Topological Hausdorff dimension and level sets of generic continuous functions on fractals

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Abstract

In an earlier paper we introduced a new concept of dimension for metric spaces, the so called topological Hausdorff dimension. For a compact metric space K let $\dim_H K$ and $\dim_{tH} K$ denote its Hausdorff and topological Hausdorff dimension, respectively. We proved that this new dimension describes the Hausdorff dimension of the level sets of the generic continuous function on K, namely $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$. We also proved that if K is sufficiently homogeneous then $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ for the generic $f \in C(K)$ and the generic $y \in f(K)$. The most important goal of this paper is to make these theorems more precise.

As for the first result, we prove that the supremum is actually attained, and also show that there may only be a unique level set of maximal Hausdorff dimension.

As for the second result, we characterize those compact metric spaces for which for the generic $f \in C(K)$ and the generic $y \in f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. We also generalize a result of B. Kirchheim by showing that if K is self-similar then for the generic $f \in C(K)$ for every $y \in \operatorname{int} f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$.

Finally, we prove that the graph of the generic $f \in C(K)$ has the same Hausdorff and topological Hausdorff dimension as K.

1 Introduction

We recall first the definition of the (small inductive) topological dimension.

Definition 1.1. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

For more information on this concept see [3] or [6].

We introduced the topological Hausdorff dimension for compact metric spaces in [1]. It is defined analogously to the topological dimension. However, it is not inductive, and it can attain non-integer values as well. The Hausdorff dimension of a metric space X is denoted by $\dim_H X$, see e.g. [5] or [9]. In this paper we adopt the convention that $\dim_H \emptyset = -1$.

Definition 1.2. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Both notions of dimension can attain the value ∞ as well.

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Let K be a compact metric space, and let $C(K)$ denote the space of continuous real-valued functions equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments. If $\dim_t K = 0$ then the generic $f \in C(K)$ is well-known to be one-to-one, so every non-empty level set is a singleton.

Assume $\dim_t K > 0$. The following results from [1] show the connection between the topological Hausdorff dimension and the level sets of the generic $f \in C(K)$.

Theorem 1.3. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

(i) $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every* $y \in \mathbb{R}$,

(ii) for every $\varepsilon > 0$ there exists an interval $I_{f,\varepsilon}$ such that $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon$ *for every* $y \in I_{f,\varepsilon}$.

Corollary 1.4. *If K is a compact metric space with $\dim_t K > 0$ then* $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ *for the generic* $f \in C(K)$.

If K is also sufficiently homogeneous, for example self-similar, then we can actually say more.

Theorem 1.5. *If K is a self-similar compact metric space with $\dim_t K > 0$ then* $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ *for the generic* $f \in C(K)$ *and the generic* $y \in f(K)$.

Theorems 1.3 and 1.5 are the starting points of this paper, our primary aim is to make these theorems more precise.

In the Preliminaries section we introduce some notation and definitions, cite some important properties of the topological Hausdorff dimension and prove several technical lemmas.

In Section 3 we prove a partial converse of Theorem 1.5. We show that for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ iff K is homogeneous for the topological Hausdorff dimension, that is for every non-empty closed ball $B(x,r) \subseteq K$ we have $\dim_{tH} B(x,r) = \dim_{tH} K$. If K is (weakly) self-similar then much more is true: For the generic $f \in C(K)$ for every $y \in \operatorname{int} f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. This generalizes a result of B. Kirchheim. He proved in [8] that for the generic $f \in C\left([0,1]^d\right)$ for every $y \in \operatorname{int} f\left([0,1]^d\right)$ we have $\dim_H f^{-1}(y) = d - 1$.

In Section 4 we prove that the generic $f \in C(K)$ has at least one level set of maximal Hausdorff dimension. Hence the supremum is attained in Corollary 1.4. We construct an attractor of an iterated function system $K \subseteq \mathbb{R}^2$ such that the generic $f \in C(K)$ has a unique level set of Hausdorff dimension $\dim_{tH} K - 1$. This shows that the above theorem is sharp.

Finally, in Section 5 we prove that the graph of the generic $f \in C(K)$ has the same Hausdorff and topological Hausdorff dimension as K. This generalizes a result of R. D. Mauldin and S. C. Williams which states that the graph of the generic $f \in C([0,1])$ is of Hausdorff dimension one, see [11].

2 Preliminaries

2.1 Notation and definitions

Let (X, d) be a metric space, and let $A, B \subseteq X$ be arbitrary sets. We denote by $\operatorname{int} A$ and ∂A the interior and boundary of A. The diameter of A is denoted by $\operatorname{diam} A$. We use the convention $\operatorname{diam} \emptyset = 0$. The distance of the sets A and B is defined by $\operatorname{dist}(A, B) = \inf\{d(x, y) : x \in A,\, y \in B\}$. Let $B(x, r) = \{y \in X : d(x, y) \leq r\}$ and $U(x, r) = \{y \in X : d(x, y) < r\}$. More generally, we define $B(A, r) = \{x \in X : \operatorname{dist}(x, A) \leq r\}$ and $U(A, r) = \{x \in X : \operatorname{dist}(x, A) < r\}$.

For two metric spaces (X, d_X) and (Y, d_Y) a function $f \colon X \to Y$ is *Lipschitz* if there exists a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The smallest such constant C is called the Lipschitz constant of f and denoted by $\operatorname{Lip}(f)$. If $\operatorname{Lip}(f) < 1$ then f is a *contraction*. A function $f \colon X \to Y$ is called *bi-Lipschitz* if f is a bijection and both f and f^{-1} are Lipschitz.

If $s \geq 0$ and $\delta > 0$, then

$$\mathcal{H}^s_\delta(X) = \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : X \subseteq \bigcup_i U_i,\ \forall i \operatorname{diam} U_i \leq \delta\right\},$$

$$\mathcal{H}^s(X) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(X).$$

The *Hausdorff dimension of* X is defined as

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\},$$

we adopt the convention that $\dim_H \emptyset = -1$ throughout the paper. For more information on these concepts see [5] or [9].

We define on $X \times Y$ the following metric. For all $(x_1, y_1), (x_2, y_2) \in X \times Y$ set

$$d_{X \times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

The metric space X is *totally disconnected* if every connected component is a singleton.

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is of *second category* if it is not meager. A set is called *co-meager* if its complement is meager. By Baire's Category Theorem co-meager sets are dense. It is not difficult to show that a set is co-meager iff it contains a dense G_δ set. We say that the *generic* element $x \in X$ has property $\mathcal{P}$, if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The term '*typical*' is also used instead of 'generic'. Our main example will be $X = C(K)$ endowed with the supremum metric (for some compact metric space K).

Let X, Y be Polish spaces. We call the set $A \subseteq X$ *analytic*, if it is a continuous image of a Polish space. We call it *co-analytic* if its complement is analytic. The set A has the *Baire property* if $A = U \Delta M$ where U is open and M is meager. Both analytic and co-analytic sets have the Baire property. If a set is of second category in every non-empty open set and has the Baire property then it is co-meager. If $E \subseteq X \times Y$, $x \in X$ and $y \in Y$ then let $E_x = \{y \in Y : (x,y) \in E\}$ and $E^y = \{x \in X : (x,y) \in E\}$. Let $\mathrm{pr}_X \colon X \times Y \to X$, $\mathrm{pr}_X(x,y) = x$ be the projection of $X \times Y$ onto X. If $E \subseteq X \times Y$ is Borel, then $\mathrm{pr}_X(E)$ is analytic. For more information see [7].

If K is a non-empty compact metric space then we say that K is an *attractor of an iterated function system* (IFS) if there exist contractions $\Psi_i \colon K \to K$, $i \in \{1, \dots, m\}$ such that $K = \cup_{i=1}^m \Psi_i(K)$. If the Ψ_i's are also similarities then K is *self-similar*.

For every $\alpha \in (0,1)$ we construct the *middle-α Cantor set* C_α in the following way. In the first step we remove the middle-α open interval $((1-\alpha)/2, (1+\alpha)/2)$ from $[0,1]$. After the $(n-1)$st step we have 2^{n-1} disjoint, closed $(n-1)$st level intervals. In the nth step we remove the middle-α open intervals from each of them. We continue this procedure for all $n \in \mathbb{N}^+$, and the limit set is the middle-α Cantor set. It is well-known that $\dim_H C_\alpha = \log 2 / \log(2/(1-\alpha))$.

Let us define the *Smith-Volterra-Cantor set* S in the following way. In the first step we remove the open interval of length $1/4$ from the middle of $[0,1]$. After the $(n-1)$st step we have 2^{n-1} disjoint, closed $(n-1)$st level intervals. In the nth step we remove the middle open intervals of length $1/2^{2n}$ from each of them. We continue this procedure for all $n \in \mathbb{N}^+$, and the limit set is the Smith-Volterra-Cantor set. Elementary computation shows that S has positive Lebesgue measure (more precisely its measure is $1/2$).

The *nth level elementary pieces* of C_α are the intersections of C_α with the nth level intervals of C_α. This definition is also analogous for S.

We adopt the convention that *intervals* are non-degenerate.

2.2 Properties of the topological Hausdorff dimension

The next theorems are from [1].

