DECOMPOSING THE REAL LINE INTO BOREL SETS CLOSED UNDER ADDITION

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ABSTRACT. We consider decompositions of the real line into pairwise disjoint Borel pieces so that each piece is closed under addition. How many pieces can there be? We prove among others that the number of pieces is either at most 3 or uncountable, and we show that it is undecidable in ZFC and even in the theory $ZFC + c = \omega_2$ if the number of pieces can be uncountable but less than the continuum. We also investigate various versions: what happens if we drop the Borelness requirement, if we replace addition by multiplication, if the pieces are subgroups, if we partition $(0, \infty)$, and so on.

1. INTRODUCTION

We consider decompositions of the real line into Borel subsets with some additional algebraic structure. In this paper the words 'decomposition' and 'partition' will always refer to writing the real line as a union of pairwise disjoint sets. The main question is weather the number of pieces can be strictly between ω and $\mathfrak{c}$. First, the question is interesting if we impose no algebraic structure at all. Lebesgue was the first one to show that the real line can be decomposed into ω_1 Borel sets, hence it is consistent with ZFC that the number of pieces can be strictly between ω and $\mathfrak{c}$. Later Hausdorff [?] showed that there is in fact a partition into ω_1 many $F_{\sigma\delta}$ sets. Whether the real line can be partitioned into ω_1 many $G_{\delta\sigma}$ sets is already independent from ZFC, and even from $ZFC + c = \omega_2$, see [?] for the details. Finally, J. Stern [?] and independently A. W. Miller [?] proved that consistently $\mathfrak{c} = \omega_2$ and the real line can be partitioned into ω_1 many compact sets.

It is important to mention the following remarkable theorem of Silver: If an equivalence relation on the line (considered as a subset of the plane) is Borel then there are countably many or continuum many equivalence classes [?]. This shows that in all these decompositions into $\kappa \in (\omega, \mathfrak{c})$ many Borel pieces the relation of being in the same piece is fairly complicated. We will return to this issue in the Open questions section.

In this paper the algebraic assumption about the pieces will mostly be that each piece is closed under addition. In other words, we partition the real line into Borel additive semigroups. Since $(-\infty, 0), \{0\}$ and $(0, \infty)$ are closed under addition, we can trivially decompose $\mathbb{R}$ into one, two or three Borel sets that are closed under addition. We show (Theorem **??**) that in any other decomposition of the real line into Borel additive semigroups each semigroup must have Lebesgue measure zero and must be of first category. Therefore the number of pieces is either $1, 2, 3$ or uncountable (in fact, at least $\max(\operatorname{cov} \mathcal{M}, \operatorname{cov} \mathcal{N})$). The sets of the form $c \cdot \mathbb{Q}^+$ clearly decompose $\mathbb{R}$ into continuum many sets that are closed under addition. So

it is consistent with ZFC that the set of the possible number of pieces is $\{1, 2, 3, \mathfrak{c}\}$. Can this be proved in ZFC?

Our main result (Theorem **??**) is that it is consistent with ZFC that $\mathfrak{c} = \omega_2$ and $\mathbb{R}$ can be decomposed into ω_1 Borel sets that are closed under addition.

Now, in the remaining part of the introduction we say a few words about certain natural variations of the problem.

If we do not require the sets to be Borel then the problem is much easier and the answer is much simpler. Throughout the paper κ and λ will denote cardinal numbers.

Proposition 1.1. *For any $1 \leq \kappa \leq \mathfrak{c}$ there exists a decomposition $\mathbb{R} = \bigcup_{\alpha < \kappa} A_\alpha$ such that each A_α is nonempty and closed under addition.*

Proof. Let $H = \{h_\alpha \ : \alpha < \mathfrak{c}\}$ be a Hamel basis of $\mathbb{R}$. For every $x \in \mathbb{R} \setminus \{0\}$ take its unique representation in the form $x = r_1 h_{\alpha_1} + \ldots + r_n h_{\alpha_n}$, where $r_1, \ldots, r_n \in \mathbb{Q} \setminus \{0\}$ and $\alpha_1 < \ldots < \alpha_n < \mathfrak{c}$ and let $f(x) = \alpha_1$ and $g(x) = r_1$. For any $\alpha < \mathfrak{c}$ let

$$B_\alpha = \{x \ : f(x) = \alpha, g(x) > 0\}.$$

Clearly every B_α is nonempty and closed under addition. The same is true for $C_\beta = \mathbb{R} \setminus (\cup_{\alpha < \beta} B_\alpha)$ for any $\beta \leq \mathfrak{c}$ since $x \in C_\beta \Leftrightarrow f(x) \geq \beta$ or $g(x) < 0$ or $x = 0$.

