DECOMPOSITIONS OF EDGE-COLORED INFINITE COMPLETE GRAPHS INTO MONOCHROMATIC PATHS

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ABSTRACT. For $r \in \mathbb{N}\setminus\{0\}$ an *r-edge coloring* of a graph or hypergraph $G = (V, E)$ is a map $c : E \to \{0, \ldots, r-1\}$. Extending results of Rado and answering questions of Rado, Gyárfás and Sárközy we prove that

- every r-edge colored complete k-uniform hypergraph on $\mathbb{N}$ can be partitioned into r monochromatic tight paths with distinct colors (a tight path in a k-uniform hypergraph is a sequence of distinct vertices such that every set of k consecutive vertices forms an edge),
- for all natural numbers r and k there is a natural number M such that the every r-edge colored complete graph on $\mathbb{N}$ can be partitioned into M monochromatic k^{th} powers of paths apart from a finite set (a k^{th} power of a path is a sequence $v_0, v_1, \ldots$ of distinct vertices such that $|i-j| \leq k$ implies that $\{v_i, v_j\}$ is an edge),
- every 2-edge colored complete graph on $\mathbb{N}$ can be partitioned into 4 monochromatic squares of paths, but not necessarily into 3,
- every 2-edge colored complete graph on ω_1 can be partitioned into 2 monochromatic paths with distinct colors.

1. Introduction

Our goal is to find partitions of edge-colored infinite graphs and hypergraphs into nice monochromatic subgraphs. In particular, we are interested in partitioning the vertices of complete graphs and hypergraphs into monochromatic paths and powers of paths.

An *r-edge coloring* of a graph or hypergraph $G = (V, E)$ is a map $c : E \to \{0, \ldots, r-1\}$, where $r \in \mathbb{N} \setminus \{0\}$. Investigations began in the '80s with a result of Rado [8] implying that every r-edge colored complete graph on $\mathbb{N}$ can be partitioned into r monochromatic paths with distinct colors. We will abbreviate this statement as

$$K_{\mathbb{N}} \sqsubset (\mathfrak{Path}, \ldots, \mathfrak{Path})_r.$$

In Section 3, answering a question of Gyárfás and Sárközy from [3] we extend this result for hypergraphs by proving that every r-edge colored complete k-uniform

Date: February 16, 2015.

2010 *Mathematics Subject Classification.* 05C63, 05C70.

Key words and phrases. graph partition, monochromatic path, path square, infinite complete graph, uncountable complete graph, complete hypergraph, edge coloring.

hypergraph on $\mathbb{N}$ can be partitioned into r monochromatic tight paths with distinct colors (Theorem 3.3):

$$K^k_{\mathbb{N}} \sqsubset (\mathfrak{TightPath}, \dots, \mathfrak{TightPath})_r.$$

Furthermore, Erdős, Gyárfás and Pyber [1] conjectured that the vertices of every r-edge colored complete graph can be covered with r disjoint monochromatic cycles.

This conjecture was disproved by Pokrovskiy [6]. However, the case $k = 2$ of Theorem 3.3(2) yields that the corresponding version of the conjecture above holds for countable infinite graphs: *Given an r-edge coloring of $K_{\mathbb{N}}$, we can partition the vertices into r disjoint cycles and 2-way infinite paths of distinct colors.*

In Section 4, we prove that for all natural numbers r and k there is a natural number M such that the every r-edge colored complete graph on $\mathbb{N}$ can be partitioned into M monochromatic k^{th} powers of paths apart from a finite set (Theorem 4.6):

$$K_{\mathbb{N}} \sqsubset^* (k^{th} - \mathfrak{Power\ of\ Path})_{r,M}.$$

Using a recent result of Pokrovskiy on finite graphs we show that every 2-edge colored complete graph on $\mathbb{N}$ can be partitioned into 4 monochromatic squares of paths:

$$K_{\mathbb{N}} \sqsubset (\mathfrak{PathSquare})_{2,4}.$$

Finally, in Section 5, we give a partial answer to a question of Rado from [8] (the definitions are postponed to the section): every 2-edge colored complete graph on ω_1 can be partitioned into 2 monochromatic paths of distinct colors:

$$K_{\omega_1} \sqsubset (\mathfrak{Path}, \mathfrak{Path})_2.$$

The paper ends with the short Section 6 on further results (without proofs) and open problems.

2. Notations, preliminaries

The cardinality of a set X is denoted by $|X|$. For a set X and $k \in \mathbb{N}$ we will denote the set of k-element subsets of X by $[X]^k$. The set of all subsets of X is denoted by $\mathcal{P}(X)$. A *graph* is an ordered pair $G = (V, E)$ so that $E \subset [V]^2$, and a *hypergraph* is an ordered pair $H = (V, E)$ so that $E \subset \mathcal{P}(V)$; we will use the notation $V(G)$, $E(G)$ for the vertices and edges of a graph or hypergraph G. A hypergraph $H = (V, E)$ is *k-uniform* if $E \subset [V]^k$.

For a graph $G = (V, E)$ and $v \in V$ we write

$$N_G(v) = \{w \in V : \{v, w\} \in E\},$$

and for $F \subset V$

$$N_G[F] = \bigcap \{N_G(v) : v \in F\}.$$

For a graph G and $X \subset V(G)$ we let $G[X]$ denote the *induced subgraph* $(X, [X]^2 \cap E(G))$.

Let $c : E \to \{0, \ldots, r-1\}$ be an r-edge coloring of a graph $G = (V, E)$. For an edge $\{v, w\} \in E$ we will simply write $c(v, w)$ instead of $c(\{v, w\})$. For $v \in V$ and $i < r$ also let

$$N_G(v, i) = \{w \in N_G(v) : c(v, w) = i\},$$

and for $F \subset V$ and $i < r$ let

$$N_G[F, i] = \bigcap \{N_G(v, i) : v \in F\}.$$

As we always work with a fixed coloring, this notation will lead to no misunderstanding (and sometimes we will even drop the subscript G).

We will use $K_{\mathbb{N}}$ to denote $(\mathbb{N}, [\mathbb{N}]^2)$, that is, the complete graph on $\mathbb{N}$. A *path* in a graph is a finite or one-way infinite sequence of distinct vertices such that each pair of consecutive vertices is connected by an edge. If P is a finite path and Q is a disjoint path such that the end-point of P is connected by an edge with the starting point of Q then $P^\frown Q$ denotes their concatenation. We say that Q *end extends* P if P is an initial segment of Q.

Definition 2.1. Let $G = (V, E)$ be a graph and $A \subset V$. We say that A is *infinitely linked* iff there are infinitely many disjoint finite paths between any two distinct points of A. We say that A is *infinitely connected* iff there are infinitely many disjoint finite paths *inside* A between any two distinct points of A.

Remark. An easy recursive construction shows that A is infinitely linked iff for every two distinct members v and w of A and every finite set $F \subset V(G) \setminus \{v, w\}$ there is a path connecting the two points and avoiding F. Similarly, A is infinitely connected if we can additionally require that the path is *inside* A.

If we fix an edge coloring c of G with r colors, $i < r$, $\mathcal{P}$ is a graph property (e.g. being a path, being infinitely connected...) and $A \subset V$ then we say that A *has property* $\mathcal{P}$ *in color* i (with respect to c) iff A has property $\mathcal{P}$ in the graph $(V, c^{-1}(i))$. In particular, by a *monochromatic path* we mean a set P which is a path in some color.

Lemma 2.2. *Let $G = (V, [V]^2)$ be a complete countably infinite graph. Given any edge coloring $c : [V]^2 \to \{0, \ldots, r-1\}$, there is a partition $d_c : V \to \{0, \ldots, r-1\}$ and a color $i_c < r$ so that*

$$N[F, i] \cap V_{i_c} \text{ is infinite for all } i < r \text{ and finite set } F \subset V_i = d_c^{-1}\{i\}.$$

In particular, V_i is infinitely linked in color i for all $i < r$ and V_{i_c} is infinitely connected in color i_c.

Proof. Let U be a non-trivial ultrafilter on V, see e.g. [5]. (In other words, take a finitely additive $0/1$-measure on V assigning measure 0 to singletons, and let U be the class of sets of measure 1.) For $i < r$ define $V_i = \{v \in V : N(v, i) \in U\}$ (e.g. $d_c \restriction V_i \equiv i$), and let i_c be the unique element of $\{0, \ldots, r-1\}$ with $V_{i_c} \in U$. It is not hard to check that this works. $\square$

The next lemma looks slightly technical at first sight. However, note that for our first application, that is for the proof of Rado's theorem we can ignore the sets A_j, as well as the last clause.

Lemma 2.3. *Suppose that $G = (V, E)$ is a countably infinite graph and c is an edge coloring. Suppose that $\{C_j : j < k\}$ is a finite family of subsets of V and that each C_j is infinitely linked in some color i_j. Moreover, for $j < k$ let $A_j \subseteq C_j$ be arbitrary subsets.*

Then we can find disjoint sets P_j so that

(*a*) *P_j is a path (either finite or one-way infinite) in color i_j for all $j < k$,*
(*b*) *if A_j is infinite then so is $A_j \cap P_j$,*
(*c*) $\bigcup\{P_j : j < k\} \supset \bigcup\{C_j : j < k\}$.

Moreover, if a C_j is infinite then we can choose the first point of P_j freely from C_j.

Proof. Let $v_0, v_1, \ldots$ be a (possibly finite) enumeration of $\bigcup\{C_j : j < k\}$.

For all the infinite C_j, fix distinct $x_j \in C_j$ as starting points for the P_js. We define disjoint finite paths $\{P_j^n : j < k\}$ by induction on $n \in \mathbb{N}$ so that

(i) P_j^n is a path of color i_j with first point x_j,
(ii) P_j^{n+1} end extends P_j^n (as a path of color i_j),
(iii) the last point of the path P_j^n is in C_j,
(iv) if A_j is infinite then the last point of P_j^{2n} is in A_j,

for all $j < k$, and

(v) if $v_n \notin \bigcup_{j<k} P_j^{2n}$ and $v_n \in C_j$ then v_n is the last point of P_j^{2n+1}.