Fact 2.1. *For every metric space X*

$$\dim_{tH} X = 0 \Longleftrightarrow \dim_t X = 0.$$

Theorem 2.2. *For every metric space X*

$$\dim_t X \le \dim_{tH} X \le \dim_H X.$$

Theorem 2.3. *The topological Hausdorff dimension satisfies the following properties.*

(i) ***Extension of the classical dimension.*** *The topological Hausdorff dimension of a countable set equals zero, and for open subspaces of* $\mathbb{R}^d$ *and for smooth* d*-dimensional manifolds the topological Hausdorff dimension equals* d*.*

(ii) ***Monotonicity.*** *If* $X \subseteq Y$ *are metric spaces then* $\dim_{tH} X \leq \dim_{tH} Y$*.*

(iii) ***Lipschitz-invariance.*** *Let* X, Y *be metric spaces. If* $f \colon X \to Y$ *is a Lipschitz homeomorphism then* $\dim_{tH} X \leq \dim_{tH} Y$*. If* f *is bi-Lipschitz then* $\dim_{tH} X = \dim_{tH} Y$*.*

(iv) ***Countable stability for closed sets.*** *Let* X *be a separable metric space and* $X = \cup_{n\in\mathbb{N}} X_n$*, where* X_n*,* $n \in \mathbb{N}$ *are closed subsets of* X*. Then* $\dim_{tH} X = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$*.*

Theorem 2.4. *If* X *is a non-empty separable metric space then*

$$\dim_{tH} (X \times [0,1]) = \dim_H X + 1.$$

For compact metric spaces the infimum is attained in the definition of the topological Hausdorff dimension.

Theorem 2.5. *If* K *is a non-empty compact metric space then*

$$\dim_{tH} K = \min\{d : K \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

2.3 Technical lemmas

The next lemma and its consequence will be very useful throughout the paper.

Lemma 2.6. *Let* X, Y *be complete metric spaces and let* $R \colon X \to Y$ *be a continuous, open and surjective mapping.*

(i) If $A \subseteq X$ *is of second category/co-meager then* $R(A) \subseteq Y$ *is of second category/co-meager.*

(ii) If $B \subseteq Y$ *is of second category/co-meager then* $R^{-1}(B) \subseteq X$ *is of second category/co-meager.*

Proof. *(i)* First we show that if $B \subseteq Y$ is meager then $R^{-1}(B) \subseteq X$ is also meager. Clearly it is enough to prove that if $B \subseteq Y$ is closed and nowhere dense then $R^{-1}(B) \subseteq X$ is nowhere dense. Since R is continuous $R^{-1}(B)$ is closed. We show that $R^{-1}(B)$ is nowhere dense. Assume to the contrary that there is a non-empty open set $U \subseteq R^{-1}(B)$. Since the map R is open the set $R(U)$ is non-empty and open. Then $R(U) \subseteq B$ implies that B is of second category, a contradiction.

Let $A \subseteq X$ be of second category. Assume to the contrary that $R(A) \subseteq Y$ is meager. Then by the previous argument $R^{-1}(R(A))$ is meager and $A \subseteq R^{-1}(R(A))$, a contradiction.

Suppose that $A \subseteq X$ is co-meager. We want to prove that $R(A) \subseteq Y$ is also co-meager. We may assume that A is a dense G_δ set. Assume to the contrary that $R(A)$ is not co-meager. As a continuous image of a Borel set $R(A)$ is analytic, and hence has the Baire property. Thus there exists a non-empty open set $U \subseteq Y$ such that $R(A) \cap U$ is meager. Since R is continuous and surjective $R^{-1}(U)$ is open and non-empty. The map $\widehat{R} = R|_{R^{-1}(U)} : R^{-1}(U) \to U$ is clearly continuous, open and surjective. Since $R(A) \cap U$ is meager $\widehat{R}^{-1}(R(A) \cap U)$ is meager in $R^{-1}(U)$. The set $A \cap R^{-1}(U)$ is co-meager in $R^{-1}(U)$, and clearly $A \cap R^{-1}(U) \subseteq \widehat{R}^{-1}(R(A) \cap U)$, a contradiction.

(*ii*) Let $B \subseteq Y$ be of second category. Assume to the contrary that $R^{-1}(B)$ is meager. Then $R^{-1}(B)^c$ is co-meager and its R image $R(R^{-1}(B)^c) \subseteq B^c$ is not co-meager. This contradicts part (i) of the lemma.

Let $B \subseteq Y$ be co-meager. Then B^c is meager, and hence $R^{-1}(B^c)$ is meager. This implies that $R^{-1}(B) = X \setminus R^{-1}(B^c)$ is co-meager. □

We need the following special case.

Corollary 2.7. *Let $K_1 \subseteq K_2$ be compact metric spaces and*

$$R\colon C(K_2) \to C(K_1), \;\; R(f) = f|_{K_1}.$$

(i) If $\mathcal{F}_2 \subseteq C(K_2)$ is of second category/co-meager then $R(\mathcal{F}_2) \subseteq C(K_1)$ is of second category/co-meager.

(ii) If $\mathcal{F}_1 \subseteq C(K_1)$ is of second category/co-meager then $R^{-1}(\mathcal{F}_1) \subseteq C(K_2)$ is of second category/co-meager.

Proof. Clearly $C(K_2)$ and $C(K_1)$ are complete metric spaces, R is continuous, and Tietze's Extension Theorem implies that R is surjective and open. Thus Lemma 2.6 completes the proof. □

We need the following theorem, see [10, 6.1. Thm.] for the proof.

Theorem 2.8. *Let X, Y be Polish spaces, and let $E \subseteq X \times Y$ be a Borel set. If E_x is σ-compact for all $x \in X$ then the function $h\colon X \to [-1, \infty]$ defined by $h(x) = \dim_H E_x$ is Borel measurable.*

Remark 2.9. Unlike [10], we adopt the convention that $\dim_H \emptyset = -1$, hence the level sets of h may need to be modified by the set $\{x \in X : E_x = \emptyset\} = (\operatorname{pr}_X E)^c$. Therefore we also have to check that $\operatorname{pr}_X E$ is Borel.

Lemma 2.10. *Let K be a compact metric space and $d \in \mathbb{R}$. Then the set*

$$\Delta = \left\{(f, y) \in C(K) \times \mathbb{R} : \dim_H f^{-1}(y) < d\right\}$$

is Borel.

Proof. We check that the conditions of Theorem 2.8 hold for $X = C(K) \times \mathbb{R}$, $Y = K$ and $E = \{(f, y, z) \in C(K) \times \mathbb{R} \times K : f(z) = y\} \subseteq X \times Y$. Clearly X, Y are Polish spaces and E is closed, thus Borel. For every $(f, y) \in X$ the set $E_{(f,y)} = \{z \in K : f(z) = y\} = f^{-1}(y)$ is compact. Finally, the set $\mathrm{pr}_X E = \{(f, y) \in X : y \in f(K)\}$ is closed, hence Borel. Theorem 2.8 implies that $h\colon X \to [0, \infty]$, $h((f, y)) = \dim_H E_{(f,y)} = \dim_H f^{-1}(y)$ is Borel measurable. Thus $h^{-1}((-\infty, d)) = \left\{(f, y) \in C(K) \times \mathbb{R} : \dim_H f^{-1}(y) < d\right\} = \Delta$ is Borel. □

Lemma 2.11. *Suppose (K, d) is a compact metric space such that for all $x \in K$ and $r > 0$ we have $\dim_t B(x, r) > 0$. Let $\mathcal{C}$ be the set of connected components of K. Then for the generic $f \in C(K)$*

$$\bigcup_{C \in \mathcal{C}} \operatorname{int} f(C) = \operatorname{int} f(K).$$

We remark that if K_0 is the triadic Cantor set then $K = K_0 \times [0, 1]$ has uncountably many connected components but it is a 'homogeneous' self-similar set.

Proof. Consider

$$\mathcal{F} = \left\{ f \in C(K) : \bigcup_{C \in \mathcal{C}} \operatorname{int} f(C) = \operatorname{int} f(K) \right\},$$

and for all $n \in \mathbb{N}^+$ let

$$\mathcal{F}_n = \{f \in C(K) : \forall y \in f(K) \setminus B(\partial f(K), 1/n),\ \exists C \in \mathcal{C} \text{ such that } y \in \operatorname{int} f(C)\}.$$

We must prove that $\mathcal{F}$ is co-meager in $C(K)$. Since $\mathcal{F} = \cap_{n \in \mathbb{N}^+} \mathcal{F}_n$, it is enough to show that the $\mathcal{F}_n$'s are co-meager in $C(K)$. Let us fix $n \in \mathbb{N}^+$ and let $f_0 \in C(K)$ and $\varepsilon > 0$ be arbitrary. It is sufficient to show that there is a non-empty ball $B(g_0, r_0) \subseteq \mathcal{F}_n \cap B(f_0, 4\varepsilon)$.