Let κ' be $\kappa - 1$ if κ is finite and κ otherwise. Then $\mathbb{R} = C_{\kappa'} \cup \bigcup_{\alpha < \kappa'} B_\alpha$ is a decomposition with the required properties. $\square$

If we consider multiplication instead of addition then we get similar results (both for the Borel and the non-Borel case), but the negative numbers cause some extra technical complications, see Theorem **??** and Corollary **??**. Hence it is perhaps more natural to decompose only $(0, \infty)$ into subsets that are closed under multiplication. Taking logarithm it is clear that such a decomposition is equivalent to a decomposition of $\mathbb{R}$ into sets that are closed under addition. Recently R. Freud [**?**] raised the question if $(0, \infty)$ can be decomposed into two parts that are closed under *both* addition and multiplication. It turned out that this had already been solved in 2007 by D. M. Kane, who had proved that such a decomposition exists. Another answer was given in [**?**], where the authors also describe the structure of all such decompositions. They also show that for any $1 \leq \kappa \leq \mathfrak{c}$ one can decompose $(0, \infty)$ into κ sets that are closed under addition and multiplication.

So far we considered additive or multiplicative semigroups. It is natural to ask what happens if we require groups. Then of course we cannot hope to get decompositions since all groups contain the neutral element. So we should rather require that their intersection only contains the neutral element. In this case we say that the subgroups are *essentially disjoint*. But it turns out that even in that case the answer is fairly easy, even in a more general setting. Recall that $|X|$ denotes the cardinality of a set X.

Proposition 1.2. *If an infinite group G is the union of κ many essentially disjoint subgroups then $\kappa = 1$ or $\kappa = |G|$.*

Proof. Let $G = \bigcup_{\alpha < \kappa} H_\alpha$ be a union consisting of essentially disjoint nonempty subgroups. Assume $\kappa \geq 2$. Since the H_α's are essentially disjoint, κ cannot exceed $|G|$, so we only need to show that $\kappa \geq |G|$.

Let $\alpha < \kappa$ and $g \notin H_\alpha$. Then $|gH_\alpha \cap H_\beta| \leq 1$ for every $\beta \neq \alpha$, since if $g_1, g_2 \in gH_\alpha \cap H_\beta$ then $g_1^{-1} g_2 \in H_\alpha \cap H_\beta = \{e\}$, hence $g_1 = g_2$. Since $gH_\alpha \cap H_\alpha = \emptyset$, the H_β's cover gH_α, hence $\kappa \geq |H_\alpha|$. Repeating the argument for every α yields $\kappa \geq \sup\{|H_\alpha| : \alpha < \kappa\}$. Let $\lambda = \sup\{|H_\alpha| : \alpha < \kappa\}$, then $\kappa \geq \lambda$. If $\lambda = |G|$ then $\kappa \geq \lambda = |G|$ and we are done, so let $\lambda < |G|$. Since the H_α's cover G, we obtain $\kappa \lambda \geq |G|$, in particular, κ is infinite. Then $\kappa \lambda = \kappa$, therefore $\kappa \geq |G|$, hence the proof is complete. $\square$

We remark here that it is easy to check that if $\mathbb{R}$ is the union of essentially disjoint subgroups, then these groups are actually $\mathbb{Q}$-linear vector spaces. Hence there is a close connection to the theory of so called vector space partitions, which deals with the problem of how one can write a vector space as the union of essentially disjoint proper subspaces. However, this theory mostly considers finite vector spaces, since one of the main motivations is the connection to error-correcting codes. See e.g. [**?**], [**?**], [**?**], [**?**] and the references therein for more details.

2. Non-existence results concerning Borel decompositions

In this section we prove structural results about decompositions into additive and/or multiplicative Borel semigroups, which in turn yield strong limitations about the possible number of pieces. All results of the section are proved in ZFC.

Notation 2.1. For any $A \subset \mathbb{R}$, $n \in \mathbb{N}^+$ let $(n)A = \{a_1 + \ldots + a_n : a_n \in A\}$.

The following lemma is clearly well-known, however, for the sake of completeness we include its proof.

Lemma 2.2. *Let $B \subset \mathbb{R}$ be a Borel set and suppose that B has positive Lebesgue measure or B is of second category.*
(i) Then $\bigcup_{k=1}^{\infty}(2k)B$ contains a halfline.
(ii) If we also have $B + B \subset B$ or $B + B + B \subset B$ then B contains a halfline.