It should be easy to carry out this induction applying that each A_j is infinitely linked in color i_j. Finally, we let $P_j = \cup\{P_j^n : n \in \mathbb{N}\}$ for $j < k$ which finishes the proof. □

In particular, we have the following trivial corollary:

Corollary 2.4. *If a countable graph is infinitely connected then it is a single one-way infinite path. If a countable set of vertices A is infinitely linked then it is covered by a single one-way infinite path.*

More importantly, the above lemmas yield

Theorem 2.5 (R. Rado [8])**.** *For every r-edge coloring of $K_{\mathbb{N}}$ we can partition the vertices into r disjoint paths of distinct colors.*

Proof. Apply Lemma 2.2 and find a partition $\mathbb{N} = \{V_i : i < r\}$ so that each V_i is infinitely linked in color i. Now apply Lemma 2.3 with $C_i = V_i$ (and $A_i = \emptyset$) to get the desired partition into monochromatic paths. □

To abbreviate the formulation of certain result we introduce the following notation.

Definition 2.6. Let G be a graph and $\mathfrak{F}$ be a class of graphs. We write

$$G \sqsubset (\mathfrak{F})_{r,m}$$

if given any r-edge coloring $c : E(G) \to \{0, \ldots, r-1\}$ the vertex set of G can be partitioned into m monochromatic elements of $\mathfrak{F}$.

We write

$$G \sqsubset (\mathfrak{F}, \mathfrak{F}, \ldots, \mathfrak{F})_r$$

if given any r-edge coloring $c : E(G) \to \{0, \ldots, r-1\}$ the vertex set of G can be partitioned into r monochromatic elements of $\mathfrak{F}$ in *distinct colors*.

In particular, $G \sqsubset (\mathfrak{Path})_{r,m}$ holds if given any r-edge coloring c of G the vertex set of G can be partitioned into m monochromatic paths.

We write $\sqsubset^*$ instead of $\sqsubset$ if we can partition the vertex set *apart from a finite set.*

Using our new notation, Theorem 2.5 can be formulated as follows:

$$K_{\mathbb{N}} \sqsubset (\mathfrak{Path}, \ldots, \mathfrak{Path})_r.$$

3. Partitions of hypergraphs

In this section, we briefly look at a generalization of Rado's result, Theorem 2.5 above, to hypergraphs. Let $k \in \mathbb{N} \setminus \{0\}$.

Definition 3.1. A *loose path* in a k-uniform hypergraph is a finite or one-way infinite sequence of edges, $e_1, e_2, \ldots$ such that $|e_i \cap e_{i+1}| = 1$ for all i, and $e_i \cap e_j = \emptyset$ for all i, j with $i + 1 < j$.

A *tight path* in a k-uniform hypergraph is a finite or one-way infinite sequence of distinct vertices such that every set of k consecutive vertices forms an edge.

Remark. Occasionally, we will refer to loose and tight *cycles* and *two-way infinite paths* as well, with the obvious analogous definitions.

The following result was proved recently:

Theorem 3.2 (A. Gyárfás, G. N. Sárközy [3, Theorem 3.])**.** *Suppose that the edges of a countably infinite complete k-uniform hypergraph are colored with r colors. Then the vertex set can be partitioned into monochromatic finite or one-way infinite loose paths of distinct colors.*

In the introduction of [3], the authors asked if one can find a partition into tight paths instead of loose ones. We prove the following:

Theorem 3.3. *Suppose that the edges of a countably infinite complete k-uniform hypergraph are colored with r colors. Then*

(1) the vertex set can be partitioned into monochromatic finite or one-way infinite tight paths of distinct colors,

(2) the vertex set can be partitioned into monochromatic tight cycles and two-way infinite tight paths of distinct colors.

Proof. (1) Note that the case of $k = 2$ is Rado's Theorem 2.5 above; we will imitate his original proof here.

Let $c : [\mathbb{N}]^k \to \{0, \dots, r-1\}$. A set $T \subset \{0, \dots, r-1\}$ of colors is called *perfect* iff there are disjoint finite subsets $\{P_t : t \in T\}$ of $\mathbb{N}$ and an infinite set $A \subset \mathbb{N} \setminus \bigcup_{t\in T} P_t$ such that for all $t \in T$

(a) P_t is a tight path in color t,

(b) if $1 \le i < k$ and x is the set of the last i vertices from the tight path P_t and $y \in [A]^{k-i}$, then $c(x \cup y) = t$.

Since $\emptyset$ is perfect, we can consider a perfect set T of colors with maximal number of elements.

Claim 3.3.1. *If the vertex disjoint finite tight paths $\{P_t : t \in T\}$ and the infinite set A satisfy (a) and (b) then for all $v \in \mathbb{N} \setminus \bigcup_{t\in T} P_t$ there is a color $t' \in T$, a finite sequence $v_1, v_2, \dots, v_{k-1}$ from A, and an infinite set $A' \subset A$ such that the tight paths*

$$\{P_t : t \in T \setminus \{t'\}\} \cup \{P_{t'}{}^\frown(v_1, v_2, \dots, v_{k-1}, v)\}$$

and A' satisfy (a) and (b) as well.

Proof of the Claim. Define a new coloring $d : [A]^{k-1} \to \{0, \dots, r-1\}$ by the formula $d(x) = c(x \cup \{v\})$. By Ramsey's Theorem, there is an infinite d-homogeneous set $B \subset A$ in some color t'. Then $t' \in T$, since otherwise $T \cup \{t'\}$ would be a bigger perfect set witnessed by $P_{t'} = \{v\}$,$\{P_t : t \in T\}$ and B.

Now pick distinct $v_1, v_2, \dots, v_{k-1}$ from B and let $A' = B \setminus \{v_1, \dots, v_{k-1}, v\}$. $\square$

Finally, by applying the claim repeatedly, we can cover the vertices with $|T|$ tight paths of distinct colors.

(2) Let $c : [\mathbb{N}]^k \to \{0, \dots, r-1\}$. Write $V_{-1} = \mathbb{N}$. Using Ramsey's Theorem, by induction on $n \in \mathbb{N}$ choose $d(n) < r$ and $V_n \in [V_{n-1}]^{\mathbb{N}}$ such that

$$c(\{n\} \cup O) = d(n) \text{ for all } O \in [V_n]^{k-1}. \tag{3.1}$$

For $i < r$ let

$$A_i = \{n \in \mathbb{N} : d(n) = i\}. \tag{3.2}$$

Let $K = \{i < r : A_i \text{ is finite}\}$. By induction on $i \in K$ we will define tight cycles $\{P_i : i \in K\}$ such that

$$\bigcup_{i'<i, i'\in K} A_{i'} \subseteq \bigcup_{i'<i, i'\in K} P_{i'}$$

while some of the P_i's might be empty.

Assume that $\{P_{i'} : i' < i, i' \in K\}$ is defined and suppose $i \in K$. Enumerate $A_i \setminus \bigcup_{i'<i, i'\in K} P_{i'}$ as $\{x_i^j : j < t\}$.

Choose disjoint $k-1$ element sets

$$Y_i^j \subseteq \bigcap_{j<t} V_{x_i^j} \setminus \bigcup_{i'<i, i'\in K} P_{i'} \text{ for } j < t. \tag{3.3}$$

Consider an ordering $\prec_i$ on $P_i = \{x_i^j : j < t\} \cup \bigcup_{j<t} Y_i^j$ such that

$$x_i^0 \prec_i Y_i^0 \prec_i x_i^1 \prec_i Y_i^1 \prec_i \cdots \prec_i x_i^{t-1} \prec_i Y_i^{t-1}.$$

Then $\prec_i$ witnesses that P_i is a tight cycle in color i.

Now, let

$$P = \bigcup_{i \in K} P_i$$

and for each $i \in \{0, \ldots, r-1\} \setminus K$ we define a 2-way infinite tight path P_i as follows.

By induction, for every integer $z \in \mathbb{Z}$ and $i \in \{0, \ldots, r-1\} \setminus K$ choose disjoint sets $\{x_i^z\} \in [A_i \setminus P]^1$ and $Y_i^z \in \left[\mathbb{N} \setminus P\right]^{k-1}$ such that

$$Y_i^z \subset V_{x_i^z} \cap V_{x_i^{z+1}}$$

and

$$\bigcup_{i \in \{0,\ldots,r-1\} \setminus K} A_i \subset P \cup \bigcup \{\{x_i^z\}, Y_i^z : i \in \{0, \ldots, r-1\} \setminus K, z \in \mathbb{Z}\}.$$

Consider an ordering $\prec_i$ on $P_i = \{x_i^z : z \in \mathbb{Z}\} \cup \bigcup_{z \in \mathbb{Z}} Y_i^z$ such that

$$\ldots \prec_i Y_i^{-2} \prec_i x_i^{-1} \prec_i Y_i^{-1} \prec_i x_i^0 \prec_i Y_i^0 \prec_i x_i^1 \prec_i Y_i^1 \prec_i \ldots$$

Then $\prec_i$ witnesses that P_i is a 2-way infinite tight path in color i. □

4. Covers by k^{TH} powers of paths

Our aim is to prove a stronger version of Rado's theorem; in order to state this result we need the following

Definition 4.1. Suppose that $G = (V, E)$ is a graph and $k \in \mathbb{N} \setminus \{0\}$. The k^{th} *power of* G is the graph $G^k = (V, E^k)$ where $\{v, w\} \in E^k$ iff there is a finite path of length $\leq k$ from v to w.

We will be interested in partitioning an edge colored copy of $K_{\mathbb{N}}$ into finitely many monochromatic k^{th} *powers of paths*.