Since f_0 is uniformly continuous on K there is a $\delta_1 > 0$ such that if $x, z \in K$ and $d(x, z) \leq \delta_1$ then $|f_0(x) - f_0(z)| \leq \varepsilon$. By the compactness of K there is a finite set $\{x_1, ..., x_k\}$ such that $\cup_{i=1}^k B(x_i, \delta_1) = K$. Choose $0 < \delta_2 < \delta_1$ such that the balls $B(x_i, \delta_2)$ are disjoint. The conditions of the lemma imply that for every $i \in \{1, \dots, k\}$ we have $\dim_t B(x_i, \delta_2/2) > 0$. Thus there exist non-trivial connected components C_i of $B(x_i, \delta_2/2)$ for all $i \in \{1, \dots, k\}$, see [4, 6.2.9. Thm.]. For all $i \in \{1, \dots, k\}$ let us choose $u_i, v_i \in C_i$, $u_i \neq v_i$ and select $\varepsilon_i \in [\varepsilon, 2\varepsilon]$ such that the set

$$E = \{f_0(x_i) + \varepsilon_i : i = 1, \dots, k\} \cup \{f_0(x_i) - \varepsilon_i : i = 1, \dots, k\}$$

has $2k$ many elements. Let $\theta = \min\{d(x, y) : x, y \in E, x \neq y\} > 0$. Clearly for all $x \in B(x_i, \delta_1)$, $i \in \{1, \dots, k\}$ we have $f_0(x) \in [f_0(x_i) - \varepsilon, f_0(x_i) + \varepsilon] \subseteq [f_0(x_i) - \varepsilon_i, f_0(x_i) + \varepsilon_i]$. Hence Tietze's Extension Theorem implies that there exists a $g_0 \in C(K)$ such that $g_0(x) = f_0(x)$ if $x \in K \setminus \cup_{i=1}^k U(x_i, \delta_2)$ and for all $i \in \{1, \ldots, k\}$ we have $g_0(u_i) = f_0(x_i) - \varepsilon_i$, $g_0(v_i) = f_0(x_i) + \varepsilon_i$ and

$$g_0(x) \in [f_0(x_i) - \varepsilon_i, f_0(x_i) + \varepsilon_i], \quad x \in B(x_i, \delta_1). \tag{2.1}$$

Therefore, using that the oscillations of f_0 on the $B(x_i, \delta_1)$'s are at most ε and $\varepsilon_i \leq 2\varepsilon$ for all $i \in \{1, \ldots, k\}$, we have $g_0 \in B(f_0, 3\varepsilon)$. Set $r_0 = \min\{\varepsilon, \theta/6, 1/(3n)\}$. Since $B(g_0, r_0) \subseteq B(g_0, \varepsilon) \subseteq B(f_0, 4\varepsilon)$, it is enough to prove that $B(g_0, r_0) \subseteq \mathcal{F}_n$. Let $f \in B(g_0, r_0)$ and $y_0 \in f(K) \setminus B(\partial f(K), 1/n)$, that is, $B(y_0, 1/n) \subseteq \operatorname{int} f(K)$. It is enough to verify that there is an $i \in \{1, \ldots, k\}$ such that $y_0 \in \operatorname{int} f(C_i)$. (Note that every C_i is contained in a member of $\mathcal{C}$.) Let us choose $z_0 \in K$ with $f(z_0) = y_0$ and fix $i \in \{1, \ldots, k\}$ such that $z_0 \in B(x_i, \delta_1)$. Then equation (2.1) and $f \in B(g_0, r_0)$ imply that $y_0 \in [f_0(x_i) - \varepsilon_i - r_0, f_0(x_i) + \varepsilon_i + r_0]$.

First assume that $y_0 \in (f_0(x_i) - \varepsilon_i + r_0, f_0(x_i) + \varepsilon_i - r_0) = (g_0(u_i) + r_0, g_0(v_i) - r_0)$. Then $f \in B(g_0, r_0)$ and the connectedness of C_i imply $y_0 \in (f(u_i), f(v_i)) \subseteq \operatorname{int} f(C_i)$.

Finally, suppose that $y_0 \in [f_0(x_i) - \varepsilon_i - r_0, f_0(x_i) - \varepsilon_i + r_0]$ or

$$y_0 \in [f_0(x_i) + \varepsilon_i - r_0, f_0(x_i) + \varepsilon_i + r_0]. \tag{2.2}$$

We may assume by symmetry that (2.2) holds. Since $y_0 + 3r_0 \in B(y_0, 1/n) \subseteq \operatorname{int} f(K)$, there exists $z_1 \in K$ such that $f(z_1) = y_0 + 3r_0$ and $j \in \{1, \ldots, k\}$ such that $z_1 \in B(x_j, \delta_1)$. From $f \in B(g_0, r_0)$ and (2.1) it follows that

$$y_0 + 3r_0 \in [f_0(x_j) - \varepsilon_j - r_0, f_0(x_j) + \varepsilon_j + r_0]. \tag{2.3}$$

Equation (2.2) implies $y_0 + 3r_0 > f_0(x_i) + \varepsilon_i + r_0$, thus we have $j \neq i$. Equation (2.2) also implies $|y_0 - (f_0(x_i) + \varepsilon_i)| \leq r_0$. Therefore the triangle inequality and the definition of θ yield

$$\begin{aligned} |y_0 - (f_0(x_j) - \varepsilon_j)| &\geq |(f_0(x_j) - \varepsilon_j) - (f_0(x_i) + \varepsilon_i)| - |y_0 - (f_0(x_i) + \varepsilon_i)| \\ &\geq \theta - r_0 > 4r_0. \end{aligned} \tag{2.4}$$

Then (2.3) implies $y_0 < f_0(x_j) + \varepsilon_j - r_0$ and $y_0 \geq f_0(x_j) - \varepsilon_j - 4r_0$, thus (2.4) yields $y_0 \in (f_0(x_j) - \varepsilon_j + r_0, f_0(x_j) + \varepsilon_j - r_0) = (g_0(u_j) + r_0, g_0(v_j) - r_0)$. Hence $f \in B(g_0, r_0)$ and the connectedness of C_j imply $y_0 \in (f(u_j), f(v_j)) \subseteq \operatorname{int} f(C_j)$. This completes the proof. □

Lemma 2.12. *Let K be a compact metric space with a fixed $x_0 \in K$. Let $K_n \subseteq K$, $n \in \mathbb{N}$ be compact sets such that*

(i) $\dim_t K_n > 0$ *for all* $n \in \mathbb{N}$ *and*

(ii) $\operatorname{diam}(K_n \cup \{x_0\}) \to 0$ *if* $n \to \infty$.

Then for the generic $f \in C(K)$ we have $x_0 \in f(K_n)$ for infinitely many $n \in \mathbb{N}$.

Proof. Clearly it is enough to show that the sets

$$\mathcal{F}_N = \{f \in C(K) : x_0 \notin f(K_n) \text{ for all } n \geq N\}$$

are nowhere dense in $C(K)$ for all $N \in \mathbb{N}$. Let $f_0 \in C(K)$ and $\varepsilon > 0$ be arbitrary, it is enough to find a ball in $B(f_0, 2\varepsilon) \setminus \mathcal{F}_N$. The compact K_n's have positive topological dimension, therefore they are not totally disconnected, see [4, 6.2.9. Thm.]. Let us choose a non-trivial connected component $C_n \subseteq K_n$ for every $n \in \mathbb{N}$. We can choose by (ii) an $n_0 \in \mathbb{N}$ such that $n_0 \geq N$ and $\operatorname{diam} f_0(C_{n_0} \cup \{x_0\}) < \varepsilon$. Tietze's Extension Theorem implies that there is an $f \in B(f_0, \varepsilon)$ such that $\operatorname{diam} f(C_{n_0}) > 0$ and $f(x_0)$ is the midpoint of $f(C_{n_0})$. If $\delta = \min\left\{\varepsilon, \frac{1}{4}\operatorname{diam} f(C_{n_0})\right\}$ then for all $g \in B(f, \delta)$ we have $g(x_0) \in g(C_{n_0}) \subseteq g(K_{n_0})$, so $g \notin \mathcal{F}_N$. Thus $B(f, \delta) \subseteq B(f_0, 2\varepsilon) \setminus \mathcal{F}_N$. □

The following lemma is probably known, but we could not find an explicit reference, so we outline its proof.

Lemma 2.13. *The Smith-Volterra-Cantor set S is an attractor of an IFS.*

Proof. In the nth step of the construction we remove 2^{n-1} many disjoint open intervals of length $a_n = 1/2^{2n}$, the remaining 2^n disjoint, closed nth level intervals are of length $b_n = \frac{1}{2^n}\left(1 - \sum_{i=1}^{n} 2^{i-1} a_i\right) = 1/2^{n+1} + 1/2^{2n+1}$. Let $\pi\colon S \to \{0,1\}^{\mathbb{N}}$ be the natural homeomorphism, that is, for $x \in S$ and $n \in \mathbb{N}$ we define $\pi(x)(n)$, as follows. There is a unique nth level interval I_n and a unique $(n+1)$st level interval I_{n+1} such that $x \in I_n$ and $x \in I_{n+1}$. Then I_{n+1} is either the left or the right hand side interval of I_n. If it is the left hand side interval then $\pi(x)(n) = 0$, otherwise $\pi(x)(n) = 1$. Let

$$\begin{aligned} &\varphi_1\colon S \to S \cap [0, 1/2], \ \varphi_1(x) = \pi^{-1}\left(0\hat{\ }\pi(x)\right), \\ &\varphi_2\colon S \to S \cap [1/2, 1], \ \varphi_2(x) = \pi^{-1}\left(1\hat{\ }\pi(x)\right) \end{aligned} \tag{2.5}$$

be the natural homeomorphisms onto the left and the right half of S (where $\hat{\ }$ stands for concatenation). Clearly, $S = \varphi_1(S) \cup \varphi_2(S)$, so it is sufficient to prove that φ_1 and φ_2 are contractions. By symmetry it is enough to show that φ_1 is a Lipschitz map with $\operatorname{Lip}(\varphi_1) \leq 1/2$, that is, for all $x, z \in S$

$$|\varphi_1(x) - \varphi_1(z)| \leq \frac{|x - z|}{2}. \tag{2.6}$$

The endpoints of the intervals at the construction are dense in S. Thus we may assume for the proof of (2.6) that x, z are both endpoints of some nth level intervals and $x < z$. Let us assume that in the interval $[x, z]$ there are $\beta_n = \beta_{n,x,z}$ many intervals of length b_n and there are $\alpha_i = \alpha_{i,x,z}$ many open intervals of length a_i, $i \in \{1, \ldots, n\}$. In the interval $[\varphi_1(x), \varphi_1(z)]$ there are β_n many closed intervals of length b_{n+1} and there are α_i many open intervals of length a_{i+1}, $i \in \{1, \ldots, n\}$. These intervals are disjoint, and their union is $[x, z]$ and $[\varphi_1(x), \varphi_1(z)]$ (apart from the endpoints x, z and $\varphi_1(x), \varphi_1(z)$), respectively. We obtain $|x - z| = \beta_n b_n + \sum_{i=1}^{n} \alpha_i a_i$ and $|\varphi_1(x) - \varphi_1(z)| = \beta_n b_{n+1} + \sum_{i=1}^{n} \alpha_i a_{i+1}$. Hence for (2.6) it is enough to prove that $b_{n+1} \leq b_n/2$ and $a_{i+1} \leq a_i/2$ for all $i \in \{1, \ldots, n\}$, but it is clear from the definitions of the b_n's and the a_n's. □

3 Level sets on fractals

Let K be a compact metric space. If $\dim_t K = 0$ then it is well-known that the generic continuous function is one-to-one on K, hence every non-empty level set is a single point.

