Is Lebesgue measure the only σ-finite invariant Borel measure?

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Abstract

S. Saks and recently R. D. Mauldin asked if every translation invariant σ-finite Borel measure on $\mathbb{R}^d$ is a constant multiple of Lebesgue measure. The aim of this paper is to investigate the versions of this question, since surprisingly the answer is "yes and no", depending on what we mean by Borel measure and by constant. According to a folklore result, if the measure is only defined for Borel sets then the answer is affirmative. We show that if the measure is defined on a σ-algebra *containing* the Borel sets then the answer is negative. However, if we allow the multiplicative constant to be infinity, then the answer is affirmative in this case as well. Moreover, our construction also shows that an isometry invariant σ-finite Borel measure (in the wider sense) on $\mathbb{R}^d$ can be non-σ-finite when we restrict it to the Borel sets.

Introduction

It is classical that, up to a nonnegative multiplicative constant, Lebesgue measure is the unique locally finite translation invariant Borel measure on $\mathbb{R}^d$. R. D. Mauldin [6] asked if we can replace locally finiteness by σ-finiteness. Then he himself gave an affirmative answer in the case when Borel measure means a measure defined on the σ-algebra of Borel sets, and later noticed that this is actually a folklore result, see (in a more general form) e.g. [3, Theorem B and Exercise 7]. In fact, the problem already appeared in [8], as an open question posed by Saks. For the sake of completeness we include a proof here. Let λ_d denote d-dimensional Lebesgue measure, and $B + t = \{b + t : b \in B\}$.

Theorem 0.1 *Let μ be a σ-finite translation invariant measure defined on the Borel subsets of $\mathbb{R}^d$. Then there exists $c \in [0, \infty)$ such that $\mu(B) = c\lambda_d(B)$ for every Borel set B.*

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Proof. First we prove that μ is absolutely continuous with respect to λ_d. Let $B \subset \mathbb{R}^d$ be a Borel set with $\lambda_d(B) = 0$. Define $\widetilde{B} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x + y \in B\}$. This set is clearly Borel, and as both λ_d and μ are σ-finite measures, we can apply the Fubini theorem to $(\lambda_d \times \mu)(\widetilde{B})$. Note that the x-section $\widetilde{B}_x = \{y : (x, y) \in \widetilde{B}\} = B - x$, and similarly $\widetilde{B}^y = \{x : (x, y) \in \widetilde{B}\} = B - y$. So by Fubini $\lambda_d(B) = 0$ implies $(\lambda_d \times \mu)(\widetilde{B}) = 0$. Hence $\mu(B - x) = 0$ for λ_d-almost every x, but μ is translation invariant, so $\mu(B) = 0$.

Therefore by the Radon-Nikodým theorem there exists a Borel function $f : \mathbb{R}^d \to [0, \infty]$ such that $\mu(B) = \int_B f \mathrm{d}\lambda_d$ for every Borel set B. Clearly

$$\mu(B) = \mu(B + t) = \int_{B+t} f \mathrm{d}\lambda_d = \int_B f(x - t) \mathrm{d}\lambda_d(x)$$

for every t and every Borel set B. Hence the uniqueness of the Radon-Nikodým derivative implies that for every t for Lebesgue almost every x the equation

$$f(x - t) = f(x) \tag{1}$$

holds.

In order to complete the proof it is clearly sufficient to show that there is a constant $c \in [0, \infty)$ such that $f(x) = c$ holds for λ_d-almost every x. Suppose on the contrary that there are real numbers $r_1 < r_2$ such that the Borel sets $\{x : f(x) < r_1\}$ and $\{x : f(x) > r_2\}$ are of positive Lebesgue measure. Let d_1 and d_2 be Lebesgue density points of the two sets, respectively. But then equation (1) fails for $t = d_1 - d_2$, a contradiction. □

However, in the literature there are at least two different notions that are referred to as Borel measure. The first one is measures defined only for Borel sets (see e.g. [3], [7]), while the second one is measures defined on σ-algebras *containing* the Borel sets (see e.g. [1], [5]).

In the rest of the paper we investigate the question of Saks and Mauldin in the case of the more general notion. As a spin-off, we also show that σ-finiteness is also sensitive to the definition of Borel measure. This question is related to [2], and was implicitly asked there.

1 The negative result

In this section we prove somewhat more than just a negative answer to our question.

