Self-similar and self-affine sets; measure of the intersection of two copies

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Abstract. Let $K \subset \mathbb{R}^d$ be a self-similar or self-affine set, let μ be a self-similar or self-affine measure on it, and let $\mathcal{G}$ be the group of affine maps, similitudes, isometries or translations of $\mathbb{R}^d$. Under various assumptions (such as separation conditions or we assume that the transformations are small perturbations or that K is a so called Sierpiński sponge) we prove theorems of the following types, which are closely related to each other;

- *(Non-stability)*

  There exists a constant $c < 1$ such that for every $g \in \mathcal{G}$ we have either $\mu\big(K \cap g(K)\big) < c \cdot \mu(K)$ or $K \subset g(K)$.

- *(Measure and topology)*

  For every $g \in \mathcal{G}$ we have $\mu\big(K \cap g(K)\big) > 0 \iff \mathrm{int}_K(K \cap g(K)) \neq \emptyset$ (where int_K is interior relative to K).

- *(Extension)*

  The measure μ has a $\mathcal{G}$-invariant extension to $\mathbb{R}^n$.

Moreover, in many situations we characterize those g's for which $\mu\big(K \cap g(K)\big) > 0$ holds, and we also get results about those g's for which $g(K) \subset K$ or $g(K) \supset K$ holds.

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1. *Introduction*

The study of the size of the intersection of Cantor sets has been a central research area in geometric measure theory and dynamical systems lately, see e.g. the works of Igudesman [**12**], Li and Xiao [**17**], Moreira [**23**], Moreira and Yoccoz [**24**], Nekka and Li [**25**], Peres and Solomyak [**26**]. For instance J-C. Yoccoz and C. G. T. de Moreira [**24**] proved that if the sum of the Hausdorff dimensions of two regular Cantor sets exceeds one then, in the typical case, there are translations of them stably having intersection with positive Hausdorff dimension.

The main purpose of this paper is to study the measure of the intersection of two Cantor sets which are (affine, similar, isometric or translated) copies of a self-similar or self-affine set in $\mathbb{R}^d$. By measure here we mean a self-similar or self-affine measure on one of the two sets.

We get instability results stating that the measure of the intersection is separated from the measure of one copy. This strong non-continuity property is in sharp contrast with the well known fact that for any Lebesgue measurable set $H \subset \mathbb{R}^d$ with finite measure the Lebesgue measure of $H \cap (H+t)$ is continuous in t.

We get results stating that the intersection is of positive measure if and only if it contains a relative open set. This result resembles some recent deep results (e.g. in [**16**], [**24**]) stating that for certain classes of sets having positive Lebesgue measure and nonempty interior is equivalent. In the special case when the self-similar set is the classical Cantor set our above mentioned results were obtained by F. Nekka and Jun Li [**25**]. For other related results see also the work of Falconer [**5**], Feng and Wang [**8**], Furstenberg [**9**], Hutchinson [**11**], Järvenpää [**13**] and Mattila [**19**], [**20**], [**21**].

As an application we also get isometry (or at least translation) invariant measures of $\mathbb{R}^d$ such that the measure of the given self-similar or self-affine set is 1.

Feng and Wang [**8**] has proved recently "The Logarithmic Commensurability Theorem" about the similarity ratios of a homogeneous self-similar set in $\mathbb{R}$ with the open set condition and a similarity map that maps the self-similar set into itself (see more precisely after Theorem 4.9), and they also posed the problem of generalizing their result to higher dimensions. For self-similar sets with the strong separation condition we prove a higher dimensional generalization without assuming homogeneity.

1.1. *Self-affine sets.*

Let $K \subset \mathbb{R}^d$ be a *self-affine set* with the *strong separation condition*; that is, $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ is a compact set, where $r \geq 2$ and $\varphi_1, \ldots, \varphi_r$ are injective and contractive $\mathbb{R}^d \to \mathbb{R}^d$ affine maps and $\cup^*$ denotes disjoint union.

For any $p_1, \ldots, p_r \in (0,1)$ such that $p_1 + \ldots + p_r = 1$ let μ be the corresponding *self-affine measure*; that is, the image of the infinite product of the discrete probability measure $p(\{i\}) = p_i$ on $\{1, \ldots, r\}$ under the representation map $\pi : \{1, \ldots, r\}^{\mathbb{N}} \to K, \quad \{\pi(i_1, i_2, \ldots)\} = \cap_{n=1}^{\infty} (\varphi_{i_1} \circ \ldots \circ \varphi_{i_n})(K)$.

In Section 3 we show (Theorem 3.2) that small affine perturbations of K cannot intersect a very large part of K; that is, there exists a $c < 1$ and a neighborhood U

of the identity map in the space of affine maps such that for any $g \in U \setminus \{\text{identity}\}$ we have $\mu\big(K \cap g(K)\big) < c$. We also prove (Theorem 3.5) that no isometric but nonidentical copy of K can intersect a very large part of K; that is, there exists a constant $c < 1$ such that for any isometry g either $\mu\big(K \cap g(K)\big) < c$ or $g(K) = K$.

1.2. *Self-similar sets.* Now let $K \subset \mathbb{R}^d$ be a *self-similar set* with the *strong separation condition* and μ a self-similar measure on it; that is, K and μ are defined as above with the extra assumption that $\varphi_1, \dots, \varphi_r$ are similitudes.

In Section 4 we prove (Theorem 4.1) that for any given self-similar set $K \subset \mathbb{R}^d$ with the strong separation condition and self-similar measure μ on K there exists a $c < 1$ such that for any similitude g either $\mu\big(K \cap g(K)\big) < c \cdot \mu(K) = c$ or $K \subset g(K)$. In other words, the intersection of a self-similar set with the strong separation condition and its similar copy cannot have a really big non-trivial intersection.

Let K, μ and g be as above. An obvious way of getting $\mu\big(K \cap g(K)\big) > 0$ is when $g(K)$ contains a nonempty (relative) open set in K. The main result (Theorem 4.5) of Section 4, which will follow from the above mentioned Theorem 4.1, shows that this is the only way. That is, for any self-similar set $K \subset \mathbb{R}^d$ with the strong separation condition and self-similar measure μ on K a similar copy of K has positive μ measure in K if and only if it has nonempty relative interior in K.

An immediate consequence (Corollary 4.6) of the above result is that for any fixed self-similar set with the strong separation condition and for any two self-similar measures μ_1 and μ_2 we have $\mu_1\big(g(K) \cap K\big) > 0 \iff \mu_2\big(g(K) \cap K\big) > 0$ for any similitude g. As an other corollary (Corollary 4.7) we get that for any given self-similar set $K \subset \mathbb{R}^d$ with the strong separation condition and self-similar measure μ on K there exist only countably many (in fact exactly countably infinitely many) similitudes $g : A_K \to \mathbb{R}^d$ (where A_K is the affine span of K) such that $g(K) \cap K$ has positive μ-measure.

Let $K \subset \mathbb{R}^d$ be a self-similar set with the strong separation condition and let s be its Hausdorff dimension, which in this case equals its similarity and box-counting dimension. Then the s-dimensional Hausdorff measure is a constant multiple of a self-similar measure (one has to choose $p_i = a_i^s$, where a_i is the similarity ratio of φ_i). Therefore all the above results hold when μ is s-dimensional Hausdorff measure.

In Section 4 we also need and get results (Proposition 4.3, Lemma 4.8, Theorem 4.9 and Corollary 4.10) stating that only very special similarity maps can map a self-similar set with the strong separation condition into itself. Theorem 4.9 and Corollary 4.10 are the already mentioned generalizations of The Logarithmic Commensurability Theorem of Feng and Wang [**8**].

In Section 5 we apply the main result (Theorem 4.5) and some of the above mentioned results (Lemma 4.8 and Theorem 4.9) of Section 4 to characterize those self-similar measures on a self-similar set with the strong separation condition that can be extended to $\mathbb{R}^d$ as an isometry invariant Borel measure. It turns out that, unless there is a clear obstacle, any self-similar measure can be extended to $\mathbb{R}^d$ as an isometry invariant measure. Thus, for a given self-similar set with the strong separation condition, there are usually many distinct isometry invariant Borel measures for which the set is of measure 1.

Let us simply call a measure defined on K isometry invariant if it can be extended to an isometry invariant measure on $\mathbb{R}^d$. Many different collections of similitudes can define the same self-similar set. We call $\{\varphi_1, \varphi_2, \dots, \varphi_r\}$ a *presentation* of K if $K = \varphi_1(K) \cup^* \dots \cup^* \varphi_r(K)$ holds; in other words, K is the attractor of the iterated function system $\{\varphi_1, \varphi_2, \dots, \varphi_r\}$ with the extra condition of disjointness.

The notion of a self-similar measure on K depends on the particular presentation. However, we show that the notion of isometry invariant self-similar measure on K is indifferent of the presentations (Theorem 5.5). By this theorem we can define a natural number for each self-similar set (satisfying the strong separation property), an invariant, which does not depend on the presentation (Theorem 5.7). This invariant is equal to the dimension of the space of isometry invariant self-similar measures, and is related to the algebraic dependence of the similitudes of some (any) presentation of K.

In Section 6 we show that the connection between different presentations of a self-similar set can be very complicated. This sheds some light on why results and their proofs in Section 5 are complicated. The structure of different presentations of a self-similar set in $\mathbb{R}$ has been also studied recently and independently by Feng and Wang in [**8**], where a similar example is presented.

1.3. *Self-affine sponges.* Take the $[0,1]^n$ unit cube in $\mathbb{R}^n$ ($n \in \mathbb{N}$) and subdivide it into $m_1 \times \dots \times m_n$ boxes of same size ($m_1, \dots, m_n \geq 2$) and cut out some of them. Then do the same with the remaining boxes using the same pattern as in the first step and so on. What remains after infinitely many steps is a self-affine set, which is called *self-affine Sierpiński sponge*. (A more precise definition will be given in Definition 2.14.)

For $n = 2$ these sets were studied in several papers (in which they were called self-affine carpets or self-affine carpets of Bedford and McMullen). Bedford [**2**] and McMullen [**22**] determined the Hausdorff and Minkowski dimensions of these self-affine carpets. (The Hausdorff and Minkowski dimension of self-affine Sierpiński sponges was determined by Kenyon and Peres [**15**]). Gatzouras and Lalley [**10**] proved that except in some relatively simple cases such a set has zero or infinity Hausdorff measure in its dimension (and so in any dimension). Peres extended their results by proving that (except in the same rare simple cases) for any gauge function neither the Hausdorff [**28**] nor the packing [**27**] measure of a self-affine carpet can be positive and finite (in fact, the packing measure cannot be σ-finite either), and remarked that these results extend to self-affine Sierpiński sponges of higher dimensions.

Recently the first and the second listed authors of the present paper showed [**4**] that some nice sets – among others the set of Liouville numbers – have zero or non-σ-finite Hausdorff and packing measure for any gauge function by proving that these sets have zero or non-σ-finite measure for any translation invariant Borel measure. (Much earlier Davies [**3**] constructed a compact subset of $\mathbb{R}$ with this property.) So it was natural to ask whether the self-affine carpets of Bedford and McMullen have this stronger property.

In Section 7 we prove (Corollary 7.7) that for any self-affine Sierpiński sponge $K \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) with the natural Borel probability measure μ (see in Definition 2.15) on K and $t \in \mathbb{R}^n$, the set $K \cap (K+t)$ has positive μ measure if and only if it has non-empty interior relative to K.

For this we prove (Theorem 7.4) that for any self-affine Sierpiński sponge $K \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) and translation vector $t \in \mathbb{R}^n$ we have $\mu\big(K \cap (K+t)\big) = 0$ unless K or t are of very special form.

We also characterize (Theorem 7.9) those Sierpiński sponges for which we do not have instability result for translations and the natural probability measure μ. In fact, we get that $\mu\big(K \cap (K+t)\big)$ can be close to 1 only for the same special sponges that appear in the above mentioned result.

In Section 8 we show (Theorem 8.1) that for any self-affine Sierpiński sponge $K \subset \mathbb{R}^n$ the natural probability measure μ on K can be extended as a translation invariant Borel measure ν on $\mathbb{R}^n$. We also extend this result (Theorem 8.2, Corollary 8.3) to slightly larger classes of self-affine sets.

2. *Notation, basic facts and some lemmas*

In this section we collect several notions and well known or fairly easy statements that we will need in the sequel. Some of these might be interesting in their own right. Of course, only a few of them are needed for each specific section. Though some of these statements may be well known, for the sake of completeness we included the proofs.

Notation 2.1. We shall denote by $\cup^*$ the disjoint union and by dist the (Euclidean) distance.

2.1. *Affine maps, similitudes, isometries.*

Definition 2.2. A mapping $g : \mathbb{R}^d \to \mathbb{R}^d$ is called a *similitude* if there is a constant $r > 0$, called *similarity ratio*, such that $\operatorname{dist}(g(a), g(b)) = r \cdot \operatorname{dist}(a, b)$ for any $a, b \in \mathbb{R}^d$.

The *affine maps* of $\mathbb{R}^d$ are of the form $x \mapsto Ax + b$, where A is an $n \times n$ matrix and $b \in \mathbb{R}^d$ is a translation vector. Thus the set of all affine maps of $\mathbb{R}^d$ can be considered as $\mathbb{R}^{d^2+d}$ and so it can be considered as a metric space.

It is easy to check that a sequence (g_n) in this metric space converges to an affine map g if and only if g_n converges to g uniformly on any compact subset of $\mathbb{R}^d$.

Definition 2.3. For a given set $K \subset \mathbb{R}^d$ with affine span A_K let $\mathcal{A}_K$, $\mathcal{S}_K$ and $\mathcal{I}_K$ denote the metric space (with the above metric) of the injective affine maps, similitudes and isometries of A_K, respectively.

Note also that all these three metric spaces with the composition can be also considered as topological groups.

2.2. *Self-similar and self-affine sets and measures.*

DEFINITION 2.4. A $K \subset \mathbb{R}^d$ compact set is a *self-similar/self-affine set* if $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$, where $r \geq 2$ and $\varphi_1, \ldots, \varphi_r$ are similitudes/injective and contractive affine maps.

By the *n-th generation elementary pieces of* K we mean the sets of the form $(\varphi_{i_1} \circ \ldots \circ \varphi_{i_n})(K)$, where $n = 0, 1, 2, \ldots$.

We shall use multi-indices. By a *multi-index* we mean a finite sequence of indices; for $I = (i_1, i_2, \ldots, i_n)$ let $\varphi_I = \varphi_{i_1} \circ \ldots \circ \varphi_{i_n}$ and $p_I = p_{i_1} p_{i_2} \ldots p_{i_n}$. We shall consider $I = \emptyset$ as a multi-index as well: $\varphi_\emptyset$ is the identity map and $p_\emptyset = 1$.

Note that the elementary pieces of K are the sets of the form $\varphi_I(K)$. These sets are also self-similar/self-affine; and if h is an injective affine map then $h(K)$ is also self-similar/self-affine and its elementary pieces are the sets of the form $h(\varphi_I(K))$.

DEFINITION 2.5. Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-similar/self-affine set, and let $p_1 + \ldots + p_r = 1$, $p_i > 0$ for all i. Consider the symbol space $\Omega = \{1, \ldots, r\}^{\mathbb{N}}$ equipped with the product topology and let ν be the Borel measure on Ω which is the countable infinite product of the discrete probability measure $p(\{i\}) = p_i$ on $\{1, \ldots, r\}$. Let

$$\pi : \Omega \to K, \quad \{\pi(i_1, i_2, \ldots)\} = \cap_{n=1}^{\infty} (\varphi_{i_1} \circ \ldots \circ \varphi_{i_n})(K)$$

be the continuous addressing map of K. Let μ be the image measure of ν under the projection π; that is,

$$\mu(H) = \nu\big(\pi^{-1}(H)\big) \quad \text{for every Borel set } H \subset K. \tag{1}$$

Such a μ is called a *self-similar/self-affine measure* on K.

One can also define (see e.g. in [**7**]) self-similar or self-affine measures as the unique probability measure μ on K such that

$$\mu(H) = \sum_{i=1}^{r} p_i \mu\big(\varphi_i^{-1}(H)\big)$$

holds for every Borel set $H \subset K$. It was already proved by Hutchinson [**11**] that the two definition agrees.

LEMMA 2.6. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine set, $p_1 + \ldots + p_r = 1$, $p_i > 0$ for all i, and let μ be the self-affine measure on K corresponding to the weights p_i.*

Then for every affine subspace A either $\mu(A \cap K) = 0$ or $A \supset K$.

Proof. Let $\{x_1, x_2, \ldots, x_k\}$ be a maximal collection of affine independent points in K. Choose $U_1, \ldots, U_k$ convex open sets such that $x_j \in U_j$ $(j = 1, \ldots, k)$ and whenever we choose one point from each U_j they are affine independent. Since $K \cap U_i$ is a nonempty relative open subset of K, we may choose an elementary piece $\varphi_{I_j}(K)$ in U_j for each j. Let $\varepsilon = \min_{1 \leq j \leq k} p_{I_j} > 0$.

We shall use the notation we introduced in Definition 2.5. For $1 \le i \le r$ and $\omega = (i_0, i_1, \ldots) \in \Omega$, let $\sigma_i(\omega) = (i, i_0, i_1, \ldots)$. Thus $\nu\big(\sigma_i(H)\big) = p_i \nu(H)$ for all Borel subset H of Ω.

Suppose that A is an affine subspace such that $\mu(A \cap K) > 0$. Thus $\nu\big(\pi^{-1}(A)\big) > 0$. It is easy to prove (see a possible argument later in the proof of Lemma 2.12) that this implies that there exists an elementary piece $\sigma_J(\Omega)$ such that

$$\nu\big(\pi^{-1}(A) \cap \sigma_J(\Omega)\big) > (1-\varepsilon)\nu\big(\sigma_J(\Omega)\big) = (1-\varepsilon)p_J.$$

Since $\nu\big((\sigma_J \circ \sigma_{I_j})(\Omega)\big) = p_J p_{I_j} \ge p_J \varepsilon$ $(j = 1, \ldots, k)$, the set $\pi^{-1}(A)$ must intersect the sets $(\sigma_J \circ \sigma_{I_j})(\Omega)$. Therefore the set A must intersect the sets $\pi((\sigma_J \circ \sigma_{I_j})(\Omega)) = (\varphi_J \circ \varphi_{I_j})(K)$ $(j = 1, \ldots, k)$.