Thus in the sequel we assume that $\dim_t K > 0$.

Definition 3.1. If K is a compact metric space then let

$$\operatorname{supp} K = \{x \in K : \forall r > 0,\ \dim_{tH} B(x,r) = \dim_{tH} K\}.$$

We say that K is *homogeneous for the topological Hausdorff dimension* if $\operatorname{supp} K = K$.

Remark 3.2. The stability of the topological Hausdorff dimension for closed sets clearly yields $\operatorname{supp} K \neq \emptyset$. If K is self-similar then it is also homogeneous for the topological Hausdorff dimension.

We proved in [1] that if K is homogeneous for the topological Hausdorff dimension then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. Now we prove the opposite direction.

Theorem 3.3. *Let K be a compact metric space with $\dim_t K > 0$. The following statements are equivalent.*

(i) For the generic $f \in C(K)$ for the generic $y \in f(K)$ we have $\dim_H f^{-1}(y) = \dim_{tH} K - 1$.

(ii) K is homogeneous for the topological Hausdorff dimension.

Proof. $(ii) \Rightarrow (i)$: See [1, Thm. 6.22.].

$(i) \Rightarrow (ii)$: Assume to the contrary that $K \setminus \operatorname{supp} K \neq \emptyset$. Then there exist $f_0 \in C(K)$ and $\varepsilon_0 > 0$ such that for all $f \in B(f_0, \varepsilon_0)$ we have $f(K) \setminus f(\operatorname{supp} K) \neq \emptyset$. Let us choose for all $f \in B(f_0, \varepsilon_0)$ an interval I_f such that $I_f \cap f(\operatorname{supp} K) = \emptyset$ and $I_f \cap f(K \setminus \operatorname{supp} K) \neq \emptyset$. Let us define for all $n \in \mathbb{N}^+$

$$K_n = \{x \in K : \operatorname{dist}(x, \operatorname{supp} K) \geq 1/n\}.$$

Then the K_n's are compact and $\cup_{n\in\mathbb{N}^+} K_n = K \setminus \operatorname{supp} K$. The definition of $\operatorname{supp} K$ and the compactness of K_n imply that K_n can be covered with finitely many closed balls of topological Hausdorff dimension less than $\dim_{tH} K$. Then the stability of the topological Hausdorff dimension for closed sets implies

$$\dim_{tH} K_n < \dim_{tH} K \quad (n \in \mathbb{N}^+). \tag{3.1}$$

For all $n \in \mathbb{N}^+$ let

$$\mathcal{F}_n = \left\{f \in C(K_n) : \dim_H f^{-1}(y) \leq \dim_{tH} K_n - 1 \text{ for all } y \in \mathbb{R}\right\}.$$

Define $R_n \colon K \to K_n$, $R_n(f) = f|_{K_n}$ and let $\mathcal{F} = \cap_{n\in\mathbb{N}^+} R_n^{-1}(\mathcal{F}_n)$. Theorem 1.3 yields that the $\mathcal{F}_n$'s are co-meager in $C(K_n)$, and it follows from Corollary

2.7 that the $R_n^{-1}(\mathcal{F}_n)$'s are co-meager in $C(K)$. As $\mathcal{F}$ is the intersection of countable many co-meager sets, it is also co-meager in $C(K)$. If $f \in B(f_0, \varepsilon)$ and $y \in I_f \cap f(K)$ then the definition of I_f and the compactness of $f^{-1}(y)$ imply that there is an $n_{f,y} \in \mathbb{N}^+$ such that $f^{-1}(y) \subseteq K_{n_{f,y}}$. If $f \in \mathcal{F}$ then for all $y \in I_f \cap f(K)$ the definition of $n_{f,y}$, the definition of $\mathcal{F}$ and (3.1) imply

$$\begin{aligned}\dim_H f^{-1}(y) &= \dim_H \left(f^{-1}(y) \cap K_{n_{f,y}}\right) \leq \dim_{tH} K_{n_{f,y}} - 1 \\ &< \dim_{tH} K - 1.\end{aligned}$$

This contradicts (i), and the proof is complete. □

B. Kirchheim showed in [8] that for the generic $f \in C\left([0,1]^d\right)$ for every $y \in \operatorname{int} f\left([0,1]^d\right)$ we have $\dim_H f^{-1}(y) = d - 1$. We generalize this result for weakly self-similar compact metric spaces.

Definition 3.4. Let K be a compact metric space. We say that K is *weakly self-similar* if for all $x \in K$ and $r > 0$ there exist a compact set $K_{x,r} \subseteq B(x,r)$ and a bi-Lipschitz map $\phi_{x,r} \colon K_{x,r} \to K$.

Remark 3.5. If K is self-similar then it is also weakly self-similar. If K is weakly self-similar then it is also homogeneous for the topological Hausdorff dimension.

Theorem 3.6. *Let K be a weakly self-similar compact metric space. Then for the generic $f \in C(K)$ for any $y \in \operatorname{int} f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Proof. If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, and $f(K)$ is nowhere dense. Thus $\operatorname{int} f(K) = \emptyset$, and the statement is obvious.

Next we assume $\dim_t K > 0$. Theorem 1.3 implies that for the generic $f \in C(K)$ for all $y \in \mathbb{R}$ we have $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$, thus we only need to verify the opposite inequality.

Fact 2.1 implies $\dim_{tH} K > 0$. It follows from the weak self-similarity of K that for all $x \in K$ and $r > 0$ we have $\dim_{tH} B(x,r) = \dim_{tH} K > 0$. Then applying Fact 2.1 again we obtain that $\dim_t B(x,r) > 0$. If $\mathcal{C}$ denotes the set of connected components of K then Lemma 2.11 yields that for the generic $f \in C(K)$ we have $\cup_{C \in \mathcal{C}} \operatorname{int} f(C) = \operatorname{int} f(K)$.

Thus it is enough to prove that for the generic $f \in C(K)$ for every $y \in \cup_{C \in \mathcal{C}} \operatorname{int} f(C)$ we have $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1$.

Let us choose a sequence $0 < d_n \nearrow \dim_{tH} K$ and let us fix $n \in \mathbb{N}^+$. Theorem 1.3 implies that for the generic $f \in C(K)$ there exists an interval $I(f,n) = I_{f, \dim_{tH} K - d_n}$ such that for all $y \in I(f,n)$ we have $\dim_H f^{-1}(y) \geq d_n - 1$. By Baire's Category Theorem there are $m_2 < m_1 < M_1 < M_2$ such that

$$\mathcal{H}_n = \{f \in C(K) : f(K) \subseteq [m_2, M_2],\ \forall y \in [m_1, M_1],\ \dim_H f^{-1}(y) \geq d_n - 1\}$$

is of second category. Note that $d_n > 0$ implies that for every $f \in \mathcal{H}_n$ we have $[m_1, M_1] \subseteq f(K)$. Let us also define the following set.

$$\mathcal{G}_n = \left\{ f \in C(K) : \forall y \in \bigcup_{C \in \mathcal{C}} (f(C) \setminus B(\partial f(C), 1/n)), \ \dim_H f^{-1}(y) \geq d_n - 1 \right\}.$$

It is sufficient to verify that $\mathcal{G}_n$ is co-meager, since by taking the intersection of the sets $\mathcal{G}_n$ for all $n \in \mathbb{N}^+$ we obtain the desired co-meager set in $C(K)$. In order to prove this we show that $\mathcal{G}_n$ contains 'certain copies' of $\mathcal{H}_n$. First we need the following lemma.

Lemma 3.7. *$\mathcal{H}_n$ and $\mathcal{G}_n$ have the Baire property.*

Proof of Lemma 3.7. Lemma 2.10 implies that $\Gamma_n = \{(f, y) \in C(K) \times \mathbb{R} : \dim_H f^{-1}(y) < d_n - 1\}$ is Borel. Then $\mathcal{H}_n = \{f \in C(K) : f(K) \subseteq [m_2, M_2]\} \cap \{f \in C(K) : \forall y \in [m_1, M_1], \dim_H f^{-1}(y) \geq d_n - 1\}$. The first term of the intersection is clearly closed. It is sufficient to prove that the second one has the Baire property. It equals $\left(\mathrm{pr}_{C(K)} \left((C(K) \times [m_1, M_1]) \cap \Gamma_n \right)\right)^c$, which is the complement of the projection of a Borel set. Hence it is co-analytic, and therefore has the Baire property.

The set

$$\Delta_n = \left\{ (f, y) \in C(K) \times \mathbb{R} : y \in \bigcup_{C \in \mathcal{C}} \Big(f(C) \setminus B(\partial f(C), 1/n) \Big) \right\}$$

is clearly open. Then $\mathcal{G}_n = \left(\mathrm{pr}_{C(K)} (\Gamma_n \cap \Delta_n)\right)^c$, which is the complement of the projection of a Borel set. Thus it is co-analytic, and therefore has the Baire property. □

Now we return to the proof of Theorem 3.6. Consider $\mathcal{G}_n$ (note that we already fixed n), our aim is to show that $\mathcal{G}_n$ is co-meager. Since $\mathcal{G}_n$ has the Baire property, it is enough to prove that $\mathcal{G}_n$ is of second category in every non-empty open subset of $C(K)$. Let $f_0 \in C(K)$ and $0 < \varepsilon < 1/n$ be fixed. We want to show that $\mathcal{G}_n \cap B(f_0, \varepsilon)$ is of second category.

