# RECONSTRUCTING GEOMETRIC OBJECTS FROM THE MEASURES OF THEIR INTERSECTIONS WITH TEST SETS

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Abstract. Let us say that an element of a given family $\mathcal{A}$ of subsets of $\mathbb{R}^d$ can be reconstructed using $n$ test sets if there exist $T_1, \dots, T_n \subset \mathbb{R}^d$ such that whenever $A, B \in \mathcal{A}$ and the Lebesgue measures of $A \cap T_i$ and $B \cap T_i$ agree for each $i = 1, \dots, n$ then $A = B$. Our goal will be to find the least such $n$.

We prove that if $\mathcal{A}$ consists of the translates of a fixed reasonably nice subset of $\mathbb{R}^d$ then this minimum is $n = d$. In order to obtain this result we reconstruct a translate of a fixed function using $d$ test sets as well, and also prove that under rather mild conditions the measure function $f_{K,\theta}(r) = \lambda^{d-1}(K \cap \{x \in \mathbb{R}^d : \langle x, \theta \rangle = r\})$ of the sections of $K$ is absolutely continuous for almost every direction $\theta$. These proofs are based on techniques of harmonic analysis.

We also show that if $\mathcal{A}$ consists of the magnified copies $rE+t$ ($r \geq 1, t \in \mathbb{R}^d$) of a fixed reasonably nice set $E \subset \mathbb{R}^d$, where $d \geq 2$, then $d+1$ test sets reconstruct an element of $\mathcal{A}$. This fails in $\mathbb{R}$: we prove that an interval, and even an interval of length at least 1 cannot be reconstructed using 2 test sets.

Finally, using randomly constructed test sets, we prove that an element of a reasonably nice $k$-dimensional family of geometric objects can be reconstructed using $2k+1$ test sets. A example from algebraic topology shows that $2k+1$ is sharp in general.

## 1. Introduction

There is a vast literature devoted to various kinds of geometric reconstruction problems. Part of the reasons why these are so popular is their connection with geometric tomography.

The set of reconstruction problems we will study is the following. Given a family of subsets of $\mathbb{R}^d$ we would like to find "test sets" so that whenever someone picks a set from the family and hands us the Lebesgue measure of the chunk of this set in the test sets, then we can tell which the chosen set is. In other words, the measures of the intersection of the set with our test sets uniquely determine the member of the family. Our aim is to use as few test sets as possible. If it is enough to use $n$ test sets then we say that *an element of the given family can be reconstructed using $n$ test sets*. The formal definition is the following. We denote the Lebesgue measure on $\mathbb{R}^d$ by $\lambda^d$.

**Definition 1.1.** Let $\mathcal{A}$ be a family of Lebesgue measurable subsets of $\mathbb{R}^d$ of finite measure. We say that *an element of $\mathcal{A}$ can be reconstructed using $n$ (test) sets* if there exist measurable sets $T_1, \dots, T_n$ so that whenever $A, B \in \mathcal{A}$ and $\lambda^d(A \cap T_i) = \lambda^d(B \cap T_i)$ for every $i = 1, \dots, n$ then $A = B$.

The first question of this form we are aware of is the following folklore problem, which asks, using the above terminology, weather an axis parallel unit subsquare of $[0, 10] \times [0, 10]$ can be reconstructed using two test sets. We leave this question to

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the reader as an exercise. There are numerous natural modifications of the problem: Can a unit segment of $[0, 10]$ (or of $\mathbb{R}$) be reconstructed using 1 test set? Can a unit disc be reconstructed using 2 test sets? What happens in higher dimensions? And so on.

In each of the above problems the given family $\mathcal{A}$ is the set of translates of a fixed set. Since in $\mathbb{R}^d$ this means $d$ parameters, we might hope that we can reconstruct a translate of a fixed set using $d$ test sets. One of our main goals is to show that this is indeed true, at least under some mild assumptions on the set. For $d \geq 3$ we prove (Corollary 5.8) this for any bounded measurable set of positive measure, in the plane (Theorem 5.11) for any bounded measurable set of positive measure with rectifiable boundary of finite length, and in the real line for any finite union of intervals.

The first idea behind all of the above results for $d \geq 2$ is the following. Suppose we want to reconstruct a translate of $E \subset \mathbb{R}^d$. Let $T$ be a test set of the form $T = S \times \mathbb{R}^{d-1}$, where $S \subset \mathbb{R}$. Then clearly $\lambda^d((E + x) \cap T)$ depends only on the first coordinate of $x$: in fact, one can easily check that

$$\lambda^d((E + x) \cap T) = \int_S f(t - x_1)dt, \tag{1}$$

where $x_1$ denotes the first coordinate of $x$ and $f(a)$ is the $(d-1)$-dimensional Lebesgue measure of the intersection of $E$ and the hyperplane that intersects the first axis orthogonally at $(a, 0, \ldots, 0)$. Thus if $\int_S f(t - x_1)dt$ uniquely determines $t$, in other words if we can reconstruct a translate of the $\mathbb{R} \to [0, \infty]$ function $f$ using the test set $S \subset \mathbb{R}$ then $\lambda^d((E + x) \cap T)$ determines $x_1$. Therefore, if we can do this in $d$ linearly independent directions then we are done.

In order to carry out the above program, first we show (Theorem 3.4) that a translate of a fixed non-negative not identically zero compactly supported absolutely continuous function can be reconstructed using one test set in the above sense. Then we get the above results by finding many directions in which the above defined function $f$ (which we will call section measure function) is absolutely continuous. For concrete sets (for example if we want to reconstruct a translate of the unit ball) this is immediate, for more general sets we use Fourier transforms.

We also consider the problem of reconstruction of a magnified copy of fixed set $E \subset \mathbb{R}^d$ $(d \geq 2)$, where by a magnified copy of $E$ we mean a set of the form $rE + x$, where $r \geq 1$ and $x \in \mathbb{R}^d$. Since we have $d+1$ parameters, one can hope to reconstruct using $d+1$ test sets. Using $\mathbb{R}^d$ as a test set we can reconstruct $r$ since $\lambda^d(rE + x)$ depends only on $r$. The reconstruction of $x$ is done similarly as in the above case of translations but now we need to consider not only translations of the section measure functions but also the translations of their rescaled copies, since instead of (1) we need here the more general $\lambda^d((rE + x) \cap T) = \int_S f(\frac{t}{r} - x_1)dt$. Therefore we need to choose the set $S \subset \mathbb{R}$ so that for *every* $r \geq 1$ the integral $\int_S f(\frac{t}{r} - x_1)dt$ determines $x_1$. So this is a harder task than in the case of translations (where we needed this only for $r = 1$) and we can only prove a positive result (Theorem 4.1) under some additional assumption on $f'$: we also require that $f'$ can be approximated well by a $g \in C^1$ function with small $\|g'\|_1$. For functions obtained from concrete nice sets of $\mathbb{R}^d$ $(d \geq 2)$ (for example, if we want to reconstruct a ball of radius at least 1) this condition can be checked. For the more general case again we have to find many directions in which this condition is satisfied, for which we use Fourier transforms here as well. This way we get (Corollary 6.7) that for any fixed bounded set $E \subset \mathbb{R}^d$ $(d \geq 4)$, a set of the form $rE + x$, where $r \geq 1$ and $x \in \mathbb{R}^d$ can be reconstructed using $d + 1$ sets.

In all of the above mentioned results the reconstruction is impossible using fewer test sets, since that would mean a continuous injective map from the parameter space into a smaller dimensional Euclidean space: if we attempt to reconstruct an element of $\{A_\alpha : \alpha \in \Lambda\}$, where the parametrization is chosen so that $\alpha \mapsto \lambda^d(A_\alpha \cap T)$ is continuous for any measurable set $T \subset \mathbb{R}^d$ then reconstruction using $T_1, \dots, T_n$ would yield that $\alpha \mapsto (\lambda^d(A_\alpha \cap T_1), \dots, \lambda^d(A_\alpha \cap T_n))$ is a continuous injective $\Lambda \to \mathbb{R}^n$ map, which is impossible if $\Lambda$ contains an open subset of $\mathbb{R}^{n+1}$, or more generally an $(n+1)$-dimensional manifold.

The above argument also shows that sometimes we need more functions than the number of parameters: if the above $\Lambda$ cannot be embedded continuously into $\mathbb{R}^n$ then one cannot reconstruct an element of $\{A_\alpha : \alpha \in \Lambda\}$ using $n$ test sets.

**Example 1.2.** Let $\Lambda$ be the $k$-skeleton of a $2k+2$ simplex (that is, $\Lambda$ is the union of the $k$-dimensional faces of a simplex in $\mathbb{R}^{2k+2}$). By the van Kampen - Flores Theorem (see e.g. in [5]) $\Lambda$ cannot be embedded continuously into $R^{2k}$, so the above argument shows that a unit ball in $\mathbb{R}^{2d+2}$ centered at a point of $\Lambda$ cannot be reconstructed using $2k$ sets, although the parameter space is $k$-dimensional, which means that we only have $k$ parameters.

In Section 7 we prove (Theorem 7.2) that reasonably nice geometric objects parametrized reasonably nicely by $k$ parameters can be reconstructed using $2k+1$ test sets, which is sharp according to the above example. The test sets are given using a random construction. As applications we get for example that an $n$-gon in the plane can be reconstructed using $4n+1$ test sets, an ellipsoid in $\mathbb{R}^3$ can be reconstructed using 19 test sets, and a ball in $\mathbb{R}^d$ can be reconstructed using $2d+3$ test sets, in particular an interval in $\mathbb{R}$ can be reconstructed using 5 test sets.

One might be tempted to say that Example 1.2 is quite artificial and in natural situations the number of parameters should suffice, and probably an interval can be reconstructed using 2 test sets. But this is false, we prove (Theorem 2.2) that an interval cannot be reconstructed using 2 test sets, not even if we consider only intervals of length more than 1. Like above, the obstacle is again of topological nature, although it is more complicated since the parameter space can be embedded into $\mathbb{R}^2$.

**Remark 1.3.** It is natural to ask what happens if we try to reconstruct a set using test functions by considering the integrals of the functions over the set, or even test measures by considering the measures of the set.

First we describe why a nice geometric object can be reconstructed using the following single test measure. Let

$$\mu = \sum_{i=1}^{\infty} \frac{\delta_{q_i}}{3^i},$$

where $\{q_1, q_2, \dots\} = \mathbb{Q}^d$ and $\delta_x$ denotes the Dirac measure at $x$. Suppose that the symmetric difference of any two distinct sets of $\mathcal{A}$ contains a ball, which is always the case for families of geometric objects. Then $\mu$ reconstructs a member of $\mathcal{A}$, that is, whenever $A, B \in \mathcal{A}$ are distinct sets then $\mu(A) \neq \mu(B)$.