By picking one point from each $A \cap (\varphi_J \circ \varphi_{I_j})(K)$, we get a maximal collection of affine independent points in K since φ_J is an invertible affine mapping. As this collection is contained in the affine subspace A, we get that K is also contained in A. □

Remark 2.7. In this paper one of our main goals is to study $\mu\big(K \cap g(K)\big)$, where g is an affine map of $\mathbb{R}^d$. By the above lemma if the affine map g does not map the affine span A_K of K onto itself then $\mu\big(g(K) \cap K\big) = 0$ since $K \not\subset g(A_K)$. The other property of affine maps we are interested in is $K \subset g(K)$, which also implies that g maps A_K onto itself. Thus it is enough to consider those affine maps g of $\mathbb{R}^d$ that map the affine span A_K of K onto itself. Since then both K and $g(K)$ are in A_K, only the restriction of g to A_K matters. This is why in the next section we shall study $\mathcal{A}_K$, $\mathcal{S}_K$ and $\mathcal{I}_K$ (the injective affine maps, similitudes and isometries of A_K) instead of all affine maps, similitudes and isometries of $\mathbb{R}^d$.

Therefore if we state something (about $\mu\big(g(K) \cap K\big)$ or about the property $K \subset g(K)$) for every affine map, similitude or isometry g, it will be enough to prove them for $g \in \mathcal{A}_K$, $g \in \mathcal{S}_K$ or $g \in \mathcal{I}_K$, respectively.

Note also that self-similar sets and measures are self-affine as well, so results about self-affine sets and measures also apply for self-similar sets and measures.

2.3. *Separation properties.*

Definition 2.8. A self-similar/self-affine set $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ (or more precisely, the collection $\varphi_1, \ldots, \varphi_r$ of the representing maps) satisfies the

- *strong separation condition* (SSC) if the union $\varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ is disjoint;
- *open set condition* (OSC) if there exists a nonempty bounded open set $U \subset \mathbb{R}^d$ such that $\varphi_1(U) \cup^* \ldots \cup^* \varphi_r(U) \subset U$;
- *strong open set condition* (SOSC) if there exists a nonempty bounded open set $U \subset \mathbb{R}^d$ such that $U \cap K \ne \emptyset$ and $\varphi_1(U) \cup^* \ldots \cup^* \varphi_r(U) \subset U$;
- *convex open set condition* (COSC) if there exists a nonempty bounded open convex set $U \subset \mathbb{R}^d$ such that $\varphi_1(U) \cup^* \ldots \cup^* \varphi_r(U) \subset U$;

- *measure separation condition* (MSC) if for any self-similar/self-affine measure μ on K we have $\mu\big(\varphi_i(K) \cap \varphi_j(K)\big) = 0$ for any $1 \le i < j \le r$.

We note that the first three definitions are standard but we have not seen any name for the last two in the literature.

It is easy to check the well known fact that we must have $K \subset \overline{U}$ (where $\overline{E}$ denotes the closure of a set E) for the open set U in the definition of OSC (and SOSC, COSC).

It is easy to see (U can be chosen as a small ε-neighborhood of K for the first implication) that for any self-affine set

$$SSC \Longrightarrow SOSC \Longrightarrow OSC.$$

Using the methods of C. Bandt and S. Graf [**1**], A. Schief proved in [**30**] that, in fact, $SOSC \Longleftrightarrow OSC$ holds for self-similar sets.

In [**30**] for self-similar sets $SOSC \Longrightarrow MSC$ is also proved. Since the proof works for self-affine sets as well we get that for any self-affine set

$$SOSC \Longrightarrow MSC.$$

It seems to be also true that $COSC \Longrightarrow SOSC$ and so $COSC \Longrightarrow MSC$ but we do not prove this, since we do not need the first implication and the following lemma is stronger than the second implication.

Lemma 2.9. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine set in $\mathbb{R}^d$ with the convex open set condition and let μ be a self-affine measure on it. Then for any affine map $\Psi : \mathbb{R}^d \to \mathbb{R}^d$ we have*

$$\mu\Big(\Psi\big(\varphi_i(K) \cap \varphi_j(K)\big)\Big) = 0 \qquad (\forall\, 1 \le i < j \le r).$$

Proof. Let $1 \le i < j \le r$ and U be the convex open set given in the definition of COSC. Let A_K be the affine span of K. Since $\varphi_i(U \cap A_K)$ and $\varphi_j(U \cap A_K)$ are disjoint convex open sets in A_K, $\overline{\varphi_i(U \cap A_K)} \cap \overline{\varphi_j(U \cap A_K)}$ must be contained in a proper affine subspace A of A_K. Since $K \subset \overline{U} \cap A_K$, this implies that $\varphi_i(K) \cap \varphi_j(K) \subset A$, and so

$$\Psi\big(\varphi_i(K) \cap \varphi_j(K)\big) \subset \Psi(A). \tag{2}$$

Since $\Psi(A)$ is an affine subspace, which is smaller dimensional than the affine span A_K of K, we cannot have $K \subset \Psi(A)$, so by Lemma 2.6 we must have $\mu\big(K \cap \Psi(A)\big) = 0$. By (2) this implies that $\mu\big(\Psi(\varphi_i(K) \cap \varphi_j(K))\big) = 0$. □

We also note that one can find a self-similar set in $\mathbb{R}$ that satisfies even the SSC but does not satisfy the COSC [**8**, Example 5.1], so SSC and COSC are independent even for self-similar sets of $\mathbb{R}$.

Notation 2.10. Given a fixed measure μ, we shall say that two sets are *almost disjoint* if their intersection has μ-measure 0. The almost disjoint union will be denoted by $\cup^{**}$.

It is very easy to prove one by one each of the following facts.

FACTS 2.11. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine/self-similar set with the measure separation condition and let μ be a self-affine/self-similar measure on it, which corresponds to the weights $p_1, \ldots, p_r$. Then the following statements hold.*

1. *Any two elementary pieces of K are either almost disjoint or one contains the other.*
2. *Any union of elementary pieces can be replaced by an almost disjoint countable union.*
3. *For any multi-index I we have $\mu \circ \varphi_I = p_I \cdot \mu$; that is, $\mu \circ \varphi_I(B) = p_I \cdot \mu(B)$ for any Borel set $B \subset K$.*
4. *We have $\mu\big(\varphi_I(K)\big) = p_I$ for any multi-index I.*
5. *For any Borel set $B \subset K$ we have*
$$\mu(B) = \inf\Big\{ \sum_{i=1}^{\infty} p_{I_i} : B \subset \bigcup_{i=1}^{\infty}{}^{**} \varphi_{I_i}(K) \Big\}.$$

Since SOSC and COSC are both stronger than MSC and one of them will be always assumed in this paper, the statements of this lemma will often be tacitly used. Sometimes, for example, we shall even handle the above almost disjoint sets as disjoint sets and often consider Fact 5 as the definition of self-affine/self-similar measures.

LEMMA 2.12. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine set with the measure separation property (or in particular with the SSC or SOSC or COSC) and let μ be a self-affine measure on it. Then for every $\varepsilon > 0$ and for every Borel set $B \subset K$ with positive μ-measure there exists an elementary piece $a(K)$ of K of arbitrarily large generation such that $\mu\big(B \cap a(K)\big) > (1-\varepsilon)\mu\big(a(K)\big)$.*

Proof. Since $\mu(B) > 0$, using Fact 5, B can be covered by countably many elementary pieces $\varphi_{I_i}(K)$ $(i \in \mathbb{N})$ such that
$$(1+\varepsilon)\,\mu(B) > \sum_i \mu\big(\varphi_{I_i}(K)\big).$$
By subdividing the elementary pieces if necessary, we can suppose that each is of large generation.

If there exists an $i \in \mathbb{N}$ such that
$$(1+\varepsilon)\,\mu\big(B \cap \varphi_{I_i}(K)\big) > \mu\big(\varphi_{I_i}(K)\big)$$
then we can choose φ_{I_i} as a.

Otherwise we have $(1+\varepsilon)\,\mu\big(B \cap \varphi_{I_i}(K)\big) \le \mu\big(\varphi_{I_i}(K)\big)$ for each $i \in \mathbb{N}$, hence
$$(1+\varepsilon)\,\mu(B) = (1+\varepsilon)\,\mu\Big(\bigcup_i B \cap \varphi_{I_i}(K)\Big) \le \sum_i (1+\varepsilon)\mu\big(B \cap \varphi_{I_i}(K)\big) \le \sum_i \mu\big(\varphi_i(K)\big),$$
contradicting the above inequality. □

LEMMA 2.13. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine set with the measure separation property (or in particular with the SSC or SOSC or COSC) and let μ be a self-affine measure on it. Then for any Borel set $B \subset K$ and $\varepsilon > 0$ there exist countably many pairwise almost disjoint elementary pieces $a_i(K)$ such that $\mu\big(B \cap a_i(K)\big) > (1-\varepsilon)\mu\big(a_i(K)\big)$ and $\mu\big(B \setminus \cup_i^{**} a_i(K)\big) = 0$.*

Proof. The elementary pieces $a_i(K)$ will be chosen by greedy algorithm. In the n^{th} step ($n = 0, 1, 2, \ldots$) we choose the largest elementary piece $a_n(K)$ such that $\mu\big(a_n(K) \cap a_i(K)\big) = 0 \quad (0 \leq i < n)$ and $\mu\big(B \cap a_n(K)\big) > (1-\varepsilon)\mu\big(a_n(K)\big)$. If there is no such $a_n(K)$ then the procedure terminates.

We claim that $\mu\big(B \setminus \cup_i^{**} a_i(K)\big) = 0$. Suppose that $\mu\big(B \setminus \cup_i^{**} a_i(K)\big) > 0$. Then by Lemma 2.12 there exists an elementary piece $a(K)$ such that

$$\mu\big((B \setminus \cup_i^{**} a_i(K)) \cap a(K)\big) > (1-\varepsilon)\mu\big(a(K)\big).$$

Then $\mu\big(B \cap a(K)\big) > (1-\varepsilon)\mu\big(a(K)\big)$ but $a(K)$ was not chosen in the procedure. This could happen only if $a(K)$ intersects a chosen elementary piece $a_i(K)$ in a set of positive measure. But then either $a_i(K) \supset a(K)$ or $a_i(K) \subset a(K)$, which are both impossible. □

2.4. *Self-affine Sierpiński sponges.*

DEFINITION 2.14. By *self-affine Sierpiński sponge* we mean self-affine sets of the following type. Let $n, r \in \mathbb{N}$, $m_1, m_2, \ldots, m_n \geq 2$ integers, M be the linear transformation given by the diagonal $n \times n$ matrix

$$M = \begin{pmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_n \end{pmatrix},$$

and let

$$D = \{d_1, \ldots, d_r\} \subset \{0, 1, \ldots, m_1 - 1\} \times \ldots \times \{0, 1, \ldots, m_n - 1\}$$

be given. Let $\varphi_j(x) = M^{-1}(x + d_j)$ $(j = 1, \ldots, r)$. Then the self-affine set $K(M, D) = K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ is a Sierpiński sponge.

We can also define the self-affine Sierpiński sponge as

$$K = K(M, D) = \left\{ \sum_{k=1}^{\infty} M^{-k} \alpha_k \ : \ \alpha_1, \alpha_2, \ldots \in D \right\},$$

or equivalently K is the unique compact set in $\mathbb{R}^n$ (in fact, in $[0,1]^n$) such that

$$M(K) = K + D = \bigcup_{j=1}^{r} K + d_j;$$

that is,

$$K = M^{-1}(K) + M^{-1}(D).$$

By iterating the last equation we get

$$\begin{aligned} K &= M^{-k}(K) + M^{-k}(D) + M^{-k+1}(D) + \ldots + M^{-1}(D) \\ &= \bigcup_{\alpha_1,\ldots,\alpha_k \in D} M^{-k}(K) + M^{-k}\alpha_k + \ldots + M^{-1}\alpha_1. \end{aligned}$$

Note that the k-th generation elementary pieces of K are the sets of the form $M^{-k}(K)+M^{-k}(\alpha_k)+\ldots+M^{-1}(\alpha_1)$ ($\alpha_1,\ldots,\alpha_k \in D$) and the only 0-th generation elementary part of K is K itself.

Definition 2.15. By the *standard* (or sometimes *natural*) probability measure on a self-affine sponge $K = K(M,D)$ we shall mean the self-affine measure on K obtained by using equal weights $p_j = \frac{1}{r}$ ($j = 1,\ldots,r$).

Since the first generation elementary pieces of K are translates of each other (in fact, so are the k-th generation elementary parts), this is indeed the most natural self-affine measure on K. Using (5) of Facts 2.11 we get that

$$\mu(B) = \inf\left\{\sum_{i=1}^{\infty}\mu(S_i) : B \subset \cup_{i=1}^{\infty} S_i,\ S_i \text{ is an elementary part of } K\ (i \in \mathbb{N})\right\}$$

for every Borel set $B \subset K$.

Let $\tilde\mu$ be the $\mathbb{Z}^n$-invariant extension of μ to $\mathbb{R}^n$; that is, for any Borel set $B \subset \mathbb{R}^n$ let

$$\tilde\mu(B) = \sum_{t\in\mathbb{Z}^n} \mu\big((B+t)\cap K\big).$$

One can check that

$$\tilde\mu\big(M^l(H)+v\big) = r^l\mu(H) \quad \text{for any } H \subset K \text{ Borel set, } v \in \mathbb{Z}^n,\ l = 0,1,2,\ldots. \tag{3}$$

Lemma 2.16. *Let $m_1,\ldots,m_n \geq 2$ and M like in Definition 2.14 and let $t \in \mathbb{R}^n$ be such that $\|M^k t\| > 0$ for every $k = 0,1,2,\ldots$, where $\|.\|$ denotes the distance from $\mathbb{Z}^n$.*

Then there exists infinitely many $k \in \mathbb{N}$ such that $\|M^k t\| > \frac{1}{2\max(m_1,\ldots,m_n)}$.

Proof. This lemma immediately follows from the following clear fact:

$$\|u\| \leq \frac{1}{2\max(m_1,\ldots,m_n)} \Longrightarrow \|Mu\| \geq \min(m_1,\ldots,m_n)\|u\| \geq 2\|u\|.$$

□

2.5. *Invariant extension of measures to larger sets.*

Lemma 2.17. *Suppose that the group G acts on a set X, $\mathcal{M}$ is a G-invariant σ-algebra on X, $A \in \mathcal{M}$, $\mathcal{M}_A = \{B \in \mathcal{M} : B \subset A\}$ and μ is a measure on $(A, \mathcal{M}_A)$.*

Then the following two statements are equivalent:

(i) $\mu\big(g(B)\big) = \mu(B)$ *whenever* $g \in G$ *and* $B, g(B) \in \mathcal{M}_A$.

(ii) There exists a G-invariant measure $\tilde{\mu}$ on $(X, \mathcal{M})$ such that $\tilde{\mu}(B) = \mu(B)$ for every $B \in \mathcal{M}_A$.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious. For proving the other implication we construct $\tilde{\mu}$ as follows.

If H is a set of the form

$$H = \cup_{i=1}^{*\infty} B_i, \text{ where } g_1, g_2, \ldots \in G \text{ and } g_1(B_1), g_2(B_2), \ldots \in \mathcal{M}_A \tag{4}$$

then let

$$\tilde{\mu}(H) = \sum_{i=1}^{\infty} \mu\big(g_i(B_i)\big)$$

and let $\tilde{\mu}(H) = \infty$ if $H \in \mathcal{M}$ cannot be written in the above form.

First we check that $\tilde{\mu}$ is well defined; that is, if we have (4) and $H = \cup_{j=1}^{*\infty} C_j$, $h_1, h_2, \ldots \in G$ and $h_1(C_1), h_2(C_2), \ldots \in \mathcal{M}_A$ then

$$\sum_{i=1}^{\infty} \mu\big(g_i(B_i)\big) = \sum_{j=1}^{\infty} \mu\big(h_j(C_j)\big). \tag{5}$$

Using that $B_i \subset H = \cup_{j=1}^{*\infty} C_j$ we get that $g_i(B_i) = g_i(\cup_{j=1}^{*\infty} B_i \cap C_j) = \cup_{j=1}^{*\infty} g_i(B_i \cap C_j)$ and so

$$\sum_{i=1}^{\infty} \mu\big(g_i(B_i)\big) = \sum_{i=1}^{\infty} \mu\big(\cup_{j=1}^{*\infty}\, g_i(B_i \cap C_j)\big) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \mu\big(g_i(B_i \cap C_j)\big),$$

and similarly

$$\sum_{j=1}^{\infty} \mu\big(h_j(C_j)\big) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \mu\big(h_j(B_i \cap C_j)\big).$$

Thus, using condition (i) for $B = g_i(B_i \cap C_j)$ and $g = h_j g_i^{-1}$, we get (5).

Using the freedom in (4) and that whenever $H \in \mathcal{M}$ can be written in the form (4) then the same is true for any $H \supset H' \in \mathcal{M}$, it is easy to check that $\tilde{\mu}$ is a G-invariant measure on $(X, \mathcal{M})$ such that $\tilde{\mu}(B) = \mu(B)$ for every $B \in \mathcal{M}_A$. □

We will need only the following special case of this lemma.

Lemma 2.18. *Let μ be a Borel measure on a Borel set $A \subset \mathbb{R}^n$ ($n \in \mathbb{N}$), G is group of affine transformations of $\mathbb{R}^n$ and suppose that*

$$\mu\big(g(B)\big) = \mu(B) \text{ whenever } b \in G,\ B, g(B) \subset A \text{ and } B \text{ is a Borel set.} \tag{6}$$

Then there exists a G-invariant Borel measure $\tilde{\mu}$ on $\mathbb{R}^n$ such that $\tilde{\mu}(B) = \mu(B)$ for any $B \subset A$ Borel set. □

Remark 2.19. The extension we get in the above proof do not always give the measure we expect – it may be infinity for too many sets. For example, if $A \subset \mathbb{R}$ is a Borel set of first category with positive Lebesgue measure, G is the group of translations and μ is the restriction of the Lebesgue measure to A then the Lebesgue measure itself would be the natural translation invariant extension of μ, however the extension $\tilde{\mu}$ as defined in the proof is clearly infinity for every Borel set of second category.

DEFINITION 2.20. Let μ be a Borel measure on a compact set K. We say that μ is isometry invariant if given any isometry g and a Borel set $B \subset K$ such that $g(B) \subset K$, then $\mu(B) = \mu\big(g(B)\big)$.

This definition makes sense since (by Lemma 2.18) exactly the isometry invariant measures on K can be extended to be isometry invariant measures on $\mathbb{R}^n$ in the usual sense.

As an illustration of Lemma 2.18 we mention the following special case with a peculiar consequence.