The continuity of f_0 and the compactness of K imply that there are finitely many distinct $x_1, ..., x_k \in K$ and positive $r_1, ..., r_k$ such that

$$K = \bigcup_{i=1}^{k} B(x_i, r_i) \tag{3.2}$$

and for each $i \in \{1, \dots, k\}$ the oscillation of f_0 on $B(x_i, r_i)$ is less than

$$\omega = \frac{\varepsilon(M_1 - m_1)}{2(M_2 - m_2)} < \frac{\varepsilon}{2}. \tag{3.3}$$

Choose positive $r'_1, ..., r'_k$ such that the balls $B(x_i, r'_i) \subseteq B(x_i, r_i)$ are disjoint. Using the weak self-similarity property we can choose for every $i \in \{1, \dots, k\}$ a set $K_i \subseteq B(x_i, r_i')$ and a bi-Lipschitz map $\phi_i \colon K_i \to K$. Let us fix $i \in \{1, \dots, k\}$. We define the affine function $\psi_i \colon \mathbb{R} \to \mathbb{R}$ such that

$$\psi_i([m_1, M_1]) = [f_0(x_i) - \omega, f_0(x_i) + \omega]. \tag{3.4}$$

Suppose $f \in \mathcal{H}_n$ and consider $\widehat{f}_i \in C(K_i)$ defined by

$$\widehat{f}_i = \psi_i \circ f \circ \phi_i.$$

The form of ψ_i, (3.4) and (3.3) imply

$$\begin{aligned} \operatorname{diam} \widehat{f}_i(K_i) &= \operatorname{diam} \psi_i(f(K)) \le \operatorname{diam} \psi_i([m_2, M_2]) \\ &= \frac{M_2 - m_2}{M_1 - m_1} \operatorname{diam} \psi_i([m_1, M_1]) \\ &= \frac{M_2 - m_2}{M_1 - m_1} 2\omega = \varepsilon. \end{aligned}$$

Then $f_0(K_i) \subseteq \widehat{f}_i(K_i)$ and the above inequality yield for all $x \in K_i$

$$\left| f_0(x) - \widehat{f}_i(x) \right| \le \varepsilon. \tag{3.5}$$

Set

$$\widehat{\mathcal{F}}_i = \{\psi_i \circ f \circ \phi_i : f \in \mathcal{H}_n\}.$$

It follows from (3.5) that $\widehat{\mathcal{F}}_i \subseteq B(f_0|_{K_i}, \varepsilon)$. The maps $\phi_i \colon K_i \to K$ and $\psi_i \colon \mathbb{R} \to \mathbb{R}$ are homeomorphisms, hence the map $G_i \colon C(K) \to C(K_i)$, $G_i(f) = \psi_i \circ f \circ \phi_i$ is also a homeomorphism. Since $\mathcal{H}_n$ is of second category in $C(K)$ we obtain that $\widehat{\mathcal{F}}_i = G_i(\mathcal{H}_n)$ is of second category in $C(K_i)$. Set

$$\mathcal{F}_i = \left\{ f \in B(f_0, \varepsilon) : f|_{K_i} \in \widehat{\mathcal{F}}_i \right\}.$$

The map $\widehat{R}_i \colon B(f_0, \varepsilon) \to B(f_0|_{K_i}, \varepsilon)$, $\widehat{R}_i(f) = f|_{K_i}$ is clearly continuous, and by Tietze's Extension Theorem it is also surjective and open. Thus Lemma 2.6 (ii) implies that $\mathcal{F}_i = \widehat{R}_i^{-1}\left(\widehat{\mathcal{F}}_i\right)$ is of second category in $B(f_0, \varepsilon)$. Set

$$\mathcal{F} = \bigcap_{i=1}^{k} \mathcal{F}_i.$$

Clearly $\mathcal{F} \subseteq B(f_0, \varepsilon)$.

Lemma 3.8. *$\mathcal{F}$ is of second category in $B(f_0, \varepsilon)$.*

Proof of Lemma 3.8. Let

$$R \colon B(f_0, \varepsilon) \to B\left(f_0|_{\bigcup_{i=1}^k K_i}, \varepsilon\right), \ R(f) = f|_{\bigcup_{i=1}^k K_i}$$

and for all $i \in \{1, \dots, k\}$

$$R_i \colon B\left(f_0|_{\bigcup_{i=1}^k K_i}, \varepsilon\right) \to B\left(f_0|_{K_i}, \varepsilon\right), \ R_i(f) = f|_{K_i}.$$

Clearly the map R is continuous, open and surjective. Since $\mathcal{F} = R^{-1}\left(\cap_{i=1}^k R_i^{-1}\left(\widehat{\mathcal{F}}_i\right)\right)$, it follows from Lemma 2.6 (ii) that it is enough to prove that $\cap_{i=1}^k R_i^{-1}\left(\widehat{\mathcal{F}}_i\right)$ is of second category in $B\left(f_0|_{\cup_{i=1}^k K_i}, \varepsilon\right)$. Lemma 3.7 implies that $\mathcal{H}_n$ and hence $\widehat{\mathcal{F}}_i$ has the Baire property for every $i \in \{1, \dots, k\}$. Thus there is a non-empty open set $\mathcal{U}_i \subseteq C(K_i)$ such that $\widehat{\mathcal{F}}_i$ is co-meager in $\mathcal{U}_i$. The sets K_i, $i \in \{1, \dots, k\}$ are disjoint. Hence $\cap_{i=1}^k R_i^{-1}(\mathcal{U}_i) \subseteq B\left(f_0|_{\cup_{i=1}^k K_i}, \varepsilon\right)$ is a non-empty open set, and $\cap_{i=1}^k R_i^{-1}\left(\widehat{\mathcal{F}}_i\right)$ is co-meager in $\cap_{i=1}^k R_i^{-1}\left(\mathcal{U}_i\right)$. Therefore, it is of second category in $B\left(f_0|_{\cup_{i=1}^k K_i}, \varepsilon\right)$. □

Now we return to the proof of Theorem 3.6. We prove that $\mathcal{F} \subseteq \mathcal{G}_n$ and then Lemma 3.8 will imply that $\mathcal{G}_n$ is of second category in $B(f_0, \varepsilon)$. Assume that $g \in \mathcal{F}$. Let $y_0 \in \bigcup_{C \in \mathcal{C}} \left(g(C) \setminus B(\partial g(C), 1/n)\right)$ be arbitrary. Then there is a $C_0 \in \mathcal{C}$ such that $B(y_0, 1/n) \subseteq \operatorname{int} g(C_0)$. The connectedness of C_0 and $g \in B(f_0, \varepsilon)$ yield $y_0 \in f_0(C_0) \subseteq f_0(K)$. Hence the definition of ω and (3.2) imply that there is an $i \in \{1, \dots, k\}$ such that $y_0 \in [f_0(x_i) - \omega, f_0(x_i) + \omega]$. The definition of $\mathcal{F}$ yields that there exists an $f \in \mathcal{H}_n$ such that $g|_{K_i} = \psi_i \circ f \circ \phi_i = \widehat{f}_i$. Then (3.4) implies $\psi_i^{-1}(y_0) \in [m_1, M_1]$, and $f \in \mathcal{H}_n$ implies $\dim_H f^{-1}\left(\psi_i^{-1}(y_0)\right) \geq d_n - 1$. By the bi-Lipschitz property of ϕ_i we infer

$$\begin{aligned}\dim_H g^{-1}(y_0) &\geq \dim_H \widehat{f}_i^{-1}(y_0) = \dim_H \phi_i^{-1}\left(f^{-1}\left(\psi_i^{-1}(y_0)\right)\right) \\ &= \dim_H f^{-1}\left(\psi_i^{-1}(y_0)\right) \geq d_n - 1.\end{aligned}$$

Therefore $g \in \mathcal{G}_n$, and hence $\mathcal{F} \subseteq \mathcal{G}_n$. This completes the proof. □

It is natural to ask what we can say about the level sets of *every* $f \in C(K)$. Clearly we cannot hope that for every $y \in \operatorname{int} f(K)$ the level set $f^{-1}(y)$ is of small Hausdorff dimension, since f can be constant on a large set. The opposite direction is less trivial, it is easy to prove that for every $f \in C\left([0,1]^2\right)$ for every $y \in \operatorname{int} f\left([0,1]^2\right)$ we have $\dim_H f^{-1}(y) \geq 1 = \dim_{tH} [0,1]^2 - 1$. This is not true in general even for connected self-similar metric spaces. We have the following counterexample.

Example 3.9. Set $K = [-1,0]^2 \cup [0,1]^2$. Clearly K is a connected compact metric space, and Figure 1 shows that K is self-similar with 4 contractions. Let $f \colon K \to \mathbb{R}$, $f(x,y) = x + y$. It is straightforward that $f \in C(K)$, $0 \in \operatorname{int} f(K)$ and $f^{-1}(0) = (0,0)$. Clearly $\dim_{tH} K = 2$, but $\dim_H f^{-1}(0) = 0 < 1 = \dim_{tH} K - 1$.

Does at least some weaker statement hold?

Question 3.10. *Let K be a connected self-similar compact metric space. Is it true that for every $f \in C(K)$ there exists a $y_f \in \mathbb{R}$ such that $\dim_H f^{-1}(y_f) \geq \dim_{tH} K - 1$?*

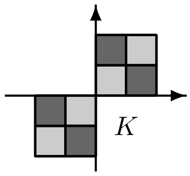

Figure 1: Illustration to Example 3.9

4 Level sets of maximal dimension

Let K be a compact metric space. If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, and every non-empty level set is a single point.