Similarly to the case of test sets, it is not possible to reconstruct a member of $\mathcal{A}$ with fewer bounded test functions than the dimension of the parameter space of $\mathcal{A}$. However, as opposed to the case of test sets, it is almost obvious to reconstruct a translate of a fixed bounded measurable set $E \subset \mathbb{R}^d$ using $d$ bounded test functions: it is easy to see that the functions $\arctan x_1, \dots, \arctan x_d$ reconstruct a translate of $E$.

It is also very easy to reconstruct a magnified copy $rE + t$ of a fixed bounded measurable set $E \subset \mathbb{R}^d$ using $d+1$ bounded test functions: the constant 1 function determines $r$, and then $\arctan x_1, \dots, \arctan x_d$ determine $t$. As it was mentioned earlier, finding $d+1$ test sets that reconstruct a magnified copy of a fixed set is much harder in higher dimensions, and it is even impossible in $\mathbb{R}$: an interval cannot be reconstructed using two test sets.

Therefore the reconstruction problem for intervals nicely show the difference between test measures, test functions and test sets: an interval of $\mathbb{R}$ can be reconstructed using 1 test measure, it can be reconstructed using 2 test functions (but 1 does not suffice) and it cannot be reconstructed using less than 3 test sets.

## 2. Reconstruction of an interval

The following simple lemma is the key tool to reconstruct a translate of a fixed interval or a finite union of intervals.

**Lemma 2.1.** *Suppose that $G \subset (0, \infty)$, $h \in G$ and $G + h \subset G$. Let $A \subset \mathbb{R}$ be a measurable set that has positive Lebesgue measure in every nonempty interval. Suppose that $A \cap (A + G) = \emptyset$. Then an interval of length $h$ can be reconstructed using the test set $T = A \cup (A+G)$; in fact, $x \mapsto \lambda([x, x+h] \cap T)$ is strictly increasing.*

*Proof.* Clearly it is enough to prove that $\lambda([u, u + h] \cap T) < \lambda([v, v + h] \cap T)$ for any $u < v < u + h$. So let $u < v < u + h$. Then

$$\lambda([v, v + h] \cap T) - \lambda([u, u + h] \cap T) = \lambda([u + h, v + h] \cap T) - \lambda([u, v] \cap T)$$

$$= \lambda([u + h, v + h] \cap A) + \lambda([u + h, v + h] \cap (A + G)) - \lambda([u, v] \cap (A \cup (A + G))).$$

The first term is positive since $A$ has positive measure in every nonempty interval. By the translation invariance of $\lambda$, the second term can be written as $\lambda([u, v] \cap (A + G - h))$. Hence it is enough to prove that $A \cup (A + G) \subset A + G - h$. We have $A \subset A + G - h$ since $h \in G$, and we have $A + G \subset A + G - h$ since $G + h \subset G$. $\square$

**Theorem 2.2.** *A translate of a fixed interval can be reconstructed using 1 test set; that is, for any $h$ there exists a set $T \subset \mathbb{R}$ such that $\lambda([x, x+h] \cap T) \neq \lambda([y, y+h] \cap T)$ if $x \neq y$.*

*Proof.* We can clearly suppose that $h = 1$. It is not hard to see that one can choose countably many pairwise disjoint measurable subsets of $[0, 1]$, so that each of them has positive measure in each nonempty subinterval of $[0, 1]$. Using $\mathbb{Z}$ as the index set we denote them by $A_k$ ($k \in \mathbb{Z}$). Then Lemma 2.1 applied to $A = \cup_{k \in \mathbb{Z}} (A_k + k)$, $G = \{1, 2, \ldots\}$, $h = 1$ completes the proof. $\square$

To reconstruct a translate of a fixed finite union of intervals, Lemma 2.1 has to be applied for a more complicated $G$, for which it is a bit harder to construct suitable $A$. This is done in the following lemma.

We call a set $E \subset \mathbb{R}$ *locally finite* if it has finitely many elements in every bounded interval.

**Lemma 2.3.** *For any locally finite set $G \subset (0, \infty)$ there exists a measurable set $A \subset \mathbb{R}$ such that $A$ has positive Lebesgue measure in every nonempty interval and $A \cap (A + G) = \emptyset$.*

*Proof.* Since the lemma is scaling invariant we can suppose that the minimal element of $G$ is bigger than 2. Let $I_1, I_2, \ldots$ be an enumeration of the intervals with rational endpoints such that $I_j \subset (-j, j)$ holds for every $j$.

By induction we define nowhere dense closed sets $A_1, A_2, \ldots$ with positive measure so that for every $n$, $A_n \subset I_n$ and

$$\left( \bigcup_{j=1}^{n} A_j \right) \cap \left( \bigcup_{j=1}^{n} A_j + G \right) = \emptyset. \tag{2}$$

This will complete the proof since then we can choose $A = \cup_{j=1}^{\infty} A_j$.

We can take $A_1$ as an arbitrary nowhere dense closed subset of $I_1$ with positive measure since then (2) is guaranteed by $I_1 \subset (-1,1)$ and $\min G > 2$.

Suppose that we already chose $A_1, \dots, A_{n-1}$ with all the requirements up to $n-1$. To complete the proof we need to choose a nowhere dense closed set $A_n \subset I_n$ disjoint to $(\cup_{j=1}^{n-1} A_j) + G$ and $(\cup_{j=1}^{n-1} A_j) - G$. Since $G$ is locally finite, we need to avoid only finitely many translates of the nowhere dense closed set $\cup_{j=1}^{n-1} A_j$, which can clearly be done. $\square$

**Theorem 2.4.** *Let $E$ be a finite union of intervals in $\mathbb{R}$. Then a translate of $E$ can be reconstructed using 1 set; that is, there exist a measurable set $T$ so that $\lambda((E+t)\cap T) \neq \lambda((E+t')\cap T)$ if $t \neq t'$.*

*Proof.* Let $E = \cup_{j=1}^{n} I_j$, where $I_j$ is an interval of length $a_j$ and the intervals are pairwise disjoint. Let $G$ be the additive semigroup generated by $a_1, \dots, a_n$; that is, $G = \{\sum_{i=1}^{n} k_i a_i \ : \ k_1, \dots, k_n \in \{0,1,2,\dots\}\} \setminus \{0\}$. Then $G \subset (0,\infty)$ is a locally finite set and it contains every $a_i$. Let $A$ be the set obtained by Lemma 2.3 from $G$ and let $T = A \cup (A+G)$. Then by Lemma 2.1 each function $x \mapsto \lambda((I_j + x)\cap T)$ is strictly increasing, so their sum $x \mapsto \lambda((E+x)\cap T)$ is also strictly increasing, which completes the proof. $\square$

An interval has two parameters, so one cannot reconstruct an interval using 1 test test, but one might expect that 2 test sets should be enough. We show that this is false. The following lemma concerns the topological obstacle, which excludes reconstruction using two sets. The lemma is surely well known for topologists, but for completeness we present a short proof.

**Lemma 2.5.** *Let $U \subset \mathbb{R}^2$ be a path connected open set and let $f : U \to \mathbb{R}^2$ be continuous and injective. Suppose that $f$ is differentiable at two points $a$ and $b$ such that the determinant of the Jacobi matrix at $a$, $\det f'(a) > 0$. Then $\det f'(b) \geq 0$.*

*Proof.* Suppose that $\det f'(b) < 0$. Let $C : [0, 2\pi] \to \mathbb{R}^2$ be the curve $C(t) = e^{it}$. If $r$ is small enough, the winding number of the curve $f(a + rC)$ around $f(a)$ is 1, while the winding number of $f(b + rC)$ around $f(b)$ is $-1$. However, since $U$ is path-connected and the winding number is homotopy invariant, which yields a contradiction. $\square$

**Theorem 2.6.** *An interval in $\mathbb{R}$ cannot be reconstructed using two measurable sets. Moreover, even an interval of length bigger than 1 cannot be reconstructed using two sets; that is, for any pair of measurable sets $A, B \subset \mathbb{R}$ there exist two distinct intervals $I$ and $I'$ of length bigger than 1 such that $\lambda(I \cap A) = \lambda(I' \cap A)$ and $\lambda(I \cap B) = \lambda(I' \cap B)$.*

*Proof.* Suppose to the contrary that $A$ and $B$ reconstruct an interval of length bigger than 1. Let $U = \{(x,y) \in \mathbb{R}^2 : y - x > 1\}$. Let $f : U \to [0,\infty)^2$ be defined by

$$f((x,y)) = (\lambda(A \cap [x,y]), \lambda(B \cap [x,y])).$$

The map $f$ is Lipschitz, and since $A$ and $B$ reconstruct, it is also injective.

Let $d_H(x) = \lim_{r \to 0+} \lambda(H \cap [x-r, x+r])/2r$ denote the density of a set $H$ at a point $x$ if the limit exists. Suppose that $y - x > 1$ and $d_A(x), d_B(x), d_A(y), d_B(y)$ all exist. Using the $o$ notation,

$$\begin{aligned} f(x+t_x, y+t_y) = (&\lambda(A \cap [x,y]) - d_A(x)t_x + d_A(y)t_y + o(t_x) + o(t_y), \\ &\lambda(B \cap [x,y]) - d_B(x)t_x + d_B(y)t_y + o(t_x) + o(t_y)). \end{aligned}$$

This means that $f$ is differentiable at $(x,y)$ and its derivative (Jacobian) is

$$\begin{pmatrix} -d_A(x) & d_A(y) \\ -d_B(x) & d_B(y) \end{pmatrix}.$$

Let $I_1$ and $I_2$ be two non-empty intervals so that their distance is bigger than 1 and $I_1$ is on the left-hand side of $I_2$. Then none of $A, B, A^c, B^c$ and $(A \triangle B)$ can have zero measure intersection both with $I_1$ and $I_2$, since otherwise $f$ maps $I_1 \times I_2 \subset U$ injectively and continuously into a (vertical, horizontal or diagonal) line, which is impossible. This implies that for any interval of length bigger than 1 all of $A, B, A^c, B^c$ and $(A \triangle B)$ must have positive measure. In particular, $\lambda(A \triangle B) > 0$ and both $A$ and $B$ have positive measure in any halfline.

