y).

The strongest possible decomposition result is formulated in the following terminology.

Definition 2.3 Let $\mathbf{H}$ be a cover of X and let $Y \subseteq X$. Let $\mathbf{h}\colon \mathrm{Card} \to \mathrm{On} \setminus \{0\}$ be a partial function satisfying $\mathbf{h}(\kappa) < \kappa^+$ $(\kappa \in \mathrm{dom}(\mathbf{h}))$. A partial function $c : \mathbf{H} \to \mathrm{On}$ is an *$\mathbf{h}$-maximal coloring of $\mathbf{H}$ over Y*, or simply an *$\mathbf{h}$-maximal coloring over Y*, if for every $x \in Y$,

(m1) if $\mathbf{H}(x) \neq \emptyset$ then $0 \in c(\mathbf{H}(x))$,

(m2) if $|\mathbf{H}(x)| \in \mathrm{dom}(\mathbf{h}) \setminus \omega$ then $\mathbf{h}(|\mathbf{H}(x)|) \subseteq c(\mathbf{H}(x))$.

Similarly, $c : \mathbf{H} \to \mathrm{On}$ is an *$\mathbf{h}$-maximal coloring of $\mathbf{H}$*, or simply an *$\mathbf{h}$-maximal coloring*, if it is an $\mathbf{h}$-maximal coloring over $\cup\mathcal{H}$.

In particular, if λ is an infinite cardinal then $c : \mathbf{H} \to \mathrm{On}$ is a *λ-maximal coloring of $\mathbf{H}$ over Y* if for every $x \in Y$ we have (m1) and

(m2') $|\mathbf{H}(x)| \geq \lambda$ implies $|\mathbf{H}(x)| \subseteq c(\mathbf{H}(x))$.

The notions *λ-maximal coloring of $\mathbf{H}$* and *λ-maximal coloring* are defined accordingly. For $\lambda = \omega$ we simply write *maximal coloring* instead of ω-maximal coloring.

In the remaining part of this section we prove the following reduction theorems.

Proposition 2.4 *Let X, Y, $\mathbf{H}$ and $\mathbf{h}$ be as in Definition 2.3 and suppose for each $\mu \in \mathrm{Card}$ the set $\{\nu < \mu\colon \nu < \mathbf{h}(\nu)\}$ is not stationary in μ. Set $\mathbf{i}\colon \mathrm{dom}(\mathbf{h}) \to \mathrm{On}$, $\mathbf{i}(\kappa) = \kappa$ $(\kappa \in \mathrm{dom}(\mathbf{h}))$. If there exists an $\mathbf{i}$-maximal coloring of $\mathbf{H}$ over Y then there exists an $\mathbf{h}$-maximal coloring of $\mathbf{H}$ over Y, as well.*

Proposition 2.5 *Let X be a set, let $Y \subseteq X$ and let $\mathcal{H}$, m, $\mathbf{H}$ and $\mathbf{h}$ be as in Definition 2.1. If there is an $\mathbf{h}$-maximal coloring of $\mathcal{H}$ over Y then there is an $\mathbf{h}$-maximal coloring of $\mathbf{H}$ over Y, as well.*

Proposition 2.4 reduces the quest for $\mathbf{h}$-maximal colorings to the special $\mathbf{h} = \mathbf{i}$ case, while Proposition 2.5 allows us to consider only simple covers. We will frequently use these reduction steps in inductive proofs: they allow us to use the inductive assumption for multicovers or for $\mathbf{h} \neq \mathbf{i}$ while we prove our statement in the special case of simple covers or $\mathbf{h} = \mathbf{i}$. We will always state explicitly when these reductions are used.

Before proving the two propositions, we need a lemma in advance.

Lemma 2.6 *Let λ be an arbitrary cardinal, let $\mathbf{h}\colon \mathrm{Card} \cap \lambda^+ \to \mathrm{On}$ satisfy $\kappa \leq \mathbf{h}(\kappa) < \kappa^+$ for every $\kappa \in \mathrm{Card} \cap \lambda^+$ and suppose for each $\mu \in \mathrm{Card}$ the set $\{\nu < \mu \colon \nu < \mathbf{h}(\nu)\}$ is not stationary in μ. Then there exists a mapping $\chi\colon \lambda \to \mathrm{On}$ such that for every $\kappa \in \mathrm{Card} \cap [\omega, \lambda]$, $[0, \mathbf{h}(\kappa)) \subseteq \chi([0, \kappa))$.*

Proof. We prove the statement by induction on λ. For $\lambda = \omega$ a bijection $\chi\colon \omega \to \mathbf{h}(\omega)$ does the job.

Let now $\lambda > \omega$ and suppose that the statement holds for every cardinal $\kappa < \lambda$. If λ is a successor, say$\lambda = \kappa^+$, set $\mathbf{h}_\kappa = \mathbf{h}|_{\kappa^+}$ and let χ_κ satisfy $[0, \mathbf{h}_\kappa(\kappa')) \subseteq \chi_\kappa([0, \kappa'))$ for every $\kappa' \in \mathrm{Card} \cap [\omega_1, \kappa]$. Define $\chi\colon \lambda \to \mathrm{On}$ by $\chi|_\kappa = \chi_\kappa$ and $\chi|_{[\kappa,\kappa^+)}\colon [\kappa, \kappa^+) \to [\kappa, \mathbf{h}(\kappa^+))$ by taking any bijection. Then $\mathbf{h}$ clearly fulfills the requirements.

If λ is a limit cardinal, take a strictly increasing continuous cofinal sequence $\langle \lambda_\alpha < \lambda \colon \alpha < \mathrm{cf}(\lambda) \rangle$ of cardinals such that $\mathbf{h}(\lambda_\alpha) = \lambda_\alpha$ for every $\alpha < \mathrm{cf}(\lambda)$. Let $\mathbf{h}(\lambda) \setminus \lambda = \bigcup\{K_\alpha \colon \alpha < \mathrm{cf}(\lambda)\}$ such that for every $\alpha \leq \alpha' < \mathrm{cf}(\lambda)$ we have $K_\alpha \subseteq K_{\alpha'}$ and $|K_\alpha| \leq \lambda_\alpha$. For every $\alpha < \mathrm{cf}(\lambda)$ define $\mathbf{h}_\alpha \colon \mathrm{Card} \cap \lambda_\alpha^+ \to \mathrm{On}$, $\mathbf{h}_\alpha = \mathbf{h}|_{\mathrm{Card} \cap \lambda_\alpha^+}$. By the inductive hypothesis, for every $\alpha < \mathrm{cf}(\lambda)$ we have $\chi_\alpha \colon \lambda_\alpha \to \mathrm{On}$ satisfying $[0, \mathbf{h}_\alpha(\kappa)) \subseteq \chi_\alpha([0, \kappa))$ ($\kappa \in \mathrm{Card} \cap [\omega_1, \lambda_\alpha]$).

For every $\alpha < \mathrm{cf}(\lambda)$ fixed we define $\chi\colon [\lambda_\alpha, \lambda_{\alpha+1}) \to \mathrm{On}$ as follows. Let $\vartheta\colon [\lambda_\alpha, \lambda_\alpha \dot{+} |K_\alpha|) \to K_\alpha$ be a bijection and let $\varepsilon\colon [\lambda_\alpha \dot{+} |K_\alpha|, \lambda_{\alpha+1}) \to \lambda_{\alpha+1}$ be the enumeration. Observe that for every $\kappa \in \mathrm{Card} \cap (\lambda_\alpha, \lambda_{\alpha+1})$ we have $\varepsilon(\kappa) = \kappa$. For $\eta \in [\lambda_\alpha, \lambda_\alpha \dot{+} |K_\alpha|)$ set $\chi(\eta) = \vartheta(\eta)$ while for $\eta \in [\lambda_\alpha \dot{+} |K_\alpha|, \lambda_{\alpha+1})$ set $\chi(\eta) = \chi_{\alpha+1}(\varepsilon(\eta))$. Then for every $\kappa \in \mathrm{Card} \cap (\lambda_\alpha, \lambda_{\alpha+1}]$ we have $\chi([0, \kappa)) \supseteq K_\alpha$ and

$$\chi([0, \kappa)) \supseteq \chi_{\alpha+1}(\varepsilon([\lambda_\alpha \dot{+} |K_\alpha|, \kappa))) \supseteq \chi_{\alpha+1}([0, \kappa)) \supseteq [0, \mathbf{h}_{\alpha+1}(\kappa)) = [0, \mathbf{h}(\kappa)).$$

For every limit ordinal $\alpha < \mathrm{cf}(\lambda)$ we have $\lambda_\alpha = \sup_{\beta < \alpha} \lambda_\beta$, hence $\lambda_\alpha \subseteq \chi([0, \lambda_\alpha])$ and we have $\chi([0, \lambda)) \supseteq [0, \mathbf{h}(\lambda))$ as well. This completes the proof. $\square$

In Lemma 2.6, the technical assumption on the set $\{\nu \in \mathrm{Card} \colon \nu < \mathbf{h}(\nu)\}$ cannot be left out since e.g. for a Mahlo cardinal λ and for the function $\mathbf{h}(\nu) = \nu \dot{+} 1$ ($\nu \in \mathrm{Card} \cap \lambda$) no function χ can be found with the properties in Lemma 2.6. This explains the assumption on $\mathbf{h}$ in Proposition 2.4.

Proof of Proposition 2.4. Let $c_{\mathbf{i}}\colon \mathbf{H} \to \mathrm{On}$ be an $\mathbf{i}$-maximal coloring of $\mathbf{H}$ over Y. Set $\lambda = \sup \mathrm{dom}(\mathbf{h})$,

$$\mathbf{h}^+(\kappa) = \begin{cases} \max\{\mathbf{h}(\kappa), \kappa\}, & \kappa \in \mathrm{dom}(\mathbf{h}), \\ \kappa, & \kappa \in \lambda^+ \setminus \mathrm{dom}(\mathbf{h}); \end{cases}$$

and let $\chi\colon \lambda \to \mathrm{On}$ be the function of Lemma 2.6 for $\mathbf{h}^+$. Define $c\colon \mathbf{H} \to \mathrm{On}$ by $c = \chi \circ c_{\mathbf{i}}$. Then c is clearly an $\mathbf{h}$-maximal coloring of $\mathbf{H}$ over Y. $\square$

Proof of Proposition 2.5. Let $c_0\colon \mathcal{H} \to \mathrm{On}$ be a λ-maximal coloring of $\mathcal{H}$ over Y. We define

$$c(\langle H, \alpha \rangle) = \begin{cases} c_0(H) & \text{if } H \in \mathcal{H},\ \alpha = 0, \\ \alpha & \text{if } 0 < \alpha < m(H) \end{cases}$$

Then c is clearly a λ-maximal coloring of $\mathbf{H}$ over Y.

$\square$

3 Arbitrary sets

In this section we briefly recall some easy coloring results which hold for covers by arbitrary sets. We admit the following notation.

Definition 3.1 Let $\mu, \nu, \kappa, \lambda$ and ρ be cardinals. We write $\mathbb{G}_\rho(\nu, \lambda, \mu, \kappa)$ if

$$\text{for every } X \text{ and for every } \mathbf{H} \subseteq \left[X\right]^{\le\lambda},\ \text{if } |X| \le \nu,\ |\mathbf{H}| \le \mu \text{ and } \mathbf{H} \text{ is a } \kappa\text{-fold simple cover of } X \text{ then there is a good } \rho\text{-coloring } c\colon \mathbf{H} \to \rho.$$

Let us recall that Axiom Stick is the statement

$$\text{there is a family } \mathcal{S} \subseteq \left[\omega_1\right]^{\omega} \text{ such that } |\mathcal{S}| = \omega_1 \text{ and } \forall X \in \left[\omega_1\right]^{\omega_1}\ \exists S \in \mathcal{S}\ (S \subseteq X).$$

Theorem 3.2

1. *$\mathbb{G}_\kappa(\mu, \kappa, \mu, \kappa)$ if $\mu \ge \kappa$ are infinite cardinals.*
2. *$\neg\mathbb{G}_2(2^\mu, 2^\mu, \mu, \mu)$.*
3. *$MA_{Cohen}(\mu)$ implies $\mathbb{G}_\omega(\mu, \mu, \mu, \omega)$.*
4. *$MA_{countable}(\mu)$ implies $\mathbb{G}_\omega(\mu, \mu, \omega, \omega)$.*
5. *Axiom Stick implies $\neg\mathbb{G}_2(\omega_1, \omega_1, \omega_1, \omega)$.*
6. *It is consistent that 2^ω is arbitrarily large, $\neg\mathbb{G}_2(\omega_1, \omega_1, \omega_1, \omega)$ but $\mathbb{G}_\omega(\mu, \mu, \omega, \omega)$ for every $\mu < 2^\omega$.*

Proof. 1. Let first $\mu = \kappa$. Let $\varphi\colon X \times \kappa \to \kappa$ be a bijection. By induction, for every $\alpha < \kappa$ we define partial colorings $c_\alpha\colon \mathbf{H} \to \kappa$ satisfying $|c_\alpha| < \kappa$, as follows. If $\alpha < \kappa$ and c_β is defined for $\beta < \alpha$, let $x \in X$, $\chi < \kappa$ satisfy $\varphi(x, \chi) = \alpha$ and pick a $H \in \mathbf{H} \setminus \bigcup_{\beta<\alpha} \operatorname{dom}(c_\beta)$ satisfying $x \in H$ and set $c_\alpha(H) = \chi$. This completes the α^{th} step of the construction. Then $c = \bigcup_{\alpha<\kappa} c_\alpha$ is a good κ-coloring of $\mathbf{H}$.

Suppose now $\mu > \kappa$. Let $\equiv$ be the equivalence relation on $\mathbf{H}$ generated by the relation $\{(H, H')\colon H, H' \in \mathbf{H},\ H \cap H' \ne \emptyset\}$. Since $\mathbf{H} \subseteq [X]^{\le\kappa}$ and $\mathbf{H}$ is a κ-fold simple cover, the equivalence classes $\{\mathbf{H}_i\colon i \in I\}$ of $\equiv$ have cardinalities κ. Set $X_i = \cup\mathbf{H}_i$ $(i \in I)$. Then $|X_i| \le |\mathbf{H}_i| = \kappa$ and $\mathbf{H}_i$ is a κ-fold cover of X_i. So by Proposition 2.5 and the induction hypothesis, there is a good κ-coloring $c_i\colon \mathbf{H}_i \to \kappa$ over X_i $(i \in I)$. Then $c = \bigcup_{i\in I} c_i$ is a good κ-coloring over X.