LEMMA 2.21. *Let $A \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ be a Borel set such that $A \cap (A + t)$ is at most countable for any $t \in \mathbb{R}^n$. Then any continuous Borel measure μ on A (continuous here means that the measure of any singleton is zero) can be extended to a translation invariant Borel measure on $\mathbb{R}^n$.* □

Note that although the condition that $A \cap (A + t)$ is at most countable for any $t \in \mathbb{R}^n$ seems to imply that A is very small, such a set can be still fairly large. For example there exists a compact set $C \subset \mathbb{R}$ with Hausdorff dimension 1 such that $C \cap (C + t)$ contains at most one point for any $t \in \mathbb{R}$ [**14**]. Combining this with Lemma 2.21 we get the following.

COROLLARY 2.22. *There exists a compact set $C \subset \mathbb{R}$ with Hausdorff dimension 1 such that any continuous Borel measure μ on C can be extended to a translation invariant Borel measure on $\mathbb{R}$.* □

2.6. *Some more lemmas.* The following simple lemmas might be known but for completeness (and because it is easier to prove them than to find them) we present their proof.

Recall that the support of a measure is the smallest closed set with measure zero complement.

LEMMA 2.23. *Let μ be a finite Borel measure on $\mathbb{R}^n$ with compact support K. Then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$|u| \geq \varepsilon \Longrightarrow \mu\big(K \cap (K + u)\big) \leq (1 - \delta)\mu(K).$$

Proof. We prove by contradiction. Assume that there exists an $\varepsilon > 0$ and a sequence $u_1, u_2, \ldots \in \mathbb{R}^n$ such that $|u_n| \geq \varepsilon$ (for every $n \in \mathbb{N}$) and $\mu\big(K \cap (K+u)\big) \to \mu(K) > 0$ $(n \to \infty)$. By omitting some (at most finitely many) zero terms we can guarantee that every u_n is in the compact annulus $\{x : \varepsilon \leq |x| \leq \mathrm{diam}(K)\}$ (where diam denotes the diameter), so by taking a subsequence we can suppose that (u_n) converges, say to u. Since $K \cap (K + u)$ is a proper compact subset of K (since K is compact and $u \neq 0$, $K + u \supset K$ is impossible) and K is the support of μ, we must have $\mu(K) > \mu\big(K \cap (K + u)\big) = \mu(K + u)$.

It is well known (see e.g. [**29**], 2.18. Theorem) that any finite Borel measure is outer regular in the sense that the measure of any Borel set is the infimum of the measures of the open sets that contain the Borel set. Thus $\mu(K+u) < \mu(K)$ implies that there exists an open set $G \supset K + u$ such that $\mu(G) < \mu(K)$. Then whenever $|u_n - u|$ is less than the (positive) distance between K and the complement of G, G contains $K + u_n$ and so $\mu(K) > \mu(G) \geq \mu(K + u_n)$. This is a contradiction since $u_n \to u$ and $\mu(K + u_n) = \mu\big(K \cap (K + u_n)\big) \to \mu(K)$. $\square$

LEMMA 2.24. *Let $K \subset \mathbb{R}^d$ be compact and μ be a probability Borel measure on K such that any nonempty relative open subset of K has positive μ measure. Then if the sequence (g_n) of affine maps converges to an affine map g and $\mu\big(g_n(K) \cap K\big) \to 1$ then $\mu\big(g(K) \cap K\big) = 1$. Moreover, $K \subset g(K)$.*

Proof. Suppose that $\mu\big(g(K) \cap K\big) = q < 1$. Let $g(K)_\varepsilon$ denote the ε-neighborhood of $g(K)$. Since $\bigcap_{n=1}^{\infty}(g(K)_{1/n} \cap K) = g(K) \cap K$ and μ is a finite measure we have $\mu\big(g(K)_{1/n} \cap K\big) \to \mu\big(g(K) \cap K\big) = q$. Thus there exists an $\varepsilon > 0$ for which $\mu\big(g(K)_\varepsilon \cap K\big) \leq \frac{1+q}{2} < 1$. Since g_n converges uniformly on K, for n large enough we have $g_n(K) \subset g(K)_\varepsilon$ and so $\mu\big(g_n(K) \cap K\big) \leq \frac{1+q}{2}$, contradicting $\mu\big(g_n(K) \cap K\big) \to 1$. Therefore we proved that $\mu\big(g(K) \cap K\big) = 1$.

Then $K \setminus g(K)$ is relative open in K and has μ measure zero, so it must be empty, therefore $K \subset g(K)$. $\square$

3. *Self-affine sets with the strong separation condition*

PROPOSITION 3.1. *For any self-affine set $K \subset \mathbb{R}^d$ with the strong separation condition there exists an open neighborhood $U \subset \mathcal{A}_K$ of the identity map such that for any $g \in U$,*

$$g(K) \supset K \iff g = \textit{identity}.$$

Proof. Let n denote the dimension of the affine span of K.

We shall prove that there exists a small open neighborhood $V \subset \mathcal{A}_K$ of the identity map such that for any $g \in V$ we have $g(K) \subset K \iff g = \text{identity}$. This would be enough since then for any $g \in V$ we get $K \subset g^{-1}(K) \iff g = \text{identity}$, therefore $U = V^{-1} = \{g^{-1} : g \in V\}$ has all the required properties.

Similarly as in the proof of Lemma 2.6, choose $n + 1$ elementary pieces $\varphi_{I_1}(K), \ldots, \varphi_{I_{n+1}}(K)$ of K so that if we pick one point from the convex hull of each of them then we get a maximal collection of affine independent points in the affine span of K.

Let $d = \min_{1 \leq i \leq n+1} \operatorname{dist}(\varphi_{I_i}(K), K \setminus \varphi_{I_i}(K))$, then $d > 0$. Let V be a so small neighborhood of the identity map that $\operatorname{dist}(x, g(x)) < d$ for any $g \in V$ and $x \in K$.

Let $g \in V$ and $g(K) \subset K$. Then, by the definition of d and V we have $g(\varphi_{I_i}(K)) \subset \varphi_{I_i}(K)$ for every $1 \leq i \leq n + 1$. Then the convex hulls of these elementary pieces are also mapped into themselves. Since each of these convex hulls is homeomorphic to a ball, by Brouwer's fixed point theorem we get a fixed point of g in each of these elementary pieces. So we obtained $n + 1$ fixed points of

g such that their affine span is exactly the affine span of K. Since g is an affine map, the set of its fixed points form an affine subspace, thus the set of fixed points of g contains the affine span of K. Since $g \in \mathcal{A}_K$, g is defined exactly on the affine span of K, therefore g must be the identity map. $\square$

THEOREM 3.2. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-affine set satisfying the strong separation condition and let μ be a self-affine measure on K. Then there exists a $c < 1$ and an open neighborhood $U \subset \mathcal{A}_K$ of the identity map such that $g \in U \setminus \{identity\} \Longrightarrow \mu\big(K \cap g(K)\big) < c$.*

Proof. Using Proposition 3.1 we can choose a small open neighborhood $U \subset \mathcal{A}_K$ of the identity map such that even in the closure of U the only affine map g for which $g(K)$ contains K is the identity map and so that

$$\operatorname{dist}(x, g(x)) < 1 \text{ for any } g \in U \text{ and } x \in K. \tag{7}$$

Since $\mathcal{A}_K$ is locally compact, we may also assume that the closure of U is compact.

We claim that we can choose an even smaller open neighborhood $V \subset U$ of the identity map such that $\varphi_i^{-1} \circ V \circ \varphi_i \subset U$ for $i = 1, \ldots, r$ and that $g(\varphi_i(K)) \cap \varphi_j(K) = \emptyset$ for any $i \neq j$ and $g \in V$. Indeed, the first property can be satisfied since $\mathcal{A}_K$ is a topological group and those g's for which the second property do not hold are far from the identity map.

Now we claim that there exists a $c < 1$ such that $g \in \overline{U} \setminus V \Longrightarrow \mu\big(g(K) \cap K\big) < c$. Suppose that there exists a sequence $(g_n) \subset \overline{U} \setminus V$ such that $\mu\big(K \cap g_n(K)\big) \to 1$. Since $\overline{U} \setminus V$ is compact there exists a subsequence g_{n_i} such that $g_{n_i} \to h \in \overline{U} \setminus V$. By Lemma 2.24 this implies that $h(K) \supset K$ but in $\overline{U} \setminus V$ there is no such affine map h.

We prove that this U and this c have the required properties; that is, $g \in U \setminus \{\text{identity}\} \Longrightarrow \mu\big(K \cap g(K)\big) < c$.

If $g \in \overline{U} \setminus V$ then we are already done, so suppose that $g \in V \setminus \{\text{identity}\}$. Let F denote the set of fixed points of g.

The heuristics of the remaining part of the proof is the following. The affine map g moves K too slightly. We zoom in on small elementary pieces $a(K)$ of K so that each $g(a(K))$ intersects only $a(K)$ in K, but g moves $a(K)$ far enough (compared to its size). Technically this second requirement means that $a^{-1} \circ g \circ a \in U \setminus V$, so we can use the $g \in U \setminus V$ case for the elementary piece $a(K)$. We find such an elementary piece around each point of K that is not a fixed point of g, and so we get a partition of $K \setminus F$ into elementary pieces with the above property. Finally, by adding up the estimates for these elementary pieces we derive $\mu\big(g(K) \cap K\big) < c$.

CLAIM 3.3. *For any $x \in K \setminus F$ there exists a largest elementary piece $\varphi_{I_x}(K)$ of K that contains x and for which $\varphi_{I_x}^{-1} \circ g \circ \varphi_{I_x} \in U \setminus V$.*

Proof. Let $(i_1, i_2, \ldots)$ be the sequence of indices for which

$$\{x\} = \bigcap_{n=1}^{\infty} (\varphi_{i_1} \circ \varphi_{i_2} \circ \ldots \circ \varphi_{i_n})(K),$$

and let $I_n = (i_1, \dots, i_n)$. Since $g \in V$, we have $\varphi_{i_1}^{-1} \circ g \circ \varphi_{i_1} \in U$ by the definition of V. If for some n we have $\varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n} \in V$ then by the definition of V we have

$$\varphi_{I_{n+1}}^{-1} \circ g \circ \varphi_{I_{n+1}} = \varphi_{i_{n+1}}^{-1} \circ \varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n} \circ \varphi_{i_{n+1}} \in U.$$

Therefore it is enough to find an n such that $\varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n} \notin V$ since then taking the smallest such n, $I_x = I_n$ has the desired property. Letting $y_n = \varphi_{I_n}^{-1}(x)$ we have $y_n \in K$ (since $\{x\} = \bigcap_{n=1}^{\infty} \varphi_{I_n}(K)$) and $(\varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n})(y_n) = \varphi_{I_n}^{-1}(g(x))$. Since x is not a fixed point of g, for n large enough we have

$$\operatorname{dist}\bigl(g(x), \varphi_{I_n}(K)\bigr) > \frac{\operatorname{dist}(g(x), x)}{2} \overset{\text{def}}{=} t > 0.$$

Since each φ_i is a contractive affine map, there exists an $\alpha_i < 1$ such that $\operatorname{dist}(\varphi_i(a), \varphi_i(b)) \le \alpha_i \cdot \operatorname{dist}(a, b)$ for any a, b. Then, using the multi-index notation $\alpha_{I_n} = \alpha_{i_1} \cdot \ldots \cdot \alpha_{i_n}$, we clearly have $\operatorname{dist}(\varphi_{I_n}(a), \varphi_{I_n}(b)) \le \alpha_{I_n} \cdot \operatorname{dist}(a, b)$ for any a, b. Then $\operatorname{dist}\bigl(\varphi_{I_n}^{-1}(g(x)), K\bigr) > t/\alpha_{I_n}$, hence $\operatorname{dist}\bigl((\varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n})(y_n), K\bigr) > t/\alpha_{I_n}$, which is bigger than 1 if n is large enough. Thus for n large enough, $\varphi_{I_n}^{-1} \circ g \circ \varphi_{I_n}$ is not in V, since it is not even in U by (7). □

Claim 3.4. *For any $x \in K \setminus F$ we have $g(\varphi_{I_x}(K)) \cap K \subset \varphi_{I_x}(K)$, where $I_x = I_n = (i_1, \dots, i_n)$ is the multi-index we got in Claim 3.3.*

Proof. Let $k \in \{0, 1, \dots, n-1\}$ be arbitrary and let $I_k = (i_1, \dots, i_k)$. Then $\varphi_{I_k}^{-1} \circ g \circ \varphi_{I_k} \in V$, hence for any $l \ne i_{k+1}$ we have $(\varphi_{I_k}^{-1} \circ g \circ \varphi_{I_k} \circ \varphi_{i_{k+1}})(K) \cap \varphi_l(K) = \emptyset$, which is the same as $(g \circ \varphi_{I_{k+1}})(K) \cap (\varphi_{I_k} \circ \varphi_l)(K) = \emptyset$ $(l \ne i_{k+1})$. Since $(g \circ \varphi_{I_n})(K) \subset (g \circ \varphi_{I_{k+1}})(K)$, this implies that

$$(g \circ \varphi_{I_n})(K) \cap (\varphi_{I_k} \circ \varphi_l)(K) = \emptyset \quad (k \in \{0, 1, \dots, n-1\},\ l \ne i_{k+1}).$$

Since $K \setminus \varphi_{I_n}(K) = \cup_{k=0}^{n-1} \cup_{l \ne i_{k+1}} (\varphi_{I_k} \circ \varphi_l)(K)$, this implies that $g(\varphi_{I_n}(K)) \cap K \subset \varphi_{I_n}(K)$. □

The elementary pieces $\{\varphi_{I_x}(K) : x \in K \setminus F\}$ clearly cover $K \setminus F$. Since for any $x \ne y$ we have $\varphi_{I_x}(K) \cap \varphi_{I_y}(K) = \emptyset$ or $\varphi_{I_x}(K) \subset \varphi_{I_y}(K)$ or $\varphi_{I_x}(K) \supset \varphi_{I_y}(K)$, one can choose a

$$K \setminus F \subset \bigcup_{i=1}^{\infty}{}^{*} \varphi_{J_i}(K) \tag{8}$$

countable disjoint subcover. By Claim 3.4 we have

$$g(\varphi_{J_i}(K)) \cap K \subset \varphi_{J_i}(K). \tag{9}$$

Since g is not the identity map (of the affine span of K) and F is the set of fixed points of the affine map g, the dimension of the affine subspace F is smaller than the dimension of the affine span of K, and so we cannot have $g(F) \supset K$. By Lemma 2.6 this implies that $\mu\bigl(g(F) \cap K\bigr) = 0$. Using this last equation, (8), (9), and finally the definition of a self-affine measure we get that

$$\begin{aligned}\mu\big(g(K)\cap K\big) &\le \mu\big(g(F)\cap K\big)+\mu\big(g(K\setminus F)\cap K\big)=\mu\big(g(K\setminus F)\cap K\big)\\ &\le \mu\Big(g\Big(\bigcup_{i=1}^{\infty}{}^{*}\varphi_{J_i}(K)\Big)\cap K\Big)=\sum_{i=1}^{\infty}\mu\Big(g\big(\varphi_{J_i}(K)\big)\cap K\Big)\\ &=\sum_{i=1}^{\infty}\mu\Big(g\big(\varphi_{J_i}(K)\big)\cap\varphi_{J_i}(K)\Big)=\sum_{i=1}^{\infty}\mu\Big(\varphi_{J_i}\big((\varphi_{J_i}^{-1}\circ g\circ\varphi_{J_i})(K)\cap K\big)\Big)\\ &=\sum_{i=1}^{\infty}p_{J_i}\,\mu\Big((\varphi_{J_i}^{-1}\circ g\circ\varphi_{J_i})(K)\cap K\Big).\end{aligned}$$

Since $\varphi_{J_i}^{-1}\circ g\circ\varphi_{J_i}\in U\setminus V$, the measures in the last expression are less than c. Thus $\mu\big(g(K)\cap K\big)<\sum p_{J_i}\cdot c=\sum\mu\big(\varphi_{J_i}(K)\big)\cdot c=\mu\big(\bigcup_i^{*}\varphi_{J_i}(K)\big)\cdot c=c$, which completes the proof. □

THEOREM 3.5. *Let $K\subset\mathbb{R}^d$ be a self-affine set with the strong separation condition and let μ be a self-affine measure on K. Then there exists a constant $c<1$ such that for any isometry g we have $\mu\big(K\cap g(K)\big)<c$ unless $g(K)=K$.*

Proof. Suppose that $g_n\in\mathcal{I}_K$ (that is, g_n is an isometry of the affine span of K) such that $g_n(K)\ne K$ $(n\in\mathbb{N})$ and $\mu\big(K\cap g_n(K)\big)\to 1$. We can clearly assume that $K\cap g_n(K)\ne\emptyset$ for each n and so the whole sequence (g_n) is in a compact subset of $\mathcal{I}_K$. Thus, after choosing a subsequence if necessary, we can also assume that g_n converges to an $h\in\mathcal{I}_K$. By Lemma 2.24 we must have $K\subset h(K)$. It is well known and not hard to prove that no compact set in $\mathbb{R}^d$ can have an isometric proper subset, so $K\subset h(K)$ implies that $h(K)=K$.

Applying Theorem 3.2 we get a $c<1$ and an open neighborhood $U\subset\mathcal{A}_K$ of the identity such that $g\in U\setminus\{\text{identity}\}\Longrightarrow\mu\big(K\cap g(K)\big)<c$.

Since $g_n\to h$ we get $g_n\circ h^{-1}\to$ identity. Let n be large enough to have $g_n\circ h^{-1}\in U$ and $\mu\big(K\cap g_n(K)\big)>c$. Since $g_n(K)\ne K$ but $h(K)=K$ we cannot have $g_n=h$ and so $g_n\circ h^{-1}\in U\setminus\{\text{identity}\}$. Then, by the previous paragraph, we get $\mu\big(K\cap g_n(K)\big)<c$, contradicting $\mu\big(K\cap g_n(K)\big)>c$. □

4. *Self-similar sets with the strong separation property*

Our first goal in this section is to prove the following theorem.

THEOREM 4.1. *Let $K=\varphi_1(K)\cup^*\ldots\cup^*\varphi_r(K)$ be a self-similar set satisfying the strong separation condition and μ be a self-similar measure on it. There exists $c<1$ such that for every similitude g either $\mu\big(g(K)\cap K\big)<c$ or $K\subset g(K)$.*

Now, for the sake of transparency we outline the proof. At first we need a new notation.