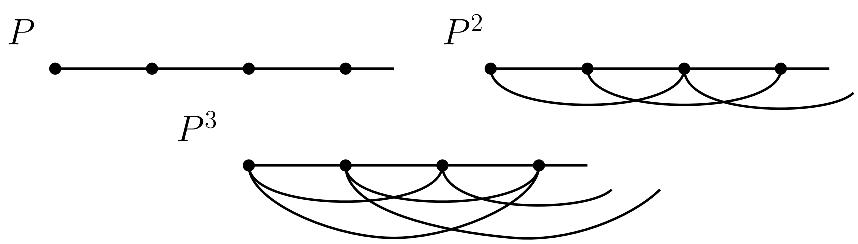

FIGURE 1. Powers of paths.

We will investigate this problem by introducing the following game.

Definition 4.2. Assume that H is a graph, $W \subset V(H)$ and $k \in \mathbb{N}$. The game $\mathfrak{G}_k(H,W)$ is played by two players, Adam and Bob, as follows. The players choose disjoint finite subsets of $V(H)$ alternately:

$$A_0, B_0, A_1, B_1, \ldots$$

Bob wins the game $\mathfrak{G}_k(H,W)$ iff

(A) $W \subset \bigcup_{i\in\mathbb{N}} A_i \cup B_i$, and
(B) $H[\bigcup_{i\in\mathbb{N}} B_i]$ contains the k^{th} power of a (finite or one way infinite) Hamiltonian path (that is, a path covering all the vertices).

For $k = 1$, we have the following

Observation 4.3. *If $H = (V,E)$ is a countable graph and $W \subset V$ then the following are equivalent:*

(1) W is infinitely linked,
(2) Bob wins $\mathfrak{G}_1(H,W)$.

Proof. (1) ⇒ (2): By our assumption, Bob can always connect an uncovered point of W to the end-point of the previously constructed path while avoiding vertices played so far. This shows the existence of a winning strategy for Bob.

(2) ⇒ (1): Fix any two distinct points $v, w \in W$ and a finite set $F \subset V \setminus \{v,w\}$. Let Adam start with $A_0 = F$ and continue with $A_i = \emptyset$; the Hamiltonian path P constructed by Bob's strategy will go through a and b while $P \cap F = \emptyset$. □

Now, we show how to produce a partition of the vertices into k^{th} powers of paths using winning strategies of Bob:

Lemma 4.4. *Suppose that $H = (V,E)$, $V = \bigcup\{W_i : i < M\}$ with $M \in \mathbb{N}$ and let $H_i = (V, E_i)$ for some $E_i \subset E$. If Bob wins $\mathfrak{G}_k(H_i, W_i)$ for all $i < M$ then V can be partitioned into $\{P_i : i < M\}$ so that P_i is a k^{th} power of a path in H_i.*

Proof. We will conduct M games simultaneously as follows: the plays of Adam and Bob in the i^{th} game will be denoted by $A^i_0, B^i_0, A^i_1, B^i_1, \ldots$ for $i < M$. Let σ^i denote the winning strategy for Bob in $\mathfrak{G}_k(H_i, W_i)$, that is, if we set $B^i_n = \sigma^i(A^i_0, B^i_0, \ldots, A^i_n)$ then Bob wins the game.

Now, we define A^i_n, B^i_n by induction using the lexicographical ordering $<_{lex}$ on $\{(n,i) : n \in \mathbb{N}, i < M\}$. First, let $A^0_0 = \emptyset$ and $B^0_0 = \sigma^0(A^0_0)$. In general, assume that A^j_m and B^j_m are defined for $(m,j) <_{lex} (n,i)$, and we let

$$A^i_n = \bigcup\{B^j_m : (m,j) <_{lex} (n,i)\} \setminus \left(\bigcup\{A^i_m, B^i_m : m < n\}\right) \tag{4.1}$$

and

$$B^i_n = \sigma^i(A^i_0, B^i_0, \ldots, A^i_n).$$

One easily checks that the above defined plays are valid; indeed, for a fix $i < M$ the finite sets $\{A^i_n, B^i_n : n \in \mathbb{N}\}$ defined above are disjoint.

Next, let $P_i = \bigcup\{B^i_n : n \in \mathbb{N}\}$ for $i < M$. As Bob wins the i^{th} game we have that P_i is a k^{th} power of path in H_i. Note that $P_i \cap P_j = \emptyset$ if $i \neq j < M$. Indeed, if $(m, j) <_{lex} (n, i)$, then

$$B^i_n \cap B^j_m \subset B^i_n \cap (A^i_n \cup \Big(\bigcup\{A^i_m, B^i_m : m < n\}\Big) = \emptyset$$

by (4.1).

To finish the proof, we prove

$$V = \{P_i : i < M\}. \tag{4.2}$$

Indeed, first note that $W_i \subset \bigcup_{n\in\mathbb{N}} A^i_n \cup B^i_n$ as Bob wins the i^{th} game and hence

$$V = \bigcup_{n\in\mathbb{N}, i<M} A^i_n \cup B^i_n.$$

Second, by (4.1), we have

$$A^i_n \subset \bigcup\{B^j_m : (m, j) <_{lex} (n, i)\}$$

and so

$$\bigcup_{n\in\mathbb{N}, i<M} A^i_n \subset \bigcup_{n\in\mathbb{N}, i<M} B^i_n$$

and hence $V = \{P_i : i < M\}$. □

The next theorem provides conditions under which Bob has a winning strategy:

Theorem 4.5. *Assume that H is a countably infinite graph, $W \subset V(H)$ is non-empty and $k \in \mathbb{N}$. If there are subsets $W_0, \ldots, W_k$ of $V(H)$ such that $W_0 = W$ and*

$$W_{j+1} \cap N_H[F] \text{ is infinite for each } j < k \text{ and finite } F \subset \bigcup_{i\le j} W_i \tag{4.3}$$

then Bob wins $\mathfrak{G}_k(H, W)$.

Proof. We can assume that $\mathrm{V}(H) = \mathbb{N}$.

Consider first the easy case when W_0 is finite. Adam plays a finite set A_0 in the first round. Write $N = |W_0 \setminus A_0|$. Let Bob play $B_0 = W_0 \setminus A_0 = \{b_{n,0} : n < N\}$. In the j^{th} round for $1 \le j \le k$, let Bob play an N-element set

$$B_j = \{b_{n,j} : n < N\} \subset W_j \cap N_H\big[\bigcup_{i<j} B_i\big] \tag{4.4}$$

which avoids all previous choices, i.e. $B_j \cap \bigcup\{A_{i'}, B_i : i' \le j, i < j\} = \emptyset$. For $j > k$ let Bob play $B_j = \emptyset$.

We claim that

(A) $W_0 \subseteq \bigcup\{A_n, B_n : n \in \mathbb{N}\}$, and
(B) $P = \{b_{n,j} : n < N, j \le k\}$ is the k^{th}-power of a path.

(A) is clear because $W_0 \subseteq A_0 \cup B_0$.

To check (B) consider the lexicographical order of the indexes. Let $(m,i) \neq (n,j) \in \{0,\dots,N-1\} \times \{0,\dots,k\}$. Then $b_{m,i}$ and $b_{n,j}$ are the $((k+1)m+i)^{th}$ and $((k+1)n+j)^{th}$ elements, respectively, in the lexicographical order.

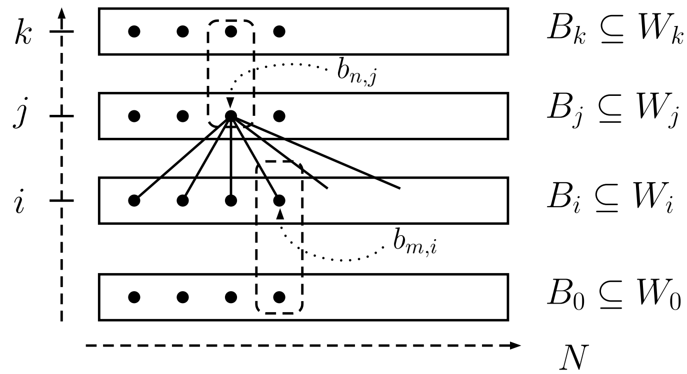

FIGURE 2. $b_{n,j}$ and its k successors.

Assume that $|((k+1)m+i) - ((k+1)n+j)| \leq k$; then $i \neq j$ and, without loss of generality, we can suppose that $i < j$. Then we have $b_{m,i} \in \bigcup_{i'<j} B_{i'}$, so $b_{n,j} \in N_H(b_{m,i})$ by (4.4). In other words, $\{b_{m,i}, b_{n,j}\}$ is an edge in H which yields (B).

Consider next the case when W_0 is infinite; let us outline the idea first in the case when $k = 2$. Bob will play one element sets at each step and aims to build a one-way infinite square of a path following the lexicographical ordering on $\mathbb{N} \times \{0,1,2\}$. However, he picks the vertices in a different order, denoted by $\trianglelefteq$ later, which is demonstrated in Figure 3.

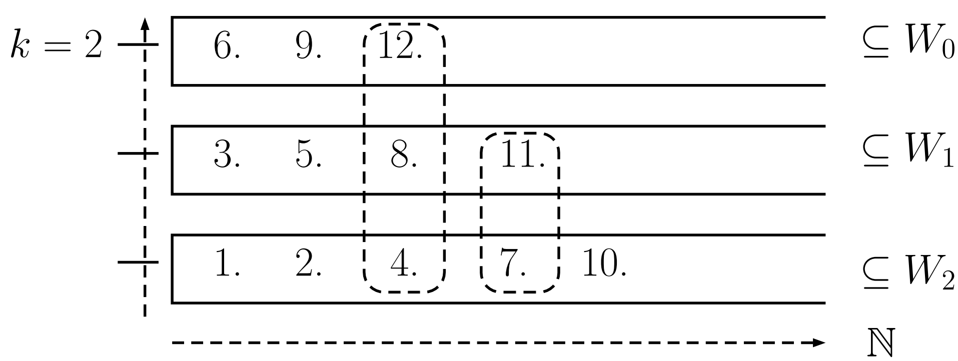

FIGURE 3. The two orderings.