2. Set $X = 2^\mu$, and for every $\alpha < \mu$ let $H_\alpha = \{x \in 2^\mu\colon x(\alpha) = 1\}$. Then $\mathbf{H} = \{H_\alpha\colon \alpha < \mu\}$ witnesses $\neg\mathbb{G}_2(2^\mu, 2^\mu, \mu, \mu)$.

3. By applying $MA_{\text{cohen}}(\mu)$ to the poset $\{(\mathcal{K}, c)\colon \mathcal{K} \in [\mathbf{H}]^{<\omega},\ c\colon \mathcal{K} \to \omega\}$ with partial order $(\mathcal{K}', c') \le (\mathcal{K}, c)$ if and only if $\mathcal{K} \subseteq \mathcal{K}'$ and $c \subseteq c'$, the statement follows. The argument for 4. is similar.

5. Let $\mathcal{S} \subseteq [\omega_1]^\omega$ be a stick sequence. For every $\alpha < \omega_1$ let $H_\alpha = \{S \in \mathcal{S}\colon \alpha \in S\}$. Then $\mathbf{H} = \{H_\alpha\colon \alpha < \omega_1\}$ is a cover of $\mathcal{S}$ witnessing $\neg\mathbb{G}_2(\omega_1, \omega_1, \omega_1, \omega)$.

6. In [5] it was proved that it is consistent that 2^ω is arbitrarily large, Axiom Stick holds and $MA_{countable}$ also holds. $\square$

Theorem 3.2 is to be compared with the results of Section 6 on splitting closed covers.

4 Graphs

Now we investigate the interesting special case of graphs, that is, when each covering set has 2 elements. In this section "graph" means an undirected possibly infinite graph where multiple edges are allowed, but we exclude loops. We follow the standard notation, i.e. $G = (V, E)$ denotes the graph with vertex set V and edge set E. According to our convention, for $v, w \in V$, the set of edges containing v is denoted by $E(v)$ and $E(v, w)$ stands for the edges connecting v to w. For every $v \in V$, $d_G(v)$ stands for the degree of v in G, i.e. $d_G(v) = |E(x)|$ where multiple edges are counted with multiplicity. Set $\Delta(G) = \sup\{d_G(v) \colon v \in V\}$; the supremum of edge multiplicities is denoted by $\mu(G)$. For every $E' \subseteq E$, $V[E']$ is the set of vertices of the edge set E'.

As we mentioned in the introduction, the splitting problem for finite graphs is much studied (see e.g. [15, Chapter 28]) and the following result, originally due to R. P. Gupta [6, Theorem 2.2 pp. 500], solves our problem for finite graphs.

Theorem 4.1 *Let $1 \leq n < \omega$. Let $G = (V, E)$ be a finite graph and let $X \subseteq V$ be such that for every $x \in X$ we have $d_G(x) \geq n + \mu(G)$. Then E has a good n-coloring over X.*

The main result of this section is the extension of Theorem 4.1 for infinite graphs such that, in addition, for the existence of good 2-colorings a necessary and sufficient condition is given. First we show that even for simple graphs, in order to ensure the existence of a good n-coloring, the condition on the degree of vertices cannot be weakened to $d_G(x) \geq n$. We will use the following constructions proposed by Gyula Pap.

For every $n < \omega$ let K_n denote the complete graph on the vertex set $\{v_0, v_1, \ldots, v_{n-1}\}$. For odd n let K_n^- denote the graph obtained from K_n by *deleting* the edges $\{v_0, v_{n-1}\}$ and $\{v_{2k}, v_{2k+1}\}$ $(k < (n-1)/2)$. Take two disjoint copies of K_{n+2}^-, say on the vertex sets $\{v_0, v_1, \ldots, v_{n+1}\}$ and $\{v'_0, v'_1, \ldots, v'_{n+1}\}$ and let D_n denote the graph obtained as the union of the two copies of K_{n+2}^- and the edge $\{v_0, v'_0\}$.

Proposition 4.2 *Let $n < \omega$.*

1. *If n is even, K_{n+1} is an n-regular graph with no good n-coloring.*

2. *If n is odd, D_n an n-regular graph with no good n-coloring.*

Proof. We prove the statements simultaneously. It is obvious that K_{n+1} and D_n are n-regular graphs. Hence a good n-coloring of any of these graphs is a partition of their edge set into n disjoint complete matchings.

Now for even n, K_{n+1} has no complete matchings since its vertex set has odd cardinality. Also for cardinality reasons, if n is odd any complete matching of D_n must contain the edge $\{v_0, v'_0\}$. Hence D_n has no two disjoint complete matchings. □

We also note that Theorem 3.2.1 completely solves the splitting problem for infinite-fold edge-covers.

Theorem 4.3 *Let $G = (V, E)$ be a graph, let $X \subseteq V$ be arbitrary and suppose that for every $x \in X$ we have $d_G(x) \geq \omega$. Then E has a good ω-coloring over X.*

Proof. The statement follows from Theorem 3.2.1 as $|e| = 2 \leq \omega$ for every $e \in E$. □

From now on we work for n-colorings with $n < \omega$. The case $n = 1$ is trivial, so we start with $n = 2$. We have the following characterization for the existence of good 2-colorings.

Theorem 4.4 *Let $G = (V, E)$ be a graph, let $X \subseteq V$ be arbitrary and suppose that for every $x \in X$ we have $d_G(x) \geq 2$. Then E has a good 2-coloring over X if and only if no connected component of G is an odd cycle with vertex set contained in X.*

Proof. It is easy to see that the condition is necessary, as an odd cycle has no good 2-coloring.

To prove sufficiency, observe that we can assume G is connected since it is sufficient to color the connected components separately. So let $G = (V, E)$ be a connected graph and let $X \subseteq V$ such that

(i) for every $x \in X$ we have $d_G(x) \geq 2$

(ii) if G is an odd cycle then $V \neq X$.

Lemma 4.5 *There exists a nonempty (not necessarily spanned) subgraph $G' = (V', E')$ of G such that E' has a good 2-coloring over $V' \cap X$.*

Proof. If $X \neq V$ then $G' = (V \setminus X, \emptyset)$ works, so we can assume $X = V$. Depending on the subgraphs of G we distinguish several cases.

Case I: G contains an even cycle, or a path which is infinite in both directions. Since even cycles and paths infinite in both directions have good 2-colorings we can choose G' to be one of these subgraphs. Note that a pair of multiple edges is an even cycle.

From now on we assume that G contains no such subgraphs. Pick an arbitrary vertex $v \in V$ and start a path from v until it first fails to be vertex-disjoint.

Case II: we get an infinite path in this direction. Let us start another path from v until it is disjoint from itself and from the previous infinite path. As we have no doubly infinite paths, the second path has to terminate, say at $w \in V$. We obtained a cycle on w and an infinite path starting from w. Let us call such a configuration an *infinite lasso*. It is easy to check that we get a good 2-coloring of our infinite lasso if we color its edges by the alternating coloring in such a way that we start the coloring at w and first we color the edges of the cycle on w. Thus in this case we can set G' to be an infinite lasso.

Case III: our (first) path from v reaches a point visited before. Since G contains no even cycles we get an odd cycle C. By (ii) and by $X = V$, no component of G is an odd cycle so there is a vertex w of C with $d_G(w) \geq 3$. Let us start a path from w disjoint from C. If we get an infinite lasso then we are done by Case II. Otherwise the path reaches either a vertex of C or a vertex of the path itself. If it reaches C then it has to reach it at w: else G would contain an even cycle since for two odd cycles intersecting each other in a finite path, removing the intersection results an even cycle.

Hence we obtain two disjoint cycles connected by a path, possibly of length 0. Let G' be this graph. As for the infinite lasso, color the edges of this graph the alternating way, starting from w and coloring first a cycle containing w. It is easy to check that this is a good 2-coloring of G'. □

Now we go back to the proof of Theorem 4.4. For an ordinal ξ to be specified later, we define a sequence of partial colorings $c_\alpha \colon E \to 2$ $(\alpha < \xi)$ such that

(i) $\mathrm{dom}(c_\alpha) \subsetneq \mathrm{dom}(c_{\alpha'})$ and $c_{\alpha'}|_{\mathrm{dom}(c_\alpha)} = c_\alpha$ $(\alpha < \alpha' < \xi)$,

(ii) $e \in \mathrm{dom}(c_\alpha)$ and $v \in e \cap X \Rightarrow c_\alpha(E(v)) = 2$ $(\alpha < \xi)$,

(iii) $X \subseteq V\left[\bigcup_{\alpha<\xi} \mathrm{dom}(c_\alpha)\right]$.

Once this done the function $c \colon E \to 2$, $c = \bigcup_{\alpha<\xi} c_\alpha$ is a 2-coloring of E by (i) which is good over X by (ii) and (iii).

To start the construction, by the previous lemma there exists a subgraph G' of G which has a good 2-coloring over $G' \cap X$. By adding $V \setminus X$ to V', we can assume $V \setminus X \subseteq V'$ and $E \cap [V \setminus X]^2$ is colored. Let c_0 be this partial coloring. Let α be an ordinal and suppose that c_β is defined for every $\beta < \alpha$. If $X \subseteq V\left[\bigcup_{\beta<\alpha} \mathrm{dom}(c_\alpha)\right]$ set $\xi = \alpha$ and the construction is done. Else set $c_\alpha^- = \bigcup_{\beta<\alpha} c_\beta$. We have $X \setminus V[\mathrm{dom}(c_\alpha^-)] \neq \emptyset$. As G is connected, there exists an edge $\{u,v\} \in E \setminus \mathrm{dom}(c_\alpha^-)$ such that $u \in V[\mathrm{dom}(c_\alpha^-)]$ and $v \in X \setminus V[\mathrm{dom}(c_\alpha^-)]$. Start a path P from u whose first edge is $\{u,v\}$ until it is vertex-disjoint and disjoint from $V[\mathrm{dom}(c_\alpha^-)] \setminus \{u\}$. This P can be infinite, or it can end either in $V[\mathrm{dom}(c_\alpha^-)]$ or in a point of P. Let c_α be the partial coloring extending c_α^- where we color the edges of P the alternating way starting with $\{u,v\}$. It is easy to see that c_α satisfies (i) and (ii).

Since $\mathrm{dom}(c_\alpha)$ ($\alpha < \xi$) are strictly increasing this transfinite procedure terminates at some ordinal ξ. The resulting sequence $(c_\alpha)_{\alpha<\xi}$ satisfies (i-iii), so the proof is complete. □

Clearly, an n-regular graph has a good n-coloring if and only if its edge chromatic number is n. It is a well-known theorem of Vizing that the edge chromatic number of a simple finite graph is either $\Delta(G)$ or $\Delta(G)+1$ (see e.g. [15, Theorem 28.2 pp. 467]). But to decide e.g. whether a 3-regular graph is 3-chromatic or not is an NP-complete problem (see e.g. [15, Theorem 28.3 pp. 468]). Hence we cannot hope for a very simple analogue of Theorem 4.4 for $n \geq 3$.

It remains to extend the Theorem of R. P. Gupta to infinite graphs.

Theorem 4.6 *Let $1 \leq n < \omega$. Let $G = (V,E)$ be a graph and let $X \subseteq V$ be such that for every $x \in X$ we have $d_G(x) \geq n + \mu(G)$. Then E has a good n-coloring over X.*

Proof. For finite graphs this is Theorem 4.1 due to Gupta. If G is locally finite, i.e. all degrees are finite, then an easy compactness argument yields the result. If $\mu(G) \geq \omega$ we are done by Theorem 4.3.

If G is arbitrary with $\mu(G) < \omega$ we construct a locally finite graph $G' = (V', E')$ as follows. First we obtain V' by adding a vertex $v(0)$ for every $v \in V$ with $d_G(v) < \omega$, and by replacing every $v \in V$ with $d_G(v) \geq \omega$ by $d_G(v)$ many distinct vertices $(v(\alpha))_{\alpha<d_G(v)}$. Then for every $v \in V$ with $d_G(v) \geq \omega$, using $d_G(v) = d_G(v) \times (n + \mu(G))$, we "distribute evenly" the edges $E(v)$ onto $(v(\alpha))_{\alpha<d_G(v)}$: that is E' is constructed in such a way that

(i) for every $v, w \in V$, $|E(v,w)| = \sum_{\alpha,\beta} |E'(v(\alpha), w(\beta))|$,

(ii) for $v \in V$ with $d_G(v) < \omega$, $d_{G'}(v(0)) = d_G(v)$,

(iii) for $v \in V$ with $d_G(v) \geq \omega$, $d_{G'}(v(\alpha)) = n + \mu(G)$.

This is clearly possible. The resulting graph G' is locally finite and each of its vertices has degree at least $n + \mu(G)$. Hence there is a good n-coloring of G'. By merging, for every $v \in V$, the vertices $v(\alpha)$ to one vertex we get a graph isomorphic to G, thus the good n-coloring of G' yields a good n-coloring of G. □

5 Intervals in linearly ordered sets

Let $\mathcal{L} = (L, \leq)$ be a linearly ordered set and let $\mathrm{conv}(\mathcal{L})$ denote the family of convex subsets of L. In this section we prove the following two results. The first establishes maximal coloring for convex covers.

Theorem 5.1 *Let $(L, \leq)$ be an ordered set and let* $\mathbf{H}$ *be a cover of L with $\mathcal{H} \subseteq \mathrm{conv}(L)$. Then* $\mathbf{H}$ *has a maximal coloring.*

The second gives the splitting of k-covers.

Theorem 5.2 *Let $(L, \leq)$ be an ordered set and, for some (finite or infinite) cardinal k, let* $\mathbf{H}$ *be a k-fold cover of L with $\mathcal{H} \subseteq \mathrm{conv}(L)$. Then* $\mathbf{H}$ *has a good k-coloring.*

As we noted in the introduction, Theorem 5.2 was obtained in [1] much earlier than our investigations. But [1] remained so unnoticed that it was one of its authors, namely A. Hajnal, who asked us whether Theorem 5.2 holds in the special $L = \mathbb{R}$ and $k = 2$ case. In spite of [1], we decided to treat the splitting of convex covers of linearly ordered sets in the present paper because Theorem 5.1 is new, and we found a significantly simpler proof of Theorem 5.2 than the one in [1].