From S_K we excluded those similarity maps which map everything to a single point. So let S_K^* be the metric space of all degenerate and all non-degenerate similarity maps in the affine span A_K of K; that is,

$$S_K^* = S_K \cup \{f \mid f : A_K \to \{y\},\, y \in A_K\}. \tag{10}$$

First we show that there exists a compact set $\mathcal{G} \subset S_K^*$ of similarity maps such that for every $g' \in S_K^*$ there exists $g \in \mathcal{G}$ such that $g'(K) \cap K = g(K) \cap K$. Then it is easy to see that it suffices to prove the theorem for $g \in \mathcal{G}$. (It is easy to see that no such compact set G in $\mathcal{S}_K$ exists.)

Let μ_H be a constant multiple of Hausdorff measure of appropriate dimension so that $\mu_H(K) = 1$. The restriction of this measure to K is a self-similar measure. Let us consider those $h \in \mathcal{G}$ for which $K \subset h(K)$ holds. Using Hausdorff measures and Theorem 3.2 we prove that there are only finitely many such h, and also that the theorem holds in small neighbourhoods of each such h for the measure μ_H. The maximum of the corresponding finitely many values c is still strictly smaller than 1. Let us now cut these small neighbourhoods out of $\mathcal{G}$. Using upper semicontinuity of our measure (Lemma 2.24) we produce a $c < 1$ such that for the remaining similarity maps g we have $\mu_H\big(g(K) \cap K\big) < c$. Then clearly the same holds for all elements of $\mathcal{G}$, possibly with a larger $c < 1$, finishing the proof for the measure μ_H.

Applying the theorem for μ_H, and also in a small open neighbourhood U of the identity for every self-similar measure μ, we show that if $h \in \mathcal{G}$, $K \subset h(K)$, and g is in a small neighbourhood of h then $\mu\big(g(K) \cap K\big) < c$. Then the same argument as above (using upper semicontinuity) yields the theorem, possibly with a larger constant again.

Proposition 4.2. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set satisfying the strong separation condition. Then there exists a compact set $\mathcal{G} \subset S_K^*$ such that for every similarity map $g' \in S_K^*$ there is a $g \in \mathcal{G}$ for which $g'(K) \cap K = g(K) \cap K$ holds.*

Proof. Let D denote the diameter of K, let $\delta = \min_{1 \le i < j \le r} \operatorname{dist}(\varphi_i(K), \varphi_j(K))$ and let

$$\mathcal{G} = \{g \in S_K^* : g(K) \cap K \neq \emptyset, \text{ the similarity ratio of } g \text{ is at most } D/\delta\} \cup \{g_0\},$$

where $g_0 \in S_K^*$ is an arbitrary fixed similarity map such that $g(K) \cap K = \emptyset$. It is easy to check that $\mathcal{G} \subset S_K^*$ is compact.

Let $g' \in S_K^*$. If $g' \in \mathcal{G}$ or $g'(K) \cap K = \emptyset$ then we can choose $g = g'$ or $g = g_0$, respectively. So we can suppose that $g'(K) \cap K \neq \emptyset$ and the similarity ratio of g' is greater than D/δ. Then the minimal distance between the first generation elementary pieces $g'(\varphi_j(K))$ of $g'(K)$ is larger than D. So there exists φ_i such that $g'(K) \cap K = g'(\varphi_i(K)) \cap K$. Therefore g' can be replaced by $g' \circ \varphi_i$, which has similarity ratio α_i times smaller than the similarity ratio of g', where a_i denotes the similarity ratio of φ_i. Since $\max(\alpha_1, \ldots, \alpha_r) < 1$, this way in finitely many steps we get a g with similarity ratio at most D/δ such that $g(K) \cap K = g'(K) \cap K \neq \emptyset$, which completes the proof. ◻

Proposition 4.3. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set satisfying the strong separation condition.*

(i) Then $\{g \in \mathcal{S}_K : g(K) \supset K\}$ *is discrete in* $\mathcal{S}_K$*, hence countable, and also closed in* $\mathcal{S}_K$*.*

(ii) Let μ_H *be a constant multiple of Hausdorff measure of appropriate dimension so that* $\mu_H(K) = 1$*. There exists* $c < 1$ *such that for every similitude* g *either* $\mu_H\big(g(K) \cap K\big) < c$ *or* $K \subset g(K)$*.*

Proof. By Lemma 2.24 $\{g \in \mathcal{S}_K : g(K) \supset K\}$ is closed. Since every discrete subset of a subspace of $\mathbb{R}^{d^2+d}$ is countable, in order to prove (i) it is enough to prove that $\{g \in \mathcal{S}_K : g(K) \supset K\}$ is discrete.

Let ε be a positive number to be chosen later, and h be a similitude for which $K \subset h(K)$. Denote by K_δ the δ-neighbourhood of K. As $\mu_H\big(h(K)\big)$ is finite, there is a small $\delta > 0$ such that $\mu_H\big(K_\delta \cap (h(K) \setminus K)\big) < \varepsilon$. Applying Theorem 3.2 to K and μ_H we obtain an open neighbourhood $U \subset \mathcal{A}_K$ and a constant c_H. There exists an open neighbourhood $W_\varepsilon \subset \mathcal{S}_K$ of the identity such that

(a) $W_\varepsilon = W_\varepsilon^{-1} \subset U$,

(b) $\operatorname{dist}(g(x), x) < \delta$ for every $x \in K$,

(c) $\mu_H\big(g(B)\big) \le (1+\varepsilon)\mu_H(B)$ for every $g \in W_\varepsilon$ and Borel set B,

where for (c) we use that a similitude of ratio α multiplies the s-dimensional Hausdorff measure by α^s.

Let $g \in W_\varepsilon h$ and $g \neq h$. Clearly $W_\varepsilon h$ is an open neighbourhood of h and $g \circ h^{-1}$, $h \circ g^{-1} \in W_\varepsilon \setminus \{\text{identity}\}$, and $(h \circ g^{-1})(K) \subset K_\delta$. Hence

$$\begin{aligned}
\mu_H\big(K \cap g(K)\big) &\le (1+\varepsilon)\mu_H\big((h \circ g^{-1})(K \cap g(K))\big) = \\
&= (1+\varepsilon)\mu_H\big((h \circ g^{-1})(K) \cap h(K)\big) = \\
&= (1+\varepsilon)\mu_H\big((h \circ g^{-1})(K) \cap K\big) + (1+\varepsilon)\mu_H\big((h \circ g^{-1})(K) \cap (h(K) \setminus K)\big) \le \\
&\le (1+\varepsilon)c_H + (1+\varepsilon)\mu_H\big(K_\delta \cap (h(K) \setminus K)\big) \le (1+\varepsilon)c_H + (1+\varepsilon)\varepsilon. \quad (11)
\end{aligned}$$

The last expression is clearly smaller than 1 if ε is small enough, so let us fix such an ε. Therefore if $g \in W_\varepsilon h$ and $g \neq h$ then $g(K) \not\supset K$, which shows that $\{g \in \mathcal{S}_K : g(K) \supset K\}$ is discrete finishing the proof of (i).

In order to prove (ii) suppose towards a contradiction that $\sup\{\mu_H(g(K) \cap K) : g \in S_K^*, g(K) \not\supset K\} = 1$. Then we also have $\sup\{\mu_H(g(K) \cap K) : g \in \mathcal{G}, g(K) \not\supset K\} = 1$. Let (g_n) be a convergent sequence in $\mathcal{G}$ so that $g_n(K) \not\supset K$, $\mu_H\big(g_n(K) \cap K\big) \to 1$, $g_n \to h$. Lemma 2.24 yields $h(K) \supset K$, hence $g_n \neq h$. If n is large enough then $g_n \in W_\varepsilon h$ and, by (11),

$\mu_H\big(K \cap g_n(K)\big) \le (1+\varepsilon)c_H + (1+\varepsilon)\varepsilon$, contradicting $\mu_H\big(g_n(K) \cap K\big) \to 1$. □

Proof of Theorem 4.1. By Proposition 4.2 we can assume $g \in \mathcal{G}$. Let c_H be the constant yielded by Proposition 4.3 (ii). Fix $h \in \mathcal{G}$ with $h(K) \supset K$. There are only finitely many such h by Proposition 4.3 (i) and the compactness of $\mathcal{G}$.

Let us now apply Lemma 2.12 to the self-similar set $h(K)$, μ_H, $0 < \varepsilon \le 1 - c_H$ and $B = K \subset h(K)$. We obtain φ_I such that

$$\mu_H\big(K \cap h(\varphi_I(K))\big) \ge (1-\varepsilon)\,\mu_H\big(h(\varphi_I(K))\big).$$

Hence Proposition 4.3 (ii) applied to the self-similar set $h(\varphi_I(K))$ and the similitude $(h \circ \varphi_I)^{-1}$ gives $K \supset h(\varphi_I(K))$.

Since $h(\varphi_I(K))$ is open in $h(K)$, it is also open in K and so it can be written as a union of elementary pieces of K. Since $h(\varphi_I(K))$ is compact this implies that $h(\varphi_I(K))$ is a finite union of elementary pieces of K. Let $\varphi_J(K)$ be one of these elementary pieces. So $\varphi_J(K) \subset h(\varphi_I(K)) \subset K \subset h(K)$. As $\varphi_J(K)$ is open in K, it is also open in $h(\varphi_I(K))$, hence also in $h(K)$. Therefore $\operatorname{dist}(\varphi_J(K), h(K) \setminus \varphi_J(K)) > 0$, and so for every g that is close enough to h we have

$$\big(g \circ h^{-1}\big)\big(h(K) \setminus \varphi_J(K)\big) \cap \varphi_J(K) = \emptyset.$$

Thus, as $\varphi_J(K) \subset h(K)$, for every such g we have

$$g(K) \cap \varphi_J(K) = \big(g \circ h^{-1}\big)\big(h(K)\big) \cap \varphi_J(K) = \big(g \circ h^{-1}\big)\big(\varphi_J(K)\big) \cap \varphi_J(K).$$

On the other hand, Theorem 3.2 yields that there exists a $c < 1$ such that if g is close enough to h and $g \ne h$ then

$$\mu\big((g \circ h^{-1})(\varphi_J(K)) \cap \varphi_J(K)\big) < c \cdot \mu\big(\varphi_J(K)\big) = c \cdot p_J.$$

Therefore $\mu\big(g(K) \cap \varphi_J(K)\big) = \mu\big((g \circ h^{-1})(\varphi_J(K)) \cap \varphi_J(K)\big) < c \cdot p_J$ and

$$\begin{aligned}\mu\big(g(K) \cap K\big) &= \mu\big(g(K) \cap \varphi_J(K)\big) + \mu\big(g(K) \cap (K \setminus \varphi_J(K))\big) \\ &< c \cdot p_J + 1 - p_J = 1 - (1-c)p_J. \qquad (12)\end{aligned}$$

As we only considered finitely many h's, there exists $c' < 1$ such that if g is close to one of these h's, but distinct from it, then $\mu\big(g(K) \cap K\big) < c'$. This, together with Lemma 2.24 provides a $c'' < 1$ such that for every $g \in \mathcal{G}$ either $\mu\big(g(K) \cap K\big) < c''$ or $g(K) \supset K$. (Just like at the end of the proof of Proposition 4.3.) Finally, by Proposition 4.2 this also holds outside $\mathcal{G}$. □

We will apply this theorem to elementary pieces of K instead of K itself. It is easy to see that the same c works for every elementary piece; that is, we have the following corollary of Theorem 4.1.

COROLLARY 4.4. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set satisfying the strong separation condition and μ be a self-similar measure on it. There exists $c < 1$ such that for every similitude g and every elementary piece $a(K)$ of K either $\mu\big(g(K) \cap a(K)\big) < c \cdot \mu\big(a(K)\big)$ or $a(K) \subset g(K)$.* □

Now we are ready to prove the second main result of this section.

THEOREM 4.5. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set satisfying the strong separation condition, μ be a self-similar measure on it, and g be a similitude. Then $\mu\big(g(K) \cap K\big) > 0$ if and only if the interior (in K) of $g(K) \cap K$ is nonempty. Moreover, $\mu\big(int_K(g(K) \cap K)\big) = \mu\big(g(K) \cap K\big)$.*

Proof. If the interior (in K) of $g(K) \cap K$ is nonempty then clearly it is of positive measure, since the measure of every elementary piece is positive.

Let c be the constant given by Corollary 4.4, and let g be a similitude such that $\mu\big(g(K) \cap K\big) > 0$. Applying Lemma 2.13 for $B = g(K) \cap K$ and $\varepsilon = 1 - c$ we obtain countably many disjoint elementary pieces $a_i(K)$ of K such that

$$\mu\big(g(K) \cap a_i(K)\big) = \mu\big((g(K) \cap K) \cap a_i(K)\big) > c \cdot \mu\big(a_i(K)\big) \tag{13}$$

and $\big(g(K) \cap K\big) \setminus \bigcup_i^* a_i(K)$ is of μ-measure zero. By Corollary 4.4, (13) implies that $a_i(K) \subset g(K)$. Since $a_i(K)$ is open in K, it is open in $g(K) \cap K$, so $\bigcup_i^* a_i(K) \subset \mathrm{int}_K(g(K) \cap K)$. Hence

$$\begin{aligned}\mu\big(g(K) \cap K\big) &= \mu\big(g(K) \cap K \cap \bigcup_i^* a_i(K)\big) + \mu\big((g(K) \cap K) \setminus \bigcup_i^* a_i(K)\big) \\ &= \mu\big(\bigcup_i^* a_i(K)\big) \leq \mu\big(\mathrm{int}_K(g(K) \cap K)\big),\end{aligned}$$

proving the theorem. □

As an immediate consequence we get the following.

Corollary 4.6. *Let $K \subset \mathbb{R}^d$ be a self-similar set satisfying the strong separation condition, and let μ_1 and μ_2 be self-similar measure on K. Then for any similitude g of $\mathbb{R}^d$,*

$$\mu_1\big(g(K) \cap K\big) > 0 \Longleftrightarrow \mu_2\big(g(K) \cap K\big) > 0.$$

We also get the following fairly easily.

Corollary 4.7. *Let $K \subset \mathbb{R}^d$ be a self-similar set satisfying the strong separation condition, let A_K be the affine span of K and let μ be a self-similar measure on K. Then the set of those similitudes $g : A_K \to \mathbb{R}^d$ for which $\mu\big(g(K) \cap K\big) > 0$ is countably infinite.*

Proof. It is clear that there exist infinitely many similitudes g such that $\mu\big(g(K) \cap K\big) > 0$ since the elementary pieces of K are similar to K and have positive μ measure.

By Lemma 2.6, $\mu\big(g(K) \cap K\big) > 0$ implies that $g \in \mathcal{S}_K$ and, by Theorem 4.5, that $g(K)$ contains an elementary piece of K. Therefore it is enough to show that for each fixed elementary piece $a(K)$ of K there are only countably many $g \in \mathcal{S}_K$ such that $g(K) \supset a(K)$, which is the same as $(a^{-1} \circ g)(K) \supset K$. By the first part of Proposition 4.3 there are only countably many such $a^{-1} \circ g \in \mathcal{S}_K$, so there are only countably many such $g \in \mathcal{S}_K$. □

From the first part of Proposition 4.3 we get more results about those similarity maps that map a self-similar set into itself. These results will be used in the next section and they are also related to a theorem and a question of Feng and Wang [**8**] as it will be explained before Corollary 4.10.

Lemma 4.8. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with strong separation condition. There exists only finitely many similitudes g for which $g(K) \subset K$ holds and $g(K)$ intersects at least two first generation elementary pieces of K.*

Proof. The similarity ratios of these similitudes g are strictly separated from zero. Thus the similarity ratio of their inverses have some finite upper bound, and also $K \subset g^{-1}(K)$ holds. The set of similitudes with the latter property form a discrete and closed set according to the first part of Proposition 4.3.

Those $h \in S_K^*$ similarity maps (cf. (10)) whose similarity ratio is under some fixed bound and for which $h(K) \cap K \neq \emptyset$ holds form a compact set in S_K^* (see proof of Proposition 4.2). Since a discrete and closed subspace of a compact set is finite, the proof is finished. □

Theorem 4.9. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with strong separation condition and let λ be a similitude for which $\lambda(K) \subset K$. There exist an integer $k \geq 1$ and multi-indices I, J such that $\lambda^k \circ \varphi_I = \varphi_J$.*

Proof. For every integer $k \geq 1$ there exists a smallest elementary piece $\varphi_I(K)$ which contains $\lambda^k(K)$. For this multi-index I, $(\varphi_I^{-1} \circ \lambda^k)(K)$ is a subset of K and intersects at least two first generation elementary pieces of K. There are only finitely many similitudes with this property according to Lemma 4.8, hence there exist $k < k'$, I, I' such that $\varphi_I^{-1} \circ \lambda^k = \varphi_{I'}^{-1} \circ \lambda^{k'}$. By rearrangement we obtain $\varphi_{I'} \circ \varphi_I^{-1} = \lambda^{k'-k}$ and $\lambda^{k'-k} \circ \varphi_I = \varphi_{I'}$. □

Feng and Wang [**8**, Theorem 1.1 (The Logarithmic Commensurability Theorem)] proved that if $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ is a self-similar set in $\mathbb{R}$ satisfying the open set condition with Hausdorff dimension less than 1 and such that each similarity map φ_i is of the form $\varphi_i(x) = bx + c_i$ with a fixed b and $aK + t \subset K$ for some $a, t \in \mathbb{R}$ then $\log|a| / \log|b| \in \mathbb{Q}$. They also posed the problem (Open Question 2) of generalizing this result to higher dimensions. If we assume the strong open set condition instead of the open set condition then the above Theorem 4.9 tells much more about the maps $\varphi_1, \ldots, \varphi_r$ and $ax + t$ and immediately gives the following higher dimensional generalization of the Logarithmic Commensurability Theorem of Feng and Wang, in which we can also allow non-homogeneous self-similar sets.

Corollary 4.10. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with strong separation condition and suppose that λ is a similitude for which $\lambda(K) \subset K$. If $a_1, \ldots, a_r$ and b denote the similarity ratios of $\varphi_1, \ldots, \varphi_r$ and λ, respectively, then $\log b$ must be a linear combination of $\log a_1, \ldots, \log a_r$ with rational coefficients.*

5. *Isometry invariant measures*

In this section all self-similar sets we consider will satisfy the strong separation condition, even if we do not mention it every time.

Before we start to study and characterize the isometry invariant measures on a self-similar set of strong separation condition, we have to pay some attention to the connection of a self-similar set and the self-similar measures living on it.

We have called a compact set K self-similar with SSC if $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ holds for some similitudes $\varphi_1, \ldots, \varphi_r$. A *presentation* of K is a finite collection of similitudes $\{\psi_1, \ldots, \psi_s\}$, such that $K = \psi_1(K) \cup^* \ldots \cup^* \psi_s(K)$ and $s \geq 2$. Clearly, a self-similar set with SSC has many different presentations. For example, if $\{\varphi_1, \varphi_2, \ldots, \varphi_r\}$ is a presentation of K, then $\{\varphi_i \circ \varphi_j : 1 \leq i, j \leq r\}$ is also a presentation.