Theorem 1.1 *There exists an isometry invariant σ-finite measure μ defined on an isometry invariant σ-algebra $\mathcal{A}$ containing the Borel subsets of $\mathbb{R}^d$ such that, for every Borel set B, if $\lambda_d(B) = 0$ then $\mu(B) = 0$, while if $\lambda_d(B) > 0$ then $\mu(B) = \infty$.*

Before the proof we need a lemma, which resembles some results proven by various authors, but we were unable to find this version in the literature.

We also need some notation: $\mathrm{Isom}(\mathbb{R}^d)$ is the group of isometries of $\mathbb{R}^d$, the symbol $|X|$ denotes the cardinality of a set X, the continuum cardinality is denoted by 2^ω, Δ stands for symmetric difference of two sets, and a set $P \subset \mathbb{R}^d$ is perfect if it is nonempty, closed and has no isolated points. Throughout the proof we use the fact that a countable union of sets of cardinality $< 2^\omega$ is itself of cardinality $< 2^\omega$ (see e.g. [4, Cor. I.10.41]).

Lemma 1.2 *There exists a disjoint decomposition* $\mathbb{R}^d = \cup_{n=0}^\infty A_n$ *such that* $|\varphi(A_n) \Delta A_n| < 2^\omega$ *for every* $n \in \mathbb{N}$ *and every* $\varphi \in \mathrm{Isom}(\mathbb{R}^d)$*, and such that* $|A_n \cap P| = 2^\omega$ *for every* $n \in \mathbb{N}$ *and every perfect set* $P \subset \mathbb{R}^d$*.*

Proof. We say that a set $A \subset \mathbb{R}^d$ is $< 2^\omega$-invariant, if $|\varphi(A) \Delta A| < 2^\omega$ for every $\varphi \in \mathrm{Isom}(\mathbb{R}^d)$. As $\mathrm{Isom}(\mathbb{R}^d)$ is closed under inverses, this is equivalent to $|\varphi(A) \setminus A| < 2^\omega$ for every $\varphi \in \mathrm{Isom}(\mathbb{R}^d)$.

It is enough to construct a sequence A_n of disjoint $< 2^\omega$-invariant sets such that $|A_n \cap P| = 2^\omega$ for every $n \in \mathbb{N}$ and every perfect set $P \subset \mathbb{R}^d$, since then clearly $\mathbb{R}^d \setminus \cup_{n=0}^\infty A_n$ is also $< 2^\omega$-invariant, hence we can simply replace A_0 by $A_0 \cup (\mathbb{R}^d \setminus \cup_{n=0}^\infty A_n)$.

Now we construct such a sequence by transfinite induction. Let us enumerate $\mathrm{Isom}(\mathbb{R}^d) = \{\varphi_\alpha : \alpha < 2^\omega\}$ and define G_α to be the group generated by $\{\varphi_\beta : \beta < \alpha\}$. Note that $|G_\alpha| < 2^\omega$. For $x \in \mathbb{R}^d$ let $G_\alpha(x) = \{\varphi(x) : \varphi \in G_\alpha\}$. Let us also enumerate the perfect subsets of $\mathbb{R}^d$ as $\{P_\alpha : \alpha < 2^\omega\}$ such that each perfect set P is listed 2^ω many times.

Define $A_n^0 = \emptyset$ for every $n \in \mathbb{N}$. At step α we recursively construct a sequence $x_n^\alpha \in P_\alpha$ $(n \in \mathbb{N})$ such that for every $k \neq l$

$$\left[\cup_{\beta<\alpha} A_k^\beta \cup G_\alpha(x_k^\alpha)\right] \cap \left[\cup_{\beta<\alpha} A_l^\beta \cup G_\alpha(x_l^\alpha)\right] = \emptyset. \tag{2}$$

To see that this is possible, note first that (2) holds whenever for every n the point x_n^α is not in the set

$$\cup_{\varphi \in G_\alpha} \varphi^{-1} \left(\cup_{m \neq n} \cup_{\beta<\alpha} A_m^\beta \cup \cup_{i=0}^{n-1} G_\alpha(x_i^\alpha)\right),$$

which is of cardinality $< 2^\omega$. As every perfect set is of cardinality 2^ω, this set cannot cover P_α, so we can find an x_n^α with the required property and define $A_n^\alpha = \cup_{\beta<\alpha} A_n^\beta \cup G_\alpha(x_n^\alpha)$. Clearly, $|A_n^\alpha| < 2^\omega$. Finally, define $A_n = \cup_{\alpha<2^\omega} A_n^\alpha$ for every n. These sets are clearly disjoint, they all intersect every perfect set in a set of cardinality 2^ω.