Since $\lambda(A \triangle B) > 0$, we have $\lambda(A \setminus B) > 0$ or $\lambda(B \setminus A) > 0$. We may suppose that the first one holds. Recall that Lebesgue's density theorem states that the density of a measurable set is 1 at almost all of its points and 0 at almost all of the points of its complement. Since $\lambda(A \setminus B) > 0$, this implies that there exists a point $z$ for which $d_{A \setminus B}(z) = 1$. Then $d_A(z) = 1$ and $d_B(z) = 0$. Since $B \cap (-\infty, z-1)$ and $B \cap (z+1, \infty)$ have positive measure we can pick $u < z-1$ and $v > z+1$ so that $d_B(u) = d_B(v) = 1$ and both of $d_A(u)$ and $d_A(v)$ exist.

Then
$$f'(z,u) = \begin{pmatrix} -1 & d_A(u) \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f'(v,z) = \begin{pmatrix} -d_A(v) & 1 \\ -1 & 0 \end{pmatrix},$$
thus $\det f'(z,u) = -1$, $\det f'(v,z) = 1$. This contradicts Lemma 2.5. $\square$

In Corollary 7.4 we will see that 5 test sets are enough. We do not know weather 3 or 4 are enough or not.

## 3. Reconstruction of a translate of a fixed function

As it is explained in the Introduction, we can reconstruct a translate of a fixed set in $\mathbb{R}^d$ $(d \geq 2)$ using $d$ test sets if we can reconstruct a translate of a specific function using 1 test set.

To reconstruct a translate of a fixed function, the following definition will be crucial.

**Definition 3.1.** Let $f : \mathbb{R} \to \mathbb{R}$ be an $L^1$ function with $\operatorname{supp}(f) \subset [-1,1]$ and $\varepsilon > 0$. Define
$$K(\varepsilon, f) = \inf \{Var(g) : g \text{ is of bounded variation, } \operatorname{supp}(g) \subset [-1,1],\ \|f-g\|_1 < \varepsilon\},$$
where $Var(g)$ denotes the total variation of $g$.

Clearly, $K(\varepsilon, f)$ is monotone in $\varepsilon$. Since the piecewise constant functions are dense in $L^1$, we also obtain that $K(\varepsilon, f) < \infty$ for every such $f$ and $\varepsilon$.

The following lemma shows that we can replace functions of bounded variation with $C^1$ functions.

**Lemma 3.2.** *Let $f : \mathbb{R} \to \mathbb{R}$ be an $L^1$ function with* $\operatorname{supp}(f) \subset [-1,1]$ *and $\varepsilon > 0$. Then*
$$K(\varepsilon, f) = \inf \left\{Var(g) : g \in C^1, \operatorname{supp}(g) \subset [-1,1],\ \|f-g\|_1 < \varepsilon\right\}.$$

Note that in this case $Var(g) = \|g'\|_1$.

*Proof.* It suffices to prove that if $g$ is of bounded variation with $\operatorname{supp}(g) \subset [-1,1]$ and $\varepsilon > 0$ then there exists a $g_1 \in C^1$ with $\operatorname{supp}(g_1) \subset [-1,1]$, $\|g-g_1\|_1 < \varepsilon$ and $Var(g_1) = Var(g)$.

Let us first assume instead that $g$ is constant on $(-\infty,-1)$ and $(1,\infty)$, and it is also monotone. It is not hard to find a piecewise constant monotone function $g_0$ such that $\|g-g_0\|_1 < \varepsilon$. Then clearly $Var(g_0) = Var(g)$. Finally, we can easily approximate $g_0$ by a monotone $g_1 \in C^1$ such that $\|g_0-g_1\|_1 < \varepsilon$ and $Var(g_1) = Var(g)$.

Let now $g$ be a general function of bounded variation. It is well-known that it can be decomposed as $g = g_+ - g_-$, where $g_+$ and $g_-$ are non-decreasing and $Var(g) = |g_+(1) - g_+(-1)| + |g_-(1) - g_-(-1)|$ (indeed, let $g_+(x)$ be the positive variation of $g$ on $[-1, x]$). Applying the above approximation gives the result. $\square$

Recall that $f * g$ stands for the convolution of the two functions, and also that a function $f$ is locally absolutely continuous iff there exists a function $f^* \in L^1$ such that $f(y) - f(x) = \int_x^y f^*(t)\,dt$ for every $x, y \in \mathbb{R}$. Moreover, in that case $f^* = f'$ almost everywhere. The following lemma is rather well-known, but we were unable to find a suitable reference so we include a proof.

**Lemma 3.3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be locally absolutely continuous, $g : \mathbb{R} \to \mathbb{R}$ be locally in $L^1$ and assume that one of them is compactly supported. Then $f * g$ is also locally absolutely continuous and $(f * g)' = f' * g$ almost everywhere.*

*Proof.* Since we are only interested in the local behaviour of $f * g$, and one of them is compactly supported, we may actually assume (using the formula defining $f * g$) that both of them are compactly supported. This justifies the use of Fubini's Theorem in the following computation.

$$(f*g)(y)-(f*g)(x) = \int_{\mathbb{R}} [f(y-u)-f(x-u)]g(u)\,du = \int_{\mathbb{R}} \int_{x-u}^{y-u} f'(t)\,dt\; g(u)\,du =$$
$$\int_{\mathbb{R}} \int_x^y f'(t-u)\,dt\; g(u)\,du = \int_x^y \int_{\mathbb{R}} f'(t-u)g(u)\,du\,dt = \int_x^y (f' * g)(t)\,dt,$$

hence we are done. $\square$

**Theorem 3.4.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative not identically zero compactly supported absolutely continuous function. Then a translate of $f$ can be reconstructed using one test set; that is, there exists a measurable set $T$ such that if $b \neq b'$ then $\int_T f(x-b)\,dx \neq \int_T f(x-b')\,dx$.*

*In fact, $\int_T f(x-b)\,dx$ is strictly increasing in $b$, and we can choose $T$ to be a locally finite union of intervals.*

*Proof.* Since $f$ is absolutely continuous, $f'$ exists almost everywhere, $f' \in L^1$ and $f(x) = \int_{-\infty}^x f'(t)\,dt$ for every $x \in \mathbb{R}$. We may suppose that $f$ (and $f'$) is supported in $[-1, 1]$ and that $\int_{\mathbb{R}} f = 1$.

Let $\Phi : \mathbb{R} \to [0, 1]$ be an arbitrary $C^1$ function with $\Phi' > 0$, and $h : [0, \infty) \to (0, 1)$ be an arbitrary decreasing continuous function (which we will specify later). It is easy to see that we can choose $T$ so that its characteristic function (denoted by $\chi_T$) is close to $\Phi$ in the following sense:

$$\left|\lambda([0,r]\cap T) - \int_0^r \Phi\right| \le h(r) \quad \text{and} \quad \left|\lambda([-r,0]\cap T) - \int_{-r}^0 \Phi\right| \le h(r) \tag{3}$$

and

$$\left|\lambda([a,b]\cap T) - \int_a^b \Phi\right| \le h(r) \tag{4}$$

for every $r \geq 0$ and for every $[a, b] \subset [r-1, r+1]$ and every $[a, b] \subset [-r-1, -r+1]$. Clearly, $T$ can be chosen to be a locally finite union of intervals. Note that (4) implies that

$$\left\|\int (\chi_T - \Phi)\right\|_{L^\infty[x-1,x+1]} \le h(|x|) \quad \text{for every } x \in \mathbb{R}. \tag{5}$$

Here $\int(\chi_T - \Phi)$ is an abbreviation for the function $t \mapsto \int_x^t (\chi_T - \Phi)(u)\,du$.

We have to show that $f * \chi_T$ is strictly increasing. As this function is locally absolutely continuous by Lemma 3.3, it suffices to show that $(f * \chi_T)' > 0$ almost everywhere. Let $g$ be a $C^1$ function with $\operatorname{supp}(g) \subset [-1, 1]$. Then, using Lemma 3.3 several times, we obtain

$$\begin{aligned}
(f * \chi_T)'(x) &= (f * \Phi)'(x) + (f * (\chi_T - \Phi))'(x) = \\
&= (f * \Phi')(x) + (f' * (\chi_T - \Phi))(x) = \\
&= (f * \Phi')(x) + ((f' - g) * (\chi_T - \Phi))(x) + (g * (\chi_T - \Phi))(x) = \\
&= (f * \Phi')(x) + ((f' - g) * (\chi_T - \Phi))(x) + \left(g' * \int (\chi_T - \Phi)\right)(x) \geq \\
&\geq \min_{[x-1,x+1]} \Phi' - \|f' - g\|_1 \|\chi_T - \Phi\|_\infty - \|g'\|_1 \left\| \int (\chi_T - \Phi) \right\|_{L^\infty[x-1,x+1]}.
\end{aligned}$$

Using Lemma 3.2 and then (5), for every $\varepsilon > 0$ we obtain

$$\begin{aligned}
(f * \chi_T)'(x) &\geq \min_{[x-1,x+1]} \Phi' - \varepsilon \|\chi_T - \Phi\|_\infty - K(\varepsilon, f') \left\| \int (\chi_T - \Phi) \right\|_{L^\infty[x-1,x+1]} \\
&\geq \min_{[x-1,x+1]} \Phi' - 2\varepsilon - K(\varepsilon, f') h(|x|).
\end{aligned}$$

Choosing $\varepsilon = \varepsilon(x) = 1/4 \min_{[x-1,x+1]} \Phi'$ we see that if we fix $h$ such that

$$h(|x|) \leq \frac{\min_{[x-1,x+1]} \Phi'}{4K(\varepsilon(x), f')}$$

for every $x \in \mathbb{R}$ then

$$(f * \chi_T)'(x) \geq 1/4 \min_{[x-1,x+1]} \Phi' > 0.$$

□

## 4. Reconstruction of a function of the form $f(\frac{x}{a} + b)$

The reconstruction of a magnified copy of a fixed set in $\mathbb{R}^d$ ($d \geq 2$) will be based on the following result.

**Theorem 4.1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative not identically zero compactly supported absolutely continuous function. Suppose that*

$$\frac{K(\varepsilon, f')}{\exp(\varepsilon^{-1/3})} \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{6}$$

*Then a function of the form $f(\frac{x}{a} + b)$ ($a \geq 1$, $b \in \mathbb{R}$) can be reconstructed using two test sets; that is, there are two measurable sets $T_1, T_2 \subset \mathbb{R}$ such that if $(a, b) \neq (a', b')$ are as above then $\int_{T_i} f(\frac{x}{a} + b)\,dx \neq \int_{T_i} f(\frac{x}{a'} + b')\,dx$ for some $i \in \{0, 1\}$.*

*In fact, we will have $T_1 = \mathbb{R}$ and $T_2$ a locally finite union of intervals, and hence $T_1$ determines the value of $a$, and $T_2$ will be a set, not depending on $a$, that reconstructs a member of $\{f(\frac{x}{a} + b) : b \in \mathbb{R}\}$ for every fixed $a \geq 1$.*

We remark here that the condition $a \geq 1$ looks somewhat artificial, but we will show after the proof that it does play an important role.