The heart of the proof of Theorem 5.1 is the following general statement on maximal colorings.

Theorem 5.3 *Let X be a set and let $\mathbf{H}$ be a simple cover of X. If for each $\mathbf{K} \subseteq \mathbf{H}$ there is $\mathbf{J} \subseteq \mathbf{K}$ such that*

(e1) $\cup\mathbf{J} = \cup\mathbf{K}$,

(e2) $\mathbf{J}$ *has a maximal coloring,*

then $\mathbf{H}$ *has a maximal coloring.*

Proof. Let $\{H_\alpha\colon \alpha < |\mathbf{H}|\}$ be an enumeration of $\mathbf{H}$. By transfinite recursion on $\alpha < |\mathbf{H}|$ we define families $\mathbf{J}_\alpha \subseteq \mathbf{H}$ satisfying $H_\alpha \in \bigcup_{\beta\le\alpha} \mathbf{J}_\beta$, and maximal colorings $c_\alpha : \mathbf{J}_\alpha \to \mathrm{On}$, as follows. Let $\alpha < |\mathbf{H}|$ be arbitrary and suppose $\mathbf{J}_\nu$ and c_ν are constructed for $\nu < \alpha$. Set $\mathbf{K}_\alpha = \mathbf{H} \setminus \cup\{\mathbf{J}_\nu : \nu < \alpha\}$. Since $\mathbf{K}_\alpha \subseteq \mathbf{H}$, by assumption we have a family $\mathbf{J}_\alpha \subseteq \mathbf{K}_\alpha$ with $\cup\mathbf{J}_\alpha = \cup\mathbf{K}_\alpha$ and a maximal coloring c_α of $\mathbf{J}_\alpha$. If $H_\alpha \notin \bigcup_{\beta\le\alpha} \mathbf{J}_\beta$, we put H_α into $\mathbf{J}_\alpha$ and we set $c_\alpha(H_\alpha) = 0$. So we can assume $H_\alpha \in \bigcup_{\beta\le\alpha} \mathbf{J}_\beta$. This completes the α^{th} step of the construction. Then we have $\mathbf{H} = \bigcup_{\alpha<|\mathbf{H}|} \mathbf{J}_\alpha$.

Let $c : \mathbf{H} \to \mathrm{On}$, $c(H) = \alpha + c_\alpha(H)$ for $H \in \mathbf{J}_\alpha$ ($\alpha < |\mathbf{H}|$). By $\mathbf{K}_0 = \mathbf{H}$ we have $\cup\mathbf{J}_0 = \cup\mathbf{H}$ and $c|_{\mathbf{J}_0} = c_0$, so condition $(m1)$ of Definition 2.3 is satisfied.

Before turning to $(m2)$, observe that if $x \in L$ and κ is a cardinal such that $\mathbf{J}_\beta(x) \ne \emptyset$ for each $\beta < \kappa$, then $0 \in c_\beta(\mathbf{J}_\beta(x))$ and so $\beta \in c(\mathbf{J}_\beta(x))$ ($\beta < \kappa$). Hence $\kappa \subseteq c(\mathbf{H}(x))$.

To see $(m2)$, pick $x \in L$ and suppose $\kappa = |\mathbf{H}(x)| \ge \omega$. We distinguish several cases. If $\mathbf{J}_\beta(x) \ne \emptyset$ for each $\beta < \kappa$, then by the previous observation $\kappa \subseteq c(\mathbf{H}(x))$, as required.

So suppose $\mathbf{J}_\beta(x) = \emptyset$ for some $\beta < \kappa$; fix a minimal such β. Then $\mathbf{H}(x) = \bigcup\{\mathbf{J}_\alpha(x) : \alpha < \beta\}$. Thus for each cardinal $\lambda < \kappa$ there is an $\alpha(\lambda) < \beta$ such that $|\mathbf{J}_{\alpha(\lambda)}(x)| \ge \max\{\omega, \lambda^+\}$. Then $\mathbf{J}_\gamma(x) \ne \emptyset$ ($\gamma < \alpha(\lambda)$) and so $\alpha(\lambda) \subseteq c(\mathcal{H}(x))$ by our observation. Moreover $\max\{\omega, \lambda^+\} \subseteq c_{\alpha(\lambda)}(\mathbf{J}_{\alpha(\lambda)})$ and so $[\alpha(\lambda), \alpha(\lambda) + \lambda^+) \subseteq c(\mathbf{H}(x))$, as well. By putting these together we obtain $\alpha(\lambda) + \lambda^+ \subseteq c(\mathbf{H}(x))$, so since $\kappa = \sup\{\lambda^+ : \lambda < \kappa\}$ we concluded $\kappa \subseteq c(\mathbf{H}(x))$, as required. □

In the following two lemmas, for any linearly ordered set $\mathcal{L}$, we establish the maximal colorability of special subfamilies of $\mathrm{conv}(L)$. For $x \in L$ set $(-\infty, x] = \{y \in L : y \le x\}$ and $[x, +\infty) = \{y \in L : x \le y\}$. We define

$$\mathrm{tail}(\mathcal{L}) = \{I \subseteq L : [x, +\infty) \subseteq I \text{ for each } x \in I\}.$$

Clearly, $\mathrm{tail}(\mathcal{L}) \subsetneq \mathrm{conv}(\mathcal{L})$ provided $|L| \ge 2$.

Lemma 5.4 *Every simple cover* $\mathbf{H} \subseteq \mathrm{tail}(\mathcal{L})$ *has a maximal coloring.*

Proof. We intend to apply Theorem 5.3. To this end, it is enough to show that for every $\mathbf{K} \subseteq \mathrm{tail}(\mathcal{L})$ there is a $\mathbf{J} \subseteq \mathbf{K}$ satisfying $\cup\mathbf{K} = \cup\mathbf{J}$ such that $\mathbf{J}$ has a maximal coloring.

Let $\mathbf{K} \subseteq \mathrm{tail}(\mathcal{L})$ be a cover. For some regular cardinal κ there is a strictly increasing chain $\mathbf{J} = \{J_\nu : \nu < \kappa\}$ of elements of $\mathbf{K}$ such that $\cup\mathbf{J} = \cup\mathbf{K}$. Note that we may have $\kappa = 1$.

Let $f : \kappa \to \kappa$ be a κ-abundant map, i.e. for every $\lambda < \kappa$ we have $|f^{-1}(\lambda)| = \kappa$. Define $c : \mathbf{J} \to \mathrm{On}$ by $c(J_\lambda) = f(\lambda)$ ($\lambda < \kappa$). Clearly, c is a maximal coloring of $\mathbf{J}$, which completes the proof. □

Lemma 5.5 *Fix $a \in L$ and let $\mathbf{H} \subseteq \mathrm{conv}(\mathcal{L})(a)$ be a simple cover. Then $\mathbf{H}$ has a maximal coloring.*

Proof. Again, we intend to apply Theorem 5.3, thus it is enough to show that for every $\mathbf{K} \subseteq \mathrm{conv}(\mathcal{L})(a)$ there is a $\mathbf{J} \subseteq \mathbf{K}$ satisfying $\cup\mathbf{K} = \cup\mathbf{J}$ such that $\mathbf{J}$ has a maximal coloring.

So let $\mathbf{K} \subseteq \mathrm{conv}(\mathcal{L})(a)$ be a cover. By the definition of maximal coloring we can assume $\cup\mathbf{K} = L$. For some regular cardinal κ there is a family $\mathbf{J}^+ = \{J_\nu : \nu < \kappa\}$ of elements of $\mathbf{K}$ such that $\{J_\nu \cap [a, +\infty) : \nu < \kappa\}$ is strictly increasing and $\mathbf{J}^+$ covers $[a, +\infty)$. Note that we may have $\kappa = 1$. By Proposition 2.5, we can apply Lemma 5.4 for $\mathbf{J}^+$ as a cover on $(-\infty, a]$ to obtain a coloring $c : \mathbf{J}^+ \to \mathrm{On}$ such that

(1) $\cup\mathbf{J}^+ \cap (-\infty, a] \subseteq \cup c^{-1}\{0\}$.

(2) for each $x \le a$ if $|\mathbf{J}^+(x)| \ge \omega$ then $|\mathbf{J}^+(x)| \subseteq c(\mathbf{J}^+(x))$.

Let $f : \kappa \to \kappa$ be a κ-abundant map. Define $h : \kappa \to \kappa$ by

$$h(\beta) = \begin{cases} f(\xi + n) & \text{if } \beta = \xi + 2n + 1 \text{ for some } \xi \in \mathrm{Lim}(\kappa), \\ \xi + n & \text{if } \beta = \xi + 2n \text{ for some } \xi \in \mathrm{Lim}(\kappa), \end{cases}$$

and let $d^+ : \mathbf{J}^+ \to \mathrm{On}$, $d^+(J_\nu) = h(c(J_\nu))$. Then d^+ is a maximal coloring of $\mathbf{J}^+$.

If $\mathbf{J}^+$ covers L, $\mathbf{J} = \mathbf{J}^+$ satisfies the requirements. If not, take $\mathbf{K}' = \mathbf{K} \setminus \mathbf{J}^+$. Then $\mathbf{K}'$ covers $(-\infty, a]$, so by repeating the previous argument for $(-\infty, a]$ instead of $[a, +\infty)$ we can find a family $\mathbf{J}^- \subseteq \mathbf{K}'$ covering $(-\infty, a]$ with a maximal coloring d^-.

Put $\mathbf{J} = \mathbf{J}^+ \cup \mathbf{J}^-$ and $d = d^- \cup d^+$. Then $\cup\mathbf{J} = L$ and d is a maximal coloring of $\mathbf{J}$, which completes the proof. $\square$

Proof of Theorem 5.1. By Proposition 2.5 we can assume $\mathbf{H}$ is simple. By Theorem 5.3 it is enough to prove that for every cover $\mathbf{K}$ with $\mathbf{K} \subseteq \mathrm{conv}(\mathcal{L})$ there is a subfamily $\mathbf{J} \subseteq \mathbf{K}$ such that

(a) $\cup\mathbf{J} = \cup\mathbf{K}$,

(b) $\mathbf{J}$ has a maximal coloring $c_{\mathbf{J}}$.

Consider the equivalence relation R on L generated by the relation $\bigcup\{I \times I : I \in \mathbf{K}\}$. The equivalence classes of R give a partition of L and every $I \in \mathbf{K}$ is contained in some equivalence class. Hence we can construct $\mathbf{J}$ and $c_{\mathbf{J}}$ for each equivalence class separately. Therefore we can assume we have only one equivalence class. Let $z \in L$ be arbitrary.

Proposition 5.6 *If*

($\circ$) *for each* $x \in [z, +\infty)$ *there is* $y \in [z, +\infty)$ *such that* $\cup\mathbf{K}(x) \subseteq (-\infty, y]$

then there is $\mathbf{J}^+ \in \left[\mathbf{K}\right]^\omega$ *such that*

($\circ\circ$) $\mathbf{J}^+$ *covers* $[z, +\infty)$ *and* $|\mathbf{J}^+(x)| < \omega$ *for each* $x \in L$

Proof. We define recursively a partition $\{L(n) : n \in \omega\}$ of $[z, +\infty)$ by setting $L(0) = \{z\}$, and for $0 < n < \omega$,

$$L(n) = \{y \in [z, +\infty) : I \cap L(n-1) \ne \emptyset \text{ for some } I \in \mathbf{K}(y)\} \setminus \bigcup_{k<n} L(k).$$

Since L is one equivalence class of R, $L = \bigcup_{n<\omega} L(n)$, indeed. Note that some $L(n)$ can be empty, e.g. if L has a maximal element.

By ($\circ$), for each $n < \omega$ there is an $\mathbf{I}_n \in \left[\mathbf{K}\right]^{\le 2}$ such that $\mathbf{I}_n$ covers $L(n)$. Let $\mathbf{J}^+ = \cup\{\mathbf{I}_n : n < \omega\}$.

Since L is one equivalence class of R, $\mathbf{J}^+$ covers $[z, +\infty)$. To see $|\mathbf{J}^+(x)| < \omega$ ($x \in L$), observe that for each $n < \omega$ and $I \in \mathbf{K}$, $I \cap L(n) \ne \emptyset$ implies $I \cap L(n+2) = \emptyset$. Hence we have $|\{n \in \omega : I \cap L(n) \ne \emptyset\}| \le 2$, and so $|\mathbf{J}^+(x)| < \omega$, as required. $\square$

Let us return to the proof of Theorem 5.1. If ($\circ$) holds then let $z^+ = z$ and fix a family $\mathbf{J}^+ \in \left[\mathcal{I}\right]^\omega$ satisfying ($\circ\circ$). Otherwise pick $z^+ \in [z, +\infty)$ such that $\mathbf{K}(z^+)$ covers $[z^+, +\infty)$ and let $\mathbf{J}^+ = \mathbf{K}(z^+)$.

By applying Proposition 5.6 to L with reversed order, we can show that if

($\diamond$) for each $x \in (-\infty, z]$ there is $y \in (-\infty, z]$ such that $\cup\mathbf{K}(x) \subseteq [y, +\infty)$

then there is a family $\mathbf{J}^- \in \big[\mathcal{I}\big]^{\omega}$ such that

($\diamond\diamond$) $\mathbf{J}^-$ covers $(-\infty, z]$ and $|\mathbf{J}^-(x)| < \omega$ for each $x \in L$.

If ($\diamond$) holds let $z^- = z$ and fix a family $\mathbf{J}^-$ satisfying ($\diamond\diamond$). Otherwise pick $z^- \in (-\infty, z]$ such that $\mathbf{K}(z^-)$ covers $(-\infty, z^-]$ and let $\mathbf{J}^- = \mathbf{K}(z^-)$. Finally pick $\mathbf{J}^0 \in \big[\mathbf{K}\big]^{<\omega}$ which covers $[z^-, z^+]$. Let $\mathbf{J} = \mathbf{J}^- \cup \mathbf{J}^0 \cup \mathbf{J}^+$. Then $\mathbf{J}$ covers L.

The families $\mathbf{J}^+$, $\mathbf{J}^- \setminus \mathbf{J}^+$, and $\mathbf{J}^0 \setminus (\mathbf{J}^+ \cup \mathbf{J}^-)$ have maximal colorings c^+, c^- and c^0 respectively, because they are either "locally finite" or Lemma 5.5 can be applied. Thus $c_{\mathbf{J}} = c^+ \cup c^- \cup c^0$ is a maximal coloring of $\mathbf{J}$. $\square$

We close this section with the proof of Theorem 5.2.