As we shall see in the next section, it can even happen that a self-similar set has no "smallest" presentation. We say that a presentation $\mathcal{F}_1 = \{\psi_1, \psi_2, \ldots, \psi_s\}$ is smaller than the presentation $\mathcal{F} = \{\varphi_1, \varphi_2, \ldots, \varphi_r\}$, if for every $1 \leq i \leq r$ there exists a multi-index I, such that $\varphi_i = \psi_I$. This defines a partial ordering on the presentations; let us denote by $\mathcal{F}_1 \leq \mathcal{F}$ if $\mathcal{F}_1$ is smaller than $\mathcal{F}$. We call a presentation *minimal*, if there is no smaller presentation (excluding itself). We call a presentation *smallest*, if it is smaller than any other presentation.

There exists a self-similar set with SSC which has more than one minimal presentations; that is, it has no smallest presentation (see Section 6).

The notion of a self-similar measure on a self-similar set depends on the presentation. Thus, when we say that μ is a self-similar measure on K, we always mean that μ is self-similar measure with respect to the given presentation of K. Clearly if $\mathcal{F}_1 \leq \mathcal{F}$, then there are less self-similar measures with respect to $\mathcal{F}_1$ than to $\mathcal{F}$. It will turn out that the isometry invariant self-similar measures are the same independently of the presentations.

NOTATION 5.1. For the sake of simplicity, for a similitude λ with $\lambda(K) \subset K$ let $\mu(\lambda)$ denote $\mu\big(\lambda(K)\big)$. In the composition of similitudes we might omit the mark $\circ$, so $g_1 g_2$ stands for $g_1 \circ g_2$, and by g^k we shall mean the composition of k many g's.

Clearly, given any self-similar measure μ, $\mu \circ \varphi_I = \mu(\varphi_I) \cdot \mu$ holds for the similitudes φ_I arising from the presentation of K. According to the next proposition, if for a given self-similar measure μ the congruent elementary pieces are of equal measure, then the same holds for any similitude λ satisfying $\lambda(K) \subset K$; that is, we have $\mu \circ \lambda = \mu(\lambda) \cdot \mu$ as well.

PROPOSITION 5.2. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be self-similar set with strong separation condition, and μ be a self-similar measure on K for which the congruent elementary pieces are of equal measure.*

1. *Then for every similitude λ with $\lambda(K) \subset K$, $\mu \circ \lambda = \mu\big(\lambda(K)\big) \cdot \mu$ holds; that is, for any Borel set $H \subset K$ we have $\mu\big(\lambda(H)\big) = \mu\big(\lambda(K)\big) \cdot \mu\big(H\big)$.*

2. *For every elementary piece $\varphi_I(K)$ and for every isometry g for which $g(\varphi_I(K)) \subset K$ holds, we have $\mu\big(\varphi_I(K)\big) = \mu\big(g(\varphi_I(K))\big)$.*

Proof. According to Lemma 4.8 there are only finitely many similitudes λ for which $\lambda(K) \subset K$ holds and $\lambda(K)$ intersects at least two first generation elementary pieces. Denote these by $\lambda_0, \lambda_1, \ldots, \lambda_t$, where λ_0 should stand for the identity.

We claim that it is enough to prove the first part of the proposition for these similitudes only. Let λ be a similitude for which $\lambda(K) \subset K$. Let $\varphi_I(K)$ the smallest elementary piece which contains $\lambda(K)$. Then the similitude $\varphi_I^{-1} \circ \lambda$ maps K into itself and the image intersects at least two first generation elementary pieces, hence it is equal to a similitude λ_i for some i. Thus $\lambda = \varphi_I \circ \lambda_i$. The measure μ being self-similar we have $\mu \circ \varphi_J = p_J \cdot \mu = \mu\big(\varphi_J(K)\big) \cdot \mu$ for every multi-index J, hence for any Borel set $H \subset K$ we obtain

$$\begin{aligned}\mu\big(\lambda(H)\big) = \mu\big((\varphi_I \circ \lambda_i)(H)\big) = \mu\big(\varphi_I(K)\big) \cdot \mu\big(\lambda_i(H)\big) = \mu\big(\varphi_I(K)\big) \cdot \mu\big(\lambda_i(K)\big) \cdot \mu\big(H\big) \\ = \mu\big((\varphi_I \circ \lambda_i)(K)\big) \cdot \mu\big(H\big) = \mu\big(\lambda(K)\big) \cdot \mu\big(H\big),\end{aligned}$$

as we stated.

According to Theorem 4.9 for every integer i with $0 \le i \le t$ there exist multi-indices I_i, J_i and a positive integer k_i, for which $\lambda_i^{k_i} \circ \varphi_{I_i} = \varphi_{J_i}$. Let $b_i \stackrel{\text{def}}{=} \varphi_{I_i}$, $c_i \stackrel{\text{def}}{=} \varphi_{J_i}$, hence $\lambda_i^{k_i} b_i = c_i$.

Let

$$\mu^*(\lambda_i) \stackrel{\text{def}}{=} \sqrt[k_i]{\frac{\mu(c_i)}{\mu(b_i)}}.$$

Our aim is to show that $\mu^*(\lambda_i) = \mu(\lambda_i)$.

For every integer i with $0 \le i \le t$ and for every multi-index I there exists an integer j, $0 \le j \le t$, and a multi-index J such that $\lambda_i \circ \varphi_I = \varphi_J \circ \lambda_j$ (let $\varphi_J(K)$ be the smallest elementary piece which contains $(\lambda_i \circ \varphi_I)(K)$).

We define the *congruency* equivalence relation among similitudes: for similitudes g_1 and g_2 let $g_1 \approx g_2$ denote that $g_1 \circ g_2^{-1}$ is an isometry; that is, for every set H the sets $g_1(H)$ and $g_2(H)$ are congruent. This is the same as that the similarity ratio of g_1 and g_2 are equal. Hence congruency is independent of the order of the composition, so $g_1 \circ g_2 \approx g_3 \iff g_2 \circ g_1 \approx g_3$. Using the equalities $\lambda_i \varphi_I = \varphi_J \lambda_j$, $\lambda_i^{k_i} b_i = c_i$ and $\lambda_j^{k_j} b_j = c_j$ we obtain

$$\underbrace{\lambda_i^{k_i k_j}\, \varphi_I^{k_i k_j}}_{\approx \varphi_J^{k_i k_j} \lambda_j^{k_i k_j}}\, b_i^{k_j}\, b_j^{k_i} \;\approx\; \varphi_J^{k_i k_j}\, b_i^{k_j}\, \underbrace{\lambda_j^{k_i k_j}\, b_j^{k_i}}_{\approx c_j^{k_i}} \;\approx\; \varphi_J^{k_i k_j}\, b_i^{k_j}\, c_j^{k_i},$$

$$\lambda_i^{k_i k_j}\, \varphi_I^{k_i k_j}\, b_i^{k_j}\, b_j^{k_i} \;\approx\; \underbrace{\lambda_i^{k_i k_j}\, b_i^{k_j}}_{\approx c_i^{k_j}}\, \varphi_I^{k_i k_j}\, b_j^{k_i} \;\approx\; c_i^{k_j}\, \varphi_I^{k_i k_j}\, b_j^{k_i}.$$

Comparing these we get

$$\varphi_J^{k_i k_j}\, b_i^{k_j}\, c_j^{k_i} \;\approx\; c_i^{k_j}\, \varphi_I^{k_i k_j}\, b_j^{k_i}.$$

Since all the similitudes b_i, b_j, c_i, c_j are some composition of similitudes of the presentation, the elementary pieces $\big(\varphi_J^{k_i k_j}\, b_i^{k_j}\, c_j^{k_i}\big)(K)$ and $\big(c_i^{k_j}\, \varphi_I^{k_i k_j}\, b_j^{k_i}\big)(K)$ are congruent, so they are of equal measure. The measure is self-similar, thus

$$\mu(\varphi_J)^{k_i k_j} \mu(b_i)^{k_j} \mu(c_j)^{k_i} = \mu(c_i)^{k_j} \mu(\varphi_I)^{k_i k_j} \mu(b_j)^{k_i},$$

hence by the definition of μ^* we get

$$\mu(\varphi_J)^{k_i k_j}\mu^*(\lambda_j)^{k_i k_j} = \mu^*(\lambda_i)^{k_i k_j}\mu(\varphi_I)^{k_i k_j},$$
$$\mu^*(\lambda_j)\mu(\varphi_J) = \mu^*(\lambda_i)\mu(\varphi_I).$$

Therefore

$$\mu(\lambda_i\varphi_I) = \mu(\varphi_J\lambda_j) = \mu(\varphi_J)\mu(\lambda_j) = \frac{\mu^*(\lambda_i)\mu\big(\varphi_I\big)}{\mu^*\big(\lambda_j\big)}\mu\big(\lambda_j\big).$$

Altering this we got the following: for every i and I there exists j such that

$$\mu(\lambda_i\varphi_I) = \mu^*(\lambda_i)\frac{\mu(\lambda_j)}{\mu^*(\lambda_j)}\mu(\varphi_I).$$

Note that $\mu^*(\lambda_j) \neq 0$.

Let m be an index for which

$$\frac{\mu(\lambda_m)}{\mu^*(\lambda_m)} \dot{\leq} \frac{\mu(\lambda_i)}{\mu^*(\lambda_i)}$$

for every index $0 \leq i \leq t$. We label some inequalities by a dot so we can refer to them later. Then for any φ_I,

$$\mu(\lambda_m\varphi_I) = \mu^*(\lambda_m)\frac{\mu(\lambda_j)}{\mu^*(\lambda_j)}\mu(\varphi_I) \dot{\geq} \mu^*(\lambda_m)\frac{\mu(\lambda_m)}{\mu^*(\lambda_m)}\mu(\varphi_I) = \mu(\lambda_m)\mu(\varphi_I)$$

for some index j with $0 \leq j \leq t$.

Let $\{\varphi_{I_i}(K)\}$ be a finite partition of K with elementary pieces such that the partition includes $\varphi_I(K)$. Then

$$\begin{aligned}\mu\big(\lambda_m(K)\big) = \mu\Big(\lambda_m\Big(\bigcup\nolimits^* \varphi_{I_i}(K)\Big)\Big) = \mu\Big(\bigcup\nolimits^* \lambda_m(\varphi_{I_i}(K))\Big) &= \sum \mu(\lambda_m\varphi_{I_i})\\ &\dot{\geq} \sum \mu(\lambda_m)\mu(\varphi_{I_i}) = \mu(\lambda_m),\end{aligned}$$

hence equality holds everywhere, so $\mu(\lambda_m\varphi_I) = \mu(\lambda_m)\mu(\varphi_I)$ for every multi-index I.

Let $H \subset K$ be a Borel set. By the definition of the measure μ, there exist elementary pieces $a_{ij}(K)$ for which $H \subset \bigcap_j \bigcup_i^* a_{ij}(K)$ and $\mu(H) = \inf_j \mu\big(\bigcup_i^* a_{ij}(K)\big) = \mu\big(\bigcap_j \bigcup_i^* a_{ij}(K)\big)$ hold. Then

$$\begin{aligned}\mu(\lambda_m(H)) &\leq \mu\Big(\lambda_m\Big(\bigcap_j \bigcup_i{}^* a_{ij}(K)\Big)\Big) = \mu\Big(\bigcap_j \bigcup_i{}^* \lambda_m(a_{ij}(K))\Big)\\ &\leq \inf_j \mu\Big(\bigcup_i{}^* \lambda_m(a_{ij}(K))\Big) = \inf_j \sum_i \mu(\lambda_m a_{ij}) = \inf_j \sum_i \mu(\lambda_m)\mu(a_{ij})\\ &= \mu(\lambda_m)\inf_j \sum_i \mu(a_{ij}) = \mu(\lambda_m)\mu\Big(\bigcap_j \bigcup_i{}^* a_{ij}(K)\Big) = \mu(\lambda_m)\mu(H).\end{aligned}$$

Repeating this argument for $H^c \stackrel{\text{def}}{=} K \setminus H$ we obtain $\mu\big(\lambda_m(H^c)\big) \leq \mu(\lambda_m)\mu(H^c)$. Summing these we get $\mu\big(\lambda_m(H))\big) + \mu\big(\lambda_m(H^c)\big) \leq \mu(\lambda_m)\mu(H) + \mu(\lambda_m)\mu(H^c)$, in fact this is an equality, so we have $\mu\big(\lambda_m(H)\big) = \mu(\lambda_m)\mu(H)$. Thus $\mu \circ \lambda_m = \mu(\lambda_m) \cdot \mu$.

From this we obtain that for any Borel set $H \subset K$,

$$\mu\big(\lambda_m^n(H)\big) = \mu\big(\lambda_m(\lambda_m^{n-1}(H))\big) = \mu(\lambda_m)\mu\big(\lambda_m^{n-1}(H)\big),$$

and by induction we get that $\mu\big(\lambda_m^n(H)\big) = \mu(\lambda_m)^n\mu(H)$, hence $\mu(\lambda_m^n) = \mu(\lambda_m)^n$.

Therefore $\mu(\lambda_m^{k_m}b_m) = \mu(\lambda_m)^{k_m}\mu(b_m)$ holds. From the definition of $\mu^*(\lambda_m)$ we have $\mu(c_m) = \mu^*(\lambda_m)^{k_m}\mu(b_m)$ and $c_m = \lambda_m^{k_m}b_m$, thus

$$\mu(\lambda_m)^{k_m}\mu(b_m) = \mu(\lambda_m^{k_m}b_m) = \mu(c_m) = \mu^*(\lambda_m)^{k_m}\mu(b_m).$$

Since $\mu(b_m) > 0$, we get $\mu(\lambda_m) = \mu^*(\lambda_m)$. Since m was chosen to be that index i for which $\frac{\mu(\lambda_i)}{\mu^*(\lambda_i)}$ is minimal, we get that $\mu^*(\lambda_i) \mathrel{\dot{\le}} \mu(\lambda_i)$ for every $0 \le i \le t$.

Now we can repeat the whole argument for such an index m for which $\frac{\mu(\lambda_m)}{\mu^*(\lambda_m)} \ge \frac{\mu(\lambda_i)}{\mu^*(\lambda_i)}$ holds for every index i $(0 \le i \le t)$. We just have to reverse the inequalities labelled with a dot, and we obtain that for every index i $(0 \le i \le t)$, $\mu^*(\lambda_i) \ge \mu(\lambda_i)$ holds. Thus for every i $(0 \le i \le t)$ we have $\mu^*(\lambda_i) = \mu(\lambda_i)$. Therefore we could choose any i $(0 \le i \le t)$ as m, so for every i the equality $\mu \circ \lambda_i = \mu(\lambda_i) \cdot \mu$ holds. By the observation we made at the beginning of the proof we get that for every similitude λ with $\lambda(K) \subset K$, $\mu \circ \lambda = \mu(\lambda) \cdot \mu$ holds, thus $\mu \circ \lambda^n = \mu(\lambda)^n \cdot \mu$ holds as well for any positive integer n.

Now we shall prove the second part of the proposition. Suppose that the isometry g maps the elementary piece $\varphi_L(K)$ into K, so $g(\varphi_L(K)) \subset K$. By Theorem 4.9 there exist multi-indices I, J and a positive integer k such that $(g \circ \varphi_L)^k \circ \varphi_I = \varphi_J$. Using the first part of this proposition (which is already proven) we get

$$\mu(\varphi_J) = \mu\big((g \circ \varphi_L)^k \circ \varphi_I\big) = \mu(g \circ \varphi_L)^k\mu(\varphi_I). \tag{14}$$

Clearly $\varphi_J = (g \circ \varphi_L)^k \circ \varphi_I \approx (\varphi_L)^k\varphi_I$, thus

$$\mu(\varphi_J) = \mu\big((\varphi_L)^k\varphi_I\big) = \mu(\varphi_L)^k\mu(\varphi_I). \tag{15}$$

By (14) and (15) we obtain

$$\mu(g \circ \varphi_L)^k\mu(\varphi_I) = \mu(\varphi_L)^k\mu(\varphi_I),$$

$$\mu(g \circ \varphi_L) = \mu(\varphi_L),$$

which proves the proposition. □

THEOREM 5.3 (CHARACTERIZATION OF ISOMETRY INVARIANT MEASURES)
Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with the strong separation condition and μ a self-similar measure on K for which congruent elementary pieces are of equal measure. Then μ is an isometry invariant measure on K.

Proof. We have to show that for any isometry g and Borel set $H \subset K$ if $g(H) \subset K$ then $\mu(H) = \mu\big(g(H)\big)$.

Let $c < 1$ be the constant given by Theorem 4.1. At first consider a set $H \subset K$ of positive measure. Applying Lemma 2.12 for the set H with $\varepsilon = 1 - c$ we obtain that there exists an elementary piece $a(K)$ for which $\mu\big(H \cap a(K)\big) > c \cdot \mu\big(a(K)\big)$. Since $H \subset g^{-1}(K)$, we have $\mu\big(g^{-1}(K) \cap a(K)\big) > c \cdot \mu\big(a(K)\big)$, so applying Theorem 4.1 $a(K) \subset g^{-1}(K)$, $g(a(K)) \subset K$. Put $\lambda = g \circ a$. According to the second part of Proposition 5.2 we have $\mu(\lambda) = \mu(a)$ (where $\mu(\lambda)$ is an abbreviation of $\mu\big(\lambda(K)\big)$), and putting $H_0 \stackrel{\text{def}}{=} a^{-1}(a(K) \cap H)$ we have $\mu\big(\lambda(H_0)\big) = \mu(\lambda)\mu(H_0)$, thus

$$\begin{aligned} 0 < c \cdot \mu\big(a(K)\big) < \mu\big(a(K) \cap H\big) = \mu\big(a(H_0)\big) = \mu(a)\mu(H_0) = \mu(\lambda)\mu(H_0) \\ = \mu\big(\lambda(H_0)\big) = \mu\big(g(a(H_0))\big) = \mu\big(g(a(K) \cap H)\big) \le \mu\big(g(H)\big), \end{aligned}$$

so $g(H)$ is of positive measure. Thus a congruent copy of a set of positive measure is of positive measure, and a congruent copy of a negligible set is also negligible.