Assume $\dim_t(K) > 0$. Corollary 1.4 states that for the generic $f \in C(K)$ we have $\sup_{y\in\mathbb{R}} \dim_H f^{-1}(y) = \dim_{tH} K - 1$. First we prove that in this statement the supremum is attained.

Theorem 4.1. *Let K be a compact metric with $\dim_t K > 0$. Then for the generic $f \in C(K)$*

$$\max_{y\in\mathbb{R}} \dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Proof. By Theorem 1.3 it is sufficient to prove that for the generic $f \in C(K)$ there exists a level set of Hausdorff dimension at least $\dim_{tH} K - 1$. Let us fix $x_0 \in \operatorname{supp} K$. We will show that for the generic $f \in C(K)$ we have $\dim_H f^{-1}(f(x_0)) \geq \dim_{tH} K - 1$. The following lemma is the heart of the proof.

Lemma 4.2. *Let $K_1 \subseteq K$ be compact metric spaces with $x_0 \in K \setminus K_1$. Let $d \in \mathbb{R}$ be such that $\dim_{tH} B(x,r) > d$ for all $x \in K_1$ and $r > 0$. Then for the generic $f \in C(K)$ either $\dim_H f^{-1}(f(x_0)) \geq d-1$ or $f(x_0) \notin f(K_1)$.*

Proof of Lemma 4.2. If $d \leq 0$ then the statement is vacuous, so we may assume $d > 0$. We must prove that the set

$$\mathcal{F} = \left\{f \in C(K) : \dim_H f^{-1}(f(x_0)) \geq d-1 \text{ or } f(x_0) \notin f(K_1)\right\}$$

is co-meager in $C(K)$. Let $K_2 = B(K_1, \varepsilon_0)$ with such a small $\varepsilon_0 > 0$ that $x_0 \notin K_2$. Consider

$$\Gamma = \left\{(f,y) \in C(K_2) \times \mathbb{R} : \dim_H f^{-1}(y) \geq d-1 \text{ or } y \notin f(K_1)\right\}.$$

First assume that Γ is co-meager in $C(K_2) \times \mathbb{R}$. Then we prove that $\mathcal{F} \subseteq C(K)$ is also co-meager. Let $R\colon C(K) \to C(K_2) \times \mathbb{R}$, $R(f) = (f|_{K_2}, f(x_0))$. Clearly R is continuous, and Tietze's Extension Theorem implies that R is surjective and open. Thus Lemma 2.6 implies that $\mathcal{F} = R^{-1}(\Gamma)$ is co-meager.

Finally, we prove that Γ is co-meager in $C(K_2) \times \mathbb{R}$. Lemma 2.10 easily implies that Γ is Borel, thus has the Baire property. Hence it is enough to prove by the Kuratowski-Ulam Theorem [7, 8.41 Thm.] that for the generic $f \in C(K_2)$ for the generic $y \in \mathbb{R}$ we have $(f, y) \in \Gamma$. Let $\{z_n\}_{n\in\mathbb{N}^+}$ be a dense set in K_1 and for $i, j \in \mathbb{N}^+$ let us define $B_{i,j} = B(z_i, 1/j)$ if $1/j \leq \varepsilon_0$, and $B_{i,j} = K_2$ otherwise. Then for all $i, j \in \mathbb{N}^+$ we have $B_{i,j} \subseteq K_2$ and the conditions of the lemma yield $\dim_{tH} B_{i,j} > d$. Let $R_{i,j} \colon C(K_2) \to C(B_{i,j})$, $R_{i,j}(f) = f|_{B_{i,j}}$ and let

$$\mathcal{G}_{i,j} = \left\{ f \in C(B_{i,j}) : \exists I \text{ interval s.t. } \forall y \in I \ \dim_H f^{-1}(y) \geq d - 1 \right\}.$$

Set

$$\mathcal{G} = \bigcap_{i,j\in\mathbb{N}^+} R_{i,j}^{-1}(\mathcal{G}_{i,j}).$$

It follows from Theorem 1.3 that $\mathcal{G}_{i,j}$ is co-meager in $C(B_{i,j})$ for every $i, j \in \mathbb{N}^+$. Corollary 2.7 implies that $R_{i,j}^{-1}(\mathcal{G}_{i,j})$ is co-meager in $C(K_2)$, and as a countable intersection of co-meager sets $\mathcal{G}$ is also co-meager in $C(K_2)$. We fix $f \in \mathcal{G}$. It is sufficient to verify that $\Gamma_f = \{y \in \mathbb{R} : (f, y) \in \Gamma\}$ is co-meager. Let $U \subseteq \mathbb{R}$ be an arbitrary open interval. It is enough to prove that $\Gamma_f \cap U$ contains an interval. If there exists $y_0 \in U$ such that $y_0 \notin f(K_1)$ then there is a $\delta > 0$ such that $B(y_0, \delta) \cap f(K_1) = \emptyset$, so $B(y_0, \delta) \cap U$ is an interval in $\Gamma_f \cap U$. Thus we may assume $U \subseteq f(K_1)$. Then there exist $i_0, j_0 \in \mathbb{N}^+$ such that $B_0 = B_{i_0,j_0}$ satisfies $f(B_0) \subseteq U$. The definition of $\mathcal{G}$ implies that there is an interval $I_{f|_{B_0}} \subseteq U$ such that for all $y \in I_{f|_{B_0}}$ we have

$$\dim_H f^{-1}(y) \geq \dim_H (f|_{B_0})^{-1}(y) \geq d - 1.$$

Hence $I_{f|_{B_0}} \subseteq \Gamma_f \cap U$, and this completes the proof. □

Now we return to the proof of Theorem 4.1. It follows from Fact 2.1 that $\dim_{tH} K > 0$. Since $\dim_{tH} B(x_0, 1/n) = \dim_{tH} K$ for all $n \in \mathbb{N}^+$, the countable stability of the topological Hausdorff dimension for closed sets implies the following. For all $n \in \mathbb{N}^+$ there exist $r_n > 0$ such that the sets $C_n = B(x_0, 1/n) \setminus U(x_0, r_n)$ satisfy $\dim_{tH} C_n > 0$ and $\dim_{tH} C_n \to \dim_{tH} K$ as $n \to \infty$. For all $n \in \mathbb{N}^+$ we put

$$K_n = \{x \in C_n : \forall r > 0, \ \dim_{tH}(C_n \cap B(x, r)) \geq \dim_{tH} C_n - 1/n\}.$$

Clearly, the K_n's are compact. First we prove that for all $n \in \mathbb{N}^+$ we have $\dim_{tH} K_n = \dim_{tH} C_n > 0$. The definition of K_n and the Lindelöf property of $C_n \setminus K_n$ imply that there are closed balls B_i, $i \in \mathbb{N}$ in C_n such that $\dim_{tH} B_i \leq \dim_{tH} C_n - 1/n$ and $\cup_{i\in\mathbb{N}} B_i = C_n \setminus K_n$. Applying the countable stability of the topological Hausdorff dimension for the closed sets $\{B_i : i \in \mathbb{N}\} \cup \{K_n\}$ yields $\dim_{tH} K_n = \dim_{tH} C_n$.

Then Fact 2.1 implies $\dim_t K_n > 0$, and the K_n's satisfy the conditions of Lemma 2.12. Applying Lemma 2.12 for the sequence $\langle K_n \rangle_{n\in\mathbb{N}^+}$ and the compact set K, and applying Lemma 4.2 for all $K_n \subseteq K$ with $d_n = \dim_{tH} C_n - 2/n$

simultaneously imply that for the generic $f \in C(K)$ we have $x_0 \in f(K_n)$ for infinitely many $n \in \mathbb{N}^+$, and for every $n \in \mathbb{N}^+$ either $\dim_{tH} f^{-1}(f(x_0)) \geq d_n - 1$ or $x_0 \notin f(K_n)$. Hence there is a subsequence $\langle n_i \rangle_{i\in\mathbb{N}}$ (that depends on f) such that $\dim_{tH} f^{-1}(f(x_0)) \geq d_{n_i} - 1$ for all $i \in \mathbb{N}$, that is

$$\dim_{tH} f^{-1}(f(x_0)) \geq \lim_{i\to\infty} (\dim_{tH} C_{n_i} - 2/n_i - 1) = \dim_{tH} K - 1.$$

This concludes the proof. □

Remark 4.3. Note that we proved the following stronger statement. Let K be a compact metric space with $\dim_t K > 0$. Set for all $x \in K$

$$\mathcal{F}_x = \left\{ f \in C(K) : \dim_H f^{-1}(f(x)) = \dim_{tH} K - 1 \right\}.$$

Then $\mathcal{F}_x$ is co-meager in $C(K)$ for every $x \in \operatorname{supp} K$.

The following example shows that the sets $\mathcal{F}_x$, $x \in \operatorname{supp} K$ depend on x indeed in general.

Example 4.4. Let K be a self-similar compact metric space with $\dim_t K > 1$. It is well-known and easy to prove that for the generic $f \in C(K)$ the maximum is attained at a unique point, say x_f. By Theorem 2.2 for the generic $f \in C(K)$ we have $\dim_H f^{-1}(f(x_f)) = 0 < \dim_t K - 1 \leq \dim_{tH} K - 1$, thus $f \notin \mathcal{F}_{x_f}$. Clearly, $\operatorname{supp} K = K$, therefore $\cap_{x\in \operatorname{supp} K} \mathcal{F}_x = \cap_{x\in K} \mathcal{F}_x$ is of first category in $C(K)$.