*Proof.* The proof will be based on the previous one. Let $f_a(x) = f(x/a)/a$ ($a \geq 1$); thus $f_a$ is supported in $[-a, a]$, $\int f_a = 1$. Let $\Phi : \mathbb{R} \to (0, 1)$ be a $C^1$ function such that

$$\Phi'(x) = \frac{c}{|x| \log^2 |x|}$$

when $|x| \geq 2$ (for some positive constant $c$) and let $\Phi' > 0$ everywhere.

We denote by $f'_a$ the $L^1$ function for which $f_a(x) = \int_{-\infty}^x f'_a(t)\,dt$. Clearly $f'_a = f'(x/a)/a^2$. Since $f_a$ is supported in $[-a,a]$, similarly as before we get

$$\begin{aligned}(f_a * \chi_{T_2})'(x) &= (f_a * \Phi')(x) + ((f'_a - g) * (\chi_{T_2} - \Phi))(x) + \left(g' * \int(\chi_{T_2} - \Phi)\right)(x)\\ &\geq \min_{[x-a,x+a]} \Phi' - \|f'_a - g\|_1 \|\chi_{T_2} - \Phi\|_\infty - \|g'\|_1 \left\|\int(\chi_{T_2} - \Phi)\right\|_{L^\infty[x-a,x+a]}\end{aligned}$$

for every $C^1$ function $g$ supported in $[-a,a]$.

Let $\delta > 0$ be arbitrary and choose a $C^1$ function $g_0$ supported on $[-1,1]$ such that $\|f' - g_0\|_1 < \varepsilon$ and $\|g'_0\|_1 \leq K(\varepsilon, f') + \delta$. Let $g(x) = g_0(x/a)/a^2$ (thus $g'(x) = g'_0(x/a)/a^3$). Then

$$\|f'_a - g\|_1 < \varepsilon/a$$

and

$$\|g'\|_1 \leq K(\varepsilon, f')/a^2 + \delta/a^2.$$

Therefore

$$(f_a * \chi_{T_2})'(x) \geq \min_{[x-a,x+a]} \Phi' - 2\varepsilon/a - K(\varepsilon, f')/a^2 \left\|\int(\chi_{T_2} - \Phi)\right\|_{L^\infty[x-a,x+a]}$$

for every $\varepsilon > 0$.

We may suppose that $x \geq 0$ as one can deal with the other case similarly.

First let us suppose that $a \geq x/2$. For some $c' > 0$ we have

$$\min_{[x-a,x+a]} \Phi' \geq \frac{c'}{3a\log^2(3a)}.$$

Clearly

$$\left\|\int(\chi_{T_2} - \Phi)\right\|_{L^\infty[x-a,x+a]} \leq 2h(0).$$

Choosing $\varepsilon = \frac{c'}{300\log^2(3a)}$ we obtain

$$(f_a * \chi_{T_2})'(x) \geq \frac{c'}{6a\log^2(3a)} - 2h(0)K(\frac{c'}{300\log^2(3a)}, f')/a^2.$$

Using the bound on the magnitude of $K$ we obtain

$$K(\frac{c'}{300\log^2(3a)}, f')/a^2 \leq 2\exp((300\log^2(3a)/c')^{1/3})/a^2 = o(a^{-1.99})$$

as $a \to \infty$. Therefore, if we choose $h(0)$ small enough, we have

$$(f_a * \chi_{T_2})'(x) \geq \frac{c'}{12a\log^2(3a)} > 0.$$

Now let us suppose that $a < x/2$. For some $c'' > 0$ we have

$$\min_{[x-a,x+a]} \Phi' \geq \frac{c''}{2x\log^2(2x)}.$$

Clearly

$$\left\|\int(\chi_{T_2} - \Phi)\right\|_{L^\infty[x-a,x+a]} \leq 2h(x/2).$$

Choosing $\varepsilon = \frac{ac''}{200x\log^2(2x)}$ we obtain

$$\begin{aligned}(f_a * \chi_{T_2})'(x) &\geq \min_{[x-a,x+a]} \Phi' - 2\varepsilon/a - K(\varepsilon, f')/a^2 \left\|\int(\chi_{T_2} - \Phi)\right\|_{L^\infty[x-a,x+a]}\\ &\geq \frac{c''}{4x\log^2(2x)} - K(\frac{ac''}{200x\log^2(2x)}, f')2h(x/2).\end{aligned}$$

Here
$$K(\frac{ac''}{200x\log^2(2x)}, f') \leq K(\frac{c''}{200x\log^2(2x)}, f'),$$
thus choosing
$$h(x/2) \leq \frac{c''}{16x\log^2(2x)} / K(\frac{c''}{200x\log^2(2x)}, f')$$
we get that
$$(f_a * \chi_{T_2})'(x) \geq \frac{c''}{8x\log^2(2x)} > 0.$$
□

**Remark 4.2.** We do not know if a function of the form $f(\frac{x}{a}+b)$ $(a>0, b\in\mathbb{R})$ can be reconstructed using two test sets. However, we now outline why one cannot choose $T_1 = \mathbb{R}$ in this case. If such a pair of sets existed then the continuous function $b \mapsto \int_{T_2} f(\frac{x}{a}+b)\,dx$ would be strictly monotone for every $a>0$. But $T_2$ cannot be of full or zero measure on any interval, so both $T_2$ and its complement has density points on any interval, and choosing a small enough $a$ easily shows that monotonicity fails.

**Remark 4.3.** It is easy to see that the same proof works if we only assume that
$$\frac{K(\varepsilon, f')}{\exp(\varepsilon^{-(1-\delta)})} \to 0 \quad \text{as} \quad \varepsilon \to 0$$
for some $\delta > 0$, but we will not need this fact.

**Definition 4.4.** We say that $x_0 \in \mathbb{R}$ is a *controlled singularity* of a function $g : \mathbb{R} \to \mathbb{R}$ if $|g(x)| \leq \frac{1}{|x-x_0|^{1-\delta}}$ in a neighbourhood of $x_0$ for some $\delta > 0$, and $g$ is monotone on $(x_0-\varepsilon, x_0)$ and $(x_0, x_0+\varepsilon)$ for some $\varepsilon > 0$.

**Lemma 4.5.** *If $g \in L^1$ is supported in $[-1,1]$, and locally is in $C^1$ except for a finite number of controlled singularities then*
$$\frac{K(\varepsilon, g)}{\exp(\varepsilon^{-1/3}))} \to 0 \qquad \textit{as} \qquad \varepsilon \to 0.$$

*Proof.* Let us approximate $g$ by $g_n = \min(n, \max(-n, g))$ for a large enough $n$. An easy computation shows that we need $n = C\varepsilon^{-\frac{1-\delta}{\delta}}$ to achieve $\|g - g_n\|_1 < \varepsilon$, and then $Var(g_n) \leq C'\varepsilon^{-\frac{1-\delta}{\delta}}$. Therefore $K(\varepsilon, g) \leq C'\varepsilon^{-\frac{1-\delta}{\delta}}$ is subexponential and we are done. □

**Corollary 4.6.** *In Theorem 4.1 one can replace (6) by the condition that $f'$ is locally in $C^1$ except for a finite number of controlled singularities.*

## 5. Absolute continuity of the section measure function and reconstruction of a translate of a fixed set

**Notation 5.1.** For a measurable set $E \subset \mathbb{R}^d$ $(d \geq 2)$ of finite Lebesgue measure and a unit vector $\theta \in S^{d-1}$ we define the *section measure function* $f_{E,\theta}$ as the measure function of the sections of $E$ in direction $\theta \in S^{d-1}$; that is,
$$f_{E,\theta}(r) = \lambda^{d-1}(E \cap \{x \in \mathbb{R}^d : \langle x, \theta\rangle = r\}),$$
where $\langle\cdot,\cdot\rangle$ denotes scalar product. Note that $f_{E,\theta}$ is almost everywhere well defined.

**Theorem 5.2.** *Suppose that $E \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded measurable set with positive Lebesgue measure, $\theta_1, \dots, \theta_d \in S^{d-1}$ are linearly independent and the section measure functions $f_{E,\theta_1}, \dots, f_{E,\theta_d}$ are absolutely continuous. Then a translate of $E$ can be reconstructed using $d$ sets.*

*Proof.* By applying Theorem 3.4 to the functions $f_{E,\theta_1}, \dots, f_{E,\theta_d}$ we get measurable test sets $T_1, \dots, T_d \subset \mathbb{R}$ so that

$$\int_{T_i} f_{E,\theta_i}(x-b)\,dx \neq \int_{T_i} f_{E,\theta_i}(x-b')\,dx \qquad (b \neq b',\ i \in \{1,\dots,d\}). \tag{7}$$

For each $i$ let

$$V_i = \{a \in \mathbb{R}^d \ :\ \langle a, \theta_i\rangle \in T_i\}. \tag{8}$$

One can easily check that

$$\lambda^d((E+v)\cap V_i) = \int_{T_i} f_{E,\theta_i}(x - \langle v, \theta_i\rangle)\,dx$$

for any $v \in \mathbb{R}^p$. Combining this with (7) we get that $\lambda^d((E+v)\cap V_i)$ determines $\langle v, \theta_i\rangle$. Since $\theta_1, \dots, \theta_d$ are linearly independent this implies that the numbers $\lambda^d((E+v)\cap V_1), \dots, \lambda^d((E+v)\cap V_d)$ determine $v$, which completes the proof. $\square$

**Remark 5.3.** Since in Theorem 3.4 every test set can be chosen to be finite union of intervals and the test sets of the above proof are defined by (8), each test set of the above theorem (and of all of its corollaries) can be chosen as finite union of parallel layers, where by layer we mean a rotated image of a set of the form $[a,b]\times\mathbb{R}^{d-1}$.

The above theorem can clearly be applied to many concrete geometric objects.

**Corollary 5.4.**
*(1) A ball of fixed radius in $\mathbb{R}^d$ $(d \geq 1)$ can be reconstructed using $d$ sets; that is, for any $r$ there exist measurable sets $T_1, \dots, T_d \subset \mathbb{R}^d$ such that if $x \neq x'$ then $\lambda^d(B(x,r)\cap T_i) \neq \lambda^d(B(x',r)\cap T_i)$ for some $i \in \{1,\dots,d\}$.*
*(2) Let $E$ be a (not necessarily convex) polytope in $\mathbb{R}^d$ $(d \geq 2)$. Then a translate of $E$ can be reconstructed using $d$ test sets.*

*Proof.* In Theorem 2.2 we already proved the case $d = 1$ of (1).