Now let $H \subset K$ be any Borel set, g an isometry, for which $g(H) \subset K$. Apply Lemma 2.13 with some $0 < \varepsilon < 1 - c$. We obtain elementary pieces $a_i(K)$ such that

$$\mu\big(H \cap a_i(K)\big) > (1 - \varepsilon) \cdot \mu\big(a_i(K)\big) \quad \text{and} \quad \mu\big(H \setminus \bigcup_i^* a_i(K)\big) = 0.$$

Then $H \subset g^{-1}(K)$, therefore $\mu\big(g^{-1}(K) \cap a_i(K)\big) > (1 - \varepsilon) \cdot \mu\big(a_i(K)\big)$. According to Theorem 4.1, $g^{-1}(K) \supset a_i(K)$, so $g(a_i(K)) \subset K$. By the second part of Proposition 5.2 we get $\mu\big(g(a_i(K))\big) = \mu\big(a_i(K)\big)$, and using the fact that a congruent copy of a set of zero measure is also of zero measure,

$$\begin{aligned} \mu\big(g(H)\big) &= \mu\Big(g\Big(H \cap \bigcup_i^* a_i(K)\Big)\Big) + \mu\Big(g\Big(H \setminus \bigcup_i^* a_i(K)\Big)\Big) = \mu\Big(g\Big(H \cap \bigcup_i^* a_i(K)\Big)\Big) \\ &= \sum_i \mu\big(g(H \cap a_i(K))\big) \le \sum_i \mu\big(g(a_i(K))\big) = \sum_i \mu\big(a_i(K)\big) \\ &\le \frac{1}{1-\varepsilon} \cdot \sum_i \mu\big(H \cap a_i(K)\big) = \frac{1}{1-\varepsilon} \cdot \mu\Big(H \cap \bigcup_i^* a_i(K)\Big) = \frac{1}{1-\varepsilon} \cdot \mu(H). \end{aligned}$$

This is true for any $0 < \varepsilon < 1 - c$, hence $\mu\big(g(H)\big) \le \mu\big(H\big)$. Repeating this argument for $g(H)$ instead of H and for g^{-1} instead of g gives $\mu\big(H\big) \le \mu\big(g(H)\big)$, hence $\mu\big(H\big) = \mu\big(g(H)\big)$. Thus μ is isometry invariant. □

Remark 5.4. Using this theorem it is relatively easy to decide whether a self-similar measure is isometry invariant or not. Denote the similarity ratio of the similitude φ_i by α_i. It is clear that two elementary pieces are congruent if and only if they are image of K by similitudes of equal similarity ratio. Thus a self-similar measure μ is isometry invariant if and only if provided that $\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_n} = \alpha_{j_1}\alpha_{j_2}\ldots\alpha_{j_m}$ holds, the equality $p_{i_1}p_{i_2}\ldots p_{i_n} = p_{j_1}p_{j_2}\ldots p_{j_m}$ also holds for the weights of the measure μ. By switching from the similarity ratios α_i and weights p_i to the negative of their logarithm we get a system of linear equations for the variables $-\log p_i$. The solutions of this system (which also satisfy the normalizing equation $\sum_i p_i = 1$) give those weight vectors which define isometry invariant measures on K.

For example, it is easy to see that if the positive numbers $-\log \alpha_i$ $(i = 1, \ldots, r)$ are linearly independent over $\mathbb{Q}$, then every self-similar measure is isometry invariant.

So, to the r dimensional vectors, formed by the $-\log p_i$ weights of the isometry invariant measures, correspond the intersection of a linear subspace of $\mathbb{R}^r$ and the hypersurface corresponding to $\sum_i p_i = 1$. That this subspace is of dimension at least 1 and intersects the positive part of the space $\mathbb{R}^r$, we know from the existence of Hausdorff measure. (Or rather from the fact that the weights $p_i = \alpha_i^s$ automatically satisfy all the equalities.)

The notion of a self-similar measure depended on the the choice of the presentation. However, the converse is true for the notion of an isometry invariant self-similar measure.

THEOREM 5.5. *Let K be self-similar with the strong separation condition and $\{\varphi_1, \varphi_2, \ldots, \varphi_r\}$ a presentation of it. Let μ be isometry invariant and self-similar with respect to this presentation. Then μ is self-similar with respect to any presentation of K. Thus the class of isometry invariant self-similar measures is independent of the choice of presentation.*

Proof. Let $\{\psi_1, \ldots, \psi_s\}$ be an other presentation of K. According to Theorem 4.9 there exist positive integer k and elementary pieces φ_I, φ_J such that $\psi_i^k \circ \varphi_I = \varphi_J$, so applying the first part of Proposition 5.2 we get

$$0 < \mu(\varphi_J) = \mu(\psi_i^k \circ \varphi_I) = \mu(\psi_i)^k \mu(\varphi_I),$$

that is, $\mu(\psi_i) > 0$ for every $1 \leq i \leq s$.

According to the first part of Proposition 5.2, $\mu \circ \psi_i = \mu(\psi_i) \cdot \mu$, and since $\sum \mu\big(\psi_i(K)\big) = 1$ holds, this means exactly that μ is a self-similar measure with respect to the presentation $\{\psi_1, \ldots, \psi_s\}$. □

DEFINITION 5.6. Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with strong separation condition. Put $S = \{-\log \alpha_i : 1 \leq i \leq r\}$, where α_i is the similarity ratio of φ_i. The *algebraic dependence number* (of this presentation) is the dimension over $\mathbb{Q}$ of the vectorspace generated by S minus one.

By Remark 5.4 it is easy to see that the algebraic dependence number of a presentation is exactly the same as the topological dimension of the surface corresponding to the isometry invariant self-similar measures on K. Thus, by Theorem 5.5, one can prove the following.

THEOREM 5.7. *The algebraic dependence number of a self-similar set does not depend on the presentation we choose.*

We mention that it is easy to show that the algebraic dependence number is the same for two presentations $\mathcal{F}_1$ and $\mathcal{F}_2$ if $\mathcal{F}_1 \leq \mathcal{F}_2$; that is, when one of them extends the other in the trivial way we defined at the beginning of this section. However, there are self-similar sets with two presentations which have no common extension and they are not an extension of the same third presentation (see Theorem 6.4). Thus we have no direct (or trivial) proof for Theorem 5.7.

An easy consequence of the characterization theorem is the following.

COROLLARY 5.8. *Let $K = \varphi_1(K) \cup^* \ldots \cup^* \varphi_r(K)$ be a self-similar set with strong separation condition, μ be a self-similar measure on K. Then if μ is invariant under orientation preserving isometries, then it is invariant under all isometries.*

Proof. According to Theorem 5.3 it is enough to show that congruent elementary pieces are of equal measure. Let $\varphi_I(K)$ and $\varphi_J(K)$ be congruent elementary pieces. Then $\varphi_I^2(K)$ and $\varphi_J^2(K)$ are also congruent elementary pieces, φ_I^2 and φ_J^2 are orientation preserving similitudes, so $\varphi_I^2 \circ \varphi_J^{-2}$ is an orientation preserving isometry, hence by the assumption $\mu\big(\varphi_I^2(K)\big) = \mu\big(\varphi_J^2(K)\big)$. Since μ is self-similar, $\mu\big(\varphi_I^2(K)\big) = \mu\big(\varphi_I(K)\big)^2$ and $\mu\big(\varphi_J^2(K)\big) = \mu\big(\varphi_J(K)\big)^2$, thus $\mu\big(\varphi_I(K)\big) = \mu\big(\varphi_J(K)\big)$. This proves the statement. □

6. *Minimal presentations*

At first we give an example for a self-similar set on the line (with strong separation condition) which has no smallest presentation, that is, it has more than one minimal presentations. Set $\varphi_1(x) = \frac{x}{3}$, $\varphi_2(x) = \frac{x}{3} + \frac{2}{3}$, let K be the compact set for which $K = \varphi_1(K) \cup \varphi_2(K)$, apparently this is the triadic Cantor set. Set $\psi_1(x) = -\frac{x}{3} + \frac{1}{3}$. Then $K = \psi_1(K) \cup^* \varphi_2(K)$ as well, and it is clear, that both of these two different presentations are minimal, since they consist of only two similitudes.

However, these two presentations are not "essentially different": the sets $\{\varphi_1(K), \varphi_2(K)\}$ and $\{\psi_1(K), \varphi_2(K)\}$ coincide. On *essential presentation* we shall mean not the set of the similitudes but rather the set of the first generation elementary pieces. We shall say that the essential presentation $\{a_1(K), \ldots, a_r(K)\}$ is *briefer* than the essential presentation $\{b_1(K), \ldots, b_s(K)\}$, if for every $j = 1, \ldots, s$ there exists $1 \le i \le r$ such that $b_j(K) \subset a_i(K)$. We call an essential presentation minimal if the only briefer essential presentation is itself, and we call it the smallest if it is briefer than any other essential presentation. It is easy to check that the triadic Cantor set possesses a smallest essential presentation.

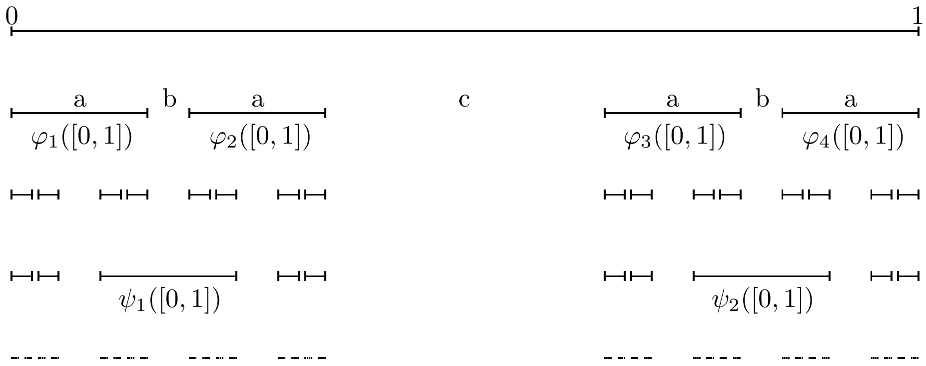
In the followings we shall present a self-similar set which has got no smallest essential presentation, that is, it has minimal essential presentations more than one.

REMARK 6.1. The following statement is true for many self-similar sets K: If λ_1 and λ_2 are similitudes for which $\lambda_1(K) \subset K$, $\lambda_2(K) \subset K$ and $\lambda_1(K) \cap \lambda_2(K) \ne \emptyset$, then $\lambda_1(K) \subset \lambda_2(K)$ or $\lambda_2(K) \subset \lambda_1(K)$. The proofs of Section 4 would have been much simpler if this statement has held for every self-similar set satisfying the strong separation condition. However this statement does not hold generally as we shall show in our following construction. We note that this statement is not necessarily equivalent to that K has only one minimal essential presentation. See also the end of Section 9 and especially Question 9.3.

Theorem 6.2. *There exists a self-similar set K with the strong separation condition which has no smallest essential presentation. Moreover, there exists similitudes λ_1 and λ_2 such that $\lambda_1(K) \cap \lambda_2(K) \neq \emptyset$, but $\lambda_1(K) \not\subset \lambda_2(K)$ and $\lambda_2(K) \not\subset \lambda_1(K)$.*

Proof. We present a figure of our construction. One may check the proof of this theorem just by looking at that figure.

Let a, b, c positive integers for which $a + b + a + c + a + b + a = 1$ and $b = a \cdot c$. It is easy to see that for every $0 < a < 1/4$ there exist a unique b and c with these conditions. Let φ_1 be the orientation preserving similitude mapping the interval $[0, 1]$ onto the interval $[0, a]$. Let φ_2 take the interval $[0, 1]$ onto $[a + b, a + b + a]$, φ_3 onto $[1 - a - b - a, 1 - a - b]$, and φ_4 onto $[1 - a, 1]$, all of them preserving the orientation. That is, $\varphi_1(x) = a \cdot x$, $\varphi_2(x) = a \cdot x + a + b$, $\varphi_3(x) = a \cdot x + 1 - a - b - a$, $\varphi_4(x) = a \cdot x + 1 - a$.



Let K be the unique compact set for which $K = \varphi_1(K) \cup^* \varphi_2(K) \cup^* \varphi_3(K) \cup^* \varphi_4(K)$. Thus the first generation elementary pieces of K are of diameter a, and there are ,,holes" between them of length b, c and b. It is clear that $K \subset [0, 1]$ and K is symmetric to $\frac{1}{2}$.

The second row of the figure symbolizes this presentation of K, more precisely it shows the intervals $\varphi_i([0, 1])$ (choosing $a = 0.15$, $c = \frac{0.4}{1.3}$). In the first row the interval $[0, 1]$ can be seen. The third row of the figure shows the intervals $\varphi_i(\varphi_j([0, 1]))$ $(1 \leq i, j \leq 4)$. The fifth row tries to present the set K.

Set $\psi_1(x) = a \cdot x + a^2 + a \cdot b + a^2 + a \cdot c$ and $\psi_2(x) = a \cdot x + 1 - a - b - a^2 - a \cdot b - a^2$. In the fourth row of the figure the images of the interval $[0, 1]$ by the similitudes φ_1^2, $\varphi_1 \circ \varphi_2$, ψ_1, $\varphi_2 \circ \varphi_3$, $\varphi_2 \circ \varphi_4$, $\varphi_3 \circ \varphi_1$, $\varphi_3 \circ \varphi_2$, ψ_2, $\varphi_4 \circ \varphi_3$, φ_4^2 are shown.

We claim that $\psi_1(K) \subset K$ and $\psi_2(K) \subset K$, moreover

$$\{\varphi_1^2, \varphi_1 \circ \varphi_2, \psi_1, \varphi_2 \circ \varphi_3, \varphi_2 \circ \varphi_4, \varphi_3 \circ \varphi_1, \varphi_3 \circ \varphi_2, \psi_2, \varphi_4 \circ \varphi_3, \varphi_4^2\}$$

is a presentation of K (see the fourth row of the figure). For this it is sufficient to prove that $\psi_1 \circ \varphi_1 = \varphi_1 \circ \varphi_3$, $\psi_1 \circ \varphi_2 = \varphi_1 \circ \varphi_4$, $\psi_1 \circ \varphi_3 = \varphi_2 \circ \varphi_1$, $\psi_1 \circ \varphi_4 = \varphi_2 \circ \varphi_2$, and $\psi_2 \circ \varphi_1 = \varphi_3 \circ \varphi_3$, $\psi_2 \circ \varphi_2 = \varphi_3 \circ \varphi_4$, $\psi_2 \circ \varphi_3 = \varphi_4 \circ \varphi_1$, $\psi_2 \circ \varphi_4 = \varphi_4 \circ \varphi_2$. These can be easily checked, all equalities rely on the choice of $b = a \cdot c$.

Now we prove that there does not exist an essential presentation $\{\varrho_1(K), \ldots, \varrho_r(K)\}$ of the self-similar set K which is briefer than both of the essential presentations corresponding to the original presentation $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ and the presentation just defined above. This would prove that K has no unique minimal essential presentation. (In fact both of these essential presentations are minimal.) Indirectly suppose that there exists an essential presentation $\{\varrho_1(K), \ldots, \varrho_r(K)\}$ of this kind. Since $\varphi_1(K) \cap \psi_1(K) \neq \emptyset$, for some i $\varphi_1(K) \cup \psi_1(K) \subset \varrho_i(K)$. For the same i we also have $\varphi_2(K) \cup \psi_1(K) \subset \varrho_i(K)$. Similarly there exists an index j such that $\varphi_3(K) \cup \psi_2(K) \cup \varphi_4(K) \subset \varrho_j(K)$. From this we conclude that $K = \varrho_i(K) \cup^* \varrho_j(K)$, but then the similitudes ϱ_i and ϱ_j could only be the ones mapping $[0,1]$ onto $[0, a+b+a]$ and $[1-a-b-a, 1]$. This yields to $b = (a+b+a) \cdot c$, which contradicts $b = a \cdot c$.

The similitudes λ_1 and λ_2 we promised can chosen to be φ_1 and ψ_1. □

Remark 6.3. This example (and many other results of the present article) is contained in the Master Thesis of the third author [**18**]. Independently, Feng and Wang in [**8**] exhibit an almost identical example. Moreover, much of their paper is devoted to the investigation of the structure of possible presentations of given self-similar sets; or, using their terminology, the structure of generating iterated function systems of self-similar sets. They also prove positive results (that is, when a smallest presentation does exist) under various assumptions.

Theorem 6.4. *There exists a self-similar set K with the strong separation condition and two (essential) presentations of K, $\mathcal{F}_1$ and $\mathcal{F}_2$, such that there is no presentation $\mathcal{G}$ which is a common extension of $\mathcal{F}_1$ and $\mathcal{F}_2$, nor there exists an (essential) presentation which is smaller (briefer) than $\mathcal{F}_1$ and $\mathcal{F}_2$.*

Thus, Theorem 5.7 cannot be proved in the trivial way (see our remarks after that theorem). We leave the proof of Theorem 6.4 to the reader, with the instructions that one should choose the self-similar set K constructed above, and the presentations of the second and fourth row of the figure should be chosen as $\mathcal{F}_1$ and $\mathcal{F}_2$.

7. *Intersection of translates of a self-affine Sierpiński sponge*

The following is the key lemma for all results of this section.

Proposition 7.1. *Let $K = K(M, D)$ and μ be like in Definition 2.14 and let $t \in \mathbb{R}^n$ be such that $\|M^k t\| > 0$ for every $k = 1, 2, \ldots$.*

Then $\mu\big(K \cap (K+t)\big) > 0$ implies that there exists a

$$w \in \{-1, 0, 1\} \times \ldots \times \{-1, 0, 1\} \setminus \{(0, \ldots, 0)\}$$

such that $D + w = D$ modulo $(m_1, \ldots, m_n)$; that is,

$$D + w + M(\mathbb{Z}^n) = D + M(\mathbb{Z}^n) = D - w + M(\mathbb{Z}^n).$$

Proof. To make the argument intuitive and precise we shall present the same proof in an informal and in a formal way separately.

The informal proof: According to Lemma 2.16 and Lemma 2.12 we can find a k such that $M^k t$ is not very close to any point of $\mathbb{Z}^n$, and a $k-1$-th generation elementary part S of K in which the density of $K+t$ is almost 1. Then in all the r k-th generation elementary parts of K that are in S the density of $K+t$ is still very close to 1.

Each of these subparts intersect some k-th generation elementary parts of $K+t$. The key observation is that there are at most 2^n possible ways how these parts can intersect each other.

Since $M^k t$ is not very close to the lattice points, these intersections are intersections of sets similar to K such that one is always a not very close translate of the other. Hence Lemma 2.23 implies that they cannot have big intersection.

Since the density of $K+t$ is very close to 1 in all k-th generation elementary parts of K that are in S, this implies that in the two directions for which the possible intersection has biggest measure, $K+t$ must have a k-th generation elementary part.