This way Bob makes sure that when he chooses the 12^{th} element he already picked its two successors (in the 7^{th} and 11^{th} plays) and two predecessors (in the 8^{th} and 4^{th} plays) in the lexicographical ordering, hence we can ensure the edge relations here.

Now, we define the strategy more precisely. In each round Bob will pick a single element $b_{n,j}$ for some $(n,j) \in \mathbb{N}\times\{0,1,\dots,k\}$ such that $\big\{b_{n,j} : (n,j) \in \mathbb{N}\times\{0,1,\dots,k\}\big\}$ will be the k^{th} power of a path in the lexicographical order of $\mathbb{N}\times\{0,1,\dots,k\}$.

As we said earlier, Bob will not choose the points $b_{n,j}$ in the lexicographical order of $\mathbb{N}\times\{0,1,\dots,k\}$, i.e. typically the $((k+1)n+j)^{th}$ move of Bob, denoted by $B_{(k+1)n+j}$, is *not* $\{b_{n,j}\}$.

To describe Bob's strategy we should define another order on $\mathbb{N}\times\{0,1,\dots,k\}$ as follows:

$$(m,i) \trianglelefteq (n,j) \ \text{ iff } \ (m+i<n+j) \text{ or } (m+i=n+j \text{ and } i\le j).$$

Write $(m,i)\triangleleft(n,j)$ iff $(m,i) \trianglelefteq (n,j)$ and $(m,i)\neq(n,j)$. Clearly every (n,j) has just finitely many $\triangleleft$-predecessors. Let $f(\ell)$ denote the ℓ^{th} element of $\mathbb{N}\times\{0,1,\dots,k\}$ in the order $\triangleleft$.

Bob will choose $B_\ell=\{b_{f(\ell)}\}$ in the ℓ^{th} round as follows: if $f(\ell)=(n,j)$, then

(a) if $j=0$ then

$$b_{n,j}=\min\Big(W_0\setminus\big(\bigcup_{s\le\ell}A_s\cup\bigcup_{t<\ell}B_t\big)\Big); \tag{4.5}$$

(b) if $j>0$ then

$$b_{n,j}\in W_j\cap N_H\big[\{b_{m,i}:(m,i)\triangleleft(n,j),i<j\}\big]. \tag{4.6}$$

Bob can choose a suitable $b_{n,j}$ by (4.3) as $\{b_{m,i}:(m,i)\triangleleft(n,j),i<j\}$ is a finite subset of $\bigcup_{i<j}W_i$.

We claim that

(A) $W_0\subseteq\bigcup\{A_n,B_n:n\in\mathbb{N}\}$, and
(B) $P=\{b_{n,j}:n\in\mathbb{N},j\le k\}$ is the k^{th}-power of a path.

(A) is clear because in (4.5) we chose the minimal possible element.

Let $(m,i)\neq(n,j)\in\mathbb{N}\times\{0,\dots,k\}$. Then $b_{m,i}$ and $b_{n,j}$ are the $((k+1)m+i)^{th}$ and $((k+1)n+j)^{th}$ elements, respectively, in the lexicographical order. Assume that $|((k+1)m+i)-((k+1)n+j)|\le k$. Then $i\neq j$ and $|m-n|\le 1$.

Without loss of generality, we can assume that $i<j$. Then $|m-n|\le 1$ implies $m+i\le n+j$ and hence $(m,i)\triangleleft(n,j)$. Since $i<j$ as well, $b_{n,j}\in N_H(b_{m,i})$ must hold by (4.6). In other words, $\{b_{m,i},b_{n,j}\}$ is an edge in H which yields (B). $\square$

We arrive at one of our main results:

Theorem 4.6. *For all positive natural numbers k,r and an r-edge coloring of $K_{\mathbb{N}}$ the vertices can be covered by $r^{(k-1)r+1}$ one-way infinite monochromatic k^{th} powers of paths and a finite set.*

Proof. The set of sequences of length m (at most m, respectively) whose members are from a set X is denoted by X^m ($X^{\leq m}$, respectively).

Recall that for each r-edge coloring c of $K_{\mathbb{N}}$ Lemma 2.2 gives a partition of the vertices, which we will denote by $d_c : \mathbb{N} \to \{0, \dots, r-1\}$, and a special color $i_c < r$. We define a set $A_s \subset \mathbb{N}$ for each finite sequence $s \in \{0, \dots, r-1\}^{\leq (k-1)r+1}$ by induction on $|s|$ as follows:

- let $A_\emptyset = \mathbb{N}$,
- if A_s is defined and finite then let
$$A_{s^\frown 0} = A_s \text{ and } A_{s^\frown i} = \emptyset \text{ for } 1 \leq i < r, \tag{4.7}$$
- if A_s is defined and infinite then let
$$A_{s^\frown i} = \{u \in A_s : d_{c \restriction A_s}(u) = i\} \text{ for } i < r. \tag{4.8}$$

Fix an arbitrary $s \in \{0, \dots, r-1\}^{(k-1)r+1}$ such that A_s is infinite. Then there is a color $i_s < r$ and a k-element subset $H_s = \{h_1 > h_2 > \dots > h_k\}$ of $\{0, \dots, (k-1)r\}$ such that
$$s(h_j) = i_s$$
for all $j = 1, \dots, k$. Let $W_0 = A_s$ and $W_j = A_{s \restriction h_j}$ for $j = 1, \dots, k$. Note that the choice of i_s ensures that
$$W_{j+1} \cap N_{G_s}[F] \text{ is infinite}$$
for each $j < k$ and finite set $F \subset \bigcup_{i \leq j} W_i$, where $G_s = (\mathbb{N}, c^{-1}\{i_s\})$. Thus, by Theorem 4.5, Bob has a winning strategy in the game $\mathfrak{G}_k(G_s, A_s)$.

Playing the games
$$\{\mathfrak{G}_k(G_s, A_s) : s \in \{0, \dots, r-1\}^{(k-1)r+1} \text{ and } A_s \text{ is infinite}\} \tag{4.9}$$
simultaneously, that is, applying Lemma 4.4 we can find at most $r^{(k-1)r+1}$ many k^{th} powers of disjoint monochromatic paths which cover $\mathbb{N}$ apart from the finite set $\bigcup\{A_s : A_s \text{ is finite}\}$. □

In the case of $k = r = 2$, we have the following stronger result:

Theorem 4.7. *(1) Given an edge coloring of $K_{\mathbb{N}}$ with 2 colors, the vertices can be partitioned into ≤ 4 monochromatic path-squares (that is, second powers of paths):*
$$K_{\mathbb{N}} \sqsubset (\mathfrak{PathSquare})_{2,4}.$$
(2) The result above is sharp: there is an edge coloring of $K_{\mathbb{N}}$ with 2 colors such that the vertices cannot be covered by 3 monochromatic path-squares:
$$K_{\mathbb{N}} \not\sqsubset (\mathfrak{PathSquare})_{2,3}.$$

To prove Theorem 4.7 we need some further preparation. First, in [7, Corollary 1.10] Pokrovskiy proved the following: *Let $k, n \geq 1$ be natural numbers. Suppose that the edges of K_n are colored with two colors. Then the vertices of K_n can be covered with k disjoint paths of color 1 and a disjoint k^{th} power of a path of color 0.*

Second, we will apply the following

Lemma 4.8. *Assume that $P = v_0, v_1, \dots$ is a finite or one-way infinite path in a graph G and there is $W \subset V(G) \setminus P$ so that*

$$(W \cap \mathrm{N}_G[\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}]) \text{ is infinite for all } v_i \in P. \tag{4.10}$$

Let $\mathcal{F}$ be a countable family of infinite subsets of W. Then G contains a square of a path R which covers P while $R \setminus P \subset W$, and $F \setminus R$ is infinite for all $F \in \mathcal{F}$. Moreover, if P is finite then R can also be chosen to be finite.

Proof. Let $F_0, F_1, \dots$ be an enumeration of $\mathcal{F}$ in which each element shows up infinitely often.

Pick distinct vertices $w_0, f_0, w_1, f_1, \dots$ from W such that

$$w_i \in \mathrm{N}_G[\{v_{2i}, v_{2i+1}, v_{2i+2}, v_{2i+3}\}] \text{ and } f_i \in F_i.$$

Then

$$R = v_0, v_1, w_0, v_2, v_3, w_1, v_4, \dots, v_{2i}, v_{2i+1}, w_i, v_{2i+2}, v_{2i+3}, w_{i+1}, \dots \tag{4.11}$$

is a square of a path which covers P, $R \setminus P \subset W$, and $\{f_n : n \in \mathbb{N}, F_n = F\} \subseteq F \setminus R$ for all $F \in \mathcal{F}$.

The last statement concerning the finiteness of R is obvious. □

Proof of Theorem 4.7(1). Fix a coloring $c : [\mathbb{N}]^2 \to \{0,1\}$ and let $G_i = (\mathbb{N}, c^{-1}\{i\})$ for $i < 2$.

We will use the notation of Lemma 2.2. Let $c_0 = c$ and let

$$A_0 = \{v \in \mathbb{N} : d_{c_0}(v) = i_{c_0}\} \text{ and } B_0 = \mathbb{N} \setminus A_0. \tag{4.12}$$

Let $c_1 = c_0 \restriction B_0$ and provided B_0 is infinite we let

$$A_1 = \{v \in B_0 : d_{c_1}(v) = i_{c_1}\} \text{ and } B_1 = B_0 \setminus A_1. \tag{4.13}$$

We can assume that $i_{c_0} = 0$ without loss of generality.

Case 1: *B_0 is finite.*

First, $G[B_0]$ can be written as the disjoint union of two paths P_0 and P_1 of color 1 and a square of a path Q of color 0 by the above mentioned result of Pokrovskiy [7, Corollary 1.10]. Applying Lemma 4.8 for $G = G_1$, $P = P_0$, $W = A_0$ and $\mathcal{F} = \emptyset$ it follows that there is a finite square of a path R_0 in color 1 which covers P_0 and $R_0 \setminus P_0 \subset A_0$. Applying Lemma 4.8 once more for $G = G_1$, $P = P_1$, $W = A_0 \setminus R_0$ and $\mathcal{F} = \emptyset$ it follows that there is a finite square of a path R_1 in color 1 which covers P_1, and $R_1 \setminus P_1 \subset A_0 \setminus R_0$. Let $A_0' = A_0 \setminus (R_0 \cup R_1)$.