Finally, in order to check that the A_n's are $< 2^\omega$-invariant, let φ_α be given. First note that $A_n^\alpha = \cup_{\beta\leq\alpha} G_\beta(x_n^\beta)$ and $A_n = \cup_{\alpha<2^\omega} G_\alpha(x_n^\alpha)$ for every n. Clearly, for $\alpha < \beta$ the set $G_\beta(x_n^\beta)$ is φ_α-invariant (for every n), hence if $x \in A_n$ is such that $\varphi_\alpha(x) \notin A_n$, then $x \in \cup_{\beta\leq\alpha} G_\beta(x_n^\beta) = A_n^\alpha$. That is; $\varphi_\alpha(A_n) \setminus A_n \subset \varphi_\alpha(A_n^\alpha)$, so the A_n's are $< 2^\omega$-invariant. This completes the proof. □

Proof. (Theorem 1.1) Let A_n be the sequence from the previous lemma. Define

$$\mathcal{A} = \{[\cup_{n=0}^\infty (A_n \cap B_n)] \,\Delta H : \forall n \; B_n \subset \mathbb{R}^d \text{ Borel}, H \subset \mathbb{R}^d, |H| < 2^\omega\}.$$

Clearly $\mathcal{A}$ contains the Borel sets, as $B = [\cup_{n=0}^{\infty}(A_n \cap B)]\,\Delta\emptyset$.

In order to check that $\mathcal{A}$ is closed under complements note that $(X\Delta H)^C = X^C\Delta H$, and therefore $([\cup_{n=0}^{\infty}(A_n \cap B_n)]\,\Delta H)^C = [\cup_{n=0}^{\infty}(A_n \cap B_n)]^C\,\Delta H = \left[\cup_{n=0}^{\infty}(A_n \cap B_n^C)\right]\Delta H$.

In order to show that $\mathcal{A}$ is closed under countable unions, we need to show $\cup_{k=0}^{\infty}(X^k\Delta H^k) \in \mathcal{A}$, where

$$X^k = \cup_{n=0}^{\infty}(A_n \cap B_n^k). \tag{3}$$

Using the identity

$$Z = W\Delta W\Delta Z \tag{4}$$

(note that Δ is associative) we obtain

$$\cup_{k=0}^{\infty}(X^k\Delta H^k) = \left[\cup_{k=0}^{\infty}X^k\right]\Delta\left[\cup_{k=0}^{\infty}X^k\right]\Delta\left[\cup_{k=0}^{\infty}(X^k\Delta H^k)\right] = \left[\cup_{k=0}^{\infty}X^k\right]\Delta Y, \tag{5}$$

where $Y = \left[\cup_{k=0}^{\infty}X^k\right]\Delta\left[\cup_{k=0}^{\infty}(X^k\Delta H^k)\right]$. As

$$\cup_{k=0}^{\infty}X^k = \cup_{n=0}^{\infty}\left(A_n \cap (\cup_{k=0}^{\infty}B_n^k)\right) \tag{6}$$

it is sufficient to check that

$$|Y| < 2^{\omega}, \tag{7}$$

but this is clear, since $Y = \left[\cup_{k=0}^{\infty}X^k\right]\Delta\left[\cup_{k=0}^{\infty}(X^k\Delta H^k)\right] \subset \cup_{k=0}^{\infty}H^k$, which is of cardinality $< 2^{\omega}$.

To show that $\mathcal{A}$ is isometry invariant, let $\varphi \in \mathrm{Isom}(\mathbb{R}^d)$. First note that

$$\varphi([\cup_{n=0}^{\infty}(A_n \cap B_n)]\,\Delta H) = [\cup_{n=0}^{\infty}(\varphi(A_n) \cap \varphi(B_n))]\,\Delta\varphi(H). \tag{8}$$

Set

$$X = \cup_{n=0}^{\infty}(\varphi(A_n) \cap \varphi(B_n)) \text{ and } Y = \cup_{n=0}^{\infty}(A_n \cap \varphi(B_n)). \tag{9}$$

We need to show that $X\Delta\varphi(H) \in \mathcal{A}$. Using (4) again, write

$$X\Delta\varphi(H) = [Y\Delta Y\Delta X]\,\Delta\varphi(H) = Y\Delta\left[(Y\Delta X)\Delta\varphi(H)\right], \tag{10}$$

where we use again the associativity of Δ. Hence it is enough to show that

$$|(Y\Delta X)\Delta\varphi(H)| < 2^{\omega}, \tag{11}$$

which follows from $Y\Delta X = [\cup_{n=0}^{\infty}(A_n \cap \varphi(B_n))]\,\Delta\,[\cup_{n=0}^{\infty}(\varphi(A_n) \cap \varphi(B_n))] \subset \cup_{n=0}^{\infty}(A_n\Delta\varphi(A_n))$, from $|\varphi(H)| < 2^{\omega}$, and the $< 2^{\omega}$-invariance of A_n.