Proof. (i) If B has positive Lebesgue measure then Steinhaus theorem [**?**], if B is of second category then Piccard theorem [**?**] (see also in [**?**]) implies that $B + B$ contains an interval (a, b), so $(2k)B \supset (ka, kb)$ for any $k \in \mathbb{N}^+$. Since for large enough k the consecutive intervals (ka, kb) overlap, and so $\cup_{k=1}^{\infty}(ka, kb)$ contains a halfline, this completes the proof of (i).

(ii) If $B + B \subset B$ then B contains $\bigcup_{k=1}^{\infty}(2k)B$ so we are done by (i).

Now suppose that $B + B + B \subset B$. Then, by induction we get that $(2k+1)B \subset B$ for any $k \in \mathbb{N}^+$. Thus we have $\bigcup_{k=1}^{\infty}(2k + 1)B \subset B$. Since $\bigcup_{k=1}^{\infty}(2k + 1)B = B + \bigcup_{k=1}^{\infty}(2k)B$ and B is nonempty, (i) implies that $\bigcup_{k=1}^{\infty}(2k + 1)B$ contains a halfline, which completes the proof of (ii). $\square$

Lemma 2.3. *Suppose that we have a decomposition $(0, \infty) = \bigcup_{\alpha < \kappa} B_\alpha$ such that every B_α is Borel and for each $\alpha < \kappa$ we have $B_\alpha + B_\alpha \subset B_\alpha$ or $B_\alpha + B_\alpha + B_\alpha \subset B_\alpha$. Then either every B_α is of Lebesgue measure zero and is of first category or $B_{\alpha_1} = (0, \infty)$ for some $\alpha_1 < \kappa$.*

Proof. Suppose that there is an $\alpha_1 < \kappa$ such that B_{α_1} has positive Lebesgue measure or is of second category. We will show that $B_{\alpha_1} = (0, \infty)$.

By Lemma **??**, B_{α_1} contains a halfline, so the other sets of the decomposition are all bounded. But clearly no bounded nonempty set $B \subset (0, \infty)$ can have the property $B + B \subset B$ or $B + B + B \subset B$, so we have $B_{\alpha_1} = (0, \infty)$. $\square$

Theorem 2.4. *Suppose that we have a decomposition $\mathbb{R} = \bigcup_{\alpha < \kappa} A_\alpha$ such that each A_α is a Borel set closed under addition. Then every A_α is of Lebesgue measure zero and is of first category, or is equal to $\mathbb{R}$, $(-\infty, 0)$, $(-\infty, 0]$, $(0, \infty)$, or $[0, \infty)$.*

Proof. If $\mathbb{R} = \bigcup_{\alpha < \kappa} A_\alpha$ is a decomposition such that each A_α is closed under addition then $(0, \infty) = \bigcup_{\alpha < \kappa} A_\alpha \cap (0, \infty)$ is also a decomposition and each $A_\alpha \cap (0, \infty)$ is closed under addition. Then by Lemma **??**, $A_\alpha \cap (0, \infty)$ is either $(0, \infty)$ or a set of Lebesgue measure zero and of first category. By symmetry, we also have that $A_\alpha \cap (-\infty, 0)$ is either $(-\infty, 0)$ or a set of Lebesgue measure zero and of first category.

It remains to prove that if A_α contains $(0, \infty)$ or $(-\infty, 0)$ then A_α equals $\mathbb{R}$, $(-\infty, 0)$, $(-\infty, 0]$, $(0, \infty)$ or $[0, \infty)$. So suppose that $A_\alpha \supset (0, \infty)$ and $a \in A_\alpha \cap$

$(-\infty, 0)$. Then, since A_α is closed under addition, we get $A_\alpha \supset (a, \infty)$, so $A_\alpha \cap (-\infty, 0)$ has positive measure, thus by the previous paragraph $A_\alpha \supset (-\infty, 0)$, therefore $A_\alpha = \mathbb{R}$. Similarly, we can prove that if $A_\alpha \supset (-\infty, 0)$ and $a \in A_\alpha \cap (0, \infty)$ then $A_\alpha = \mathbb{R}$, which completes the proof. $\qquad\square$

Recall that if $\mathcal{I}$ is an ideal on a set X then $\operatorname{cov}\mathcal{I} = \min\{|\mathcal{H}| : \mathcal{H} \subset \mathcal{I}, \cup\mathcal{H} = X\}$. Let $\mathcal{N}$ and $\mathcal{M}$ denote the ideal of Lebesgue nullsets and the ideal of sets of first category in $\mathbb{R}$, respectively.