Now let $d \geq 2$. If $B$ is a fixed ball then $f_{B,\theta}(r)$ is clearly absolutely continuous for every $\theta$. If $E$ is a polytope in $\mathbb{R}^d$ then $f_{E,\theta}(r)$ is absolutely continuous for any $\theta$ which is not orthogonal to any face of $E$. Therefor in both cases Theorem 5.2 can be applied. $\square$

In the remaining part of this section in order to apply Theorem 5.2 for a more general set $E \in \mathbb{R}^d$, we try to find a lot of angles $\theta \in S^{d-1}$ for which $f_{E,\theta}$ is absolutely continuous.

For getting a general positive result for $d \geq 3$ we use Fourier transforms. Denote the Fourier transform of a function $f$ by $\hat{f}$.

**Lemma 5.5.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a compactly supported $L^2$ function. If $r\hat{f}(r) \in L^2$ then $f$ is absolutely continuous (modulo a nullset) and $f' \in L^2$.*

*Proof.* Recall that an $L^1$ function agrees with an absolutely continuous function almost everywhere if and only if its weak derivative is an $L^1$ function. (Indeed, this is the well-known fact that the Sobolev space $W^{1,1}$ is the class of absolutely continuous function modulo nullsets, see [4, Corollary 7.14.].)

Therefore it suffices to prove that the function

$$f^*(r) = \widehat{2\pi i r\hat{f}(-r)} \quad (r \in \mathbb{R})$$

is in $L^1$, it is the weak derivative of $f$, and that $f^* \in L^2$. Clearly, $f^* \in L^2$ follows from the assumption $r\hat{f}(r) \in L^2$. Let $\varphi$ be an arbitrary $C^\infty$ function of compact support. Using the Parseval Formula twice as well as $\widehat{\varphi'}(r) = 2\pi i r\hat{\varphi}(r)$ and $\hat{\hat{f}}(r) = f(-r)$ we obtain

$$\int_{\mathbb{R}} f^* \varphi = \int_{\mathbb{R}} \widehat{2\pi i r \hat{f}(-r)} \varphi(r)\, dr = \int_{\mathbb{R}} -2\pi i r \hat{f}(r) \hat{\varphi}(r)\, dr =$$

$$-\int_{\mathbb{R}} \hat{f}(r) \widehat{\varphi'}(r)\, dr = -\int_{\mathbb{R}} f \varphi',$$

which yields that $f^*$ is the weak derivative of $f$. But it is easy to see that the support of the weak derivative of $f$ is contained in $\operatorname{supp}(f)$, hence $f^*$ is a compactly supported $L^2$ function, therefore it is in $L^1$, which concludes the proof. $\square$

**Lemma 5.6.** *Let $K \subset \mathbb{R}^d$ ($d \geq 2$) be a bounded measurable set of positive Lebesgue measure. Then for almost every $\theta \in S^{d-1}$ we have*

$$\int_{\mathbb{R}} |\widehat{f_{K,\theta}}(r)|^2 |r|^p\, dr < \infty$$

*for any $p \leq d-1$, where $f_{K,\theta}$ is the section measure function $f_{K,\theta}(r) = \lambda^{d-1}(K \cap \{x \in \mathbb{R}^d : \langle x, \theta \rangle = r\})$.*

*Proof.* By Plancherel Theorem $\int_{\mathbb{R}^d} |\hat{\chi_K}|^2 = \int_{\mathbb{R}^d} {\chi_K}^2 < \infty$. Therefore, using polar coordinates, for almost every direction $\theta$,

$$\int_{\mathbb{R}} |\widehat{\chi_K}(r\theta)|^2 |r|^{d-1}\, dr < \infty. \tag{9}$$

Fix such a $\theta$. Since $\chi_K \in L^1$, the function $\widehat{\chi_K}$ is bounded, so (9) implies that

$$\int_{\mathbb{R}} |\widehat{\chi_K}(r\theta)|^2 |r|^p\, dr < \infty \tag{10}$$

for any $p \leq d-1$. An easy computation shows the well-known fact that $\widehat{f_{K,\theta}}(r) = \widehat{\chi_K}(r\theta)$, so we are done. $\square$

**Theorem 5.7.** *If $d \geq 3$ then for any bounded measurable set $K \subset \mathbb{R}^d$ of positive Lebesgue measure the section measure functions*

$$f_{K,\theta}(r) = \lambda^{d-1}(K \cap \{x \in \mathbb{R}^d : \langle x, \theta \rangle = r\})$$

*are absolutely continuous for almost every $\theta \in S^{d-1}$.*

*Proof.* Since $d \geq 3$ we can apply Lemma 5.6 for $p = 2$ to get that $r\widehat{f_{K,\theta}}(r) \in L^2$ for almost every $\theta \in S^{d-1}$. Hence Lemma 5.5 applied to $f_{K,\theta}$ completes the proof. $\square$

Combining Theorems 5.2 and 5.7 we get the following.

**Corollary 5.8.** *Let $d \geq 3$ and let $E \subset \mathbb{R}^d$ be a bounded set of positive Lebesgue measure. A translate of $E$ can be reconstructed using $d$ sets; that is, there are measurable sets $T_1, \ldots, T_d \subset \mathbb{R}^d$ such that if $x \neq x'$ then $\lambda^d((E+x) \cap T_i) \neq \lambda^d((E+x') \cap T_i)$ for some $i \in \{1, \ldots, d\}$.*

We do not know weather Corollary 5.8 holds for $d = 1$ and $d = 2$. Our method clearly cannot work for $d = 1$. The following result shows that we cannot obtain Corollary 5.8 for $d = 2$ the same way, since Theorem 5.7 does not hold in $\mathbb{R}^2$.

**Theorem 5.9.** *There exists a bounded measurable set $K$ in $\mathbb{R}^2$ such that for every direction $\theta$ the section measure function $f_{K,\theta}$ is not equal almost everywhere to any continuous function.*

*Proof.* We call a planar set Besicovitch set if it is measurable and it contains unit line segments in every direction. It is well known that there exists a compact Besicovitch set of measure zero, let $A$ be such a set. For each $n \geq 1$, let $A_n$ be an open neighbourhood of $A$ of Lebesgue measure at most $1/2^n$. Let $p_i$ be sequence of points dense in the unit disc. Take $K = \bigcup_{n=1}^{\infty} A_n + p_n$. Then the measure of $K$ is at most 1. Since $A$ contains a unit line segment in every direction, for every $\theta$ the measure function $f_{K,\theta}(x) \geq 1$ if $x \in U_\theta$ where $U_\theta$ is an open set which is dense in an interval of length 2. Suppose that $f_{K,\theta}$ agrees with a continuous function almost everywhere. Then $f_{K,\theta}(x) \geq 1$ on an interval of length 2 (almost everywhere), thus $\int f_{K,\theta} \geq 2$, which contradicts the fact that the measure of $K$ is at most 1. $\square$

If we require only continuity of $f_{K,\theta}$ then it is enough to assume that the boundary has Hausdorff dimension less than 2. Since we do not need this result, we only sketch the proof.

**Theorem 5.10.** *Let $K$ be a compact set in $\mathbb{R}^2$ such that $\partial K$ has Hausdorff dimension less than 2. Then the measure functions $f_{K,\theta}$ of the sections of $K$ are continuous in almost every direction $\theta$.*

*Proof.* (Sketch) Since $K$ is compact, $f_{K,\theta}$ is upper semi-continuous for every $\theta$.

Suppose that $\theta$ is a direction for which $f_{K,\theta}$ is not continuous. Let $a$ be a point of discontinuity. It is easy to check that if $\partial K$ has zero one-dimensional Lebesgue measure on the line in direction $\theta$ corresponding to $a$, then $f_{K,\theta}$ is lower semi-continuous at $a$.

Therefore there are positively many directions such that $\partial K$ has positive measure on a line in these directions. It is well known that this implies that $\partial K$ has Hausdorff dimension 2. (This is a slight generalization of the fact that every planar Besicovitch set must have Hausdorff dimension 2.) $\square$

For absolute continuity of $f_{K,\theta}$ we need to assume somewhat more.

**Theorem 5.11.** *Let $K$ be a compact set in $\mathbb{R}^2$ with rectifiable boundary of finite length. Then $f_{K,\theta}$ is absolutely continuous for all but countably many $\theta$.*

We need two lemmas. The first one shows that the direct product of sets of small Lebesgue measure must meet $\partial K$ in a small set.

**Lemma 5.12.** *Let $u$ and $v$ be linearly independent directions in the plane. For every $\varepsilon' > 0$ there exists $\delta > 0$ such that for every $A, B \subset \mathbb{R}$, $\lambda(A) < \delta$, $\lambda(B) < \delta$ we have*

$$\mathcal{H}^1((uA + vB) \cap \partial K) < \varepsilon'.$$

*Proof.* Suppose to the contrary that there exist $\varepsilon' > 0$ and sets $A_i$, $B_i$ of Lebesgue measure $1/i^2$ such that $\mathcal{H}^1((uA_i + vB_i) \cap \partial K) \geq \varepsilon'$. Let

$$C_n = \bigcup_{i=n}^{\infty} uA_i + vB_i$$

and

$$C = \bigcap_n C_n.$$

Since $C$ is a direct product (in directions $u$ and $v$) of two sets of Lebesgue measure zero, $C$ is purely unrectifiable. Thus $\mathcal{H}^1(C \cap \partial K) = 0$. Using $\mathcal{H}^1(\partial K) < \infty$ and the continuity of measures we have $\mathcal{H}^1(C_n \cap \partial K) \to 0$ contradicting our assumption. $\square$

**Lemma 5.13.** *Let $J$ be a union of finitely many disjoint intervals $[x_i, y_i]$ and $\theta$ be a unit vector in $\mathbb{R}^2$. Let*

$$J^\theta = \{a \in \mathbb{R}^2 \ : \ \langle a, \theta \rangle \in J\}.$$

*Then*

$$\sum_i |f_{K,\theta}(y_i) - f_{K,\theta}(x_i)| \le \mathcal{H}^1(J^\theta \cap \partial K).$$

*Proof.* We can suppose that $\theta = (1,0)$ and so $J^\theta = J \times \mathbb{R}$. Then $|f_{K,\theta}(y_i) - f_{K,\theta}(x_i)|$ is the difference of the measure of $\{x_i\} \times \mathbb{R} \cap K$ and $\{y_i\} \times \mathbb{R} \cap K$ (two vertical lines intersected with $K$). Clearly $\partial K$ must intersect those horizontal segments $[(x_i,t),(y_i,t)]$ for which $(x_i,t) \in \{x_i\} \times \mathbb{R} \cap K$ but $(y_i,t) \notin \{y_i\} \times \mathbb{R} \cap K$ or vice-verse. The measure of these $t$ is at least $|f_{K,\theta}(y_i) - f_{K,\theta}(x_i)|$, thus the projection of $[x_i,y_i] \times \mathbb{R} \cap \partial K$ to the vertical axis has Lebesgue measure at least $|f_{K,\theta}(y_i) - f_{K,\theta}(x_i)|$. Therefore we are done. $\square$

*Proof of Theorem 5.11.* Recall that a real function $f$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every finite system of disjoint intervals $[x_j, y_j]$ satisfying $\sum_j |y_j - x_j| < \delta$ we have $\sum_j |f(y_j) - f(x_j)| < \varepsilon$.