The following theorem shows that we cannot strengthen Theorem 4.1 in general. Since the counterexample is an attractor of an iterated function system, it is 'homogeneous' to some extent.

Theorem 4.5. *There exists a compact set $K \subseteq \mathbb{R}^2$ such that K is an attractor of an iterated function system and the generic $f \in C(K)$ has a unique level set of Hausdorff dimension $\dim_{tH} K - 1$.*

Proof. Let S and C be the Smith-Volterra-Cantor set and the middle-thirds Cantor set, respectively. Let

$$\begin{aligned} &\psi_1 \colon C \to C \cap [0, 1/3], \ \psi_1(x) = x/3, \\ &\psi_2 \colon C \to C \cap [2/3, 1], \ \psi_2(x) = x/3 + 2/3 \end{aligned} \tag{4.1}$$

be the natural similarities of C.

Let us define $\alpha_n < 1$ ($n \in \mathbb{N}^+$) such that $\alpha_n \searrow 1/3$ as $n \to \infty$. Let $C_n = C_{\alpha_n}$, $n \in \mathbb{N}^+$ be the middle-α_n Cantor sets. Then clearly $\dim_H C_n \nearrow \dim_H C$ as $n \to \infty$. It is easy to verify that the natural homeomorphisms $\phi_n \colon C \to C_n$, $n \in \mathbb{N}^+$ are Lipschitz maps. For $r > 0$ we denote by C_n^r the set that is similar to C_n, furthermore $C_n^r \subseteq [0, r]$ and $\operatorname{diam} C_n^r = r$. We define positive numbers r_n, $n \in \mathbb{N}^+$ such that the following conditions hold for every $n \in \mathbb{N}^+$.

(i) There are Lipschitz maps with Lipschitz constant at most 1/2 which map the nth level elementary pieces of S onto $[0, r_n]$.

(ii) There are Lipschitz maps with Lipschitz constant at most $1/2$ which map the nth level elementary pieces of C onto $C_n^{r_n}$.

(iii) $\sum_{i=n}^{\infty} r_i \leq 1/2^{2n+2}$.

The nth level elementary pieces of S are isometric. They are of positive Lebesgue measure, since S is of positive measure. It is well-known that every measurable set with positive measure can be mapped onto $[0, 1]$ by a Lipschitz map [1, Lemma 3.10.], hence (i) can be satisfied if r_n is small enough. Moreover, (ii) follows from the Lipschitz property of ϕ_n for small enough r_n, and (iii) is straightforward.

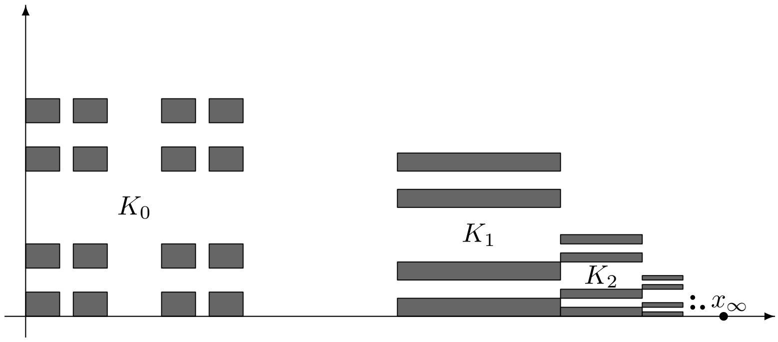

Figure 2: Illustration to the construction of K

Let $K_0 = S \times C$, $x_\infty = (2 + \sum_{i=1}^{\infty} r_i, 0)$ and for all $n \in \mathbb{N}^+$ let

$$\begin{aligned} I_n &= \left[2 + \textstyle\sum_{i=1}^{n-1} r_i, 2 + \sum_{i=1}^{n} r_i\right], \\ K_n &= I_n \times C_n^{r_n}, \\ K &= \bigcup_{n=0}^{\infty} K_n \cup \{x_\infty\}, \\ \widetilde{K}_n &= \bigcup_{i=n}^{\infty} K_i \cup \{x_\infty\}, \\ \widehat{K}_n &= \bigcup_{i=0}^{n} K_i. \end{aligned}$$

Clearly, all the sets defined above are compact.

First we prove that K is an attractor of an IFS. Recall that the φ_i's and ψ_j's are the natural homeomorphisms of S and C, respectively. For the more precise definition see (2.5) and (4.1) again. Let us define for $i, j \in \{1, 2\}$ the maps $\Psi_{i,j} : K \to K_0$ such that

$$\Psi_{i,j}(x) = \begin{cases} (0,0) & \text{if } x \in K \setminus K_0, \\ (\varphi_i(x), \psi_j(x)) & \text{if } x \in K_0. \end{cases}$$

Clearly the $\Psi_{i,j}$'s are Lipschitz maps with $\mathrm{Lip}(\Psi_{i,j}) \leq 1/2$, and $\cup_{i,j \in \{1,2\}} \Psi_{i,j}(K) = K_0$. For all $n \in \mathbb{N}^+$ and $(i,j) \in \{(1,2),(2,1),(2,2)\}$ let us define the sets $K_{i,j,n}$ to be the top left, the top right and the bottom right nth level 'elementary pieces' of the bottom left $(n-1)$st 'elementary piece' of K_0, that is,

$$K_{i,j,n} = \left(\varphi_i \circ \varphi_1^{n-1}\right)(S) \times (\psi_j \circ \psi_1^{n-1})(C).$$

These are clearly disjoint subsets of K_0. It follows from (i) and (ii) that for all $n \in \mathbb{N}^+$ and $(i,j) \in \{(1,2),(2,1),(2,2)\}$ there exist surjective Lipschitz maps

$$\varphi_{i,n} \colon \left(\varphi_i \circ \varphi_1^{n-1}\right)(S) \to I_n \quad \text{and} \quad \psi_{j,n} \colon \left(\psi_j \circ \psi_1^{n-1}\right)(C) \to C_n^{r_n}$$

with Lipschitz constant at most $1/2$. Let $\Psi \colon K \to K \setminus K_0$ be the following map.

$$\Psi(x) = \begin{cases} x_\infty & \text{if } x \in K \setminus K_0 \text{ or } x = (0,0), \\ (\varphi_{i,n}(x), \psi_{j,n}(x)) & \text{if } x \in K_{i,j,n}. \end{cases}$$

The $K_{i,j,n}$'s, $K \setminus K_0$ and $\{(0,0)\}$ are disjoint sets with union K, so Ψ is well-defined. Clearly Ψ maps $K_{i,j,n}$ onto K_n, and hence $\Psi(K) = K \setminus K_0$. Thus $K = \cup_{i,j \in \{1,2\}} \Psi_{i,j}(K) \cup \Psi(K)$. Therefore, it is enough to prove that Ψ is a Lipschitz map with $\mathrm{Lip}(\Psi) \leq 1/2$, that is for all $x, z \in K$

$$|\Psi(x) - \Psi(z)| \leq \frac{|x - z|}{2}. \tag{4.2}$$

If $x, z \in K \setminus K_0$ then $\Psi(x) = \Psi(z) = x_\infty$, thus (4.2) follows.
If $x \in K_0$ and $z \in K \setminus K_0$, then clearly $|x - z| \geq 1$. On the other hand, (iii) implies

$$\begin{aligned} |\Psi(x) - \Psi(z)| \leq \mathrm{diam}(K \setminus K_0) &\leq \sqrt{\left(\sum_{i=1}^{\infty} r_i\right)^2 + (r_1)^2} \\ &< 2\sum_{i=1}^{\infty} r_i \leq 1/8, \end{aligned}$$

therefore (4.2) follows.
If $x = (x_1, x_2) \in K_0$ and $z = (z_1, z_2) \in K_0$ then we may assume that

$$\max\{|z_1|, |z_2|\} \leq \max\{|x_1|, |x_2|\}. \tag{4.3}$$

If $x = (0,0)$ then $z = (0,0)$ and we are done. We may assume $x \in K_{i,j,n}$, where $n \in \mathbb{N}^+$ and $(i,j) \in \{(1,2),(2,1),(2,2)\}$. If $z \in K_{i,j,n}$ then (4.2) follows, since

Ψ is Lipschitz on $K_{i,j,n}$ with Lipschitz constant at most $1/2$. Hence we may assume $z \in K \setminus K_{i,j,n}$. Then (4.3) implies $\Psi(x), \Psi(z) \in \widetilde{K}_n$. By the definition of $\widetilde{K}_n$ and (iii)

$$|\Psi(x) - \Psi(z)| \leq \operatorname{diam} \widetilde{K}_n < 2\sum_{i=n}^{\infty} r_i \leq 1/2^{2n+1}.$$

The minimum distance between distinct nth level elementary pieces of S and C is $1/2^{2n}$ and $1/3^n$, respectively. Since $K_0 = S \times C$,

$$\begin{aligned} |x - z| &\geq \operatorname{dist}(K_{i,j,n}, K \setminus K_{i,j,n}) \\ &\geq \operatorname{dist}(K_{i,j,n}, K_0 \setminus K_{i,j,n}) \geq 1/2^{2n}. \end{aligned}$$

These imply (4.2), and hence K is an attractor of an IFS.

Finally, we prove that the generic $f \in C(K)$ has a unique level set of Hausdorff dimension $\dim_{tH} K - 1$.