Corollary 2.5. *Suppose that we have a decomposition $\mathbb{R} = \bigcup_{\alpha<\kappa} A_\alpha$ such that each A_α is a nonempty Borel set closed under addition. Then $\kappa \in \{1, 2, 3\}$ or $\kappa \geq \max(\operatorname{cov}\mathcal{N}, \operatorname{cov}\mathcal{M})$.*

Proof. If $\kappa < \max(\operatorname{cov}\mathcal{N}, \operatorname{cov}\mathcal{M})$ then at most κ sets of Lebesgue measure zero and of first category cannot cover $(-\infty, 0)$ or $(0, \infty)$. Then by Theorem **??**, every A_α equals to $\{0\}$, $\mathbb{R}$, $(-\infty, 0)$, $(-\infty, 0]$, $(0, \infty)$, or $[0, \infty)$, so $\kappa \leq 3$. $\qquad\square$

Theorem 2.6. *Suppose that we have a decomposition $\mathbb{R} = \bigcup_{\alpha<\kappa} M_\alpha$ such that each M_α is a Borel set closed under multiplication. Then every M_α is of Lebesgue measure zero and is of first category, or is the union of some of the sets $(-1, 0) \cup (0, 1)$, $(-\infty, -1) \cup (1, \infty)$, $\{-1, 1\}$ and $\{0\}$.*

Proof. Let $B_\alpha = M_\alpha \cap (-\infty, -1)$. Then B_α is not closed under multiplication but we still have $B_\alpha \cdot B_\alpha \cdot B_\alpha \subset B_\alpha$ since $M_\alpha \cdot M_\alpha \cdot M_\alpha \subset M_\alpha$ and $(-\infty, -1) \cdot (-\infty, -1) \cdot (-\infty, -1) \subset (-\infty, -1)$. Let $C_\alpha = \log(-B_\alpha)$. Then $\bigcup_{\alpha<\kappa} C_\alpha$ is a Borel decomposition of $(0, \infty)$ and we have $C_\alpha + C_\alpha + C_\alpha \subset C_\alpha$. Then by Lemma **??**, either every C_α is of Lebesgue measure zero and is of first category or there exists an $\alpha_1 < \kappa$ for which $C_{\alpha_1} = (0, \infty)$. In the first case every $B_\alpha = M_\alpha \cap (-\infty, -1)$ is also of Lebesgue measure zero and is of first category. In the latter case, for this α_1, we have $(-\infty, -1) = B_{\alpha_1} = M_{\alpha_1} \cap (-\infty, -1)$. Since M_{α_1} is closed under multiplication, it also contains $(1, \infty)$, so we have $M_{\alpha_1} \supset (-\infty, -1) \cup (1, \infty)$.

Repeating the above argument for $B'_\alpha = M_\alpha \cap (-1, 0)$ and $C'_\alpha = -\log(-B'_\alpha)$ we get that either every $B'_\alpha = M_\alpha \cap (-1, 0)$ is of Lebesgue measure zero and of first category or there exists an $\alpha_2 < \kappa$ such that $M_{\alpha_2} \supset (-1, 0) \cup (0, 1)$.

Since the set M_α that contains -1 also contains 1, the only fact that remains to prove is that if M_α contains one of $(-\infty, -1) \cup (1, \infty)$ and $(-1, 0) \cup (0, 1)$ then either it contains the other one or it is disjoint from the other one. So suppose that $M_\alpha \supset (-\infty, -1) \cup (1, \infty)$ and $a \in M_\alpha \cap ((-1, 0) \cup (0, 1))$. Then, since M_α is closed under multiplication, we get $M_\alpha \supset (-\infty, -a) \cup (a, \infty)$, so $M_\alpha \cap ((-1, 0) \cup (0, 1))$ has positive measure, thus by the above paragraph $M_\alpha \supset (-1, 0) \cup (0, 1)$. Similarly, we can prove that if $M_\alpha \supset (-1, 0) \cup (0, 1)$ and $a \in M_\alpha \cap (-\infty, -1) \cup (1, \infty))$ then $M_\alpha \supset (-\infty, -1) \cup (1, \infty)$, which completes the proof. $\qquad\square$

Remark 2.7. Similarly to Theorem **??**, we could also explicitly describe in Theorem **??** the possible options for M_α, in case it is not a Lebesgue measure zero set of first category. We cannot get all the $2^4 = 16$ possible unions of the sets $(-1, 0) \cup (0, 1)$, $(-\infty, -1) \cup (1, \infty)$, $\{-1, 1\}$ and $\{0\}$, since some of these unions are not closed under multiplication. We do not have to take those ones that are of measure zero and first category. But it is easy to see that M_α can be any of the remaining 10, namely $(-1, 0) \cup (0, 1)$, $[-1, 0) \cup (0, 1]$, $(-1, 1)$, $[-1, 1]$, $(-\infty, -1) \cup (1, \infty)$, $(-\infty, -1] \cup [1, \infty)$, $(-\infty, -1) \cup \{0\} \cup (1, \infty)$, $(-\infty, -1] \cup \{0\} \cup [1, \infty)$, $\mathbb{R}$ and $\mathbb{R} \setminus \{0\}$.