Proof of Theorem 5.2. If k is an infinite cardinal the statement follows immediately from Theorem 5.1. So let $k < \omega$; we prove the statement by induction on k. By Proposition 2.5 we can assume $\mathbf{H}$ is simple. For $k = 1$ the statement is trivial.

Let $k \geq 2$ and suppose the theorem is true for $k - 1$. As in the proof of Theorem 5.1, consider the equivalence relation R on L generated by the relation $\bigcup\{H \times H \colon H \in \mathbf{H}\}$. The equivalence classes of R give a partition of L and every $H \in \mathbf{H}$ is contained in some equivalence class, hence we can construct the k-good coloring of $\mathbf{H}$ for each equivalence class separately. Therefore we can assume that we have only one equivalence class.

Proposition 5.7 *Let $I \subseteq L$ be a convex set and $y \in I$. If $\mathbf{H}$ has a good k-coloring over $I \cap (-\infty, y]$ and another good k-coloring over $I \cap [y, +\infty)$ then it has a good k-coloring over I, as well.*

Proof. Fix two good k-colorings $c_- \colon \mathbf{H} \to k$ and $c_+ \colon \mathbf{H} \to k$ over $I \cap (-\infty, y]$ and $I \cap [y, +\infty)$, respectively. By thinning out the domain of c_- we can assume that for each $i < k$ the family $[c_-^{-1}(i)](y)$ has an enumeration $\{J_-^i(\gamma) \colon \gamma < \kappa_i\}$ for some regular cardinal κ_i such that $\{J_-^i(\gamma) \cap (-\infty, y] : \gamma < \kappa_i\}$ is strictly increasing and so for each cofinal subset $\Gamma \subseteq \kappa_i$ the family $(c_-^{-1}(i) \setminus [c_-^{-1}(i)](y)) \cup \{J_-^i(\gamma) \colon \gamma \in \Gamma\}$ covers $I \cap (-\infty, y]$. Let us remark that κ_i can be finite, namely 1.

Similarly, we can thin out the domain of c_+ such that for each $i < k$ the family $[c_+^{-1}(i)](y)$ has an enumeration $\{J_+^i(\gamma) \colon \gamma < \lambda_i\}$ for some regular cardinal λ_i such that for each cofinal subset $\Gamma \subseteq \lambda_i$ the family $(c_+^{-1}(i) \setminus [c_+^{-1}(i)](y)) \cup \{J_+^i(\gamma) \colon \gamma \in \Gamma\}\}$ covers $I \cap [y, +\infty)$.

Then by passing to cofinal subsets of $[c_-^{-1}(i)](y)$ and $[c_+^{-1}(i)](y)$ we can assume that for each $i, j < k$ if $[c_-^{-1}(i)](y) \cap [c_+^{-1}(j)](y) \neq \emptyset$ then $\kappa_i = \lambda_j = 1$ and so $[c_-^{-1}(i)](y) = [c_+^{-1}(j)](y)$. So there is a bijection $f : k \to k$ such that if $[c_-^{-1}(i)](y) \cap [c_+^{-1}(j)](y) \neq \emptyset$ then $j = f(i)$.

Define $c \colon \mathbf{H} \to k$ by $c(H) = i$ if $c_-(H) = i$ or $c_+(H) = f(i)$ $(H \in \mathbf{H})$. The definition of c is valid and c is a good k-coloring of $\mathbf{H}$ over I. This completes the proof. $\square$

Define the relation $\equiv$ on L by $x \equiv y$ if and only if there exists a good k-coloring of $\mathbf{H}$ over $[x, y]$. By Proposition 5.7, $\equiv$ is an equivalence relation on L. Moreover, we have the following.

Proposition 5.8 *For every $H \in \mathbf{H}$, H is contained in one equivalence class of $\equiv$.*

Proof. Let $H \in \mathbf{H}$ and $\{x, y\} \in \big[H\big]^2$. Then $\mathbf{H} \setminus \{H\}$ is a $(k-1)$-fold cover of L. Hence by the inductive hypothesis, $\mathbf{H} \setminus \{H\}$ has a good $(k-1)$-coloring $c \colon \mathbf{H} \to k - 1$ over $[x, y]$. Extend c by setting $c(H) = k$; then c is a good k-coloring over $[x, y]$. $\square$

Proposition 5.9 *Let E be an equivalence class of $\equiv$. Then there is a good k-coloring of $\mathbf{H}$ over E.*

Proof. Take an arbitrary $y \in E$. Since E is convex, by Proposition 5.7 it is enough to prove that $\mathbf{H}$ has a good k-coloring over $E \cap [y, +\infty)$ and over $E \cap (-\infty, y]$. We prove only that $\mathcal{H}$ has a good k-coloring over $E \cap [y, +\infty)$, the proof of the other statement is similar. We distinguish several cases.

Suppose first that there is $H \in \mathbf{H}$ such that H is cofinal in E. Fix $z \in H \cap [y, +\infty)$; then $[z, +\infty) \subseteq [y, +\infty)$. So $\mathbf{H} \backslash \{H\}$ is a $k-1$-fold cover of $[z, +\infty)$. So by the inductive hypothesis there is a $c\colon \mathbf{H} \backslash \{H\} \to k-1$ good $k-1$ coloring of $\mathbf{H}$ over $[z, +\infty)$. Then extending c by setting $c(H) = k$ we get a good k-coloring of $\mathbf{H}$ over $[z, +\infty)$. Since $y \equiv z$, $\mathbf{H}$ has a good k-coloring over $[y, z]$, so by Proposition 5.7 we have that $\mathbf{H}$ has a good k-coloring over $E \cap [y, +\infty)$, as well.

From now on assume that there is no $H \in \mathbf{H}$ such that H is cofinal in $E \cap [y, +\infty)$. If there is $z \in [y, +\infty)$ such that $\cup \mathbf{H}(z)$ is cofinal in E, then since for $H \in \mathbf{H}$, H is not cofinal in E, $\mathbf{H}$ has a good k-coloring over $E \cap [z, +\infty)$. Since $y \equiv z$, $\mathbf{H}$ has a good k-coloring over $[y, z]$. So by Proposition 5.7, $\mathbf{H}$ has a good k-coloring over $E \cap [y, +\infty)$, as well.

In the sequel we assume in addition that for every $z \in E \cap [y, +\infty)$, $\cup \mathbf{H}(z)$ has an upper bound in E. We define recursively a strictly increasing sequence $(x_n)_{n<\omega} \subseteq E \cap [y, +\infty)$, as follows. Let $x_0 = y$. If $n < \omega$ and x_{n-1} is already defined let b_{n-1} be an upper bound of $\cup \mathbf{H}(x_{n-1})$, and let x_n be an upper bound of $\cup \mathbf{H}(b_{n-1})$. Then $\cup \mathbf{H}(x_{n-1}) \subseteq (-\infty, b_{n-1}]$ and $\cup \mathbf{H}(x_n) \subseteq (b_{n-1}, +\infty)$ imply

$$\mathbf{H}(x_n) \cap \mathbf{H}(x_{n'}) = \emptyset \ (n < n' < \omega),$$

and by our assumption that L is one equivalence class of R, $\{x_n \colon n < \omega\}$ is cofinal in E.

For every $n < \omega$ we have $x_n \equiv x_{n+1}$ so there is a $c_n \colon \mathbf{H} \to k$ good k-coloring of $\mathbf{H}$ over $[x_n, x_{n+1}]$. Fix $n < \omega$; by thinning out the domain of c_n we can assume that for each $i < k$ the family $[c_n^{-1}(i)](x_n)$ has an enumeration $\{J_n^i(\gamma) \colon \gamma < \kappa_n^i\}$ for some regular cardinal κ_n^i such that $\{J_n^i(\gamma) \cap [x_n, x_{n+1}] \colon \gamma < \kappa_n^i\}$ is strictly increasing, and so for each cofinal subset $\Gamma \subseteq \kappa_n^i$ the family $(c_n^{-1}(i) \setminus [c_n^{-1}(i)](x_n)) \cup \{J_n^i(\gamma) : \gamma \in \Gamma\}$ covers $[x_n, x_{n+1}]$. Similarly, we can assume that for each $i < k$ the family $[c_n^{-1}(i)](x_{n+1})$ has an enumeration $\{B_n^i(\gamma) \colon \gamma < \lambda_n^i\}$ for some regular cardinal λ_n^i such that $\{B_n^i(\gamma) \cap [x_n, x_{n+1}] \colon \gamma < \lambda_n^i\}$ is strictly increasing, and so for each cofinal subset $\Gamma \subseteq \lambda_n^i$ the family $(c_n^{-1}(i) \backslash [c_n^{-1}(i)](x_{n+1})) \cup \{B_n^i(\gamma) : \gamma \in \Gamma\}$ covers $[x_n, x_{n+1}]$.

Then for every $n < \omega$ and $i < k$, we can pass to cofinal subsets of $[c_n^{-1}(i)](x_{n+1})$ and $[c_{n+1}^{-1}(i)](x_{n+1})$ in such a way that for each $i, j < k$ if $[c_n^{-1}(i)](x_{n+1}) \cap [c_{n+1}^{-1}(j)](x_{n+1}) \neq \emptyset$ then $\lambda_n^i = \kappa_{n+1}^j = 1$ and so $[c_n^{-1}(i)](x_{n+1}) = [c_{n+1}^{-1}(j)](x_{n+1})$. So there is a bijection $f_n : k \to k$ such that if $[c_n^{-1}(i)](x_{n+1}) \cap [c_{n+1}^{-1}(j)](x_{n+1}) \neq \emptyset$ then $j = f_n(i)$. Write $g_0 = \mathrm{Id}$ and $g_n = f_{n-1} \circ f_{n-2} \circ \cdots \circ f_0$ $(0 < n < \omega)$.

For every $H \in \mathbf{H}$ and $i < k$ define $c(H) = i$ if and only if for some $n < \omega$, $H \in \mathrm{dom}(c_n)$ and $c_n(H) = g_n(i)$. This definition makes sense and c is a good k-coloring of $\mathbf{H}$ over $E \cap [y, +\infty)$, which completes the proof. $\square$

We are ready to complete the proof of Theorem 5.2. By assumption, L is one equivalence class of the relation R. So by Proposition 5.8, L is one equivalence class of $\equiv$. Therefore by Proposition 5.9 there is a good k-coloring of $\mathbf{H}$, which finishes the proof. $\square$

6 Closed sets

Towards the investigation of splitting of covers with special geometric properties let us tackle closed covers, i.e. that variant of the problem where the sets in the cover are closed. The study of this special case is motivated by the facts that, apart from considering open covers, this is the simplest topological constraint one can impose; even for closed covers we get independence of ZFC by very strong means; these results will be very useful for treating the problem of covers by compact convex sets.

Obviously, we have to specify the topological spaces where closed covers are considered. Observe that similarly to the proof of Theorem 6.1 below, the construction of Theorem 3.2.2 can be carried out in such a way that the covering sets H_α are closed in 2^κ endowed with the product topology. Since our purpose is not to find suitable topologies for general constructions but to establish independence of ZFC for natural topological spaces, in this section we restrict our attention to covers of $\mathbb{R}$, or equivalently to covers of ω^ω and 2^ω.

We remark that if $\mathbf{H}$ is a simple closed cover of $\mathbb{R}$ and $|\mathbf{H}| < \mathrm{cov}(\mathcal{M})$ then $\mathbf{H}$ has a countable subcover. In particular, for $\omega < \kappa < \mathrm{cov}(\mathcal{M})$, a κ-fold closed cover *of cardinality* κ has a good κ-coloring. There are models of ZFC where even Borel covers of special cardinalities of the real line satisfy a similar Lindelöf like property. In [10], A. Miller showed that in a model obtained from a model of CH by adding ω_3 many Cohen reals, every cover of $\mathbb{R}$ by ω_2 many Borel sets has an ω_1 subcover. Here the corresponding splitting result says that if $\mathcal{H}$ is an ω_2-fold Borel cover of $\mathbb{R}$ and $|\mathcal{H}| = \omega_2$ then $\mathcal{H}$ has a good ω_2-coloring. However, these are very special settings as far as splitting is concerned, so we do not pursue our investigations in this direction. For more background on covering numbers related to closed sets see [11].

In this section our main results are the following. In Theorem 6.1 we obtain that if MA_κ(σ-centered) holds there exists a κ-fold closed cover of $\mathbb{R}$, consisting of translates of *one* compact set, which cannot be partitioned into two subcovers. In particular, we obtain in ZFC that there exists an ω-fold closed cover of $\mathbb{R}$, consisting of translates of one compact set, which cannot be partitioned into two subcovers. Finally in Theorem 6.7 we establish that in the Cohen real model every closed cover of 2^ω has a maximal coloring. In this section X denotes any of $\mathbb{R}$, ω^ω or 2^ω; and 2^ω is identified with $P(\omega)$ the usual way.

6.1 Martin's Axiom

This section is devoted to the following theorem.

Theorem 6.1 *Let κ be a cardinal satisfying $\omega \le \kappa < 2^\omega$ and assume MA_κ(σ-centered). Then there exists a κ-fold closed cover of X which cannot be decomposed into two subcovers. Moreover, in $\mathbb{R}$ the cover may consist of translates of one compact set.*

Since MA_ω(σ-centered) holds in ZFC we obtain the following corollary.

Corollary 6.2 *There exists an ω-fold closed cover of X which cannot be partitioned into two subcovers. If $X = \mathbb{R}$ the cover can consist of translates of one compact set.*

We prove Theorem 6.1 first in $\mathbb{R}$ since there we need to construct the cover using translates of one compact set. We fix some notation in advance. For a set $F \subseteq \mathbb{R}$ let $\langle F \rangle_\mathbb{Q}$ denote the linear span of F in $\mathbb{R}$ considered as a vector space over the rationals $\mathbb{Q}$. We set $\Sigma = 4^\omega$.

In order to construct a cover of $\mathbb{R}$ using translates of one compact set we need a compact set F for which $\langle F \rangle_\mathbb{Q}$ is meager. We collect some basic facts about the Hausdorff dimension of Cantor-type sets (See e.g. [3] or [9] and [12, Theorem 1 pp. 141]).