Now, by Theorem 4.5, Bob wins the game $\mathfrak{G}_2(G_0, A_0')$ witnessed by the sequence (A_0', A_0', A_0'); thus $G[A_0']$ can be covered by a single square of a path S of color 0 by Lemma 4.4. That is, G can be covered by 4 disjoint monochromatic squares of paths: R_0, R_1, Q and S.

Case 2: *B_0 is infinite and $i_{c_1} = 0$.*

Note that, by Theorem 4.5, Bob wins the games

(i) $\mathfrak{G}_2(G_0, A_0)$ witnessed by (A_0, A_0, A_0),

(ii) $\mathfrak{G}_2(G_0, A_1)$ witnessed by (A_1, A_1, A_1),
(iii) $\mathfrak{G}_2(G_1, B_1)$ witnessed by (B_1, A_1, A_0).

Hence, the vertices can be partitioned into two squares of paths of color 0 and a single square of a path of color 1 by Lemma 4.4.

Case 3: *B_0 is infinite and $i_{c_1} = 1$.*

Since we applied Lemma 2.2 twice to obtain A_0 and B_0, and A_1 and B_1, and $B_1 \subseteq B_0$ we know that

(a) Bob wins the game $\mathfrak{G}_2(G_0, A_0)$ witnessed by (A_0, A_0, A_0);
(b) Bob wins the game $\mathfrak{G}_2(G_1, A_1)$ witnessed by (A_1, A_1, A_1);
(c) $N[F, 1] \cap A_0$ is infinite for every finite set $F \subset B_1$;
(d) $N[F, 0] \cap A_1$ is infinite for every finite set $F \subset B_1$;
(e) $N[F, 0] \cap A_0$ is infinite for every finite set $F \subset A_0$;
(f) $N[F, 1] \cap A_1$ is infinite for every finite set $F \subset A_1$.

First, partition B_1 into two paths P_0 and P_1 of color 0 and 1, respectively. Indeed, if B_1 is infinite this can be done by Theorem 2.5 and if B_1 is finite one considers two disjoint paths P_0 and P_1 in B_1 of color 0 and 1 with $|P_0| + |P_1|$ maximal (as outlined in a footnote in [2]); it is easily seen that $P_0 \cup P_1$ must be B_1.

Now, our plan is to cover P_0 and P_1 with disjoint squares of paths R_0 and R_1 of color 0 and 1, respectively, such that $R_0 \setminus P_0 \subset A_1$, $R_1 \setminus P_1 \subset A_0$ while

(a') Bob wins the game $\mathfrak{G}_2(G_0, A_0 \setminus R_1)$ witnessed by $(A_0 \setminus R_1, A_0 \setminus R_1, A_0 \setminus R_1)$,
(b') Bob wins the game $\mathfrak{G}_2(G_1, A_1 \setminus R_0)$ witnessed by $(A_1 \setminus R_0, A_1 \setminus R_0, A_1 \setminus R_0)$.

Let

$$\mathcal{F}_0 = \{N[F, 0] \cap A_0 : F \subset A_0 \text{ finite}\},$$

and

$$\mathcal{F}_1 = \{N[F, 1] \cap A_1 : F \subset A_1 \text{ finite}\},$$

and note that these families consist of infinite sets by (e) and (f) above. Apply Lemma 4.8 for $G = G_0$, $W = A_1$, $P = P_0$ and $\mathcal{F} = \mathcal{F}_1$ to find a square of a path R_0 in G_0 which covers P_0, $R_0 \setminus P_0 \subset A_1$ and $F \setminus R_0$ is infinite for all $F \in \mathcal{F}_1$, that is,

$$N[F, 1] \cap (A_1 \setminus R_0) \text{ is infinite for every finite set } F \subset A_1. \tag{4.14}$$

Apply Lemma 4.8 once more for $G = G_1$, $W = A_0$, $P = P_1$ and $\mathcal{F} = \mathcal{F}_0$ to find a square of a path R_1 in G_1 with $R_1 \setminus P_1 \subset A_0$ which covers P_1 and $F \setminus R_1$ is infinite for all $F \in \mathcal{F}_0$, that is,

$$N[F, 0] \cap (A_0 \setminus R_1) \text{ is infinite for every finite set } F \subset A_0. \tag{4.15}$$

Then, by Theorem 4.5, (4.15) yields (a'), and (4.14) yields (b').

Hence $(A_0 \setminus R_1) \cup (A_1 \setminus R_0)$ can be partitioned into two monochromatic squares of paths by Lemma 4.4 which in turn gives a partition of all the vertices into 4 monochromatic squares of paths. □

Proof of Theorem 4.7(2). Fix a partition (A, B, C, D) of $\mathbb{N}$ such that A is infinite, $|B| = |C| = 4$, and $|D| = 1$. Define the coloring $c : [\mathbb{N}]^2 \to \{0, 1\}$ as follows see Figure 4:

$$c^{-1}\{1\} = \{\{a, v\} : a \in A, v \in B \cup C \cup D\} \cup \big[B\big]^2 \cup \big[C\big]^2. \tag{4.16}$$

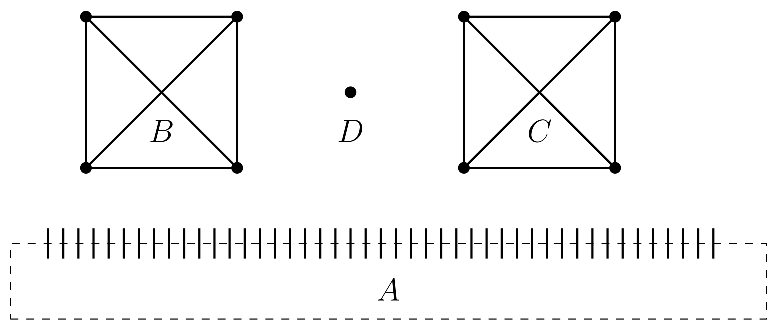

FIGURE 4.

If P is a monochromatic square of a path which intersects both A and $B \cup C \cup D$, then P should be in color 1, so $P \cap A$ should be finite. Thus every partition of $K_{\mathbb{N}}$ into monochromatic squares of paths should contain an infinite 0-monochromatic square of a path $S \subset A$.

It suffices to show now that $B \cup C \cup D$ cannot be covered by two monochromatic squares of paths. Let $D = \{d\}$.

First, if P is a 1-monochromatic square of a path then $P' = P \cap (B \cup C \cup D)$ is a 1-monochromatic path. As two 1-monochromatic paths cannot cover $B \cup C \cup D$, two 1-monochromatic squares of paths will not cover $B \cup C \cup D$ neither.

Second, if Q is a 0-monochromatic square of a path which intersects $B \cup C \cup D$ then $Q \subset B \cup C \cup D$. Assume that $d \notin Q$ and let $Q = x_1, x_2, \dots$. If $x_1 \in B$ then $x_2 \in C$ so x_3 does not exists because Q is 0-monochromatic square of a path. Hence $d \notin Q$ implies $|Q \cap B| \leq 1$ and $|Q \cap C| \leq 1$. If $d \in Q$, then cutting Q into two by d and using the observation above we yield that $|Q \cap B| \leq 2$ and $|Q \cap C| \leq 2$. In turn, two 0-monochromatic squares of paths cannot cover $B \cup C \cup D$.

Finally using just one 0-monochromatic square of a path Q we cannot cover $(B \cup C) \setminus Q$ by a single 1-monochromatic square of a path because there is no 1-colored edge between $B \setminus Q \neq \emptyset$ and $C \setminus Q \neq \emptyset$. $\square$

5. MONOCHROMATIC PATH DECOMPOSITIONS OF K_{ω_1}

The aim of this section is to extend Rado's Theorem 2.5 to 2-edge colored complete graphs of size ω_1.

First, we need to extend certain definitions to the uncountable setting.

Definition 5.1 (Rado [8])**.** We say that a graph $P = (V, E)$ is a *path* iff there is a well ordering $\prec$ on V such that

$$\{w \in N_P(v) : w \prec v\} \text{ is } \prec\text{-cofinal below } v$$

for all $v \in P$.

Observation 5.2. *Suppose that $P = (V, E)$ is a graph and $\prec$ is a well ordering of V. Then the following are equivalent:*

(1) $\prec$ witnesses that P is a path,
(2) every $v, w \in V$ are connected by a $\prec$-monotone finite path in P.

In particular, each vertex is connected to its $\prec$-successor by an edge and so this general definition of a path coincides with the usual path notion for finite graphs.

The order type of $(V, \prec)$ above is called *the order type of the path*. We will say that a path Q *end extends* the path P iff $P \subset Q$, $\prec_Q \restriction P = \prec_P$ and $v \prec_Q w$ for all $v \in P, w \in Q \setminus P$. If R and S are two paths so that the first point of S has $\prec_R$-cofinally many neighbors in R then $R \cup S$ is a path which end extends R and we denote this path by $R^\frown S$.

Let K_{ω_1} denote $(\omega_1, [\omega_1]^2)$, i.e. the complete graph on ω_1. Now we are ready to formulate the main result of this section.

Theorem 5.3.

$$K_{\omega_1} \sqsubset (\mathfrak{Path}, \mathfrak{Path})_2.$$

That is, given any coloring of the edges of K_{ω_1} with 2 colors, the vertices can be partitioned into two monochromatic paths of distinct colors.

5.1. **Further preliminaries.** In the course of the proof we need more definitions.

Definition 5.4. Let $G = (V, E)$ be a graph, κ a cardinal and let $A \subset V$. We say that A is *κ-linked* iff there are κ many disjoint finite paths between any two points of A. We say that A is *κ-connected* iff there are κ many disjoint finite paths inside A between any two points of A.