Corollary 2.8. *Suppose that we have a decomposition $\mathbb{R} = \bigcup_{\alpha<\kappa} M_\alpha$ such that each M_α is a nonempty Borel set closed under multiplication. Then $\kappa \in \{1, 2, 3, 4\}$ or $\kappa \geq \max(\operatorname{cov}\mathcal{N}, \operatorname{cov}\mathcal{M})$.*

Proof. If $\kappa < \max(\operatorname{cov}\mathcal{N}, \operatorname{cov}\mathcal{M})$ then at most κ sets of Lebesgue measure zero and of first category cannot cover any interval. Then by Theorem **??** every M_α is the union of some of the sets $(-1, 0) \cup (0, 1)$, $(-\infty, -1) \cup (1, \infty)$, $\{-1, 1\}$ and $\{0\}$, so $\kappa \leq 4$. □

Corollary 2.9. *Suppose that we have a decomposition $\mathbb{R} = \bigcup_{\alpha < \kappa} B_\alpha$ such that each B_α is a nonempty Borel set closed under* both *addition and multiplication.*

(i) Then either every B_α is of Lebesgue measure zero and is of first category or $B_{\alpha_1} = \mathbb{R}$ for some $\alpha_1 < \kappa$.

(ii) We have $\kappa = 1$ or $\kappa \geq \max(\operatorname{cov}\mathcal{N}, \operatorname{cov}\mathcal{M})$

Proof. Claim (i) follows directly from Theorems **??** and **??**. Claim (ii) follows from (i). □

3. THE MAIN RESULT: EXISTENCE OF A CERTAIN BOREL DECOMPOSITION

Theorem 3.1. *It is consistent that $\mathfrak{c} = \omega_2$ and the real line can be partitioned into ω_1 additive F_σ semigroups; that is, there exists a decomposition $\mathbb{R} = \bigcup_{\alpha < \omega_1} A_\alpha$ such that each A_α is a nonempty F_σ set closed under addition.*

Proof. Ciesielski and Pawlikowski [**?**] proved that consistently $\mathfrak{c} = \omega_2$ and there exist a Hamel basis that is the union of ω_1 pairwise disjoint Cantor sets. Fix such a Hamel basis H with such a decomposition $H = \bigcup_{\alpha < \omega_1} C_\alpha$, and in each Cantor set C_α fix a countable base $\{B_\alpha^k\}_{k \in \mathbb{N}^+}$ such that for each k and α both B_α^k and $C_\alpha \setminus B_\alpha^k$ are compact.

Every $x \in \mathbb{R}$ has a unique representation of the form $x = r_1 b_1 + \ldots + r_n b_n$, where $r_1, \ldots, r_n \in \mathbb{Q} \setminus \{0\}$ and $b_1, \ldots, b_n \in H$. For notational simplicity, let us define $B(x) = \{b_1, \ldots, b_j\}$, and for any $J \subset H$ we say that r_i is a *J-coefficient* of x if $b_i \in J$. For $x \in \mathbb{R}$ let

$$I(x) = \{\ \alpha < \omega_1\ :\ x \text{ has a } C_\alpha\text{-coefficient}\ \} = \{\ \alpha < \omega_1\ :\ B(x) \cap C_\alpha \neq \emptyset\ \}.$$

For $x \in \mathbb{R}$ and $J \subset H$ also let $S(x, J)$ be the sum of the J-coefficients of x. Note that $S(x + y, J) = S(x, J) + S(y, J)$ for every $J \subset H$ and $x, y \in \mathbb{R}$.

Finally, let

$$A^+(\alpha, k) = \{x \in \mathbb{R} \setminus \{0\} : \max(I(x)) = \alpha,\ S(x, B_\alpha^k) > 0,\ (\forall i < k)\ S(x, B_\alpha^i) = 0\},$$

$$A^-(\alpha, k) = \{x \in \mathbb{R} \setminus \{0\} : \max(I(x)) = \alpha,\ S(x, B_\alpha^k) < 0,\ (\forall i < k)\ S(x, B_\alpha^i) = 0\}.$$

In other words, $A^+(\alpha, k)$ contains $x \in \mathbb{R} \setminus \{0\}$ if and only if α is the largest ordinal so that x has a C_α-coefficient and k is the smallest integer so that the sum of the B_α^k-coefficients of x is nonzero and this sum is positive, and similarly for $A^-(\alpha, k)$.