Lemma 6.3 *Let $(k_i)_{i<\omega} \subseteq \omega$ satisfy $k_{i+1} - k_i > i$ $(i < \omega)$ and set $F = \left\{\sum_{i<\omega} \sigma(i)/4^{k_i} \colon \sigma \in 4^\omega\right\}$. Let $0 < n, n_i < \omega$ $(i < \omega)$.*

1. *$F^n \subseteq \mathbb{R}^n$ has zero Hausdorff dimension.*
2. *If $A \subseteq \mathbb{R} \times \mathbb{R}^{n-1}$ is compact and has zero Hausdorff dimension then the projection of A to the first coordinate is nowhere dense.*
3. *Suppse $R_i \subseteq \mathbb{R}^{n_i}$ $(i < \omega)$ are meager. Then there exists a nonempty perfect set $P \subseteq \mathbb{R}$ such that for every $i < \omega$, P is independent in R_i, i.e. $[F]^{n_i} \cap R_i = \emptyset$ $(i < \omega)$.*

Lemma 6.4 *Let κ be a cardinal, suppose that $\kappa < \mathrm{add}(\mathcal{M})$ and let $D \in [\mathbb{R}]^\kappa$ be arbitrary. Let the sequence $(k_i)_{i<\omega} \subseteq \omega$ satisfy $k_{i+1} - k_i > i$ $(i < \omega)$ and set $F = \left\{\sum_{i<\omega} \sigma(i)/4^{k_i} : \sigma \in \Sigma\right\}$. Then $\langle F \rangle_{\mathbb{Q}}$ is meager. In particular, there exist $(v_n)_{n<\omega} \subseteq [0,1]$ and a nonempty perfect set $W \subseteq [0,1]$ such that $\lim_{n\to\infty} v_n = 0$, $W \cup \{v_n : n < \omega\}$ are linearly independent over $\mathbb{Q}$ and*

$$\langle F \cup D \rangle_{\mathbb{Q}} \cap \langle v_n : n < \omega \rangle_{\mathbb{Q}} = \{0\}, \tag{1}$$

$$\langle F \cup D \cup \{v_n : n < \omega\} \rangle_{\mathbb{Q}} \cap \langle W \rangle_{\mathbb{Q}} = \{0\}. \tag{2}$$

Proof. By Lemma 6.3.1, F^n $(n < \omega)$ have zero Hausdorff dimension. For every $(q_i)_{i<n} \subseteq \mathbb{Q}$, $q_0 F + \cdots + q_{n-1} F \subseteq \mathbb{R}$ is the projection of $\sqrt{n}((q_0 F) \times \cdots \times (q_{n-1} F)) \subseteq \mathbb{R}^n$ to the one dimensional subspace $\{(t, \ldots, t) : t \in \mathbb{R}\} \subseteq \mathbb{R}^n$. So by Lemma 6.3.2, $\langle F \rangle_{\mathbb{Q}}$ is meager. Since $\kappa < \mathrm{add}(\mathcal{M})$ and $D \in [\mathbb{R}]^\kappa$, $\langle F \cup D \rangle_{\mathbb{Q}}$ is meager hence $\mathbb{R} \setminus \langle F \cup D \rangle_{\mathbb{Q}}$ has uncountable dimension over $\mathbb{Q}$. So we get our $(v_n)_{n<\omega} \subseteq \mathbb{R}^+$ e.g. by choosing suitable members of a Hamel basis of $\mathbb{R} \setminus \langle F \cup D \rangle_{\mathbb{Q}}$.

For every finite sequence $\underline{q} = (q_i)_{i<n} \subseteq \mathbb{Q} \setminus \{0\}$ define $R_{\underline{q}} \subseteq \mathbb{R}^n$ by

$$(x_0, \ldots, x_{n-1}) \in R_{\underline{q}} \Leftrightarrow \sum_{i<n} q_i x_i \in \langle F \cup D \cup \{v_n : n < \omega\} \rangle_{\mathbb{Q}}.$$

Since $\langle F \cup D \cup \{v_n : n < \omega\} \rangle_{\mathbb{Q}}$ is still meager, each $R_{\underline{q}}$ is meager. So we can find our perfect set $W \subseteq [0,1]$ by applying Lemma 6.3.3 to the set of relations $\{R_{\underline{q}} : \underline{q} \in [\mathbb{Q} \setminus \{0\}]^{<\omega}\}$. This completes the proof. $\square$

Corollary 6.5 *With the notation of Lemma 6.4, there exists an injective function $w : \mathbb{R} \times \kappa \to W$ such that for every $z \in \mathbb{R}$ and $\alpha < \kappa$,*

$$z - w(z, \alpha) \notin \langle F \cup D \cup \{v_n : n < \omega\} \cup W \rangle_{\mathbb{Q}}. \tag{3}$$

Proof. By (2) and since the elements of W are linearly independent over $\mathbb{Q}$, if for a $z \in \mathbb{R}$ there is a $w \in W$ such that

$$z - w \in \langle F \cup D \cup \{v_n : n < \omega\} \cup W \rangle_{\mathbb{Q}} \tag{4}$$

then $z \in \langle F \cup D \cup \{v_n : n < \omega\} \cup W \rangle_{\mathbb{Q}}$, and those $w \in W$ for which (4) holds are among the ones appearing in the expression of z over $F \cup D \cup \{v_n : n < \omega\} \cup W$ with rational coefficients. Thus $w(z, \alpha) \in W$ can be arbitrary with finitely many exceptions. So an easy transfinite definition yields the function w. $\square$

Once we have the compact set, its translates will be coded by the members of an almost disjoint family in $[\omega]^\omega$ of size κ. In the end we will need the following amended version of Solovay's Lemma.

Lemma 6.6 (MA_κ(*σ-centered*)) *Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family of size κ. Let $\mathcal{B} \subseteq \mathcal{A}$ and suppose that for every $A \in \mathcal{A}$ a set $C_A \in [A]^\omega$ is given. Then there exists $X \in [\omega]^\omega$ such that*

1. $\max(X \cap A) \in C_A$ *for $A \in \mathcal{B}$;*
2. $|X \cap A| = \omega$ *for $A \in \mathcal{A} \setminus \mathcal{B}$.*

Proof. Let

$$P = \{\langle x, b \rangle : x \in [\omega]^{<\omega},\, b \in [\mathcal{B}]^{<\omega},\ \max(x \cap B) \in C_B \text{ for } B \in b\},$$

and put $\langle x, b \rangle \leq_P \langle x', b' \rangle$ if and only if $x' \subseteq x$, $b' \subseteq b$ and $x \cap B' = x' \cap B'$ for each $B' \in b'$. Since the conditions $\langle x, b_0 \rangle, \langle x, b_1 \rangle, \ldots, \langle x, b_{n-1} \rangle$ have the common extension $\langle x, b_0 \cup b_1 \cup \cdots \cup b_{n-1} \rangle$, $P = \bigcup \{\{\langle x, b \rangle : b \in [\mathcal{B}]^{<\omega}\} : x \in [\omega]^{<\omega}\}$ shows that $\langle P, \leq_P \rangle$ is σ-centered.

For $B \in \mathcal{B}$ the set $D_B = \{\langle x, b\rangle : B \in b\}$ is dense in P since if $B \notin b$ then we have $n \in C_B \setminus \max(B \cap \cup b)$ and $\langle x \cup \{n\}, b \cup \{B\}\rangle \leq \langle x, b\rangle$ is in D_B.

For $A \in \mathcal{A} \setminus \mathcal{B}$ and $m < \omega$ the set $D_{A,m} = \{\langle x, b\rangle : \max(x \cap A) \geq m\}$ is dense in P since for $n \in (A \setminus m) \setminus \cup b$, $\langle x \cup \{n\}, b\rangle \leq \langle x, b\rangle$ is in $D_{A,m}$.

If G is a $\{D_B \colon B \in \mathcal{B}\} \cup \{D_{A,m} \colon A \in \mathcal{A} \setminus \mathcal{B}, m < \omega\}$-generic filter then $X = \bigcup\{x \colon \langle x, b\rangle \in G\}$ satisfies the requirements. $\square$

Proof of Theorem 6.1 Let $(k_i)_{i<\omega} \subseteq \omega$ satisfy $k_{i+1} - k_i > i$ $(i < \omega)$. Let $\mathcal{A} \subseteq [\omega]^\omega$ be an almost disjoint family of size κ and for every $A \in \mathcal{A}$ set $x(A) = \sum_{i<\omega} \chi_A(i)/4^{k_i}$. Recall $\Sigma = 4^\omega$, and for every $n < \omega$ let $\Sigma_n = \{\sigma \in \Sigma \colon n = \max\{i < \omega \colon \sigma(i) = 2\}\}$. Since $MA_\kappa(\sigma\text{-centered})$ implies $\mathrm{cov}(\mathcal{M}) > \kappa$ we can apply Lemma 6.4 to get $(v_n)_{n<\omega} \subseteq [0,1]$, $W \subseteq [0,1]$ such that with $F = \left\{\sum_{i<\omega} \sigma(i)/4^{k_i} \colon \sigma \in \Sigma\right\}$ and $D = \{x(A) \colon A \in \mathcal{A}\}$ we have (1) and (2) in Lemma 6.4.

For every $n < \omega$ let $F_n = \left\{\sum_{i<\omega} \sigma(i)/4^{k_i} \colon \sigma \in \Sigma_n\right\}$ and set

$$K = W \cup F \cup \bigcup\{F_n + v_n \colon n < \omega\}.$$

Note that

$$F \cup \bigcup\{F_n + v_n \colon n < \omega\} \subseteq [0,2],\ 0 \in K \subseteq [0,2], \tag{5}$$

and $\lim_{n\to\infty} v_n = 0$ implies $\lim_{n\to\infty} F_n + v_n = F$, hence K is a compact set. We define

$$K_{n,A} = K + x(A) - v_n \ (A \in \mathcal{A},\ n < \omega)$$

and $\mathbf{H}_0 = \{K_{n,A} \colon A \in \mathcal{A},\ n < \omega\}$. Set $Z = \{z \in \mathbb{R} \colon |\mathbf{H}_0(z)| < \kappa\}$ and let w be the function of Corollary 6.5. Set

$$\mathbf{H}_1 = \{K + z - w(z, \alpha) \colon z \in Z,\ \alpha < \kappa\}.$$

We show that $\mathbf{H} = \mathbf{H}_0 \cup \mathbf{H}_1$ is a κ-fold cover of $\mathbb{R}$ which has no two disjoint subcovers.

Pick an arbitrary $x \in \mathbb{R}$; since either $|\mathbf{H}_0(x)| = \kappa$ or $|\mathbf{H}_1(x)| \geq \kappa$ by definition, $\mathbf{H}$ is a κ-fold cover of $\mathbb{R}$. Let $c \colon \mathbf{H}_0 \to 2$. We find an $\varepsilon \in \{0,1\}$ and an $x \in \mathbb{R}$ such that $|\mathbf{H}_0(x)| = \kappa$, $\mathbf{H}_1(x) = \emptyset$ and for every $A \in \mathcal{A}$ and $n < \omega$, $x \in K_{n,A}$ implies $c(K_{n,A}) = \varepsilon$. This will complete the proof.

For each $A \in \mathcal{A}$ there exists an $\varepsilon_A \in \{0,1\}$ and a $C_A \in [\omega]^\omega$ such that $c(K_{n,A}) = \varepsilon_A$ for $n \in C_A$. Then there is $\varepsilon \in \{0,1\}$ and $\mathcal{B} \in [\mathcal{A}]^\kappa$ such that $\varepsilon_B = \varepsilon$ for $B \in \mathcal{B}$.

By applying Lemma 6.6 we obtain $X \in [\omega]^\omega$ satisfying $\max(X \cap A) \in C_A$ $(A \in \mathcal{B})$ and $|X \cap A| = \omega$ $(A \in \mathcal{A} \setminus \mathcal{B})$. Let $x = \sum_{i<\omega}(1 + 2\chi_X(i))/4^{k_i}$, i.e. $x \in F$ and x has digits 1 and 3 only. We show that this x fulfills the requirements.

First we show $|\mathbf{H}_0(x)| = \kappa$. For every $A \in \mathcal{A}$ and $i < \omega$ we have

$$[x - x(A)](j) = \begin{cases} 3, & \text{if } j = k_i \text{ with } i \in X \setminus A; \\ 2, & \text{if } j = k_i \text{ with } i \in X \cap A; \\ 0, & \text{else.} \end{cases} \tag{6}$$

Thus for each $A \in \mathcal{B}$, $x - x(A) \in F_{\max(X \cap A)}$, hence

$$x \in F_{\max(X\cap A)} + v_{\max(X\cap A)} + x(A) - v_{\max(X\cap A)} \subseteq K_{\max(X\cap A),A}$$

and so $|\mathbf{H}_0(x)| = \kappa$. In particular, $x \notin Z$; hence by $x \in F$ and (3), $x \notin K + z - w(z, \alpha)$ $(z \in Z,\ \alpha < \kappa)$ and so $\mathbf{H}_1(x) = \emptyset$.

It remains the show that for every $A \in \mathcal{A}$ and $n < \omega$, $x \in K_{n,A}$ implies $c(K_{n,A}) = \varepsilon$. Suppose $x \in K_{n,A}$ for some $A \in \mathcal{A}$ and $n < \omega$, i.e.

$$x \in K + x(A) - v_n = (W + x(A) - v_n) \cup (F + x(A) - v_n) \cup \bigcup\{F_m + v_m + x(A) - v_n \colon m < \omega\}.$$

By $x \in F$, (1) and (2),

$$x \notin W + x(A) - v_n,\ x \notin F + x(A) - v_n,\ x \notin F_m + v_m + x(A) - v_n\ (m \neq n)$$

hence $x \in F_n + x(A)$. By (6), for $A \in \mathcal{A} \setminus \mathcal{B}$ we have $x - x(A) \notin \bigcup_{n<\omega} F_n$. Thus $x \in K_{n,A}$ implies $A \in \mathcal{B}$. Again by (6) we have $n = \max(X \cap A) \in C_A$ so $c(K_{n,A}) = \varepsilon$. This completes the proof in $\mathbb{R}$.