By Theorem 4.1 the generic $f \in C(K)$ has at least one level set of Hausdorff dimension $\dim_{tH} K - 1$. Hence it is enough to show that for the generic $f \in C(K)$ for all $y \neq f(x_\infty)$ we have $\dim_H f^{-1}(y) < \dim_{tH} K - 1$. From Fact 2.1 follows $\dim_{tH} K_0 = 0$, clearly $\dim_{tH}\{x_\infty\} = 0$ and Theorem 2.4 implies $\dim_{tH} K_n - 1 = \dim_H C_n$. This, together with the countable stability of the topological Hausdorff dimension for closed sets and the definition of C_n yield

$$\dim_{tH} K - 1 = \sup_{n \in \mathbb{N}^+} \dim_{tH} K_n - 1 = \sup_{n \in \mathbb{N}^+} \dim_H C_n = \dim_H C.$$

Assume to the contrary that there exists $\mathcal{F} \subseteq C(K)$ such that $\mathcal{F}$ is of second category and for every $f \in \mathcal{F}$ there exists $y_f \neq f(x_\infty)$ such that $\dim_H f^{-1}(y_f) = \dim_H C$. Then $f^{-1}(y_f) \subseteq K \setminus \{x_\infty\}$, and by the compactness of $f^{-1}(y_f)$ there exists an $n_f \in \mathbb{N}^+$ such that $f^{-1}(y_f) \subseteq \widehat{K}_{n_f}$. Set

$$\mathcal{F}_n = \left\{ f \in \mathcal{F} : f^{-1}(y_f) \subseteq \widehat{K}_n \right\}.$$

Since $\mathcal{F} = \cup_{n=1}^{\infty} \mathcal{F}_n$, Baire's Category Theorem implies that there exists $n_0 \in \mathbb{N}^+$ such that $\mathcal{F}_{n_0}$ is of second category in $C(K)$. We obtain from Corollary 2.7 (i) that

$$\widehat{\mathcal{F}}_{n_0} = \left\{ f|_{\widehat{K}_{n_0}} : f \in \mathcal{F}_{n_0} \right\}$$

is of second category in $C\left(\widehat{K}_{n_0}\right)$. The definition of $\widehat{\mathcal{F}}_{n_0}$ implies that for every $f \in \widehat{\mathcal{F}}_{n_0}$ we have $\dim_H f^{-1}(y_f) = \dim_H C$. By Theorem 1.3 for the generic $f \in C\left(\widehat{K}_{n_0}\right)$ every level set is of Hausdorff dimension at most

$$\dim_{tH} \widehat{K}_{n_0} - 1 = \sup_{1 \leq n \leq n_0} \dim_{tH} K_n - 1 = \dim_H C_{n_0} < \dim_H C,$$

a contradiction. This concludes the theorem. □

Question 4.6. *Does there exist an attractor of an* injective *iterated function system K such that the generic $f \in C(K)$ has a unique level set of Hausdorff dimension* $\dim_{tH} K - 1$*?*

5 The dimension of the graph of the generic continuous function

The graph of the generic $f \in C([0,1])$ is of Hausdorff dimension one, this is a result of R. D. Mauldin and S. C. Williams [11, Thm. 2.]. We generalize the cited theorem for arbitrary compact metric spaces. Let K be a compact metric space, then for the generic $f \in C(K)$ the graph of f is of Hausdorff dimension $\dim_H K$. We prove an analogous theorem for the topological Hausdorff dimension, for the generic $f \in C(K)$ the graph of f is of topological Hausdorff dimension $\dim_{tH} K$.

Definition 5.1. If $f \in C(K)$ let us define

$$\widetilde{f} \colon K \to \operatorname{graph}(f), \quad \widetilde{f}(x) = (x, f(x)).$$

Clearly $\widetilde{f}$ is continuous and one-to-one, so it is a homeomorphism between K and $\operatorname{graph}(f)$.

Theorem 5.2. *If K is a compact metric space then for the generic $f \in C(K)$*

$$\dim_H \operatorname{graph}(f) = \dim_H K.$$

Theorem 5.2 follows from the following more general theorem applied with $E = K$. We need this slight generalization in order to prove Theorem 5.4.

Theorem 5.3. *Let K be a compact metric space and $E \subseteq K$. Then for the generic $f \in C(K)$*

$$\dim_H \operatorname{graph}(f|_E) = \dim_H E.$$

Proof of Theorem 5.3. First note that $\operatorname{graph}(f|_E) = \widetilde{f}(E)$. For every $f \in C(K)$ the map $\widetilde{f}^{-1}$ is a projection from $\widetilde{f}(E)$ onto E. Since the Hausdorff dimension cannot increase under a Lipschitz map, $\dim_H \widetilde{f}(E) \geq \dim_H E$. For the opposite direction it is enough to prove that

$$\mathcal{F} = \left\{ f \in C(K) : \dim_H \widetilde{f}(E) \leq \dim_H E \right\}$$

is a dense G_δ set in $C(K)$. We may assume $\dim_H E < \infty$. First we show that $\mathcal{F}$ is a G_δ set. Let us define for all $n \in \mathbb{N}^+$

$$\mathcal{F}_n = \left\{ f \in C(K) : \mathcal{H}^{\dim_H E + 1/n}_{1/n}\left(\widetilde{f}(E)\right) < 1/n \right\}.$$

It is straightforward that the $\mathcal{F}_n$'s are open and $\mathcal{F} = \cap_{n \in \mathbb{N}^+} \mathcal{F}_n$. Thus $\mathcal{F}$ is a G_δ set.

Finally, we show that $\mathcal{F}$ is dense in $C(K)$. If $f \in C(K)$ is Lipschitz, then clearly $\widetilde{f}$ is Lipschitz with $\operatorname{Lip}(\widetilde{f}) \leq \operatorname{Lip}(f) + 1$, and hence $\dim_H \widetilde{f}(E) \leq \dim_H E$. Therefore, it is enough to prove that $\mathcal{G} = \{f \in C(K) : f \text{ is Lipschitz}\}$ is dense in $C(K)$. This fact is well-known but one can also see it directly, since it is easy to show that $cf \in \mathcal{G}$, $f + g \in \mathcal{G}$ and $fg \in \mathcal{G}$ for all $f, g \in \mathcal{G}$ and $c \in \mathbb{R}$.

Therefore, $\mathcal{G}$ forms a subalgebra in $C(K)$. Finally, we may assume $\#K \geq 2$, and the Lipschitz functions $\{\varphi_{x_0}\}_{x_0 \in K}$, $\varphi_{x_0} \colon K \to \mathbb{R}$, $\varphi_{x_0}(x) = d_K(x_0, x)$ show that $\mathcal{G}$ separates points of K and $\mathcal{G}$ vanishes at no point of K. Hence the Stone-Weierstrass Theorem [2, 12.9] implies that $\mathcal{G}$ is dense. This completes the proof. □

Theorem 5.4. *If K is a compact metric space then for the generic $f \in C(K)$*

$$\dim_{tH} \operatorname{graph}(f) = \dim_{tH} K.$$

Proof. For every $f \in C(K)$ the map $\widetilde{f}^{-1}$ is an injective projection from $\operatorname{graph}(f)$ onto K, hence it is a Lipschitz homeomorphism. Thus Theorem 2.3 implies that $\dim_{tH} \operatorname{graph}(f) \geq \dim_{tH} K$. For the opposite direction choose a basis $\mathcal{U}$ of K such that $\dim_H \partial U \leq \dim_{tH} K - 1$ for all $U \in \mathcal{U}$, we can do this by Theorem 2.5. We may assume that $\mathcal{U}$ is countable. Suppose $U \in \mathcal{U}$ is arbitrary. By applying Theorem 5.3 for $E = \partial U$ we infer that there exists a co-meager set $\mathcal{F}_U \subseteq C(K)$ such that for all $f \in \mathcal{F}_U$ we have $\dim_H \widetilde{f}(\partial U) = \dim_H(\partial U) \leq \dim_{tH} K - 1$. The basis $\mathcal{U}$ is countable, and hence $\mathcal{F} = \cap_{U \in \mathcal{U}} \mathcal{F}_U$ is co-meager in $C(K)$. Assume $f \in \mathcal{F}$, it is enough to prove that $\dim_{tH} \operatorname{graph}(f) \leq \dim_{tH} K$. Since $\widetilde{f}$ is homeomorphism we obtain that $\mathcal{V} = \left\{ \widetilde{f}(U) : U \in \mathcal{U} \right\}$ is a basis of $\operatorname{graph}(f)$ and $\partial \widetilde{f}(U) = \widetilde{f}(\partial U)$ for all $U \in \mathcal{U}$. That is,

$$\dim_H \partial V = \dim_H \partial \widetilde{f}(U) = \dim_H \widetilde{f}(\partial U) = \dim_H \partial U \leq \dim_{tH} K - 1$$

for all $V = \widetilde{f}(U) \in \mathcal{V}$. Thus $\dim_{tH} \operatorname{graph}(f) \leq \dim_{tH} K$, and this completes the proof. □

References

[1] R. Balka, Z. Buczolich and M. Elekes, *A new fractal dimension: The topological Hausdorff dimension*, submitted.

[2] N. L. Carothers, *Real analysis*, Cambridge University Press, 2000.

[3] R. Engelking, *Dimension theory*, North-Holland Publishing Company, 1978.

[4] R. Engelking, *General topology*, Revised and completed edition, Heldermann Verlag, 1989.

[5] K. Falconer, *Fractal geometry: Mathematical foundations and applications*, Second Edition, John Wiley & Sons, 2003.

[6] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948.

[7] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.

[8] B. Kirchheim, *Hausdorff measure and level sets of typical continuous mappings in Euclidean spaces*, Trans. Amer. Math. Soc., **347**, (1995), no. 5, 1763-1777.

[9] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.

[10] P. Mattila and R. D. Mauldin, *Measure and dimension functions: Measurability and densities*, Math. Proc. Cambridge Phil. Soc., **121**, (1997), 81-100.

[11] R. D. Mauldin and S. C. Williams, *On the Hausdorff dimension of some graphs*, Trans. Amer. Math. Soc., **298**, (1986), no. 2, 793-803.