We claim that $\{0\}$ and the sets $A^+(\alpha, k)$ and $A^-(\alpha, k)$ for $\alpha < \omega_1$ and $k \in \mathbb{N}^+$ form a decomposition with all the required properties.

It is clear from the definitions that these sets are pairwise disjoint. To show that their union is $\mathbb{R}$ it is enough to check that if x has a C_α-coefficient then $S(x, B_\alpha^i)$ is nonzero for some $i \in \mathbb{N}^+$. So suppose that x has a C_α-coefficient, that is, $B(x) \cap C_\alpha$ is a nonempty finite subset of C_α. Since $\{B_\alpha^i\}_{i \in \mathbb{N}^+}$ is a base of C_α, there exists an i such that $B(x) \cap B_\alpha^i$ is a singleton, hence $S(x, B_\alpha^i)$, as the sum of a single nonzero term, cannot be zero.

Now we show that the sets $A^+(\alpha, k)$ and $A^-(\alpha, k)$ are closed under addition. By symmetry, it is enough to prove this for $A^+(\alpha, k)$. If $x, y \in A^+(\alpha, k)$ then clearly $S(x + y, B_\alpha^k) = S(x, B_\alpha^k) + S(y, B_\alpha^k) > 0$ and $S(x + y, B_\alpha^i) = S(x, B_\alpha^i) + S(y, B_\alpha^i) = 0$ for every $i < k$. Since $S(x + y, B_\alpha^k) > 0$, $x + y$ has a C_α-coefficient. On the other hand, x and y have no C_β-coefficients for $\beta > \alpha$, so $x + y$ cannot have C_β-coefficients either, thus $\max(I(x + y)) = \alpha$. Therefore $x + y \in A^+(\alpha, k)$, indeed.

So it remains to prove that the sets $A^+(\alpha, k)$ and $A^-(\alpha, k)$ are F_σ, and again it is enough to prove this for $A^+(\alpha, k)$. Fix $\alpha < \omega_1$ and $k \in \mathbb{N}^+$. Note that $A^+(\alpha, k) = D_\alpha + E^+(\alpha, k)$, where

$$D_\alpha = \left\{ x \in \mathbb{R} \setminus \{0\} \ : \ B(x) \subset \cup_{\beta < \alpha} C_\beta \right\},$$

$$E^+(\alpha, k) = \left\{ x \in \mathbb{R} \setminus \{0\} \ : \ B(x) \subset C_\alpha, \ S(x, B_\alpha^k) > 0, \ (\forall i < k) \ S(x, B_\alpha^i) = 0 \right\},$$

and we use the notation $A + B = \{a + b : a \in A, b \in B\}$. The sum of two compact sets is compact, hence the sum of two σ-compact sets is σ-compact, thus it is enough to check that both D_α and $E^+(\alpha, k)$ are σ-compact.

The set D_α is σ-compact since

$$D_\alpha = \bigcup \left\{ r_1 C_{\beta_1} + \ldots + r_m C_{\beta_m} \ : \ m \in \mathbb{N}^+, (\forall i = 1, \ldots m) \ r_i \in \mathbb{Q}, \ \beta_i < \alpha \right\}.$$

In order to show that $E^+(\alpha, k)$ is σ-compact, we rewrite it as

$$E^+(\alpha, k) = F^+(\alpha, k) \cap F^0(\alpha, 1) \cap F^0(\alpha, 2) \cap \ldots \cap F^0(\alpha, k-1),$$

where for any $n \in \mathbb{N}^+$,

$$F^0(\alpha, n) = \left\{ x \in \mathbb{R} \setminus \{0\} \ : \ B(x) \subset C_\alpha, \ S(x, B_\alpha^n) = 0 \right\},$$
$$F^+(\alpha, n) = \left\{ x \in \mathbb{R} \setminus \{0\} \ : \ B(x) \subset C_\alpha, \ S(x, B_\alpha^n) > 0 \right\}.$$