Suppose to the contrary that there are uncountably many directions for which $f_{K,\theta}$ is not absolutely continuous. Then there exists $\varepsilon > 0$ and uncountably many directions $\theta$ such that the previous description of absolute continuity fails for $\varepsilon$ and $f_{K,\theta}$. Choose $N$ such directions where $(N-1)\varepsilon > \mathcal{H}^1(\partial K)$, let these be $\theta_1, \dots, \theta_N$.

Choose $\delta > 0$ such that Lemma 5.12 holds for all the $\binom{N}{2}$ pairs of these $N$ directions for $\varepsilon' = \varepsilon/N^2$.

We fix $i \in \{1, \dots, N\}$. Since absolute continuity fails for $\varepsilon$ and $f_{\theta_i}$, we can choose a set $J_i$ so that it is a finite union of disjoint intervals $[x_j, y_j]$ with $\lambda(J_i) < \delta$ and $\sum_j |f_{\theta_i}(y_j) - f_{\theta_i}(x_j)| \ge \varepsilon$. Thus, by Lemma 5.13, we get

$$\mathcal{H}^1(J_i^{\theta_i} \cap \partial K) \ge \varepsilon, \tag{11}$$

where $J_i^{\theta_i} = \{a \in \mathbb{R}^2 \ : \ \langle a, \theta_i \rangle \in J_i\}$.

By Lemma 5.12,

$$\mathcal{H}^1(J_i^{\theta_i} \cap \partial K \cap J_j^{\theta_j}) < \varepsilon/N^2 \qquad \text{for every } i \ne j. \tag{12}$$

Combining (11) and (12) we obtain

$$\mathcal{H}^1(\partial K) \ge \mathcal{H}^1\left(\bigcup_{i=1}^N J_i^{\theta_i} \cap \partial K\right) \ge N\varepsilon - \binom{N}{2}\varepsilon/N^2 \ge (N-1)\varepsilon > \mathcal{H}^1(\partial K),$$

a contradiction. $\square$

**Corollary 5.14.** *Let $E$ be a bounded measurable set in $\mathbb{R}^2$ with positive Lebesgue measure and rectifiable boundary of finite length. Then a translate of $E$ can be reconstructed using 2 test sets.*

*Proof.* We apply Theorem 5.11 for $K = \overline{E}$. Since $K \setminus E \subset \partial E$ has Lebesgue measure zero, $f_{E,\theta}$ equals almost everywhere to $f_{K,\theta}$, so Theorem 5.2 can be applied. $\square$

From Theorem 2.4 and Corollaries 5.8 and 5.14 we get the following in any dimension.

**Corollary 5.15.** *A translate of a fixed finite union of bounded convex sets in $\mathbb{R}^d$ ($d = 1, 2, \dots$) can be reconstructed using $d$ test sets.*

## 6. Reconstruction of a magnified copy of a fixed set

The first part of this section is analogous to the first part of the previous section but here the results follow from Theorem 4.1 instead of Theorem 3.4.

**Theorem 6.1.** *Let $E \subset \mathbb{R}^d$ $(d \geq 2)$ be a bounded measurable set with positive Lebesgue measure. Suppose that $\theta_1, \dots, \theta_d \in S^{d-1}$ are linearly independent so that for each $i = 1, \dots, d$ the section measure function $f_{E,\theta_i}$ is absolutely continuous and $K(\varepsilon, f'_{E,\theta_i})/\exp(\varepsilon^{-1/3}) \to 0$ as $\varepsilon \to 0$.*

*Then a set of the form $rE + x$, where $r \geq 1$ and $x \in \mathbb{R}^d$, can be reconstructed using $d+1$ test sets.*

*Proof.* By applying Theorem 4.1 to the functions $f_{E,\theta_1}, \dots, f_{E,\theta_d}$ we get measurable sets $T_1, \dots, T_d \subset \mathbb{R}$ so that for each $i$ and $r \geq 1$ the integral $\int_{T_i} f_{E,\theta_i}(\frac{x}{r} - b)\, dx$ determines $b$. For each $i$ let $V_i = \{a \in \mathbb{R}^d \ : \ \langle a, \theta_i \rangle \in T_i\}$. One can easily check that

$$\lambda^d((rE + v) \cap V_i) = \int_{T_i} f_{E,\theta_i}\left(\frac{x}{r} - \langle v, \theta_i \rangle\right) dx$$

for any $v \in \mathbb{R}^p$.

Therefore, for any $r \geq 1$ the numbers $\lambda^d((rE + v) \cap V_1), \dots, \lambda^d((rE + v) \cap V_d)$ determine $v$. Let $V_{d+1} = \mathbb{R}^d$. Then $\lambda^d((rE + v) \cap V_{d+1})$ clearly determines $r$, which completes the proof. □

**Remark 6.2.** Since in Theorem 4.1 the test set that determines $b$ can be chosen to be finite union of intervals, we obtained that each of the first $d$ test sets of the above theorem (and of all of its corollaries) can be chosen as finite union of parallel layers (where by layer we mean a rotated image of a set of the form $[a, b] \times \mathbb{R}^{d-1}$) and one test set as $\mathbb{R}^d$.

The above theorem can clearly be applied to many concrete geometric objects.

**Corollary 6.3.**

1. *A ball of radius at least 1 in $\mathbb{R}^d$ $(d \geq 2)$ can be reconstructed using $d{+}1$ sets; that is, there are measurable sets $T_1, \dots, T_{d+1} \subset \mathbb{R}^d$ such that if $(x, r) \neq (x', r')$, $x, x' \in \mathbb{R}^d$, $r, r' \geq 1$ then $\lambda^d(B(x, r) \cap T_i) \neq \lambda^d(B(x', r') \cap T_i)$ for some $i \in \{1, \dots, d+1\}$.*
2. *Let $E$ be a (not necessarily convex) polytope in $\mathbb{R}^d$ $(d \geq 2)$. Then a magnified copy $rE + x$, where $r \geq 1$ and $x \in \mathbb{R}^d$ can be reconstructed using $d+1$ test sets.*

*Proof.* It is easy to check that the assumptions of Lemma 4.5 hold for $f'_{E,\theta}$ if $E$ is the unit ball or if $E$ is a polytope and $\theta$ is not orthogonal to any of its faces. Thus we can apply Theorem 6.1 to $E$ in both cases. □

**Remark 6.4.** By Theorem 2.6 the above corollary does not hold for $d = 1$.

For a more general theorem we need the following result concerning the section measure functions, which we can only prove for $d \geq 4$.

**Theorem 6.5.** *If $d \geq 4$ then for any bounded measurable set $E \subset \mathbb{R}^d$ of positive Lebesgue measure for almost every $\theta \in S^{d-1}$ the section measure function $f_{E,\theta}(r)$ is absolutely continuous and $K(\varepsilon, f'_{E,\theta_i}) \leq C\varepsilon^{-2}$, where $C$ depends only on $E$ and $\theta$.*

*Proof.* Applying Lemma 5.6 we get that for almost every $\theta$,

$$\int |x|^3 |\widehat{f_{E,\theta}}|^2(x) < \infty. \tag{13}$$

Fix such a $\theta$ and put $f = f_{E,\theta}$. Thus, denoting the weak derivative of $f$ by $f'$,

$$\int |x| |\widehat{f'}|^2(x) < \infty. \tag{14}$$

We may assume that $f$ (and thus $f'$) is supported in $[0, 1]$. Let $g_R = (\chi_{[-R,R]}\widehat{f'})^{\check{}}$. We will approximate $f'$ by $\chi_{[0,1]} g_R$ to get a bound on $K(\varepsilon, f')$.

We have

$$
\begin{aligned}
\left\|\chi_{[0,1]}g_R - f'\right\|_1 &= \left\|\chi_{[0,1]}(g_R - f')\right\|_1 \le \left\|\chi_{[0,1]}(g_R - f')\right\|_2 \le \|g_R - f'\|_2 \\
&\le \left\|\widehat{g_R} - \widehat{f'}\right\|_2 = \left\|(1-\chi_{B(0,R)})\widehat{f'}\right\|_2 \\
&\le \left\||x|^{1/2}R^{-1/2}\widehat{f'}\right\|_2 = R^{-1/2}\left(\int |x||\widehat{f'}|^2(x)\,dx\right)^{1/2}.
\end{aligned} \tag{15}
$$

We have to bound the total variation of $\chi_{[0,1]}g_R$. We will do this in two steps. First,

$$
\begin{aligned}
\left\|\chi_{[0,1]}g_R'\right\|_1 &\le \left\|\chi_{[0,1]}g_R'\right\|_2 \le \|g_R'\|_2 = \left\|\widehat{g_R'}\right\|_2 = \|x\widehat{g_R}\|_2 \le \left\|x\chi_{[-R,R]}\widehat{f'}\right\|_2 \\
&\le \left\||x|^{1/2}R^{1/2}\widehat{f'}\right\|_2 = R^{1/2}\left(\int |x||\widehat{f'}|^2(x)\,dx\right)^{1/2}.
\end{aligned} \tag{16}
$$

Second,

(17)

$$
\begin{aligned}
\|g_R\|_\infty &\le \|\widehat{g_R}\|_1 = \left\|\chi_{[-R,R]}\widehat{f'}\right\|_1 \le 2R\left\|\widehat{f'}\right\|_\infty \le 2R\,\|f'\|_1 \\
&\le 2R\,\|f'\|_2 = 2R\left\|\widehat{f'}\right\|_2 = 2R\left\|x\widehat{f}\right\|_2 \le 2R\left(\left\||x|^{3/2}\widehat{f}\right\|_2^2 + 2\left\|\hat{f}\right\|_\infty^2\right)^{1/2},
\end{aligned}
$$

where in the last inequality we used that $x^2 \le 1$ on $[-1,1]$ and $x^2 \le x^3$ outside $[-1,1]$.