We will apply this definition with $\kappa = \omega$ or ω_1. We leave the (straightforward) proof of the next observation to the reader:

Observation 5.5. *Let $G = (V, E)$ be a graph, κ an infinite cardinal and let $A \subset V$. The following are equivalent:*

(1) A is κ-linked (κ-connected),
(2) for every $v, w \in A$ and $F \subseteq V \setminus \{v, w\}$ of size $< \kappa$ there is a finite path P connecting v and w in $V \setminus F$ (in $A \setminus F$ respectively).

In the construction of a path longer than ω, the difficulty lies in constructing the *limit* elements. Definition 5.6 will be crucial in overcoming this difficulty; the idea is first finding *all limit vertices* of the path and then connecting these points appropriately.

Recall that a set $\mathcal{V} \subset [V]^\omega$ is a *club* (closed and unbounded) iff

(1) $\bigcup\{V_n : n \in \omega\} \in \mathcal{V}$ for every increasing sequence $\{V_n : n \in \omega\} \subset \mathcal{V}$, and
(2) for all $W \in [V]^\omega$ there is $U \in \mathcal{V}$ so that $W \subset U$.

Remark. An easy transfinite induction shows that every club on a set of size ω_1 contains a club that is a well-ordered *strictly increasing* family of the form $\{V_\alpha : \alpha < \omega_1\}$. Hence from now on we will tacitly assume that all clubs are of this form.

Definition 5.6. Suppose that $G = (V, E)$ is a graph with $|V| = \omega_1$. We say that $A \subset V$ is a *trail* iff there is a club $\{V_\alpha : \alpha < \omega_1\} \subset [V]^\omega$ so that for all $\alpha < \omega_1$ there is $v_\alpha \in A \setminus V_\alpha$ such that for all $\alpha' < \alpha$

$$N_G(v_\alpha) \cap (V_\alpha \setminus V_{\alpha'}) \cap A \text{ is infinite.} \tag{5.1}$$

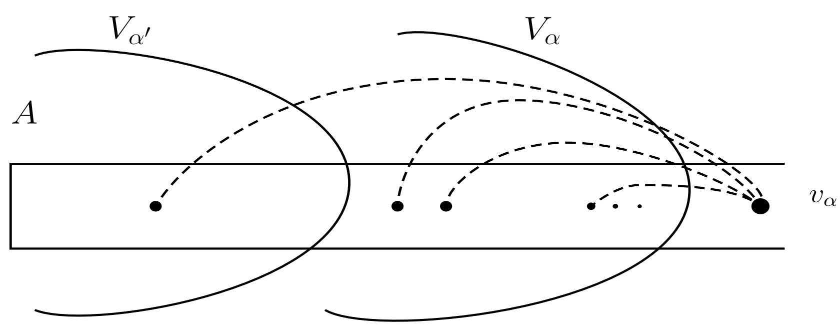

FIGURE 5. Trails.

An important example of a path is the graph H_{ω_1,ω_1} i.e. $(\omega_1 \times 2, E)$ where

$$E = \{\ \{(\alpha, 0), (\beta, 1)\} : \alpha \le \beta < \omega_1\ \}.$$

H_{ω_1,ω_1} is a bipartite graph and we call the set of vertices in H_{ω_1,ω_1} with degree ω_1, (that is, $\omega_1 \times \{0\}$) *the main class of* H_{ω_1,ω_1}.

Observation 5.7. *The main class of* H_{ω_1,ω_1} *is* ω_1*-linked and is a trail (indeed, let* $V_\alpha = \omega\alpha \times 2$*).*

If G *is any graph and* $A \subset V(G)$ *is a trail then*

(1) C *is a trail for any* $A \subset C \subset V(G)$*,*
(2) $C \setminus B$ *is a trail for any* $B \in [C]^\omega$*,*
(3) if $\{W_\alpha : \alpha < \omega_1\} \subset \{V_\alpha : \alpha < \omega_1\}$ *are clubs and* $\{V_\alpha : \alpha < \omega_1\}$ *witnesses that* A *is a trail then so does* $\{W_\alpha : \alpha < \omega_1\}$*.*

We will make use of the following lemma regularly but the reader should feel free to skip the proof when first working through this section.

Lemma 5.8. *Let* $G = (V, E)$ *be a graph with* $|V| = \omega_1$*, and let* $A \subseteq V$ *be uncountable. Then there is a club* $\{V_\alpha : \alpha < \omega_1\}$ *of* V *such that*

(1) V_α *is an initial segment of* ω_1 *and if* $\xi \in V_\alpha$ *then* $\xi + 1 \in V_\alpha$ *as well,*

(2) if A is ω_1-linked (ω_1-connected) then $A \cap (V_{\alpha+1} \setminus V_\alpha)$ is infinite and ω-linked (ω-connected) in $V_{\alpha+1} \setminus V_\alpha$,

(3) if A is a trail then $\{V_\alpha : \alpha < \omega_1\}$ witnesses this and, using the notation of Definition 5.6, the node v_α can be chosen in $V_{\alpha+1} \setminus V_\alpha$.

Proof. Let $\mathcal{M} = \{M_\alpha : 0 < \alpha < \omega_1\}$ be an $\in$-chain of countable elementary submodels of $H(\omega_2)$ such that $G, A, \prec \in M_1$, $V \subseteq \bigcup \mathcal{M}$ and let $M_0 = \emptyset$. Let $V_\alpha = V \cap M_\alpha$ for $\alpha < \omega_1$. We claim that $\mathcal{V} = \{V_\alpha : \alpha < \omega_1\}$ is a club which satisfies the above conditions.

First, $\mathcal{V}$ is a club as $\mathcal{M}$ is a continuous chain and $V \subseteq \bigcup \mathcal{M}$. Condition (1) is satisfied by elementarity.

Now, suppose that A is ω_1-linked and fix $\alpha < \omega_1$. Also, fix $v, w \in V_{\alpha+1} \setminus V_\alpha$ and a finite set $F \subset V_{\alpha+1} \setminus V_\alpha$. We prove that there is a path form v to w in $V_{\alpha+1} \setminus (V_\alpha \cup F)$; this implies that $A \cap (V_{\alpha+1} \setminus V_\alpha)$ is ω-linked by Observation 5.5. As A is ω_1-linked we have that

$$H(\omega_2) \models \text{ there is a finite path from } v \text{ to } w \text{ in } V \setminus (V_\alpha \cup F).$$

Hence, by elementarity of $M_{\alpha+1}$ and by $V_\alpha, F, v, w \in M_{\alpha+1}$ we have

$$M_{\alpha+1} \models \text{ there is a finite path from } v \text{ to } w \text{ in } V \setminus (V_\alpha \cup F).$$

We choose any such path P in $M_{\alpha+1}$ and so we have $P \subseteq V_{\alpha+1} \setminus (V_\alpha \cup F)$ as desired. The case when A is ω_1-connected is completely analogous.

Finally, suppose that A is a trail. By elementarity, since $A \in M_1$, there is a club $\mathcal{W} = \{W_\alpha : \alpha < \omega_1\} \in M_1$ which witnesses that A is a trail. First, it is easy to see that $\mathcal{V} \subseteq \mathcal{W}$ and in particular, $\mathcal{V}$ witnesses that A is a trail. Second, the node $v_\alpha \in V \setminus V_\alpha$ can be selected in $V_{\alpha+1}$ as $V_\alpha \in M_{\alpha+1}$ and

$$M_{\alpha+1} \models \text{ there is } v \in V \setminus V_\alpha \text{ such that } |N_G(v) \cap (V_\alpha \setminus V_{\alpha'}) \cap A| = \omega \text{ for all } \alpha' < \alpha. \tag{5.2}$$

This finishes the proof of the lemma. □

Finally, we state the obvious extension of Lemma 2.2.

Lemma 5.9. *Given any edge coloring $c : [\kappa]^2 \to \{0, \dots, r-1\}$ of the complete graph on κ (where $\kappa \geq \omega$), there is a partition $d_c : \kappa \to \{0, \dots, r-1\}$ and a color $i(c) < r$ so that*

$$|N[F, i] \cap V_{i(c)}| = \kappa \text{ for all } i < r \text{ and finite set } F \subset V_i = d_c^{-1}\{i\}.$$

In particular, V_i is κ-linked in color i for all $i < r$ and $V_{i(c)}$ is κ-connected in color $i(c)$.

Proof. Repeat tho proof of Lemma 2.2 but choose the ultrafilter U on κ to be uniform, that is, $|H| = \kappa$ for every $H \in U$. □

5.2. **Towards the proof of Theorem 5.3.** The following two lemmas express the connection between trails, ω_1-linked sets and paths:

Lemma 5.10. *Every path of order type ω_1 is a trail and contains an uncountable ω_1-linked subset.*

Proof. Suppose that P is a path of order type ω_1 witnessed by the well ordering $\prec$. Now, by Lemma 5.8, there is a club $\{V_\alpha : \alpha < \omega_1\}$ of vertices of P such that V_α is a $\prec$-initial segment, $v \in V_\alpha$ implies that the $\prec$-successor of v is also in V_α and $V_\alpha \subset V_\beta$ for all $\alpha < \beta < \omega_1$. Let v_α denote the $\prec$-minimal element of $V \setminus V_\alpha$. In order to prove that P is a trail it suffices to show that

Claim 5.10.1. *$N_P(v_\alpha) \cap (V_\alpha \setminus V_{\alpha'})$ is infinite for all $\alpha' < \alpha < \omega_1$.*

Proof. First, note that v_α is a $\prec$-limit. Fix $\alpha' < \alpha$. $V_{\alpha'}$ is an initial segment of the path P and has minimal bound $v_{\alpha'}$. Note that $v_{\alpha'} \prec v_\alpha$. By the definition of a path, the set $\{w \in N_P(v_\alpha) : v_{\alpha'} \prec w \prec v_\alpha\}$ is infinite and it is clearly a subset of $N_P(v_\alpha) \cap V_\alpha \setminus V_{\alpha'}$ by the choice of $v_{\alpha'}$ and v_α. □

Second, we prove

Claim 5.10.2. *The set $A = \{v \in V(P) : |N_P(v)| = \omega_1\}$ is uncountable and ω_1-linked.*

Proof. First, it suffices to show that there is a single vertex v with uncountable degree in P as every end segment of P is also a path of order type ω_1. Let $\{p_\alpha : \alpha < \omega_1\}$ enumerate P according to the path well order $\prec$. Now, for every limit $\alpha < \omega_1$ there is $\mu_\alpha < \alpha$ so that $\{p_\alpha, p_{\mu_\alpha}\} \in E(P)$; Fodor's pressing down lemma gives a stationary set $S \subset \omega_1$ and $\mu \in \omega_1$ so that $\{p_\alpha, p_\mu\} \in E(P)$ if $\alpha \in S$, that is, the degree of p_μ in P is uncountable.