Thus, to complete to proof, it is enough to show that the sets $F^0(\alpha, n)$ and $F^+(\alpha, n)$ are σ-compact. Note that these sets can be also written as

$$F^0(\alpha, n) = \bigcup \left\{ r_1 B_\alpha^n + \ldots + r_m B_\alpha^n + r_{m+1}(C_\alpha \setminus B_\alpha^n) + \ldots + r_{m+l}(C_\alpha \setminus B_\alpha^n) \ : \right.$$
$$\left. m, l \in \mathbb{N}, \ r_1, \ldots, r_{m+l} \in \mathbb{Q}, \ r_1 + \ldots + r_m = 0 \right\},$$

$$F^+(\alpha, n) = \bigcup \left\{ r_1 B_\alpha^n + \ldots + r_m B_\alpha^n + r_{m+1}(C_\alpha \setminus B_\alpha^n) + \ldots + r_{m+l}(C_\alpha \setminus B_\alpha^n) \ : \right.$$
$$\left. m, l \in \mathbb{N}, \ r_1, \ldots, r_{m+l} \in \mathbb{Q}, \ r_1 + \ldots + r_m > 0 \right\}.$$

Since the sets B_α^n and $C_\alpha \setminus B_\alpha^n$ were chosen to be compact, these sets are σ-compact indeed, which completes the proof. $\square$

The following simple observations show that Theorem **??** is sharp in the sense that F_σ cannot be replaced by G_δ.

Proposition 3.2. *Suppose that for some cardinal κ we have a decomposition $\mathbb{R} = \bigcup_{\alpha < \kappa} A_\alpha$ such that each A_α is closed under addition and nonempty.*

(i) Then each A_α is the union of sets of the form $c \cdot \mathbb{Q}^+$ $(c \in \mathbb{R})$.

(ii) If each A_α is a nonempty G_δ set then each A_α is the union of some of the sets $(-\infty, 0), \{0\}, (0, \infty)$, consequently $\kappa \leq 3$.

Proof. (i) Suppose that $x, y \in \mathbb{R}$, $x/y \in \mathbb{Q}^+$, $x \in A_\alpha$ and $y \in A_\beta$. Since $x/y \in \mathbb{Q}^+$, x and y has a common positive integer multiple z. But then z is both in A_α and A_β, so we have $\alpha = \beta$, which completes the proof of (i).

(ii) If each A_α is a nonempty G_δ set then, by (i), each A_α is residual in $(-\infty, 0)$ or in $(0, \infty)$ or equals to $\{0\}$, which completes the proof of (ii). $\square$

Combining the results of this section and the previous one, in case of $\mathfrak{c} = \omega_2$ we can exactly determine how many Borel additive semigroup one can decompose $\mathbb{R}$ into:

Corollary 3.3. *Let $\mathcal{K}$ be the set of cardinalities of all possible decompositions of $\mathbb{R}$ into Borel additive semigroup; that is, let*

$$\mathcal{K} = \{\kappa : \exists \ Borel \ decomposition \ \cup_{\alpha < \kappa} B_\alpha = \mathbb{R} \ with \ (\forall \alpha) B_\alpha + B_\alpha \subset B_\alpha, B_\alpha \neq \emptyset\}.$$

If $\mathfrak{c} = \omega_2$ then $\mathcal{K} = \{1, 2, 3, \omega_1, \omega_2\}$ or $\mathcal{K} = \{1, 2, 3, \omega_2\}$ and both possibilities are consistent.

Proof. Suppose that $\mathfrak{c} = \omega_2$. Since $\max(\operatorname{cov} \mathcal{M}, \operatorname{cov} \mathcal{N}) \geq \omega_1$, Corollary **??** implies that $\mathcal{K} \subset \{1, 2, 3, \omega_1, \omega_2\}$. The trivial decompositions $\mathbb{R} = \mathbb{R}$, $\mathbb{R} = (-\infty, 0) \cup [0, \infty)$, $\mathbb{R} = (-\infty, 0) \cup \{0\} \cup (0, \infty)$ show that $\mathcal{K} \supset \{1, 2, 3\}$. Let $\{J_\alpha : \alpha < \mathfrak{c}\}$ be the equivalence classes of the relation $x \sim y \Leftrightarrow (x = y = 0$ or $x/y \in \mathbb{Q})$. Then each J_α is countable and closed under addition, so the decomposition $\mathbb{R} = \bigcup_{\alpha < \mathfrak{c}} J_\alpha$ shows that $\omega_2 = \mathfrak{c} \in \mathcal{K}$.

Thus it remains to prove that both $\omega_1 \in \mathcal{K}$ and $\omega_1 \notin \mathcal{K}$ are consistent if $\mathfrak{c} = \omega_2$. Theorem **??** shows the consistency of $\omega_1 \in \mathcal{K}$. It is well known (see [**?**]) that it is consistent that $\operatorname{cov} \mathcal{M} = \operatorname{cov} \mathcal{N} = \omega_2 = \mathfrak{c}$. But in this case, by Corollary **??**, $\omega_1 \notin \mathcal{K}$, which completes the proof. $\qquad\square$

4. Open questions

In this final section we collect some of the numerous remaining open questions.