If $X = \omega^\omega$ or $X = 2^\omega$ take a continuous surjective map $\varphi\colon X \to [-4, 4]$ and set $\mathbf{H}_X = \{\varphi^{-1}(F)\colon F \in \mathbf{H}\}$. Then $\mathbf{H}_X$ clearly a κ-fold cover of X without two disjoint subcover. $\square$

Corollary 6.2 implies in particular that in a positive partition result for closed covers the points covered only by ω many sets must be ignored.

6.2 The Cohen real model

In this section we will prove that in the Cohen real model every closed cover of the reals has an ω_1-maximal coloring. Note that by Corollary 6.2 it is impossible to get a maximal coloring. Thus we have, in a sense, a best possible decomposition result. The proof is based on the weak Freeze-Nation property (see Proposition 6.8 below), for which we need standard additional assumptions, such as GCH and $\square_\lambda$ for cardinals λ with $\mathrm{cf}(\lambda) = \omega$.

Following [8], we recall some notation. Let V be our ground model and let κ be a cardinal. We denote by V^{C_κ} the model obtained from V by adding κ many Cohen reals the usual way. We omit the definition of the $\square_\lambda$ principle: this principle appears among the assumptions of Theorem 6.7 but it is used only for Proposition 6.8, which is the main lemma for Theorem 6.7, and which is cited from [4] without proof.

We will prove the following theorem.

Theorem 6.7 *Suppose that GCH holds in V and let κ be a cardinal. Suppose also that in V we have $\square_\lambda$ for every cardinal λ satisfying $\omega < \lambda \leq |\kappa|$, $\mathrm{cf}(\lambda) = \omega$. In V^{C_κ}, let (X, τ) be an M_2 topological space and let $\mathbf{H}$ be cover of X by closed sets. Then in V^{C_κ} there exists an ω_1-maximal coloring of $\mathbf{H}$.*

The proof of Theorem 6.7 is based on the fact that in V^{C_κ} the poset $(P(\omega), \subseteq)$ has the *weak Freese-Nation property.* We recall it in the following proposition and we introduce the corresponding notion of maximal coloring on $P(\omega)$.

Proposition 6.8 *([4, Theorem 15]) Under the assumptions of Theorem 6.7, in V^{C_κ} the poset $(P(\omega), \subseteq)$ has the weak Freese-Nation property, i.e. there is a function $f\colon P(\omega) \to [P(\omega)]^{\leq\omega}$ such that for every $A, B \in P(\omega)$ with $A \subseteq B$ there exists $C \in f(A) \cap f(B)$ satisfying $A \subseteq C \subseteq B$.*

Definition 6.9 Let $\mathcal{A}, \mathcal{B} \subseteq P(\omega)$ be arbitrary and let $\mathbf{h}$ be a function as in Definition 2.3. A $(\mathcal{B}, \mathbf{h})$-*maximal coloring* of $\mathcal{A}$ is a function $c\colon \mathcal{A} \to \mathrm{On}$ such that for every $B \in \mathcal{B}$,

1. $P(B) \cap \mathcal{A} \neq \emptyset$ implies $0 \in c(P(B))$;
2. $|P(B) \cap \mathcal{A}| \in \mathrm{dom}(\mathbf{h}) \setminus \omega_1$ implies $\mathbf{h}(|P(B) \cap \mathcal{A}|) \subseteq c(P(B) \cap \mathcal{A})$.

To get Theorem 6.7, it is enough to prove the following theorem.

Theorem 6.10 *Let $\mathcal{A}, \mathcal{B} \subseteq P(\omega)$ be arbitrary. Assume $(P(\omega), \subseteq)$ has the weak Freese-Nation property. Then $\mathcal{A}$ has a $\mathcal{B}$-maximal coloring.*

Proof of Theorem 6.7. Let $\{U_n : n < \omega\}$ be a base of X. Let $B\colon \mathbf{H} \to P(\omega)$ be defined by $B(H) = \{n :< \omega\colon U_n \cap H = \emptyset\}$ ($H \in \mathbf{H}$). Since $H \subseteq H'$ if and only if $B(H) \supseteq B(H')$, B is injective.

Let $\mathcal{A} = \{B(H)\colon H \in \mathbf{H}\}$, $\mathcal{B} = \{B(H)\colon H \in \mathbf{H}\}$. By Theorem 6.10 we have a $\mathcal{B}$-maximal coloring $c^\star\colon \mathcal{A} \to \mathrm{On}$. We show that $c\colon \mathbf{H} \to \mathrm{On}$, $c = c^\star \circ B$ is an ω_1-maximal coloring of $\mathcal{H}$.

To see this, let $H \in \mathbf{H}$ satisfy $|\mathbf{H}(H)| \geq \omega_1$. Clearly, B is a bijection between $\mathbf{H}(H)$ and $P(B(H)) \cap \mathcal{A}$. Hence $|P(B(H)) \cap \mathcal{A}| \geq \omega_1$ and so $c(\mathbf{H}(H)) \supseteq c^\star(P(B(H)) \cap \mathcal{A}) \supseteq |\mathbf{H}(H)|$, as required. $\square$

It remains to show Theorem 6.10.

Proof of Theorem 6.10. We prove the statement by induction on $\lambda = |\mathcal{A} \cup \mathcal{B}|$.

If $\lambda \leq \omega$ an arbitrary coloring $c\colon \mathcal{A} \to \mathrm{On}$ works. Consider now $\lambda = \omega_1$. Enumerate $\mathcal{B}$ as $\{B_\alpha\colon \alpha < \omega_1\}$ such that each $B \in \mathcal{B}$ occurs ω_1 many times. We define $c\colon \mathcal{A} \to \omega_1$ by transfinite induction of length ω_1, extending c to at most one further member of $\mathcal{A}$ at each step, as follows. For every $B \in \mathcal{B}$ let $I_B = \{\alpha < \omega_1\colon B_\alpha = B\}$. In the α^{th} step of the coloring if $\alpha \in I_B$ and $|P(B) \cap \mathcal{A}| = \omega_1$ pick one $A \in \mathcal{A}$ such that $c(A)$ is not defined yet and $A \in P(B)$. Define $c(A) = \mathrm{tp}(\alpha \cap I_B)$. This coloring clearly fulfills the requirements.

Assume now that $\lambda > \omega_1$ and the statement holds for $\lambda' < \lambda$. Let $\mathcal{A}, \mathcal{B} \subseteq P(\omega)$ with $|\mathcal{A} \cup \mathcal{B}| = \lambda$. Let $f\colon P(\omega) \to [P(\omega)]^{\leq\omega}$ be a function witnessing the weak Freeze-Notion property of $P(\omega)$. By closing $\mathcal{B}$ under f we can assume that $\mathcal{B}$ is f-closed.

Let $\langle M_\alpha : \alpha < \lambda\rangle$ be an increasing sequence of models of a large enough fragment of ZFC such that

1. $\mathcal{A}, \mathcal{B}, f \in M_0$,
2. $\langle M_\nu : \nu < \alpha\rangle \in M_\alpha$ $(\alpha < \lambda)$
3. $\omega_1 \dot{+} \alpha \subseteq M_\alpha$ and $|M_\alpha| = \omega_1 + |\alpha|$ $(\alpha < \lambda)$.

Set $M_\alpha^< = \bigcup\{M_\nu : \nu < \alpha\}$, let $\mathcal{A}_\alpha = \mathcal{A} \cap (M_\alpha \setminus M_\alpha^<)$, $\mathcal{B}_\alpha = \mathcal{B} \cap M_\alpha$, and let $\zeta_\alpha\colon \mathrm{Card} \cap |\mathcal{A}_\alpha|^+ \to \mathrm{On}$ be defined as $\zeta_\alpha|_{|\alpha|} = \mathrm{i}|_{|\alpha|}$, $\zeta_\alpha(|\alpha|) = \omega_1 \dot{+} \alpha$. By the inductive hypothesis and Proposition 2.4, for every $\alpha < \lambda$ we have a $(\mathcal{B}_\alpha, \zeta_\alpha)$-maximal coloring $c_\alpha\colon \mathcal{A}_\alpha \to \mathrm{On}$. Let $c = \bigcup\{c_\alpha\colon \alpha < \lambda\}$; the definition makes sense since for $\alpha \neq \beta$ we have $\mathcal{A}_\alpha \cap \mathcal{A}_\beta = \emptyset$. We show that c is a $\mathcal{B}$-maximal coloring.

Assume on the contrary that there is $B \in \mathcal{B}$ such that $|P(B) \cap \mathcal{A}| \geq \omega_1$ but $|P(B) \cap \mathcal{A}| \not\subseteq c(P(B) \cap \mathcal{A})$. Let $\alpha < \lambda$ be minimal such that we can have such a B in M_α and let $\mu \leq \lambda$ be an *uncountable regular* cardinal such that $|P(B) \cap \mathcal{A}| \geq \mu$ but $\mu \not\subseteq c(P(B) \cap \mathcal{A})$. We distinguish three cases.

Suppose first $|P(B) \cap \mathcal{A} \setminus M_\alpha^<| \geq \mu$ and $\alpha \geq \mu$. Then by $\alpha \subseteq M_\alpha$ we have $|((P(B) \cap \mathcal{A}) \setminus M_\alpha^<) \cap M_\alpha| \geq \mu$. Hence $|\mathcal{A}_\alpha \cap P(B)| \geq \mu$ and so $c(\mathcal{A} \cap P(B)) \supseteq c_\alpha(\mathcal{A}_\alpha \cap P(B)) \supseteq \mu$, a contradiction.

Suppose next $|P(B) \cap \mathcal{A} \setminus M_\alpha^<| \geq \mu$ but $\alpha < \mu$. Let $\sigma \in \mu \setminus c(P(B) \cap \mathcal{A})$ and let $\beta = \max(\alpha, \sigma + 1) < \mu$. Then $|P(B) \cap \mathcal{A} \setminus M_\beta^<| \geq \mu$ and so $\beta \subseteq M_\beta$ implies $|P(B) \cap \mathcal{A}_\beta| = \omega_1 + |\beta|$. Thus $\beta \subseteq c_\beta(P(B) \cap \mathcal{A}_\beta)$ and so $\sigma \in c_\beta(P(B) \cap \mathcal{A}_\beta) \subseteq c(P(B) \cap \mathcal{A})$, a contradiction.

Finally suppose $|P(B) \cap \mathcal{A} \setminus M_\alpha^<| < \mu$. With $\nu = |M_\alpha^< \cap f(B) \cap P(B)| \leq \omega$ enumerate $M_\alpha^< \cap f(B) \cap P(B)$ as $\{B_i\colon i < \nu\}$. For each $A \in \mathcal{A} \cap P(B) \cap M_\alpha^<$ there is $B' \in f(B) \cap f(A)$ with $A \subseteq B' \subseteq B$; and since $M_\alpha^<$ is f-closed, we have this $B' \in M_\alpha^<$ hence $B' = B_n$ for some $n < \nu$. Therefore

$$\mathcal{A} \cap P(B) \cap M_\alpha^< = \textstyle\bigcup_{n<\omega}\{A \in \mathcal{A}\colon A \in M_\alpha^<,\ A \subseteq B_n\}.$$

Since $|\mathcal{A} \cap P(B) \cap M_\alpha^<| \geq \mu$ there is $n < \nu$ such that $|\{A \in \mathcal{A} : A \in M_\alpha^<,\ A \subseteq B_n\}| \geq \mu$. Since $B_n \in M_{\alpha'}$ for some $\alpha' < \alpha$, by the minimality of α we have that $|\mathcal{A} \cap P(B_n)| \geq \mu$ implies $\mu \subseteq c(\mathcal{A} \cap P(B_n))$. But $c(\mathcal{A} \cap P(B_n)) \subseteq c(\mathcal{A} \cap P(B))$, a contradiction. This completes the proof. $\square$

7 Convex sets in $\mathbb{R}^n$

7.1 Arbitrary convex sets

In this section we observe that Theorem 6.1 and Theorem 6.7 imply that it is independent of ZFC whether an uncountable-fold cover of $\mathbb{R}^n$ $(1 < n < \omega)$ by isometric copies of one compact convex set can be split into two disjoint subcovers.

Theorem 7.1 *Let $1 < n < \omega$. Under the assumptions of Theorem 6.1, there exists a κ-fold closed of $\mathbb{R}^n$ by isometric copies of one compact convex set which cannot be decomposed into two subcovers.*

Proof. By rescaling the construction for Theorem 6.1, there is a compact set $K \subseteq [0, \pi/2]$ and a set of translations $T \subseteq [-\pi/2, \pi/2]$ such that $\mathbf{K} = \{K + t \colon t \in T\}$ is a κ-fold cover over $[0, \pi/2]$ which cannot be split into two subcovers over $[0, \pi/2]$.

Let $\mathbb{O} \in \mathbb{R}^{n-2}$ denote the origin. For every $t \in \mathbb{R}$ set

$$H(t) = \operatorname{conv}\{(\cos(\vartheta + t), \sin(\vartheta + t)) \colon \vartheta \in K\} \times \{\mathbb{O}\}$$

and let $\mathbf{H}_0 = \{H(t) \colon t \in T\}$. Set $Y = \{(\cos(\vartheta), \sin(\vartheta)) \colon \vartheta \in [0, \pi/2]\} \times \{\mathbb{O}\}$ and let $\mathbf{H}_1$ be a κ-fold cover of $\mathbb{R}^n \setminus Y$ by isometric copies $H(0)$ which do not intersect Y. Such a $\mathbf{H}_1$ clearly exists. Then $\mathbf{H} = \mathbf{H}_0 \cup \mathbf{H}_1$ fulfills the requirements. $\square$

The consistency for the existence of ω_1-maximal colorings for compact covers follows from Theorem 6.7.

7.2 Axis-parallel closed rectangles

Theorem 7.2 *There exists a countable family $\mathcal{R}$ of axis-parallel closed rectangles in $\mathbb{R}^2$ such that $\mathcal{R}$ is an ω-fold cover of $\mathbb{R}^2$ without two disjoint subcovers.*

We prove Theorem 7.2 in two steps: first we find an ω-fold cover of an abstract space without two disjoint subcovers, then we show how this cover can be realized using axis-parallel closed rectangles in $\mathbb{R}^2$.

We fix some notation in advance. For every $\sigma \in (\omega+1)^{<\omega}$, $|\sigma|$ denotes the length of σ and $\|\sigma\| = \sum\{\sigma(i) \colon i < |\sigma|,\ \sigma(i) \neq \omega\}$; if $n < \omega$, $\sigma^\frown n$ denotes the extension of σ with (n).