Combining (13), (14), (16) and (17) gives that the total variation of $\chi_{[0,1]}g_R$ is at most

$$2\,\|g_R\|_\infty + \left\|\chi_{[0,1]}g_R'\right\|_1 \le c_1 R$$

for some finite positive constant $c_1$ (depending on $f$). Comparing this to (15) gives that

$$K(c_2R^{-1/2}, f') \le c_1 R$$

for some $c_2 > 0$, and thus $K(\varepsilon, f') \le C\varepsilon^{-2}$. $\square$

**Remark 6.6.** The proof of the previous theorem is simpler for $d \ge 5$. In these dimensions we obtain from Lemma 5.6 that $r^2\widehat{f_{E,\theta}}(r) \in L^2$ and $r\widehat{f_{E,\theta}}(r) \in L^2$ for almost every $\theta$. Then by Lemma 5.5 $f_{E,\theta}$ is absolutely continuous and $f'_{E,\theta} \in L^2$. It is easy to see that the usual proof of the formula $\widehat{f'}(r) = 2\pi i r\hat{f}$ works if we only assume that $f$ is absolutely continuous. Hence $r\widehat{f'_{E,\theta}}(r) = r(2\pi i r\widehat{f_{E,\theta}}) \in L^2$, so a second application of Lemma 5.5 yields that $f'_{E,\theta}$ is absolutely continuous. Therefore $K(\varepsilon, f'_{E,\theta}) \le Var(f'_{E,\theta})$ is bounded in this case.

Theorems 6.1 and 6.5 immediately imply the following.

**Corollary 6.7.** *Let $d \ge 4$ and let $E \subset \mathbb{R}^d$ be a bounded set of positive Lebesgue measure. Then a set of the form $rE + x$, where $r \ge 1$ and $x \in \mathbb{R}^d$, can be reconstructed using $d+1$ sets; that is, there are measurable sets $T_1, \dots, T_{d+1} \subset \mathbb{R}^d$ such that if $(x,r) \ne (x',r')$, $x, x' \in \mathbb{R}^d$, $r, r' \ge 1$ then $\lambda^d((rE+x)\cap T_i) \ne \lambda^d((r'E+x')\cap T_i)$ for some $i \in \{1,\dots,d+1\}$.*

## 7. A general positive result for families with $k$ degrees of freedom

In this section we prove that nice geometric objects of $k$ degrees of freedom can be reconstructed using $2k+1$ measurable sets. We also show that this result is sharp.

**Notation 7.1.** We denote the complete metric space of non-empty compact sets of $\mathbb{R}^d$ with the Hausdorff metric by $(\mathcal{K}(\mathbb{R}^d), d_H)$.

In any metric space, let $B(A,\delta)$ denote the open $\delta$-neighborhood of the set $A$.

We recall the definition of the upper box dimension (upper Minkowski dimension) and the packing dimension in a metric space $X$. The upper box dimension of a bounded set $A \subset X$ is

$$\overline{\dim}_B(A) = \inf\{s : \limsup_{\varepsilon\to 0} N(A,\varepsilon)\varepsilon^s = 0\},$$

where $N(A,\varepsilon)$ is the smallest number of $\varepsilon$-balls in $X$ needed to cover $A$. Recall that in $\mathbb{R}^d$ this is the same as the box dimension (see e.g. in [6]); that is,

$$\overline{\dim}_B(A) = \overline{\dim}_M(A) = \inf\{s : \limsup_{\varepsilon\to 0} \lambda_n(B(A,\varepsilon))\varepsilon^{s-d} = 0\}.$$

The packing dimension (or modified upper box dimension in [1]) of $A \subset X$ is given by

$$\dim_P(A) = \inf\left\{\sup_i \overline{\dim}_B(A_i) \ : \ A_i \text{ is bounded and } A \subset \cup_{i=1}^{\infty} A_i\right\}.$$

(Alternatively, the packing dimension may be defined in terms of the radius based packing measures, see [2].)

**Theorem 7.2.** *Let $\mathcal{C}$ be a collection of compact subsets in $\mathbb{R}^d$. Suppose that $\dim_P \mathcal{C} \le k$, $k \in \{1,2,\dots\}$ and for every $K \in \mathcal{C}$, $K = \overline{\mathrm{int}\, K}$ and $\overline{\dim}_B \partial K = d-1$. Then an element of $\mathcal{C}$ can be reconstructed using $2k+1$ test sets.*

**Remark 7.3.** Example 1.2 shows that this theorem is sharp in the sense that $2k+1$ cannot be replaced by $2k$.

Before proving the theorem we show how it can be applied. In applications, the condition $\dim_P \mathcal{C} \le k$ is guaranteed by obtaining $\mathcal{C}$ as a $k$-parameter family of compact subsets of $\mathbb{R}^d$. More precisely, $\mathcal{C}$ will always be covered by finitely many sets of the form $f(G)$, where $G \subset \mathbb{R}^k$ is open and $f : G \to \mathcal{K}(\mathbb{R}^d)$ is Lipschitz. This clearly implies $\dim_P \mathcal{C} \le k$. Using this observation one can immediately apply Theorem 7.2 for any natural collection of geometric objects with finitely many parameters by counting the number of parameters. We illustrate this by the following list of applications. The reader can easily extend this list.

**Corollary 7.4.** *(1) An interval in $\mathbb{R}$ can be reconstructed using 5 test sets.*
*(2) A ball in $\mathbb{R}^d$ can be reconstructed using $2d+3$ test sets.*
*(3) An $n$-gon in $\mathbb{R}^2$ can be reconstructed using $4n+1$ test sets.*
*(4) An axis-parallel rectangle in $\mathbb{R}^2$ can be reconstructed using 9 test sets.*
*(5) An ellipsoid in $\mathbb{R}^3$ can be reconstructed using 19 test sets.*
*(6) A simplex in $\mathbb{R}^d$ can be reconstructed using $2d^2+2d+1$ test sets.*

Instead of Theorem 7.2 we prove the following even more general statement.

**Theorem 7.5.** *Let $\mathcal{C} \subset \mathcal{K}(\mathbb{R}^d)$ be such that $\dim_P \mathcal{C} < \infty$. Suppose that $K = \overline{\mathrm{int}\, K}$ and that $\overline{\dim}_B \partial K \le b < d$ for every $K \in \mathcal{C}$. Then an element of $\mathcal{C}$ can be reconstructed using $r = \left\lfloor \frac{2\dim_P \mathcal{C}}{d-b} \right\rfloor + 1$ test sets.*

*Proof.* We define a random set $A$ and we show that a set $K \in \mathcal{C}$ can be reconstructed using $r$ independent copies of $A$.

Let $1 > p_1 > p_2 > \cdots$ be a fast decreasing sequence of reals such that $\sum p_i < \infty$. Let $(n_i)$ be an increasing sequence of 2-powers converging to $\infty$ sufficiently fast. Let us also assume that $n_{i-1}$ divides $\log_2 n_i$, and $\log_2 n_i$ divides $n_i$ for each $i$, which conditions automatically hold if $n_i = 2^{2^{l_i}}$ and $l_i$ is a sufficiently fast increasing sequence of integers.

For each $i$ we take the grids of cubes $\mathcal{J}_i = \{(v + [0,1)^d)/n_i \ : v \in \mathbb{Z}^d\}$ and $\mathcal{D}_i = \{(v+[0,1)^d)/\log_2 n_i \ : v \in \mathbb{Z}^d\}$. Since $n_{i-1}$ divides $\log_2 n_i$ and $\log_2 n_i$ divides $n_i$, the partition $\mathcal{J}_i$ is finer than $\mathcal{D}_i$, which is finer than $\mathcal{J}_{i-1}$.

Now we define a random set $A_i \subset \mathbb{R}^d$ as the union of certain cubes of $\mathcal{J}_i$ in the following way. Independently for each cube $D$ of $\mathcal{D}_i$ we do the following. Choose a random integer $m_D$ between 0 and $p_i(n_i/\log_2 n_i)^d$ uniformly. Then choose randomly $m_D$ cubes of $\mathcal{J}_i$ in the cube $D$ (selecting each cube with equal probability) and let $H_D$ be their union. Finally, let $A_i = \cup_{D\in\mathcal{D}_i} H_D$.

This way each cube of $\mathcal{J}_i$ is contained in $A_i$ with probability approximately $p_i/2$, and points of distance more than $\sqrt{d}/\log_2 n_i$ are independent. (Note the major difference between this random set $A_i$ and the random set which independently chooses each cubes of $\mathcal{J}_i$ with probability $p_i/2$: The number of $\mathcal{J}_i$-cubes of our $A_i$ inside each cube of $\mathcal{D}_i$ has standard deviation $\approx n_i^d$, while in the other construction it would have standard deviation $\approx \sqrt{n_i^d}$. We ignored $p_i$ here as we will choose $n_i \gg 1/p_i$.)

Since $\sum p_i < \infty$, almost every point of $\mathbb{R}^d$ is contained only in finitely many sets $A_i$. Hence the following infinite symmetric difference makes sense (up to measure zero): let $A = A_1 \triangle A_2 \triangle \cdots$.

The key property of this random set is the following.

**Lemma 7.6.** *If $K, K' \in \mathcal{C}$, $K, K' \subset [-i,i]^d$ and $K \setminus K'$ contains a cube $D \in \mathcal{D}_i$ then*

$$|\lambda(A\cap K) - \lambda(A \cap K')| < \frac{1}{4n_i^d}$$

*with probability at most* $(\log_2 n_i)^d/(p_i n_i^d)$.

*Proof.* Let $B_i = A_1 \triangle \cdots \triangle A_i$ $(i = 1,2,\ldots)$.

First we prove that

$$|\lambda(B_i \cap K) - \lambda(B_i \cap K')| < 1/(2n_i^d) \tag{18}$$

with probability at most $(\log_2 n_i)^d/(p_i n_i^d)$.