Question 4.1. *Is it consistent that $\mathfrak{c} = \omega_3$ and the real line can be partitioned into ω_2 additive Borel (or F_σ) semigroups; that is, there exists a decomposition $\mathbb{R} = \bigcup_{\alpha < \omega_1} A_\alpha$ such that each A_α is a nonempty Borel (or F_σ) set closed under addition?*

We remark here that it is not hard to see from [**?**] or [**?**] that it is consistent that $\mathfrak{c} = \omega_3$ and the real line can be partitioned into ω_2 compact sets.

Question 4.2. *What can we say about the following cardinal invariant?*

$$\gamma = \min\{\kappa : \kappa > 3, \exists \text{ Borel dec. } \cup_{\alpha < \kappa} B_\alpha = \mathbb{R} \text{ with } (\forall \alpha) B_\alpha + B_\alpha \subset B_\alpha, B_\alpha \neq \emptyset\}.$$

By Corollary **??** we know that $\gamma \geq \max(\operatorname{cov} \mathcal{N}, \operatorname{cov} \mathcal{M})$ and by Theorem **??** we know that it is consistent that $\mathfrak{c} = \omega_2$ and $\gamma = \omega_1$. What else can we say?

As we pointed out above, by Silver's theorem the construction in Theorem **??** cannot yield a Borel equivalence relation. (Two real numbers are equivalent if they are in the same piece of the decomposition, and the equivalence relation is Borel if it is Borel when considered as a subset of the plane.) Actually, Silver proved his theorem for so called co-analytic equivalence relations, see [**?**] for the definition and some background in descriptive set theory. Now we remark that our construction is not analytic either. Indeed, by a result of Stern [**?**] if each equivalence class of an analytic equivalence relation is F_σ then there are countably many or continuum many classes. However, we do not know the answer to the following.

Question 4.3. *Is it consistent that $\mathfrak{c} = \omega_2$ and the real line can be partitioned into ω_1 additive Borel semigroups so that the resulting equivalence relation is analytic?*

It would also be interesting to check whether our construction consistently produces a projective equivalence relation.

The following question is closely related to our topic in that it also requires some extra condition using the additive structure of the real line.

Question 4.4. *Suppose that $\mathbb{R} = \bigcup_{\alpha < \kappa} B_\alpha$ is a decomposition into Borel sets that are translates of each other. Does this imply that $\kappa \leq \omega$ is or $\kappa = \mathfrak{c}$?*

First we show that $\kappa \leq \omega$ or $\kappa \geq \operatorname{cov} \mathcal{N}$, so an affirmative answer is consistent with ZFC. Indeed, if B_α has measure zero then clearly $\kappa \geq \operatorname{cov} \mathcal{N}$. If B_α has positive measure then we have κ disjoint translates of a set of positive measure, and it is easy to check that this implies $\kappa \leq \omega$.

We also claim that the answer is affirmative in ZFC if the sets B_α are F_σ. Indeed, suppose that $\omega < \kappa$. By [**?**, Thm. 1] a σ-compact set has either at most countably many or continuum many pairwise disjoint translates, hence B_0 has continuum many pairwise disjoint translates $\{B_0 + x_\alpha : \alpha < \mathfrak{c}\}$. One can easily check that this implies that no translate of B_0 can contain more than one number $-x_\alpha$, hence less than $\mathfrak{c}$ many translates of B_0 cannot cover $\mathbb{R}$, therefore $\kappa = \mathfrak{c}$.

This proof also raises the following natural question.

Question 4.5. *Suppose that a Borel subset of $\mathbb{R}$ has uncountably many pairwise disjoint translates. Does it also have continuum many pairwise disjoint translates?*

The last question is the natural continuation of the question of Freud.

Question 4.6. *Suppose that $(0, \infty) = \bigcup_{\alpha < \kappa} B_\alpha$ is a decomposition into nonempty Borel sets that are closed under both addition and multiplication. Does this imply that $\kappa = 1$ or $\kappa = \mathfrak{c}$?*

We remark here that the case $\kappa = \mathfrak{c}$ is possible indeed, as was pointed out by András Máthé [**?**]. Let $\mathcal{G}$ be the smallest family of real functions closed under addition and multiplication and also forming a group under composition (in particular, $\mathcal{G}$ contains the identity function). Note that this makes sense, since we can generate this family inside the group of strictly increasing functions. Then $\mathcal{G}$ is clearly countable and it is easy to see that the orbits of $\mathcal{G}$ form a partition of $\mathbb{R}$ into $\mathfrak{c}$ many countable sets all of which are closed under addition and multiplication.

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