Let us now define

$$\mu([\cup_{n=0}^{\infty}(A_n \cap B_n)]\,\Delta H) = \sum_{n=0}^{\infty}\lambda_d(B_n).$$

First we have to show that μ is well-defined. Let $[\cup_{n=0}^{\infty}(A_n \cap B_n)]\,\Delta H = [\cup_{n=0}^{\infty}(A_n \cap B_n')]\,\Delta H'$. We claim that $\lambda_d(B_n) = \lambda_d(B_n')$ for every n. Otherwise, without loss of generality, there exists an n_0 such that $\lambda_d(B_{n_0}) < \lambda_d(B_{n_0}')$, hence

$B'_{n_0} \setminus B_{n_0}$ contains a perfect set P (even of positive measure). But $|P \cap A_{n_0}| = 2^\omega$ and $|H \cup H'| < 2^\omega$, hence there exists an $x \in (P \cap A_{n_0}) \setminus (H \cup H')$, and then $x \in [\cup_{n=0}^\infty (A_n \cap B'_n)] \,\Delta H'$ but $x \notin [\cup_{n=0}^\infty (A_n \cap B_n)] \,\Delta H$, a contradiction. (Recall that the A_n's are disjoint.)

In order to prove that μ is σ-additive, let

$$\cup_{k=0}^\infty (X^k \Delta H^k) \tag{12}$$

be a disjoint union, where X^k is as in (3). First we claim that for every n and every $k \neq k'$ we have $\lambda_d(B_n^k \cap B_n^{k'}) = 0$. Otherwise, for some n_0 there exists a perfect set $P \subset B_{n_0}^k \cap B_{n_0}^{k'}$, and we can find $x \in (P \cap A_{n_0}) \setminus (H^k \cup H^{k'})$, hence $x \in \left[\cup_{n=0}^\infty (A_n \cap B_n^k)\right] \Delta H^k$ and $x \in \left[\cup_{n=0}^\infty (A_n \cap B_n^{k'})\right] \Delta H^{k'}$, but then the union (12) is not disjoint, a contradiction. Therefore $\lambda_d(\cup_{k=0}^\infty B_n^k) = \sum_{k=0}^\infty \lambda_d(B_n^k)$ for every n, so by (5), (6) and (7) we obtain $\mu(\cup_{k=0}^\infty (X^k \Delta H^k)) = \sum_{n=0}^\infty \lambda_d(\cup_{k=0}^\infty B_n^k) = \sum_{n=0}^\infty \sum_{k=0}^\infty \lambda_d(B_n^k) = \sum_{k=0}^\infty \sum_{n=0}^\infty \lambda_d(B_n^k) = \sum_{k=0}^\infty \mu(X^k \Delta H^k)$.

Now we show that μ is isometry invariant. By (8), (9), (10) and (11) we obtain that $\mu(\varphi([\cup_{n=0}^\infty (A_n \cap B_n)] \,\Delta H)) = \sum_{n=0}^\infty \lambda_d(\varphi(B_n))$, which clearly equals $\sum_{n=0}^\infty \lambda_d(B_n)$, which is $\mu([\cup_{n=0}^\infty (A_n \cap B_n)] \,\Delta H)$ by definition.

The fact that μ is σ-finite follows from $\mathbb{R}^d = \cup_{n=0}^\infty \cup_{K=0}^\infty (A_n \cap [-K, K]^d)$, since $\mu(A_n \cap [-K, K]^d) = \lambda_d([-K, K]^d) = (2K)^d < \infty$ for every n and K.

Finally, for a Borel set B we have $\mu(B) = \mu(\cup_{n=0}^\infty (A_n \cap B)) = \sum_{n=0}^\infty \lambda_d(B)$, which is zero if $\lambda_d(B) = 0$ and ∞ otherwise. $\square$

As an immediate corollary we obtain the following.

Corollary 1.3 *There exists an isometry invariant σ-finite measure μ defined on an isometry invariant σ-algebra $\mathcal{A}$ containing the Borel subsets of $\mathbb{R}^d$ such that μ restricted to the Borel sets is not equal to $c\lambda_d$ for every $c \in [0, \infty)$.*

As $\mathbb{R}^d$ is not the union of countably many Lebesgue nullsets, the next statement is also a corollary to Theorem 1.1.