Hence we get two periods of the pattern D such that their difference w is in $\{-1,0,1\}\times\ldots\times\{-1,0,1\}$.

The formal proof: Applying Lemma 2.23 for $\varepsilon = 1/(2\max(m_1,\ldots,m_n))$ we get a $0<\delta<1$ such that

$$\mu\big(K\cap(K+u)\big)\le 1-\delta \quad \text{whenever } |u|\ge\frac{1}{2\max(m_1,\ldots,m_n)}. \tag{16}$$

Applying Lemma 2.12 for $B=(K+t)\cap K$ and $\varepsilon=\frac{\delta}{2^n r}$ and Lemma 2.16 we get a $k\in\mathbb{N}$ and a $k-1$-th generation elementary part S of K such that

$$\mu\big(S\cap(K+t)\big)>\frac{1-\frac{\delta}{2^n r}}{r^{k-1}} \tag{17}$$

and

$$\|M^k t\|>\frac{1}{2\max(m_1,\ldots,m_n)}. \tag{18}$$

Let Φ be the similarity map which maps S to $M(K)=K+D$; that is,

$$\begin{aligned}\Phi(x) &= M^k(x-(M^{-(k-1)}\alpha_{k-1}+\ldots+M^{-1}\alpha_1))\\ &= M^k x-(M\alpha_{k-1}+M^2\alpha_{k-2}+\ldots+M^{k-1}\alpha_1),\end{aligned}$$

where $S=M^{k-1}(K)+M^{-(k-1)}\alpha_{k-1}+\ldots+M^{-1}\alpha_1$.

Using that $\Phi(S)=K+D=\cup_{j=1}^r K+d_j$, applying (3) and (17) we get

$$\begin{aligned}\tilde\mu\Big(\bigcup_{j=1}^r(K+d_j)\cap(\Phi(K+t))\Big) &= \tilde\mu\big(\Phi(S\cap(K+t))\big)=r^k\mu\big(S\cap(K+t)\big)\\ &> r^k\frac{1-\frac{\delta}{2^n r}}{r^{k-1}}=r-\frac{\delta}{2^n}.\end{aligned}$$

Since $\tilde{\mu}(K+d_j)=1$ ($j=1,\dots,r$) and the sets can intersect each other only at a set of $\tilde{\mu}$-measure zero this implies that

$$\tilde{\mu}\big((K+d_j)\cap\Phi(K+t)\big)>1-\frac{\delta}{2^n}\quad\text{for every } j=1,\dots,r. \tag{19}$$

Since $\Phi(K)=M^k(K)-(M\alpha_{k-1}+\ldots+M^{k-1}\alpha_1)$ and $M^k(K)\subset K+D+M(\mathbb{Z}^n)$, we have $\Phi(K)\subset K+D+M(\mathbb{Z}^n)$, and so $\Phi(K+t)\subset K+D+\Phi(t)+M(\mathbb{Z}^n)$. Thus

$$\begin{aligned}(K+d_j)\cap\Phi(K+t)&\\ &\subset(K+d_j)\cap\big(K+D+\Phi(t)+M(\mathbb{Z}^n)\big)\\ &=\bigcup_{i=1}^{r}\big(K\cap(K+d_i+\Phi(t)-d_j+M(\mathbb{Z}^n))\big)+d_j.\end{aligned}$$

Combining this with (19) and (3) (for $l=0$) we get

$$\begin{aligned}1-\frac{\delta}{2^n}&<\tilde{\mu}\big((K+d_j)\cap\Phi(K+t)\big)\\ &\le\sum_{i=1}^{r}\tilde{\mu}\Big(\big(K\cap(K+d_i+\Phi(t)-d_j+M(\mathbb{Z}^n))\big)+d_j\Big)\\ &=\sum_{i=1}^{r}\mu\big(K\cap(K+d_i+\Phi(t)-d_j+M(\mathbb{Z}^n))\big)\qquad(j=1,\dots,r).\end{aligned}\tag{20}$$

Clearly, we have $\mu\big(K\cap(K+d_i+\Phi(t)-d_j+M(\mathbb{Z}^n))\big)=0$ whenever

$$d_i+\Phi(t)-d_j\notin(-1,1)\times\ldots\times(-1,1)+M(\mathbb{Z}^n).$$

Hence there are at most 2^n vectors $v\in\mathbb{Z}^n$ such that $v+\Phi(t)\in(-1,1)\times\ldots\times(-1,1)$; let these vectors be $v_1,v_2,\dots,v_p$, ($p\le 2^n$).

Thus, by omitting some zero terms on the right-hand side of (20) we can rewrite (20) as

$$1-\frac{\delta}{2^n}<\sum_{l\,:\,(\exists i)\ d_i-d_j\in v_l+M(\mathbb{Z}^n)}\mu\big(K\cap(K+v_l+\Phi(t))\big)\qquad(j=1,\dots,r). \tag{21}$$

Let

$$\beta_l=\mu\big(K\cap(K+v_l+\Phi(t))\big)\qquad(l=1,\dots,p).$$

By rearranging $v_1,\dots,v_p$ if necessary, we may assume that

$$\beta_1\ge\beta_2\ge\ldots\ge\beta_p. \tag{22}$$

Since $v_l\in\mathbb{Z}^n$ and $K\subset[0,1]^n$, the sets $K+v_l+\Phi(t)$ ($l=1,\dots,p$) are pairwise disjoint and clearly $K=\cup_{l=1}^{p}K\cap(K+v_l+\Phi(t))$, we get

$$1=\mu(K)=\sum_{l=1}^{p}\beta_l. \tag{23}$$

Since, using (18), $\|M^k t\| > \frac{1}{2\max(m_1,\ldots,m_n)}$, we have $|v_1+\Phi(t)| > \frac{1}{2\max(m_1,\ldots,m_n)}$. Thus, by (16),

$$\beta_1 = \mu\big(K \cap (K + v_1 + \Phi(t))\big) \le 1 - \delta. \tag{24}$$

Clearly (22), (23) and (24) implies that $\beta_1 \ge \beta_2 \ge \frac{\delta}{p-1} > \frac{\delta}{2^n}$ and so

$$\beta_1 + \beta_3 + \beta_4 + \ldots \beta_{2^n} < 1 - \frac{\delta}{2^n} \qquad \text{and}$$
$$\beta_2 + \beta_3 + \beta_4 + \ldots \beta_{2^n} < 1 - \frac{\delta}{2^n}.$$

Combining this with (21) we get that for every $j \in \{1,\ldots,r\}$ there must be an i_1 such that $d_{i_1} - d_j \in v_1 + M(\mathbb{Z}^n)$ and an i_2 such that $d_{i_2} - d_j \in v_2 + M(\mathbb{Z}^n)$. Since $D = \{d_1,\ldots,d_r\}$, this means that for every $d \in D$ we must have $d + v_1, d + v_2 \in D + M(\mathbb{Z}^n)$.

Therefore $D + M(\mathbb{Z}^n) \supset D + v_1$ and so $D + M(\mathbb{Z}^n) \supset D + M(\mathbb{Z}^n) + v_1$. Applying this $m_1 \cdot \ldots \cdot m_n$ many times we get

$$\begin{aligned} D + M(\mathbb{Z}^n) \supset D + M(\mathbb{Z}^n) + v_1 \supset D + M(\mathbb{Z}^n) + 2v_1 \supset \ldots \\ \ldots \supset D + M(\mathbb{Z}^n) + m_1 \cdot \ldots \cdot m_n v_1 = D + M(\mathbb{Z}^n). \end{aligned} \tag{25}$$

Therefore $D + M(\mathbb{Z}^n) = D + M(\mathbb{Z}^n) + v_1$ and similarly $D + M(\mathbb{Z}^n) = D + M(\mathbb{Z}^n) + v_2$. Thus $D + M(\mathbb{Z}^n) + v_1 - v_2 = D + M(\mathbb{Z}^n) = D + M(\mathbb{Z}^n) + v_2 - v_1$. Noting that, by definition, $w = v_1 - v_2 \in \{-1,0,1\} \times \ldots \times \{-1,0,1\} \setminus \{(0,\ldots,0)\}$, the proof is complete. □

In order to use Proposition 7.1 effectively we need a discrete lemma.

LEMMA 7.2. *Let M and D be like in Definition 2.14, $l \in \{1,2,\ldots,n\}$, $i \in \mathbb{N}$,*

$$D_i = M^{i-1}(D) + M^{i-2}(D) + \ldots + M(D) + D,$$

and suppose that

$$D_i + (\underbrace{1,\ldots,1}_{l},0,\ldots,0) + M^i(\mathbb{Z}^n) = D_i + M^i(\mathbb{Z}^n). \tag{26}$$

Then at least one of the following two statements hold.
(a) We have $m_1 = \ldots = m_l$ and $a_1 = \ldots = a_l$ for every $(a_1,\ldots,a_n) \in D$.
(b) For some $l' \in \{1,2,\ldots,l-1\}$ we have

$$D_{i-1} + (\underbrace{1,\ldots,1}_{l'},0,\ldots,0) + M^{i-1}(\mathbb{Z}^n) = D_{i-1} + M^{i-1}(\mathbb{Z}^n).$$

Proof. Let $w = (\underbrace{1,\ldots,1}_{l},0,\ldots,0)$. From (26) we get

$$D_i + kw + M^i(\mathbb{Z}^n) = D_i + M^i(\mathbb{Z}^n) \qquad (k \in \mathbb{Z}). \tag{27}$$

First suppose that $a_1 = \ldots = a_l$ does not hold for some $a = (a_1,\ldots,a_n) \in D$. Then we can suppose that $a_1 = \ldots = a_j < a_{j+1} \le \ldots \le a_l$ for some

$j \in \{1, \dots, l-1\}$. Let $b = (b_1, \dots, b_n) \in D_{i-1}$ be arbitrary. Then $Mb + a \in M(D_{i-1}) + D = D_i$. Thus applying (27) for $k = -(a_1+1)$ we get

$$Mb + a - (a_1+1)w \in D_i + M^i(\mathbb{Z}^n).$$

Rewriting both sides we get

$$\begin{aligned} &M((b_1-1, \dots, b_j-1, b_{j+1}, \dots, b_n)) \\ &\quad + (m_1-1, \dots, m_j-1, a_{j+1}-a_1-1, \dots, a_l-a_1-1, a_{l+1}, \dots, a_n) \\ &\qquad\qquad\qquad\qquad\qquad\qquad \in M(D_{i-1} + M^{i-1}(\mathbb{Z}^n)) + D. \end{aligned}$$

Since the second term of the left-hand side is in $\{0, 1 \dots, m_1-1\} \times \{0, 1, \dots, m_n-1\}$, we must have

$$(b_1-1, \dots, b_j-1, b_{j+1}, \dots, b_n) \in D_{i-1} + M^{i-1}(\mathbb{Z}^n).$$

Since $b = (b_1, \dots, b_n) \in D_{i-1}$ was arbitrary we get that

$$D_{i-1} - (\underbrace{1, \dots, 1}_{j}, 0, \dots, 0) \subset D_{i-1} + M^{i-1}(\mathbb{Z}^n),$$

which implies, similarly like in (7), that

$$D_{i-1} + (\underbrace{1, \dots, 1}_{j}, 0, \dots, 0) + M^{i-1}(\mathbb{Z}^n) = D_{i-1} + M^{i-1}(\mathbb{Z}^n).$$

Thus we proved that if $a_1 = \ldots = a_l$ does not hold for some $(a_1, \dots, a_l) \in D$ then the statement (b) must hold. Exactly the same way (but ordering so that $m_1 - a_1 \le \ldots \le m_n - a_n$ and applying (27) for $k = m_1 - a_1$ instead of $k = a_1$) we get that if $m_1 - a_1 = \ldots = m_l - a_l$ does not hold for some $(a_1, \dots, a_n) \in D$ then again the statement (b) must hold. Therefore the negation of (a) implies (b), which completes the proof of the Lemma. □

LEMMA 7.3. *Let $K = K(M, D)$ be a self-affine Sierpiński sponge in $\mathbb{R}^n$ and μ the natural probability measure on it as described in Definition 2.14, let $D_n = M^{n-1}(D) + M^{n-2}(D) + \ldots + M(D) + D$ and suppose that there exists a $w_n \in \{-1, 0, 1\} \times \ldots \times \{-1, 0, 1\} \setminus \{(0, \dots, 0)\}$ such that*

$$D_n + w_n + M^n(\mathbb{Z}^n) = D_n + M^n(\mathbb{Z}^n).$$

Then K is of the form $K = L \times K_0$, where L is a diagonal of a cube $[0,1]^l$, where $l \in \{1, 2, \dots, n\}$ and K_0 is a smaller dimensional self-affine Sierpiński sponge.

Proof. Since every condition is invariant under any autoisometry of the cube $[0,1]^n$ and by such a transformation we can map w_n to a vector of the form $(1, \dots, 1, 0, \dots, 0)$ we can suppose that

$$w_n = (\underbrace{1, \dots, 1}_{l_n}, 0, \dots, 0), \qquad \text{where } l_n \in \{1, 2, \dots, n\}.$$

Now we can apply Lemma 7.2 for $i = n$, $l = l_n$. If statement (b) of Lemma 7.2 holds then let $l_{n-1} = l'$ and apply the lemma again for $i = n-1$, $l = l_{n-1}$. If (b) holds again then we continue. Since $n \geq l_n > l_{n-1} > l_{n-2} > \ldots \geq 1$ we cannot repeat this for more than $n-1$ times, hence for some $1 \leq i \leq n$ (a) of Lemma 7.2 must hold when we apply the lemma for $i, l = l_i$. This way we get $i, l \in \{1, \ldots, n\}$ such that (26) and (a) of Lemma 7.2 hold.

It is easy to see that (26) implies that

$$D + (\underbrace{1, \ldots, 1}_{l}, 0, \ldots, 0) + M(Z^n) = D + M(Z^n)$$

and also that this and (a) of Lemma 7.2 implies that D must be of the form

$$D = \{(\underbrace{a, \ldots, a}_{l}) \ : \ a \in \{0, 1, \ldots, m_1 - 1\}\} \times D',$$

where $D' \subset \{0, 1, \ldots, m_{l+1} - 1\} \times \ldots \times \{0, 1, \ldots, m_n - 1\}$ and $m_1 = \ldots = m_l$. Then $K = K(M, D)$ must be exactly of the claimed form, which completes the proof. $\square$

Now we are ready to characterize those self-affine sponges for which $\mu(K \cap (K + t))$ can be positive for "irregular" translations.

THEOREM 7.4. *Let $K = K(M, D)$ be a self-affine Sierpiński sponge in $\mathbb{R}^n$ and μ the natural probability measure on it as described in Definition 2.14 and let $t \in \mathbb{R}^n$.*

Then $\mu\big(K \cap (K + t)\big) = 0$ holds except in the following two trivial exceptional cases:

(i) There exists two elementary parts S_1 and S_2 of K such that $S_2 = S_1 + t$.

(ii) K is of the form $K = L \times K_0$, where L is a diagonal of a cube $[0, 1]^l$, where $l \in \{1, 2, \ldots, n\}$ and K_0 is a smaller dimensional self-affine Sierpiński sponge.

Proof. If $\|M^k t\| = 0$ for some $k \in \{0, 1, 2, \ldots\}$ then for any two k-th generation elementary parts S_1 and S_2 of K, S_2 and $S_1 + t$ are either identical or $\mu\big((S_1 + t) \cap S_2\big) = 0$. Therefore in this case either (i) or $\mu\big(K \cap (K + t)\big) = 0$ holds, thus we can suppose that $\|M^k t\| > 0$ for every $k = 0, 1, 2, \ldots$ and $\mu\big(K \cap (K + t)\big) > 0$.

Let $D_i = M^{i-1}(D) + M^{i-2}(D) + \ldots + M(D) + D$. Notice that, by definition, $K(M, D) = K(M^i, D_i)$ for any $i \in \mathbb{N}$. Therefore we can apply Proposition 7.1 to (M^n, D_n) to obtain $w \in \{-1, 0, 1\}^n \setminus \{(0, \ldots, 0)\}$ such that

$$D_n + w_n + M^n(\mathbb{Z}^n) = D_n + M^n(\mathbb{Z}^n).$$

Then we can apply Lemma 7.3 to get that $K = K(M, D)$ must be exactly of the form as in (ii) of Theorem 7.4, which completes the proof. $\square$

REMARK 7.5. Clearly, case (i) holds if and only if t is of the form $\sum_{j=1}^{k} M^{-j}(\alpha_j - \beta_j)$, where $k \in \{0, 1, 2, \ldots\}$ and $\alpha_1, \beta_1, \ldots, \alpha_k, \beta_k \in D$.

REMARK 7.6. It follows from the proof that in the coordinates of L every m_i must be the same hence in case (ii) we must have $l = 1$ if $m_1, \ldots, m_n$ are all distinct.

In particular, if $n = 1$ then (ii) means $K = [0, 1]$.

The following statement is the analogue of Theorem 4.5.

COROLLARY 7.7. *Let $K \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) be a self-affine Sierpiński sponge and μ the natural probability measure on it (as described in Definition 2.14) and let $t \in \mathbb{R}^n$.*

The set $K \cap (K+t)$ has positive μ-measure if and only if it has non-empty interior (relative) in K.

Proof. If $K \cap (K+t)$ has non-empty interior in K then clearly $\mu\big(K \cap (K+t)\big) > 0$.

We shall prove the converse by induction. Assume that the converse is true for any smaller dimensional self-affine Sierpiński sponge. Suppose that $\mu\big(K \cap (K+t)\big) > 0$ and apply Theorem 7.4. If (i) of Theorem 7.4 holds then clearly $K \cap (K+t)$ has non-empty interior in K, so we can suppose that (ii) holds: $K = L \times K_0$, L is a diagonal of $[0,1]^l$ and K_0 is a smaller dimensional self-affine Sierpiński sponge. Then $\mu = c\lambda \times \mu_0$, where $1/c$ is the length of L (that is, $c = 1/\sqrt{l}$), λ is the (one-dimensional) Lebesgue measure on L and μ_0 is the natural probability measure on K_0.