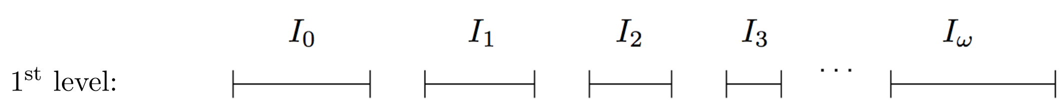
Let $P \subseteq \mathbb{R}$ denote the perfect set obtained by iterating the perfect scheme of Figure 1: that is we set $I_\emptyset = \mathbb{R}$, for $1 \leq n < \omega$ the closed intervals in the n^{th} level of the construction of P are indexed as $\{I_\sigma \colon \sigma \in (\omega+1)^n\}$ where

(I1) for every $\sigma, \sigma' \in (\omega+1)^{<\omega}$, $\sigma \subseteq \sigma'$ implies $I_{\sigma'} \subseteq I_\sigma$;

(I2) for every $\sigma \in (\omega+1)^{<\omega}$ and $n, n' \leq \omega$, $n < n'$ implies $\max I_{\sigma^\frown n} < \min I_{\sigma^\frown n'}$, and $\lim_{n<\omega} \max I_{\sigma^\frown n} = \min I_\omega$;

(I3) for every $\sigma \in (\omega+1)^{<\omega} \setminus \{\emptyset\}$, $\min I_{\sigma^\frown 0} = \min I_\sigma$ and $\max I_{\sigma^\frown \omega} = \max I_\sigma$;

and $P = \bigcap_{n<\omega} \bigcup_{\sigma \in (\omega+1)^n} I_\sigma$. We set $P_\sigma = P \cap I_\sigma$ $(\sigma \in (\omega+1)^{<\omega})$. Let $X = P \cup (\omega+1)^{<\omega}$. We define a cover $\mathcal{C}$ of X by letting $\mathcal{C} = \{C_{\sigma^\frown n} \colon \sigma \in (\omega+1)^{<\omega},\ n < \omega\}$ where $C_{\sigma^\frown n} = \{\sigma\} \cup P_{\sigma^\frown n}$ $(\sigma \in (\omega+1)^{<\omega},\ n < \omega)$.



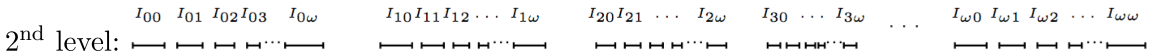


Figure 1.

Lemma 7.3 *$\mathcal{C}$ is an ω-fold cover of X which cannot be split into two disjoint subcovers.*

Proof. Pick an arbitrary $x \in X$. If $x \in P$ there is a unique $s \in (\omega+1)^\omega$ such that $x \in P_{s|_n}$ $(n < \omega)$. Then $x \in C_{s|_{n+1}}$ $(n < \omega)$ so $|\mathcal{C}(x)| = \omega$. If $x \in (\omega+1)^{<\omega}$ then $x \in C_{x^\frown n}$ $(n < \omega)$ so $\mathcal{C}$ is an ω-fold cover of X, indeed.

Split $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$ where $\mathcal{C}_0 \cap \mathcal{C}_1 = \emptyset$. We show that if $\mathcal{C}_0$ is a cover of X then $\mathcal{C}_1$ is not a cover of X. So suppose $X = \cup\mathcal{C}_0$. We define inductively a sequence $s \in (\omega+1)^\omega$ such that $C_{s|_{n+1}} \in \mathcal{C}_0$ $(n < \omega)$; then for $x = \bigcap_{n<\omega} P_{s|_{n+1}}$ we get $\mathcal{C}_1(x) = \emptyset$, which shows that $\mathcal{C}_1$ is not a cover of X.

Let $n < \omega$ and suppose that $s(i)$ is defined for $i < n$. Set $\sigma_n = s_0^\frown \ldots ^\frown s(n-1)$, we define $s(n)$ as follows. Since $\sigma_n \in \cup\mathcal{C}_0$ there is an $m < \omega$ for which $C_{\sigma_n^\frown m} \in \mathcal{C}_0$. Defining $s(n) = m$ completes the inductive step and the proof. $\square$

Proof of Theorem 7.2. By Lemma 7.3 it is enough to define an embedding $\varphi\colon \mathbb{R} \cup (\omega+1)^{<\omega} \to \mathbb{R}^2$ and a family of axis parallel closed rectangles $\mathcal{R}$ in $\mathbb{R}^2$ such that $\mathcal{R}$ in an ω-fold cover of $\mathbb{R}^2$ and on $\varphi(X)$ the cover by $\mathcal{R}$ coincides with the cover by $\{\varphi(C)\colon C \in \mathcal{C}\}$. We define φ and the rectangles iteratively, analogously to the perfect scheme in the definition of P.

Define $\varphi\colon \mathbb{R} \to \mathbb{R}^2$ by $\varphi(t) = (t, -t)$ $(t \in \mathbb{R})$. We will define φ on $(\omega+1)^{<\omega}$ in such a way that $\varphi((\omega+1)^{<\omega})$ is discrete. To start the construction, for every $n \le \omega$ pick a closed axis-parallel rectangle R_n such that $\varphi(\mathbb{R}) \cap R_n = \varphi(I_n)$, and the rectangles R_n $(n \le \omega)$ have a common upper right corner (see Figure 2.; for simplicity we do not write out φ in the figures). We define $\varphi(\emptyset)$ to be this common upper right corner. In addition, we require the abscissa of $\varphi(\emptyset)$ to be strictly greater than the abscissa of the lower right endpoint of $\varphi(I_\omega)$. This completes the first step of the construction.

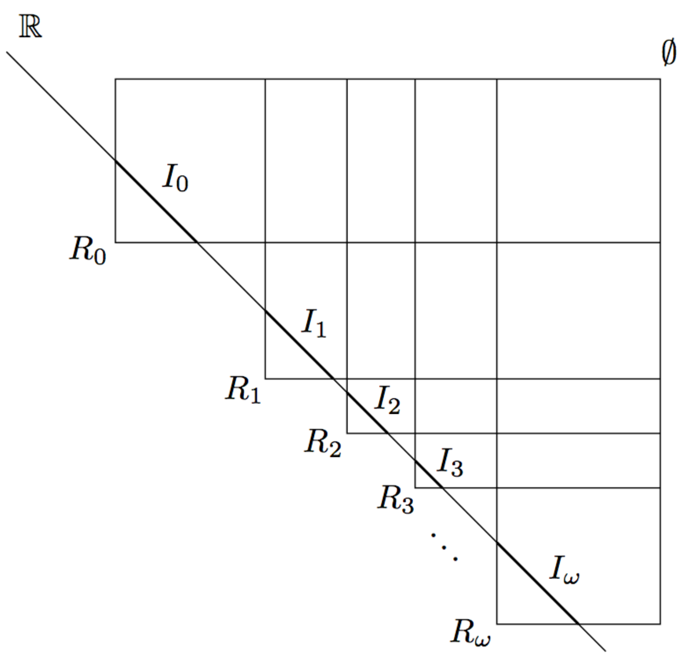


Figure 2.

Let $1 < m < \omega$ and suppose that for every $\sigma \in (\omega+1)^{<\omega}$ with $|\sigma| < m$ and for every $n \le \omega$ we defined $\varphi(\sigma)$ and $R_{\sigma^\frown n}$ such that

1. for every $\sigma \in (\omega+1)^{<\omega}$ with $|\sigma| < m$, the ordinate of $\varphi(\sigma)$ is greater than $|\sigma| + \|\sigma\|$;
2. $\varphi(\mathbb{R}) \cap R_{\sigma^\frown n} = \varphi(I_{\sigma^\frown n})$;
3. $\varphi(\sigma)$ is the upper right corner of $R_{\sigma^\frown n}$ $(n \leq \omega)$ and $\varphi(\sigma)$ is not contained in any other rectangle;
4. the abscissa of $\varphi(\sigma)$ is strictly greater than the abscissa of the lower right endpoint of $\varphi(I_{\sigma^\frown \omega})$.

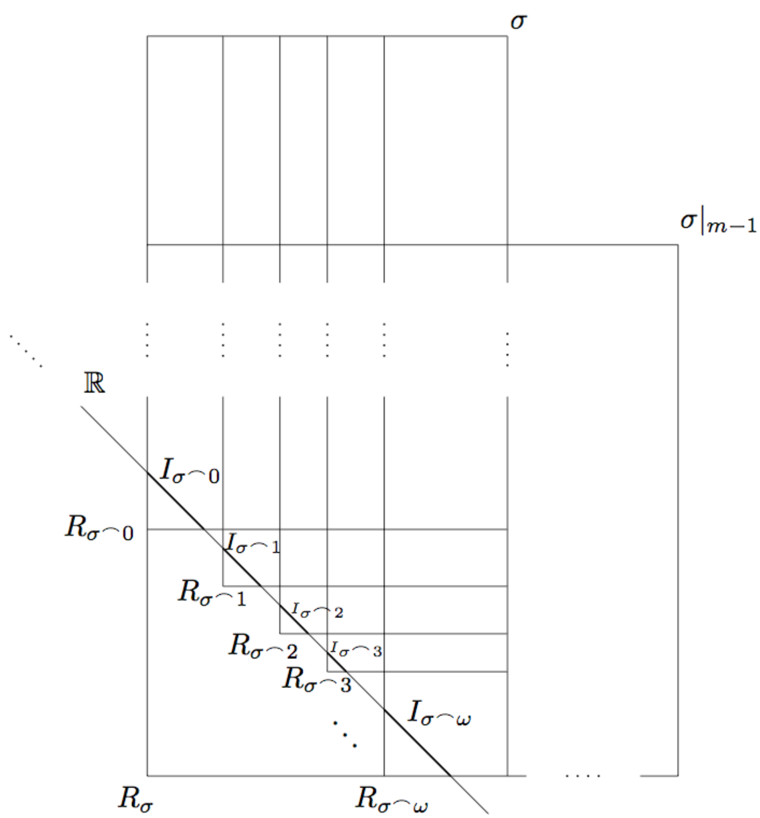


Figure 3.

Let $\sigma \in (\omega+1)^{<\omega}$ with $|\sigma| = m$. For every $n \leq \omega$ pick a closed axis-parallel rectangle $R_{\sigma^\frown n}$ such that $\varphi(\mathbb{R}) \cap R_{\sigma^\frown n} = \varphi(I_{\sigma^\frown n})$, and the rectangles $R_{\sigma^\frown n}$ $(n \leq \omega)$ have $\varphi(\sigma)$ as a common upper right corner (see Figure 3.). We require the ordinate of $\varphi(\sigma)$ to be greater than $m + \|\sigma\|$, and as above, the abscissa of $\varphi(\sigma)$ to be strictly greater than the abscissa of the lower right endpoint of $\varphi(I_{\sigma^\frown \omega})$. Moreover, we chose $R_{\sigma^\frown n}$ $(n < \omega)$ such that no $\varphi(\sigma')$ $(\sigma' \neq \sigma)$ is covered by these rectangles; this can be done by 4. Then 1.-4. are satisfied so the inductive step of the construction is complete.

Up to this point we constructed $\varphi\colon \mathbb{R} \cup (\omega+1)^{<\omega} \to \mathbb{R}^2$ and $\mathcal{R}_0 = \{R_{\sigma^\frown n}\colon \sigma \in (\omega+1)^{<\omega},\ n < \omega\}$ such that by 1, $\varphi(X)$ is a closed subset of $\mathbb{R}^2$, and by 2 and 3, $\mathcal{R}_0$ is an ω-fold cover of $\varphi(X)$ which, on $\varphi(X)$, coincides with the cover $\{\varphi(C)\colon C \in \mathcal{C}\}$. Let $\mathcal{R}_1$ be an arbitrary countable family of closed axis-parallel rectangles in $\mathbb{R}^2 \setminus \varphi(X)$ which is an ω-fold cover of $\mathbb{R}^2 \setminus \varphi(X)$. Then $\mathcal{R} = \mathcal{R}_0 \cup \mathcal{R}_1$ is an ω-fold cover of $\mathbb{R}^2$ such that on $\varphi(X)$ the cover by $\mathcal{R}$ coincides with the cover $\{\varphi(C)\colon C \in \mathcal{C}\}$, which completes the proof.$\square$

7.3 Polyhedra

The purpose of this section is to show that an uncountable-fold cover of $\mathbb{R}^n$ by polyhedra can be ω_1-maximally decomposed. We managed to obtain the following general result in this direction, which allows us to treat covers by sets with very different geometric constraints in a unified way.

We introduce some notation in advance. Let (X, τ) be a topological space and let $\mathfrak{G}$ denote the family of open subsets of X. For a $\lambda \in \mathrm{Card}$ and family $\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \lambda\} \subseteq 2^{2^X}$ we set $\bigsqcup \mathfrak{A} = \{\cup \mathcal{A}_\alpha : \alpha < \lambda\}$.

Theorem 7.4 *Let (X, τ) be a hereditarily Lindelöf space and let $\mathfrak{B} \subseteq 2^X$ be an intersection-closed family which is well-founded under $\subseteq$. Let*

$$\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \lambda\} \subseteq \left[B \cap G : B \in \mathfrak{B}, G \in \mathfrak{G}\right]^\omega$$

be arbitrary and set $\mathbf{H} = \bigsqcup \mathfrak{A}$. Then $\mathbf{H}$ has an ω_1-maximal coloring.

From Theorem 7.4 we have the following immediate corollaries.

Corollary 7.5 *Let κ be an uncountable cardinal. Any κ-fold cover of $\mathbb{R}^n$*

1. *by sets which can be obtained as countable unions of relatively open subsets of real affine varieties,*
2. *by open or closed polyhedra,*
3. *by open or closed balls,*

can be split into κ many disjoint subcovers.

Proof. Since the polynomial ring of n variables over the reals is Noetherian, the family of real affine varieties in $\mathbb{R}^n$ in intersection-closed and well-founded under $\subseteq$. So for 1, we can apply Theorem 7.4 with $\mathfrak{B}$ standing for the real affine varieties in $\mathbb{R}^n$. Statements 2 and 3 are special cases of 1. $\square$

We proceed to the proof of Theorem 7.4.

Proof of Theorem 7.4. We prove the statement by induction on λ. For $\lambda \leq \omega$ the identically zero coloring fulfills the requirements. So let first $\lambda = \omega_1$.