Now take any two distinct vertices, v and w, in A and fix an arbitrary countable set $F \subset V(P) \setminus \{v, w\}$. We will find a finite path from v to w in $V(P) \setminus F$. There is $v' \in N_P(v)$ and $w' \in N_P(w)$ so that both v' and w' are $\prec$-above all elements of F as $v, w \in A$ and $|F| \leq \omega$. Now, there is a finite $\prec$-monotone path Q between v' and w' by Observation 5.2; Q must avoid F and hence the path $(v)^\frown Q^\frown(w)$ connects v and w in $V(P) \setminus F$. By Observation 5.5, A must be ω_1-linked. □

□

Now, we show that the converse of Lemma 5.10 is true as well:

Lemma 5.11. *Suppose that $G = (V, E)$ is a graph with $|V| = \omega_1$. If V is an ω_1-connected trail then G is a path.*

Proof. Fix a club $\{V_\alpha : \alpha < \omega_1\}$ as in Lemma 5.8 and pick nodes $v_\alpha \in V_{\alpha+1} \setminus V_\alpha$ showing that V is a trail.

It suffice to construct sets $P_\alpha \subset V$ and orderings $\prec_\alpha$ for $\alpha < \omega_1$ so that

(i) $(P_\alpha, \prec_\alpha)$ is a path with last point v_α,

(ii) $P_\alpha = V_\alpha \cup \{v_\alpha\}$,
(iii) P_β end extends P_α for $\alpha < \beta < \omega_1$.

Indeed, the ordering $\bigcup\{\prec_\alpha : \alpha < \omega_1\}$ on V will witness that G is a path.

First, we set $P_0 = \{v_0\}$. Next, apply Corollary 2.4 to find a path R of order type ω on vertices V_1 with first point v_0; this can be done as V_1 is ω-connected. We let $P_1 = R^\frown(v_1)$ and note that P_1 is a path as the infinite set $N_G(v_1) \cap V_1$ is cofinal in R and hence below v_1.

In general, suppose that we have constructed P_α for $\alpha < \beta$ as above. If β is a limit then let $P_{<\beta} = \bigcup\{P_\alpha : \alpha < \beta\}$; note that $P_{<\beta} = V_\beta$ is a path. It suffices to prove

Observation 5.12. *$P_\beta = P_{<\beta}{}^\frown(v_\beta)$ is a path.*

Indeed, we know that $N_G(v_\beta) \cap (V_\beta \setminus V_\alpha)$ is infinite for all $\alpha < \beta$ by the definition of v_β.

If $\beta = \alpha + 1$ then we apply Lemma 2.4 to find a path R of order type ω on vertices $V_{\alpha+1} \setminus V_\alpha$ with first point v_α; see Figure 6.

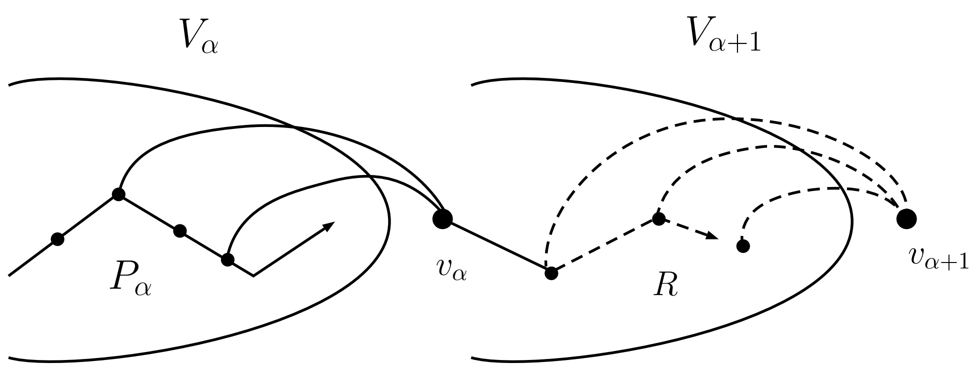

FIGURE 6. Extending P_α to $P_{\alpha+1}$.

We let $P_{\alpha+1} = P_\alpha{}^\frown R^\frown(v_{\alpha+1})$. Note that $P_{\alpha+1}$ is a path as the infinite set $N_G(v_{\alpha+1}) \cap (V_{\alpha+1} \setminus V_\alpha)$ is cofinal in R and hence below $v_{\alpha+1}$. $\square$

We note that it is proved very similarly that if a set of vertices $A \subseteq V$ is an ω_1-linked trail then A can be *covered* by a path of order type ω_1.

Finally, before the proof of Theorem 5.3, we prove a simple result about finding trails in 2-edge colored copies of K_{ω_1}.

Lemma 5.13. *Suppose that c is a 2-edge coloring of K_{ω_1} and $A \subset \omega_1$. Then either A is a trail in color 0 or we can find a copy of K_{ω_1} in color 1 (inside A).*

Proof. Suppose that A is not a trail in color 0. Let $V_\alpha = \omega\alpha \subset \omega_1$ (regarded as a set of vertices) for $\alpha < \omega_1$. Let $v_\alpha = \min(A \setminus V_\alpha)$ for $\alpha < \omega_1$. Let

$$X = \{\alpha < \omega_1 : |N(v_\alpha, 0) \cap (V_\alpha \setminus V_{\alpha'}) \cap A| = \omega \text{ for all } \alpha' < \alpha\}.$$

If there is a club C in X then $\{V_\alpha : \alpha \in C\}$ witnesses that A is a trail in color 0. Hence, as X cannot contain a club, there is a stationary set S in $\omega_1 \setminus X$. We can suppose, by shrinking S to a smaller stationary set, that $\omega\alpha = \alpha$ for all $\alpha \in S$. Now for every $\alpha \in S$ there is $\nu_\alpha < \alpha$ and finite $F_\alpha \subset V_\alpha$ such that

$$F_\alpha = N(v_\alpha, 0) \cap (V_\alpha \setminus V_{\nu_\alpha}) \cap A.$$

By Fodor's pressing down lemma we can find stationary $T \subset S$, $\nu < \omega_1$ and finite set F such that

$$F = N(v_\alpha, 0) \cap (V_\alpha \setminus V_\nu) \cap A$$

for all $\alpha \in T$. It is clear now that $B = \{v_\alpha : \alpha \in T\} \setminus (V_\nu \cup F)$ is an uncountable subset of A and $c \restriction [B]^2 \equiv 1$. □

Now, we are ready to prove the main result of this section:

Proof of Theorem 5.3. Fix an edge coloring $r : [\omega_1]^2 \to 2$ of the complete graph $K_{\omega_1} = (\omega_1, [\omega_1]^2)$. We distinguish two cases as follows:

Case 1: There is a monochromatic copy H_0 of H_{ω_1,ω_1}.

We can suppose that H_0 is 0-monochromatic by symmetry and let A denote the main class of H_0. As A is ω_1-linked in color 0, we can extend A to a *maximal* subset $C \subseteq \omega_1$ that is ω_1-linked in color 0. Note that, by the maximality of C,

$$|N(v, 0) \cap C| \leq \omega \text{ for all } v \in \omega_1 \setminus C, \tag{5.3}$$

and in particular $\omega_1 \setminus C$ is ω_1-linked in color 1.

Case 1A: $\omega_1 \setminus C$ is countable.

Find a path P^1 in color 1 and of order type $\leq \omega$ which covers $\omega_1 \setminus C$ (see Corollary 2.4).

Claim 5.13.1. *$C \setminus P^1$ is an ω_1-connected trail in color 0.*

Indeed, if $v, w \in C \setminus P^1$ and $F \subset V$ is countable then there is a path P of color 0 from v to w which avoids $F \cup P^1$ as C is ω_1-linked in color 0; in particular, P is also a subset of $C \setminus P^1$ and in turn $C \setminus P^1$ is ω_1-connected in color 0. By Observation 5.7 C is a trail in color 0 witnessed by the copy of H_{ω_1,ω_1}, and hence , using Observation 5.7 again, $C \setminus P^1$ remains a trail as well. This finishes the proof of the claim.

Hence, by Lemma 5.11, $C \setminus P^1$ is a path in color 0 which finishes the proof of Theorem 5.3 in Case 1A.

Case 1B: $\omega_1 \setminus C$ is uncountable.

Claim 5.13.2. *$\omega_1 \setminus C$ is covered by a copy of H_{ω_1,ω_1} in color 1 with main class $\omega_1 \setminus C$.*

Proof. Note that $|N[X, 1] \cap C| = \omega_1$ for all $X \in [\omega_1 \setminus C]^\omega$ by (5.3). Enumerate $\omega_1 \setminus C$ as $\{x_\alpha : \alpha < \omega_1\}$ and inductively select

$$y_\beta \in N[X_\beta, 1] \cap C \setminus \{y_\alpha : \alpha < \beta\}$$

for $\beta < \omega_1$ where $X_\beta = \{x_\alpha : \alpha \leq \beta\}$. Now $(\omega_1 \setminus C) \cup \{y_\alpha : \alpha < \omega_1\}$ is the desired copy of H_{ω_1,ω_1} in color 1. □

Let H_1 denote this copy of H_{ω_1,ω_1}. Our goal is to mimic the proof of Lemma 5.11 and, using H_0 and H_1, simultaneously construct two disjoint monochromatic paths (one in color 0 and one in color 1) which cover V.