Let $t_\alpha = (t_1, \ldots, t_l)$ and $t_\beta = (t_{l+1}, \ldots, t_n)$ and we suppose that the coordinates of L are the first l coordinates. Then

$$K \cap (K+t) = (L \times K_0) \cap ((L+t_\alpha) \times (K_0+t_\beta)) = (L \cap (L+t_\alpha)) \times (K_0 \cap (K_0+t_\beta)).$$

Therefore we have

$$0 < \mu\big(K \cap (K+T)\big) = c\lambda\big(L \cap (L+t_\alpha)\big) \cdot \mu_0\big(K_0 \cap (K_0+t_\beta)\big)$$

and so $\lambda\big(L \cap (L+t_\alpha)\big) > 0$ and $\mu_0\big(K_0 \cap (K_0+t_\beta)\big) > 0$. This implies that $L \cap (L+t_\alpha)$ has non-empty interior in L and, by our assumption, $K_0 \cap (K_0+t_\beta)$ has non-empty interior in K_0. Thus $K \cap (K+t) = (L \cap (L+t_\alpha)) \times (K_0 \cap (K_0+t_\beta))$ has non-empty interior in $K = L \times K_0$. □

For getting the analogue of Theorem 4.1 we need one more lemma.

PROPOSITION 7.8. *Let $K = K(M,D)$ and μ be like in Definition 2.14, and let $0 \neq t \in \mathbb{R}^n$ be such that $\mu\big(K \cap (K+t)\big) > 1 - \frac{1}{r^2}$.*

Then there exists a

$$w \in \{-1,0,1\} \times \ldots \times \{-1,0,1\} \setminus \{(0,\ldots,0)\}$$

such that $D + w = D$ modulo $(m_1, \ldots, m_n)$; that is,

$$D + w + M(\mathbb{Z}^n) = D + M(\mathbb{Z}^n)$$

Proof. By Proposition 7.1 we are done if $||M^k t|| > 0$ for every $k = 1, 2, \ldots$. Thus we can suppose that this is not the case and choose a minimal $k \in \{1, 2, \ldots\}$ such that $||M^k t|| = 0$. Then, letting $u = M^k t$, we have $u \in \mathbb{Z}^n \setminus M(\mathbb{Z}^n)$.

Let

$$D_k = M^{k-1}(D) + M^{k-2}(D) + \ldots + M(D) + D,$$

and define the measure μ_k so that $\mu_k(M^k A) = r^k \mu(A)$ for any Borel set $A \subset K$. Then by definition we have $M^k K = K + D_k$, and for each $d \in D_k$ we have $\mu_k(K+d)=1$. Using the above facts and definitions and the condition $\mu\big(K\cap(K+t)\big)>1-\frac{1}{r^2}$, we get

$$r^{k-2}(r^2-1)=r^k\Big(1-\frac{1}{r^2}\Big)<r^k\mu\big(K\cap(K+t)\big)=\mu_k\big(M^kK\cap(M^kK+M^kt)\big)$$
$$=\mu_k\big((K+D_k)\cap(K+D_k+u)\big)=\#(D_k\cap(D_k+u)),$$

where $\#(.)$ denotes the number of the elements of a set.

Then by the pigeonhole principle there exists an $e\in M^{k-1}(D)+M^{k-2}(D)+\cdots+M^2(D)\subset M^2(\mathbb{Z}^n)$ such that $e+M(D)+D\subset D_k+u$. This implies that $M(D)+D+M^2(\mathbb{Z}^n)\subset D_k+u+M^2(\mathbb{Z}^n)=M(D)+D+u+M^2(\mathbb{Z}^n)$. Similarly, we can prove that $M(D)+D+u+M^2(\mathbb{Z}^n)\subset M(D)+D+M^2(\mathbb{Z}^n)$. Therefore we get

$$M(D)+D+u+M^2(\mathbb{Z}^n)=M(D)+D+M^2(\mathbb{Z}^n). \tag{28}$$

In particular, we have $D+u+M(\mathbb{Z}^n)=D+M(\mathbb{Z}^n)$.

Then, starting from arbitrary $f_0\in D$ we can get a sequence $(f_i)\subset D$ so that

$$f_i+u+M(\mathbb{Z}^n)=f_{i+1}+M(\mathbb{Z}^n)\qquad(i=0,1,2,\ldots). \tag{29}$$

Since $u\notin M(\mathbb{Z}^n)$ we have $f_i\neq f_{i+1}$ for each i. This and the fact that the sequence (f_i) is contained in a finite set imply that there must be a $j\in\mathbb{N}$ such that $f_{j+1}-f_j\neq f_j-f_{j-1}$.

Let $e\in D$ be arbitrary. Applying (28) and (29) we get that there exist $e',e''\in D$ such that

$$Me'+f_{j-1}+u+M^2(\mathbb{Z}^n)=Me+f_j+M^2(\mathbb{Z}^n)$$

and

$$Me'+f_j+u+M^2(\mathbb{Z}^n)=Me''+f_{j+1}+M^2(\mathbb{Z}^n),$$

which implies

$$(f_j-f_{j-1})-(f_{j+1}-f_j)=M(e''-e)+M^2(\mathbb{Z}^n).$$

Thus there exists a $w\in\mathbb{Z}^n$ such that

$$Mw=(f_j-f_{j-1})-(f_{j+1}-f_j)=M(e''-e)+M^2(\mathbb{Z}^n). \tag{30}$$

Since $e,e'',f_{j-1},f_j,f_{j+1}\in D\subset\{0,1,\ldots,m_1-1\}\times\ldots\times\{0,1,\ldots,m_n-1\}$, (30) implies that

$$e+w+M(\mathbb{Z}^n)=e''+M(\mathbb{Z}^n)$$

and

$$w\in\{-1,0,1\}\times\ldots\times\{-1,0,1\}\setminus\{(0,\ldots,0)\}.$$

Since $e\in D$ was arbitrary, $e''\in D$ and w does not depend on e we get that

$$D+w+M(\mathbb{Z}^n)=D+M(\mathbb{Z}^n),$$

which completes the proof. □

Theorem 7.9. *Let $K = K(M, D)$ be a self-affine Sierpiński sponge in $\mathbb{R}^n$ and μ the natural probability measure on it as described in Definition 2.14 and let $t \in \mathbb{R}^n$.*

Then $\mu\big(K \cap (K + t)\big) \le 1 - \frac{1}{r^2}$ holds (where r denotes the number of elements in the pattern D) except in the following two trivial exceptional cases:
(i) $t = 0$.
(ii) K is of the form $K = L \times K_0$, where L is a diagonal of a cube $[0, 1]^l$, where $l \in \{1, 2, \dots, n\}$ and K_0 is a smaller dimensional self-affine Sierpiński sponge.

Proof. Suppose that $t \neq 0$ and $\mu(K \cap (K + t)) > 1 - \frac{1}{r^2}$. For $D_n = M^{n-1}(D) + M^{n-2}(D) + \ldots + M(D) + D$, by definition, $K(M, D) = K(M^n, D_n)$. Therefore we can apply Proposition 7.8 to (M^n, D_n) to obtain $w_n \in \{-1, 0, 1\}^n \setminus \{(0, \dots, 0)\}$ such that

$$D_n + w_n + M^n(\mathbb{Z}^n) = D_n + M^n(\mathbb{Z}^n).$$

Then Lemma 7.3 completes the proof. □

8. *Translation invariant measures for self-affine Sierpiński sponges*

As an easy application of Theorem 7.4 (and Lemma 2.18) we get the following.

Theorem 8.1. *For any self-affine Sierpiński sponge $K \subset \mathbb{R}^n$ ($n \in \mathbb{N}$) there exists a translation invariant Borel measure ν on $\mathbb{R}^n$ such that $\nu(K) = 1$.*

Proof. Let μ be the natural probability Borel measure on K (see Definition 2.14). We shall prove by induction that μ can be extended to $\mathbb{R}^n$ as a translation invariant Borel measure. Assume that this is true for any smaller dimensional self-affine Sierpiński sponge.

First suppose that K is of the form $K = K_1 \times K_2$, where K_1 and K_2 are smaller dimensional self-affine Sierpiński sponges. Then $\mu = \mu_1 \times \mu_2$, where μ_1 and μ_2 are the natural probability Borel measures on K_1 and K_2, respectively. Then, by our assumption, μ_1 and μ_2 has translation invariant extensions $\tilde{\mu}_1$ and $\tilde{\mu}_2$ and then one can easily check that $\tilde{\mu} = \tilde{\mu}_1 \times \tilde{\mu}_2$ is a translation invariant Borel measure on $\mathbb{R}^n$ and an extension of μ.

If K is not of the form $K = K_1 \times K_2$ then we shall check that condition (6) of Lemma 2.18 is satisfied, so then Lemma 2.18 will complete the proof. Fix $B \subset K$ and $t \in \mathbb{R}^n$ such that $B + t \subset K$. Then $B \subset K \cap (K - t)$ and $B + t \subset K \cap (K + t)$, so we have $\mu(B) = 0 = \mu(B + t)$ unless

$$\mu\big(K \cap (K + t)\big) > 0 \quad \text{or} \quad \mu\big(K \cap (K - t)\big) > 0 \tag{31}$$

By Theorem 7.4 and since case (ii) of Theorem 7.4 is already excluded, (31) implies (i) of Theorem 7.4. On the other hand, if (i) of Theorem 7.4 holds then the translation by t maps elementary parts of B to elementary parts of $B + t$ and then the condition (6) clearly holds.

Since we checked all cases, the proof is complete. □

We also show a more direct proof for the above theorem, which does not use Theorem 7.4 and which works for a slightly larger class of self-affine sets.

THEOREM 8.2. *Let φ be a contractive affine map, $t_1, \ldots, t_r \in \mathbb{R}^n$ and $K \subset \mathbb{R}^n$ the compact self-affine set such that $K = \cup_{i=1}^r \varphi(K) + t_i$. Suppose that the standard natural probability measure on K has the property that*

$$\mu\Big(K \cap \Big(\big((\varphi(K)+t_i) \cap (\varphi(K)+t_j)\big) + u\Big)\Big) = 0 \quad (\forall\ 1 \le i < j \le r,\ u \in \mathbb{R}^n). \tag{32}$$

(a) Then for any $t \in \mathbb{R}^n$ and elementary part S of K we have

$$\mu\big(K \cap (S+t)\big) \le \mu(S).$$

(b) There exists a translation invariant Borel measure ν on $\mathbb{R}^n$ such that $\nu(K) = 1$. In fact, ν is an extension of μ.

Proof. First we prove (a). Suppose that S is a k-th generation elementary part of K. Then K can be written as

$$K = \cup_{j=1}^{r^k} S + h_j$$

for some $h_1, \ldots, h_{r^k} \in \mathbb{R}^n$ and by (32) the sets $S + h_j$ are pairwise almost disjoint.

Using this and that $\mu(A) = \mu(A + h_j)$ for any Borel set $A \subset S$ we get that

$$\begin{aligned}
\mu\big(K \cap (S+t)\big) &= \mu\Bigg(\bigcup_{j=1}^{r^k} (S+h_j) \cap (S+t)\Bigg) \\
&= \sum_{j=1}^{r^k} \mu\Big((S+h_j) \cap (S+t)\Big) \\
&= \sum_{j=1}^{r^k} \mu\Big(\big(S \cap (S+t-h_j)\big) + h_j\Big) \\
&= \sum_{j=1}^{r^k} \mu\Big(S \cap (S+t-h_j)\Big).
\end{aligned} \tag{33}$$

Using (32) we get that for any $i \ne j$ we have

$$\begin{aligned}
&\mu\Big(\big(S \cap (S+t-h_i)\big) \cap \big(S \cap (S+t-h_j)\big)\Big) \\
&\qquad = \mu\Big(S \cap \Big(\big((S+h_j) \cap (S+h_i)\big) + t - h_i - h_j\Big)\Big) = 0.
\end{aligned} \tag{34}$$

Thus we can continue (33) as

$$\mu\big(K \cap (S+t)\big) = \sum_{j=1}^{r^k} \mu\big(S \cap (S+t-h_j)\big) = \mu\Bigg(S \cap \bigcup_{j=1}^{r^k} (S+t-h_j)\Bigg) \le \mu(S),$$

which completes the proof of (a).

For proving (b) define

$$\nu(H) = \inf\left\{ \sum_{j=1}^{\infty} \mu(S_j) : H \subset \cup_{j=1}^{\infty} S_j + u_j, S_j \text{ is an elem. part of } K, u_j \in \mathbb{R}^n \right\}$$

for any $H \subset \mathbb{R}^n$. Then ν is clearly a translation invariant outer measure on $\mathbb{R}^n$.

We claim that ν is a metric outer measure; that is, $\nu(A \cup B) = \nu(A) + \nu(B)$ if $A, B \subset \mathbb{R}^n$ have positive distance. Indeed, in this case in the cover $A \cup B \subset \cup_{j=1}^{\infty} S_j + u_j$ in the definition of $\nu(A \cup B)$ we can replace replace each S_j by its small elementary parts such that each small elementary part covers only at most one of A and B. Since this transformation does not change $\sum_{j=1}^{\infty} \mu(S_j)$ this implies that $\nu(A \cup B) \geq \nu(A) + \nu(B)$. Since ν is an outer measure we get that $\nu(A \cup B) = \nu(A) + \nu(B)$.

It is well known (see e. g. in [**6**]) that restricting a metric outer measure to the Borel sets we get a Borel measure.

So it is enough to prove that $\nu(K) = 1$. The definition of $\nu(K)$ implies that $\nu(K) \leq \mu(K) = 1$.

For proving $\nu(K) \geq 1$ let $K \subset \cup_{j=1}^{\infty} S_j + u_j$ be an arbitrary cover such that each S_j is an elementary part of K and $u_j \in \mathbb{R}^n$. Then, using the already proved (a) part we get that

$$\sum_{j=1}^{\infty} \mu(S_j) \geq \sum_{j=1}^{\infty} \mu\big(K \cap (S_j + u_j)\big) \geq \mu\Big(\bigcup_{j=1}^{\infty} (K \cap (S_j + u_j))\Big) = \mu(K),$$

which completes the proof of (b). □

Using Lemma 2.9, the above theorem has the following consequence.

Corollary 8.3. *Let $K = \varphi_1(K) \cup \ldots \cup \varphi_r(K)$ be a self-affine set with the convex open set condition and suppose that $\varphi_1(K), \ldots, \varphi_r(K)$ are translates of each other.*

Then the natural probability measure on K can be extended as a translation invariant measure on $\mathbb{R}^n$. □

9. *Concluding remarks*

Our results might be true for much larger classes of self-similar or self-affine sets. We have no counter-example even for the strongest very naive conjecture that the intersection of any two affine copies of *any* self-affine set is of positive measure (according to any self-affine measure on one of the copies) if and only if it contains a set which is open in both copies.

We do not even know whether this very naive conjecture holds at least for two isometric copies of a self-affine Sierpiński sponge. (Note that if we allow only translated copies then Corollary 7.7 provides an affirmative answer.) For generalizing our results about Sierpiński sponges from translates to isometries the following statement could help.

Conjecture 9.1. *If K is a self-affine sponge, μ is the natural probability measure on it, φ is an isometry and $\mu\big(K \cap \varphi(K)\big) > 0$ then there exists a translation t such that $K \cap \varphi(K) = K \cap (K + t)$.*

This conjecture and the above mentioned Corollary 7.7 would clearly imply that Corollary 7.7 holds for isometric copies of self-affine Sierpiński sponges as well.

Then, in the same way as Theorem 8.1 is proved, we could get an *isometry*-invariant Borel measure ν for an arbitrary Sierpiński sponge K such that $\nu(K) = 1$.

For getting this stronger version of Theorem 8.1 the other natural way could be a generalization of Theorem 8.2 for isometries at least for self-affine Sierpinśki sponges. Since part (b) of Theorem 8.2 follows from (a) for isometries as well it would be enough to show (a), that is, it would be enough to show that $\mu\big(K \cap \varphi(S)\big) \le \mu(S)$, for any elementary piece S of any self-affine Sierpiński sponge K with natural measure μ. We do not know whether this last mentioned statement holds or not.

As we saw in Theorem 7.9, the instability results are not true for arbitrary self-affine sets, not even for self-similar sets with the open set condition: the simplest counter-example is $K = C \times [0, 1]$, where C denotes the classical triadic Cantor set. Then K is self-similar (with six similitudes of ratio 1/3), the open set condition clearly holds and if μ is the evenly distributed self-similar measure on K (that is, $p_1 = \ldots = p_6$) then $\mu\big(K \cap (K + (0, \varepsilon)\big) = 1 - \varepsilon$. The instability results might be true for totally disconnected (which means that each connected component is a singleton) self-affine sets.

In the definition of self-affine sets we allowed only contractive affine maps. If we allowed non-contractive affine maps as well then the above $K = C \times [0, 1]$ set would be a self-affine set (with two affine maps) with the strong separation condition, so it would be a counter-example for both theorems (Theorem 3.2 and Theorem 3.5) about self-affine sets.

We do not know whether the analogues of Theorem 4.1, Theorem 4.5 and Corollary 4.7 hold for self-affine sets with the strong separation condition. Although Theorem 3.5 says that for self-affine sets and isometries the analogue of Theorem 4.1 holds, and Theorem 4.5 was proved from Theorem 4.1, we cannot get the same way that for self-affine sets and at least for isometries the analogue of Theorem 4.5 holds. This is because in the proof of Theorem 4.5 it was important that the maps $\varphi_1, \ldots, \varphi_r$ that generated the self-similar sets were also in the group (in this case the group of similitudes) for which we had Theorem 4.1. In order to get any analogue of Theorem 4.5 for self-affine sets in the same way we need to prove a self-affine analogue of Theorem 4.1 for a group of transformation containing the affine maps $\varphi_1, \ldots, \varphi_r$ that generates the self-affine set.

From a positive answer for the following question we could get fairly easily that the self-affine analogue of Theorem 4.1 holds at least for affine maps from any compact subset of the space of affine maps. Then, if we could also show that we can assume that the affine maps are from a compact set (as in Proposition 4.2 for similitudes) then we would get that all the main results of Section 4 also hold for self-affine sets and affine maps as well.

QUESTION 9.2. *Let $K \subset \mathbb{R}^d$ be a self-affine set satisfying the strong separation condition and let f be an affine map such that $f(K) \subset K$. Does this imply that $f(K)$ is a relative open set in K?*

Note that for $f(K)$ being a relative open set in K means that it is the union of countably many pairwise disjoint elementary pieces of K, and since $f(K)$ is compact this means that $f(K)$ is a finite union of elementary pieces of K.

A positive answer at least for the following self-similar special case of the above question could make the proof of Theorem 4.1 simpler. However, we cannot answer this question even for $d = 1$.

QUESTION 9.3. *Let $K \subset \mathbb{R}^d$ be a self-similar set satisfying the strong separation condition and let f be a similitude such that $f(K) \subset K$. Does this imply that $f(K)$ is a relative open set in K (or in other words $f(K)$ is a finite union of elementary pieces of K)?*

Note that in Section 6 we saw that self-similar set (even in $\mathbb{R}$) may contain similar copies of itself in non-trivial ways.

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