Take an arbitrary $\mathfrak{A} = \{\mathcal{A}_\alpha : \alpha < \omega_1\}$ and let

$$\mathcal{B} = \{B \in \mathfrak{B} : \exists \alpha < \omega_1,\ \exists G \in \mathfrak{G}\ (B \cap G \in \mathcal{A}_\alpha)\}.$$

We have $|\mathcal{B}| = \omega_1$, so since $\mathfrak{B}$ is well-founded, the intersection-closed hull $\mathcal{B}^\cap$ of $\mathcal{B}$ satisfies $|\mathcal{B}^\cap| = \omega_1$. Hence we can take an enumeration $\mathcal{B}^\cap = \{B_\alpha : \alpha < \omega_1\}$. We also fix a bijection $\varphi : \omega_1 \times \omega_1 \to \omega_1$.

For every $\alpha < \omega_1$ we construct inductively a countable partial coloring $c_\alpha : \lambda \to \mathrm{On}$, as follows. Let $\alpha < \omega_1$ and suppose that $\alpha = 0$ or c_η is defined for every $\eta < \alpha$. Set $\mathfrak{R}_\alpha = \lambda \setminus \bigcup_{\beta < \alpha} \mathrm{dom}(c_\beta)$. Let $\beta, \chi < \omega_1$ be such that $\varphi(\beta, \chi) = \alpha$. For $\gamma \in \mathfrak{R}_\alpha$ let

$$G(\gamma) = \bigcup \{G \in \mathfrak{G} : \exists B \in \mathfrak{B}\ (B_\beta \subseteq B,\ B \cap G \in \mathcal{A}_\gamma)\}, \tag{7}$$

and

$$C_\alpha = \{x \in B_\beta : |\{\gamma \in \mathfrak{R}_\alpha : x \in G(\gamma)\}| = \omega_1\}.$$

Since X is hereditarily Lindelöf, C_α is Lindelöf. Hence there is an $I_\alpha \in [\mathfrak{R}_\alpha]^\omega$ such that $C_\alpha \subseteq \bigcup_{\gamma \in I_\alpha} G(\gamma)$; thus by (7), $C_\alpha \subseteq \bigcup \{\cup \mathcal{A}_\gamma : \gamma \in I_\alpha\}$, as well. We define c_α by $c_\alpha(\gamma) = \chi$ ($\gamma \in I_\alpha$). This completes the α^{th} step of the construction. We set $c_{\omega_1} = \bigcup_{\alpha < \omega_1} c_\alpha$.

Define $c\colon \mathbf{H} \to \mathrm{On}$ by $c(\cup\mathcal{A}_\alpha) = c_{\omega_1}(\alpha)$ $(\alpha \in \mathrm{dom}(c_{\omega_1}))$. We show that c is an ω_1-maximal coloring of $\mathbf{H}$.

Suppose that $x \in X$ satisfies $|\mathbf{H}(x)| = \omega_1$, i.e. there are $I \in [\omega_1]^{\omega_1}$ and $B_\alpha \in \mathfrak{B}$, $G_\alpha \in \mathfrak{G}$ $(\alpha \in I)$ such that $x \in B_\alpha \cap G_\alpha \in \mathcal{A}_\alpha$ $(\alpha \in I)$. Let $B = \bigcap_{\alpha\in I} B_\alpha$, then $B \in \mathcal{B}^\cap$, that is $B = B_\beta$ for some $\beta < \omega_1$. Pick an arbitrary $\chi < \omega_1$ and let $\alpha = \varphi(\beta, \chi)$.

Recall the construction of c_α: since c_η $(\eta < \alpha)$ are countable, we have $|I \cap \mathfrak{R}_\alpha| = \omega_1$. Hence $x \in C_\alpha$ and so $x \in \cup\mathcal{A}_\gamma$ for some $\gamma \in I_\alpha$. Since $c(\cup\mathcal{A}_\gamma) = c_\alpha(\gamma) = \chi$ and $\chi < \omega_1$ was arbitrary, the proof of the $\lambda = \omega_1$ case is complete.

Let now $\lambda > \omega_1$ and suppose the statement holds for every $\kappa < \lambda$. By Proposition 2.5 we can assume $\mathbf{H}$ is simple. Let $\langle M_\alpha\colon \omega_1 \le \alpha < \lambda\rangle$ be an elementary chain of submodels such that $|M_\alpha| = |\alpha|$, $\langle M_\beta\colon \beta < \alpha\rangle \in M_\alpha$ and $(X, \tau), \mathfrak{G}, \mathfrak{B}, \mathfrak{A} \in M_{\omega_1}$. For every set y, let $\mathrm{rank}(y) = \min\{\alpha\colon y \in M_\alpha\}$.

Let $J_\alpha = \lambda \cap (M_\alpha \setminus \bigcup_{\beta<\alpha} M_\beta) = \{\eta < \lambda\colon \mathrm{rank}(\eta) = \alpha\}$. Then $|J_\alpha| = |\alpha|$. By Proposition 2.4 and the inductive hypothesis, there is a coloring $c_\alpha\colon J_\alpha \to \alpha$ such that for every $x \in X$, with $\kappa = |\{\eta \in J_\alpha\colon x \in \cup\mathcal{A}_\eta\}|$,

(1) $\omega_1 \le \kappa < |\alpha|$ implies $\kappa \subseteq c_\alpha(J_\alpha)$;

(2) $\omega_1 \le \kappa = |\alpha|$ implies $c_\alpha(J_\alpha) = \alpha$.

Set $c_\lambda = \bigcup_{\alpha<\lambda} c_\alpha$; we show $c\colon \mathbf{H} \to \mathrm{On}$, $c(\cup\mathcal{A}_\alpha) = c_\lambda(\alpha)$ $(\alpha \in \mathrm{dom}(c_\lambda))$ is an ω_1-maximal coloring of $\mathbf{H}$.

To this end, let $x \in X$ such that $\omega_1 \le |\mathbf{H}(x)|$. Let $\kappa = |\mathbf{H}(x)|$ and take $\nu_\xi \in \mathrm{On}$, $B_\xi \in \mathfrak{B}$ and $G_\xi \in \mathfrak{G}$ $(\xi < \kappa)$ such that $(\nu_\xi)_{\xi<\kappa}$ are pairwise different and $x \in B_\xi \cap G_\xi \in \mathcal{A}_{\nu_\xi}$ $(\xi < \kappa)$. Let $\rho_\xi = \mathrm{rank}(\nu_\xi)$ $(\xi < \kappa)$; we can assume $(\rho_\xi)_{\xi<\kappa}$ is an increasing sequence. Let $\rho = \sup\{\rho_\xi \dot{+} 1\colon \xi < \kappa\}$. We can also assume that if ρ is a successor ordinal $\rho = \rho' + 1$ then $\rho_\xi = \rho'$ $(\xi < \kappa)$.

If ρ is successor then $|\{\eta \in J_{\rho'}\colon x \in \cup\mathcal{A}_\eta\}| = \kappa$ so we are done by the inductive hypothesis. From now on assume ρ is a limit ordinal. By the well-foundedness of $\mathfrak{B}$ there is $F \in \left[\kappa\right]^{<\omega}$ such that

$$B = \bigcap\nolimits_{\xi<\kappa} B_\xi = \bigcap\nolimits_{\xi\in F} B_\xi.$$

Let $\sigma = \mathrm{rank}(F)$. Since ρ is a limit ordinal we have $\sigma < \rho$. Thus $|\{\xi < \kappa\colon \rho_\xi \ge \sigma\}| = \kappa$ and so

$$|\{\eta \in \lambda \setminus \bigcup\nolimits_{\sigma'<\sigma} M_{\sigma'}\colon x \in \cup\mathcal{A}_\eta\}| = \kappa. \tag{8}$$

For every $\alpha < \lambda$ let $G_\alpha = \bigcup\{G \in \mathfrak{G}\colon \exists B' \in \mathfrak{B}\ (B \subseteq B,\ B' \cap G \in \mathcal{A}_\alpha\}$. We need the following lemma.

Lemma 7.6 *Let $\mu, \delta \in \mathrm{On}$ such that $\omega_1 \le \sigma, \delta \le \mu < \lambda$. Set*

$$C = \{y \in B\colon |\{\eta \in \lambda \setminus \bigcup\nolimits_{\mu'<\mu} M_{\mu'}\colon y \in G_\eta\}| \ge \delta\}. \tag{9}$$

Then C has a δ-fold cover in $\mathbf{H} \cap M_\mu$, i.e. there are pairwise disjoint countable sets $K(\zeta) \subseteq J_\mu$ $(\zeta < \delta)$ such that $C \subseteq \bigcup_{\eta\in K(\zeta)} \cup\mathcal{A}_\eta$ $(\zeta < \delta)$.

Proof. Since C is Lindelöf, by (9)

there is a sequence $(K^\star(\zeta))_{\zeta<\delta}$ of pairwise disjoint countable subsets of $\lambda \setminus \bigcup_{\mu'<\mu} M_{\mu'}$
such that $C \subseteq \bigcup_{\eta\in K^\star(\zeta)} G_\eta$ $(\zeta < \delta)$. (10)

But $\sigma, \delta \le \mu$ implies $B, \delta \in M_\mu$, therefore $C \in M_\mu$, as well. So by elementarity (10) holds in M_μ, i.e. there is a sequence $(K(\zeta))_{\zeta<\delta}$ of pairwise disjoint countable subsets of $\lambda \cap M_\mu \setminus \bigcup_{\mu'<\mu} M_{\mu'}$ such that in M_μ, $C \subseteq \bigcup_{\eta\in K(\zeta)} G_\eta$ $(\zeta < \delta)$. For every $\zeta < \delta$ we have $K(\zeta) \subseteq M_\mu$ because $K(\zeta) \in M_\mu$ and $K(\zeta)$ is countable. So $K(\zeta) \subseteq J_\mu$ ($\zeta < \delta$). Since $G_\eta \cap B \subseteq \cup\mathcal{A}_\eta$ ($\eta < \lambda$), we have $C \subseteq \bigcup_{\eta\in K(\zeta)} \cup\mathcal{A}_\eta$ ($\zeta < \delta$). This completes the proof of the lemma. $\square$

We distinguish two cases. First suppose $\kappa \le \sigma$. We apply Lemma 7.6 with $\mu = \sigma$ and $\delta = \kappa$. By (8) we have $x \in C$, hence $|\{\eta \in J_\sigma : x \in \cup\mathcal{A}_\eta\}| = \kappa$. So by the inductive hypothesis, $\kappa \subseteq \{c_\sigma(\eta) : x \in \cup\mathcal{A}_\eta\}$, as required.

Finally suppose $\sigma < \kappa$. Fix an arbitrary $\beta \in \mathrm{On}$ satisfying $\sigma < \beta < \kappa$. We apply Lemma 7.6 with $\mu = \delta = \beta$. By (8) and by elementarity we have $x \in C$, hence $|\{\eta \in J_\beta : x \in \cup\mathcal{A}_\eta\}| = |\beta|$. So by the inductive hypothesis, $\beta \subseteq \{c_\beta(\eta) : x \in \cup\mathcal{A}_\eta\}$. Since β was arbitrary, the proof is complete. $\square$

We remark that in the proof of Theorem 7.4, formally we used only the assumption that the space X is a hereditarily ω_1-Lindelöf space. However, a space is hereditarily ω_1-Lindelöf if and only if it is hereditarily Lindelöf.

8 Open problems

It is a matter of fact that whenever we considered the splitting problem of κ-fold covers for infinite κ either we could establish the existence of a good κ-coloring or we could construct a κ-fold cover which cannot be split into two disjoint subcovers. Nevertheless, we could not prove that for κ-fold covers the existence of a good 2-coloring is equivalent with the existence of a good κ-coloring.

Problem 8.1 *Let X be a set, κ be an infinite cardinal and let $\mathcal{F} \subseteq 2^X$ be arbitrary. Suppose every κ-fold cover $\mathbf{H}$ of X satisfying $\mathcal{H} \subseteq \mathcal{F}$ has a good 2-coloring. Is it true then that every κ-fold cover $\mathbf{H}$ of X satisfying $\mathcal{H} \subseteq \mathcal{F}$ has a good κ-coloring, as well?*

In Section 4 we did not consider the splitting problem for hypergaphs.

Problem 8.2 *Examine the splitting problem of finite-fold and infinite-fold edge covers of hypergraphs.*

It would be interesting to know more on the consistency strength of the splitting of closed covers. In particular, one could examine whether maximal coloring of closed covers is possible in other well-known extensions than just the Cohen model. A special case is the following.

Problem 8.3 *Let κ be an uncountable cardinal. Is it true in a random extension of a model with GCH that every κ-fold closed cover of $\mathbb{R}$ can be split into two disjoint subcovers?*

We've seen that both under CH and under $\omega_1 < \mathrm{cov}(\mathcal{M})$, an ω_1-fold closed cover $\mathbf{H}$ of $\mathbb{R}$ with $|\mathbf{H}| = \omega_1$ has a good ω_1-coloring. However, we could not obtain it as a ZFC result.

Problem 8.4 *Is it consistent with ZFC that there exists an ω_1-fold closed cover $\mathbf{H}$ of $\mathbb{R}$ such that $|\mathbf{H}| = \omega_1$ but $\mathbf{H}$ cannot be split into two disjoint subcovers?*

As we mentioned in the introduction, there are numerous open problems concerning the splitting of finite-fold covers of $\mathbb{R}^n$ by sets with special geometric properties. The interested reader is referred to e.g. [14] for more details. Here we propose problems for ω-fold covers only.

Problem 8.5 *Is it true that every ω-fold cover of $\mathbb{R}^2$ by* translates *of one compact convex set can always be decomposed into two disjoint subcovers?*

Problem 8.6 *Is it true that every ω-fold cover of $\mathbb{R}^n$*

1. by translates or homothets of the unit cube,

2. by translates of the unit ball

can be decomposed into two disjoint subcovers?

Acknowledgement We are indebted to Péter Erdős, András Frank, János Gerlits, Ervin Győri, András Hajnal, István Juhász, László Lovász, Gyula Pap, Dömötör Pálvölgyi, Gábor Sági and Zoltán Szentmiklóssy for helpful discussions.

Our research was partially supported by the OTKA Grants K 68262, K 61600, K 49786 and F43620. We also gratefully acknowledge the support of Öveges Project of NKTH and KPI.

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