Using Lemma 5.8 twice, the observation that the intersection of two clubs is itself a club, and also Observation 5.7, we can fix a club $\{V_\alpha : \alpha < \omega_1\}$ in ω_1 such that $C \cap (V_{\alpha+1} \setminus V_\alpha)$ and $(\omega_1 \setminus C) \cap (V_{\alpha+1} \setminus V_\alpha)$ are ω-linked in color 0 and 1, respectively, inside $V_{\alpha+1} \setminus V_\alpha$ for all $\alpha < \omega_1$. Furthermore, we can suppose that V_α intersects H_0 and H_1 in initial segments of their respective H_{ω_1,ω_1} orderings for each $\alpha < \omega_1$.

Now, we inductively construct disjoint sets P^0_α, P^1_α and well orderings $\prec^0_\alpha, \prec^1_\alpha$ such that

(i) $(P^i_\alpha, \prec^i_\alpha)$ is a path in color i of order type ω for $i < 2$,
(ii) P^i_β end extends P^i_α for all $\alpha < \beta$ and $i < 2$,
(iii) $A \cap P^0_\alpha$ is cofinal in P^0_α and $(\omega_1 \setminus C) \cap P^1_\alpha$ is cofinal in P^1_α,
(iv) $P^0_\alpha \cup P^1_\alpha = V_\alpha$

for all $\alpha < \omega_1$.

First, apply Lemma 2.3 to $G[V_1]$ and the sets $C_0 = C \cap V_1$, $C_1 = (\omega_1 \setminus C) \cap V_1$, $A_0 = A \cap V_1$ and $A_1 = \emptyset$ to obtain two disjoint paths covering V_1: P^0_1 in color 0 and P^1_1 in color 1, both of order type ω. Since P^0_1 and P^1_1 are of order type ω, a subset of such a path is cofinal iff it is infinite. Lemma 2.3 makes sure that $A \cap P^0_1$ as well as $(\omega_1 \setminus C) \cap P^1_1$ are infinite (note that A_0 is infinite), hence (iii) holds.

Suppose we have constructed P^0_α, P^1_α as above for $\alpha < \beta$. Note that $P^0_{<\beta} = \bigcup\{P_\alpha : \alpha < \beta\}$ is a path in color 0, $P^1_{<\beta} = \bigcup\{P^1_\alpha : \alpha < \beta\}$ is a path in color 1 and $A \cap P^0_{<\beta}$ is cofinal in $P^0_{<\beta}$ while $\omega_1 \setminus C$ is cofinal in $P^1_{<\beta}$. Thus if β is limit we are done. Suppose that $\beta = \alpha + 1$, i.e. $P^i_{<\beta} = P^i_\alpha$ for $i < 2$.

Claim 5.13.3. *(a) There are $v^0_\alpha, w^0_\alpha \in V_{\alpha+1} \setminus V_\alpha$ such that $A \cap V_\alpha \subseteq N(v^0_\alpha, 0)$, $w^0_\alpha \in A$ and $c(v^0_\alpha, w^0_\alpha) = 0$.*
(b) There are $v^1_\alpha, w^1_\alpha \in V_{\alpha+1} \setminus (V_\alpha \cup \{v^0_\alpha, w^0_\alpha\})$ such that $(\omega_1 \setminus C) \cap V_\alpha \subseteq N(v^1_\alpha, 1)$, $w^1_\alpha \in \omega_1 \setminus C$ and $c(v^1_\alpha, w^1_\alpha) = 1$.

Proof. (a) We know that V_α intersects H_0 in an initial segment and hence any element v^0_α from $(V(H_0) \setminus A) \cap (V_{\alpha+1} \setminus V_\alpha)$ will satisfy $A \cap V_\alpha \subseteq N(v^0_\alpha, 0)$. We can now select $w^0_\alpha \in A \cap (V_{\alpha+1} \setminus V_\alpha)$ such that $c(v^0_\alpha, w^0_\alpha) = 0$.

The proof of (b) is completely analogous to (a). □

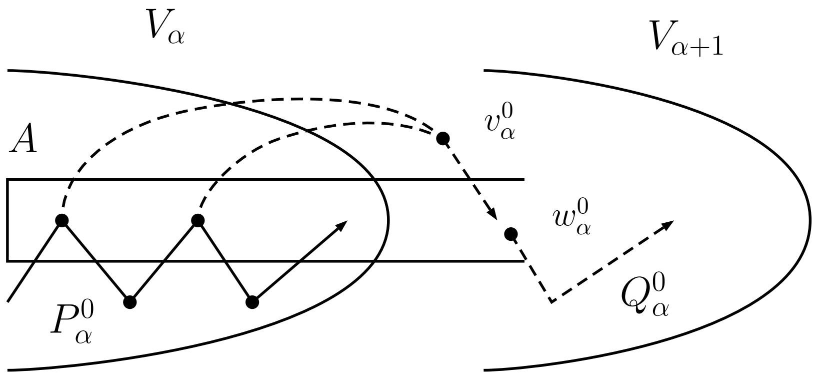

FIGURE 7. Extending P^0_α to $P^0_{\alpha+1}$.

Note that $P^i_\alpha{}^\frown(v^i_\alpha, w^i_\alpha)$ is still a path in color i for $i < 2$; see Figure 7. Now, let us find disjoint sets Q^0_α, Q^1_α such that

(1) Q^i_α is a path of color i and order type ω for $i < 2$,
(2) $Q^0_\alpha \cup Q^1_\alpha = (V_{\alpha+1} \setminus V_\alpha) \setminus \{v^0_\alpha, v^1_\alpha\}$,
(3) the first point of Q^i_α is w^i_α for $i < 2$,
(4) $A \cap Q^0_\alpha$ is cofinal in Q^0_α and $(\omega_1 \setminus C) \cap Q^1_\alpha$ is cofinal in Q^1_α.

Similarly as above, this is easily done by setting $D = (V_{\alpha+1} \setminus V_\alpha) \setminus \{v^0_\alpha, v^1_\alpha\}$ and applying Lemma 2.3 to $G[D]$ and $C_0 = C \cap D$, $C_1 = (\omega_1 \setminus C) \cap D$, $A_0 = A \cap D$ and $A_1 = \emptyset$. Note that

$$P^i_{\alpha+1} = P^i_\alpha{}^\frown(v^i_\alpha)^\frown Q^i_\alpha$$

is as desired (for $i < 2$).

Finally, let $P^i = \bigcup\{P^i_\alpha : \alpha < \omega_1\}$ for $i < 2$. Then P^0 and P^1 are monochromatic paths of distinct colors which partition ω_1.

Case 2: There is no monochromatic copy of H_{ω_1,ω_1}.

Lemma 5.13 implies that any uncountable set of vertices must be a trail in both colors. Let us find an uncountable $A \subset V$ which is ω_1-connected in some color by Lemma 5.9. We can suppose that A is ω_1-connected in color 0 and extend A to a *maximal* ω_1-connected set C in color 0.

Claim 5.13.4. *$V \setminus C$ is countable and ω_1-linked in color 1.*

Proof. Indeed, by the maximality of C, it is easy to see that $|N(v, 0) \cap C| \leq \omega$ for every $v \in V \setminus C$; this immediately gives that $V \setminus C$ is ω_1-linked in color 1. Moreover, if $V \setminus C$ is uncountable then the proof of Claim 5.13.2 shows that we can find a monochromatic copy of H_{ω_1,ω_1} which contradicts our assumption. □

Now cover $V \setminus C$ by a path P^1 of color 1 and order type ω using Corollary 2.4. By assumption, $C \setminus P^1$ is still a trail and remains ω_1-connected in color 0; that is, $C \setminus P^1$ is a path P^0 of color 0 by Lemma 5.11. We conclude the proof by noting that $P^0 \cup P^1$ is the desired partition. □

6. Further results and open problems

In general, there are two directions in which one can aim to extend our results: investigate edge colored non-complete graphs; determine the exact number of monochromatic structures (paths, powers of paths) needed to cover a certain edge colored graph.

First, for state-of-the-art results and problems concerning finite graphs and partitions into monochromatic paths, we refer the reader to A. Pokrovskiy [6]. Second, let us mention some results and problems about countably infinite graphs. Let $K_{\omega,\omega}$ denote the complete bipartite graph with two countably infinite classes. The following statements can be proved very similarly to our proof of Theorem 2.5:

Claim 6.1. *Let $c : E(K_{\omega,\omega}) \to r$ for some $r \in \mathbb{N}$. Then $K_{\omega,\omega}$ can be partitioned into at most $2r-1$ monochromatic paths. Furthermore, for every $r \in \mathbb{N}$ there is $c_r : E(K_{\omega,\omega}) \to r$ so that $K_{\omega,\omega}$ cannot be covered by less than $2r-1$ monochromatic paths.*

Claim 6.2. *For every r-edge coloring of the random graph on $\mathbb{N}$ we can partition the vertices into r disjoint paths of distinct colors.*

Regarding Theorem 4.6 we ask the following most general question:

Problem 6.3. *What is the exact number of monochromatic k^{th} powers of paths needed to partition the vertices of an r-edge colored complete graph on $\mathbb{N}$?*

Naturally, any result aside from the resolved case of $k = r = 2$ (see Theorem 4.7) would be very welcome. In particular:

Problem 6.4. *Can we bound the number of monochromatic k^{th} powers of paths needed to partition the vertices of an r-edge colored complete graph on $\mathbb{N}$ by a function of r and k?*

Finally, turning to arbitrary infinite complete graphs, we announce the following complete solution to Rado's problem from [8]:

Theorem 6.5 (D. T. Soukup, [9])**.** *The vertices of a finite-edge colored infinite complete graph can be partitioned into disjoint monochromatic paths of different colour.*

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