Corollary 1.4 *There exists an isometry invariant σ-finite measure μ defined on an isometry invariant σ-algebra $\mathcal{A}$ containing the Borel subsets of $\mathbb{R}^d$ such that μ restricted to the Borel sets is not σ-finite.*

2 The positive result

The measure μ constructed in the previous section behaves simply on Borel sets; if $\lambda_d(B) = 0$ then $\mu(B) = 0$, while if $\lambda_d(B) > 0$ then $\mu(B) = \infty$. So we can say that $\mu(B) = \infty\lambda_d(B)$ for every Borel set B. The next theorem shows that this is the only possibility.

Theorem 2.1 *Let μ be a σ-finite translation invariant measure defined on a translation invariant σ-algebra containing the Borel subsets of $\mathbb{R}^d$. Then there exists $c \in [0, \infty]$ such that $\mu(B) = c\lambda_d(B)$ for every Borel set B.*

Moreover, μ restricted to the Borel sets is σ-finite if and only if c is finite.

The proof of this theorem will be based on the following two lemmas, the second of which is well known.

Lemma 2.2 *Let μ be a σ-finite translation invariant measure defined on a translation invariant σ-algebra containing the Borel subsets of $\mathbb{R}^d$, and suppose that μ restricted to the Borel sets is not σ-finite. Then for every Borel set B we have either $\mu(B) = 0$ or $\mu(B) = \infty$.*

Proof. Let $\mathcal{B}$ be a maximal disjoint family of Borel sets of positive finite μ-measure. As μ is σ-finite (on $\mathcal{A}$), $\mathcal{B}$ is countable, hence $B_0 = \bigcup \mathcal{B}$ is a Borel set. Define

$$\mu'(B) = \mu(B_0 \cap B) \text{ for every Borel set } B.$$

Note that this measure is only defined for Borel sets. As μ' is clearly σ-finite, we can apply the Fubini theorem for $\mu' \times \mu$ and the Borel set $\widetilde{B_0^C} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x + y \in B_0^C\}$, as in the proof of Theorem 0.1. On one hand, $(\mu' \times \mu)(\widetilde{B_0^C}) = \int_{y\in\mathbb{R}^d} \mu'(B_0^C - y)\mathrm{d}\mu(y) = \int_{y\in\mathbb{R}^d} \mu(B_0 \cap (B_0^C - y))\mathrm{d}\mu(y)$. We claim that $\mu(B_0 \cap (B_0^C - y)) = 0$ for every y, hence $(\mu' \times \mu)(\widetilde{B_0^C}) = 0$. Indeed, otherwise (using that $B_0 = \bigcup \mathcal{B}$ and $\mathcal{B}$ is countable) there is a Borel set $B \in \mathcal{B}$ such that $0 < \mu(B \cap (B_0^C - y)) < \infty$. But then for $D = B \cap (B_0^C - y)$ we obtain that the Borel set $D + y$ is disjoint from B_0, hence from all elements of $\mathcal{B}$, and is of positive and finite μ-measure (since μ is translation invariant), contradicting the maximality of $\mathcal{B}$.

On the other hand, $0 = (\mu' \times \mu)(\widetilde{B_0^C}) = \int_{x\in\mathbb{R}^d} \mu(B_0^C - x)\mathrm{d}\mu'(x)$. As μ restricted to the Borel sets is not σ-finite, $\mu(B_0^C - x) = \mu(B_0^C) = \infty$ for every x. Therefore we obtain $0 = \mu'(\mathbb{R}^d) = \mu(B_0)$, so $\mathcal{B} = \emptyset$ and we are done. $\square$

Lemma 2.3 *Let μ_1 and μ_2 be σ-finite translation invariant measures defined on the (not necessarily equal) translation invariant σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ containing the Borel subsets of $\mathbb{R}^d$, and suppose that $\mu_1(\mathbb{R}^d), \mu_2(\mathbb{R}^d) > 0$. Then for every Borel set B, $\mu_1(B) = 0$ iff $\mu_2(B) = 0$.*

Proof. Apply Fubini to $\mu_1 \times \mu_2$ and $\widetilde{B} = \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x + y \in B\}$. $\square$

Proof. (Theorem 2.1) The last statement of the theorem is obvious, as countably many Lebesgue nullsets cannot cover $\mathbb{R}^d$.

Now we prove the first statement, namely that the constant $c \in [0, \infty]$ exists. If μ restricted to the Borel sets is σ-finite, then we are done by Theorem 0.1. So we can assume that this is not the case. Then applying Lemma 2.2 and Lemma 2.3 with $\mu_1 = \mu$ and $\mu_2 = \lambda_d$ the theorem follows. $\square$

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