Since $D \subset K$, we have

$$\lambda(B_i\cap K) - \lambda(B_i \cap K') = \lambda(B_i \cap D) + \lambda(B_i \cap K \cap D^c) - \lambda(B_i \cap K'). \tag{19}$$

Note that the last two terms of the right-hand side depend only on $A_1, \ldots, A_{i-1}$ and $A_i \setminus D$. Let us fix these random variables. Then the last two terms are constants, and we know the (conditional) distribution of $\lambda(B_i \cap D)$: this is $m_D/n_i^d$ if $D$ is disjoint from $B_{i-1}$, and it is $\lambda(D) - m_D/n_i^d$ if $D$ is contained in $B_{i-1}$. Hence the absolute value of the expression of (19) can be less than $1/(2n_i^d)$ only for at most one value of $m_D$. Since each value of $m_D$ was chosen with probability at most $(\log_2 n_i)^d/(p_i n_i^d)$, this implies that the conditional probability of (18) is at most $(\log_2 n_i)^d/(p_i n_i^d)$. Since this holds for each fixed choice of $A_1, \ldots, A_{i-1}$ and $A_i \setminus D$, we get that (18) holds indeed with probability at most $(\log_2 n_i)^d/(p_i n_i^d)$.

We can choose $p_{i+1}, p_{i+2}, \ldots$ such that

$$\sum_{j=i+1}^{\infty} p_j < \frac{1}{100 n_i^d (2i)^d}.$$

Then

$$\sum_{j=i+1}^{\infty} \lambda(A_j \cap K) \le \sum_{j=i+1}^{\infty} \lambda(A_j \cap [-i,i]^d) < \frac{1}{100n_i^d}$$

since $K \subset [-i,i]^d$ and by construction the density of each $A_j$ is at most $p_j$ in each cube of the form $a + [0,1]^d, a \in \mathbb{Z}^d$. Clearly the same inequality holds for $K'$.

Combining these inequalities with (18) we get that

$$|\lambda(A \cap K) - \lambda(A \cap K')| \geq \frac{1}{2n_i^d} - \frac{2}{100n_i^d} \geq \frac{1}{4n_i^d}$$

with probability at least $1 - (\log_2 n_i)^d/(p_i n_i^d)$. ☐

Let $s > \dim_P \mathcal{C}$ be such that $\left\lfloor \frac{2\dim_P \mathcal{C}}{d-b} \right\rfloor = \left\lfloor \frac{2s}{d-b} \right\rfloor$. We may suppose without loss of generality that $\overline{\dim}_B \partial K < b$ for every $K \in \mathcal{C}$ by increasing $b$ such that $\left\lfloor \frac{2s}{d-b} \right\rfloor$ does not increase. Write $\mathcal{C}$ as $\bigcup_{j=1}^\infty \mathcal{C}'_j$ such that each $\mathcal{C}'_j$ has upper box dimension less than $s$.

For every $K \in \mathcal{C}$ there exists a positive integer $m_0(K)$ such that for every $m \geq m_0(K)$,

$$\lambda(B(\partial K, 1/m)) \leq m^{b-d}$$

since the upper box dimension of $\partial K$ is less than $b$. For $K, L \in \mathcal{C}$, using that $K \triangle L \subset B(\partial K \cup \partial L, d_H(K, L))$, this implies that

$$\lambda(K \triangle L) \leq 2m^{b-d} \quad \text{if } d_H(K, L) \leq 1/m \text{ and } m_0(K), m_0(L) \leq m. \tag{20}$$

For $i \geq 1$ let

$$\mathcal{C}_i = \{K \in \bigcup_{j=1}^{i} \mathcal{C}'_j \,:\, m_0(K) \leq i,\, K \subset [-i, i]^d\}.$$

Thus $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots$, $\bigcup_i \mathcal{C}_i = \mathcal{C}$, and the upper box dimension of each $\mathcal{C}_i$ is less than $s$.

For each $i$ let

$$\widetilde{\mathcal{C}_i} = \{(K, K') \in {\mathcal{C}_i}^2 : K \setminus K' \text{ or } K' \setminus K \text{ contains a cube } D \in \mathcal{D}_i\}.$$

Then for every integer $N$, using the assumption that $K = \overline{\operatorname{int} K}$ for every $K \in \mathcal{C}$ and thus $\operatorname{int} K \triangle \operatorname{int} K' \neq \emptyset$ for every $K \neq K'$, $K, K' \in \mathcal{C}$, we get that

$$\bigcup_{i=N}^{\infty} \widetilde{\mathcal{C}_i} = \mathcal{C}^2 \setminus \{(K, K) : K \in \mathcal{C}^2\}. \tag{21}$$

Let us fix $i$. Since $\mathcal{C}_i$ has upper box dimension less than $s$, for every sufficiently large positive integer $k_i$ (say, for $k_i \geq \kappa_i$) there exist at most $k_i^s$ sets $\mathcal{C}_i^j \subset \mathcal{C}_i$ $(1 \leq j \leq k_i^s)$ with diameter at most $1/k_i$ that cover $\mathcal{C}_i$.

For each pair $(j, j') \in \{1, \ldots, k_i^s\}^2$ pick a pair

$$(K_{i,(j,j')}, K'_{i,(j,j')}) \in \widetilde{\mathcal{C}_i} \cap (\mathcal{C}_i^j \times \mathcal{C}_i^{j'})$$

whenever such a pair exists. Then

(22)

$$\forall (K, K') \in \widetilde{\mathcal{C}_i}\ \exists (j, j') \in \{1, \ldots, k_i^s\}^2 \ :\ d_H(K, K_{i,(j,j')}), d_H(K', K'_{i,(j,j')}) \leq 1/k_i.$$

Repeating the construction of $A$ independently $r$ times we obtain $A_1, \ldots, A_r$. We claim that an element $K \in \mathcal{C}$ can be reconstructed using these sets, provided we choose the sequences $(n_i)$ and $(p_i)$ appropriately.

For each picked pair $(K_{i,(j,j')}, K'_{i,(j,j')})$ we apply Lemma 7.6 to get that there exists $1 \leq t \leq r$ such that

$$|\lambda(A^t \cap K_{i,(j,j')}) - \lambda(A^t \cap K'_{i,(j,j')})| \geq \frac{1}{4n_i^d} \tag{23}$$

with probability at least $1 - (\log_2 n_i)^{rd}/(p_i^r n_i^{rd})$.

Since there are at most $k_i^{2s}$ possible pairs $(j, j')$, this implies that with probability at least $1 - k_i^{2s}(\log_2 n_i)^{rd}/(p_i^r n_i^{rd})$, for every picked pair $(K_{i,(j,j')}, K'_{i,(j,j')})$ there exists $1 \le t \le r$ such that (23) holds. If

$$k_i \ge \max(i, \kappa_i), \tag{24}$$

then using (20) and (22), this implies that with probability at least

$$1 - k_i^{2s}(\log_2 n_i)^{rd}/(p_i^r n_i^{rd}),$$

for any $(K, K') \in \widetilde{\mathcal{C}}_i$ there exists $1 \le t \le r$ such that

$$|\lambda(A^t \cap K) - \lambda(A^t \cap K')| \ge \frac{1}{4n_i^d} - 4k_i^{b-d}. \tag{25}$$

Therefore if we choose the sequences $(n_i)$ and $(k_i)$ so that (24),

$$\sum_{i=1}^{\infty} k_i^{2s}(\log_2 n_i)^{rd}/(p_i^r n_i^{rd}) < \infty \tag{26}$$

and

$$\frac{1}{4n_i^d} - 4k_i^{b-d} > 0 \qquad (i = 1, 2, \dots) \tag{27}$$

hold then by (21) and the Borel-Cantelli lemma we get that almost surely for any two distinct $K, K' \in \mathcal{C}$ we have $\lambda(A^t \cap K) \ne \lambda(A^t \cap K')$ for at least one $t \in \{1, \dots, r\}$, which is exactly what we need to prove.

Choose $k_i$ such that $k_i^{b-d} = n_i^{-d}/64$; that is, $k_i = n_i^{d/(d-b)}/64$. Then (27) clearly holds and (24) also holds if $n_i$ is large enough. Then, using that $r = \lfloor 2s/(d-b) \rfloor + 1$, we have

$$k_i^{2s}(\log_2 n_i)^{rd}/(p_i^r n_i^{rd}) = 64^{-2s} p_i^{-r} (\log_2 n_i)^{rd} n_i^{d(2s/(d-b)-r)} \le n_i^{-\delta d}$$

for $\delta = (r - 2s/(d-b))/2 > 0$, provided that we choose $n_i$ large enough compared to $1/p_i$. Since $\delta > 0$, this implies that (26) also holds if $(n_i)$ is increasing fast enough. This completes the proof of the theorem. ☐

## 8. Open questions

In this final section we collect some of the numerous remaining open questions.

**Question 8.1.** *How many test sets are needed to reconstruct an interval in $\mathbb{R}$?*

The answer is 3, 4 or 5 by Theorem 2.6 and Corollary 7.4 (1).

**Question 8.2.** *Let $d \ge 2$. How many test sets are needed to reconstruct a ball in $\mathbb{R}^d$? For example, does $d + 1$ suffice?*

We know by Corollary 7.4 (2) that $2d + 3$ test sets are enough. By Corollary 6.3 (1), if we consider only balls of radius at least 1 then the answer is $d + 1$ (for $d \ge 2$). In fact, we also do not know weather the restricition $r \ge 1$ on the magnification rate is necessary for the other two corollaries (6.3 (2) and 6.7) of Section 6.

**Question 8.3.** *Let $d = 1$ or $d = 2$. How many test sets are needed to reconstruct a translate of an arbitrary fixed bounded measurable subset of $\mathbb{R}^d$ of positive measure? For example, does $d$ suffice? Does finitely many suffice?*

For $d \ge 3$ we know by Corollary 5.8 that that $d$ sets suffice.

**Question 8.4.** *Let $d = 2$ or $d = 3$ and let $E \subset \mathbb{R}^d$ be an arbitrary bounded nondegenerate convex set. Can a set of the form $rE + x$ $(r \ge 1, x \in \mathbb{R}^d)$ be reconstructed using $d + 1$ test sets?*

Theorem 2.6 provides a negative answer for $d = 1$, whereas Corollary 6.7 shows that the answer is affirmative for $d \geq 4$.

**Question 8.5.** *Let $K$ be a compact set in $\mathbb{R}^2$ such that $\partial K$ has Hausdorff (or even upper box) dimension less than 2. Are the measure functions $f_{K,\theta}(r) = \lambda^{d-1}(K \cap \{x \in \mathbb{R}^d : \langle x, \theta \rangle = r\})$ of the sections of $K$ absolutely continuous for almost every direction $\theta$?*

Theorem 5.7 shows that this actually holds in dimension at least 3.

**Question 8.6.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth bump function. Can a function of the form $f(\frac{x}{a} + b)$ $(a > 0, b \in \mathbb{R})$ be reconstructed using 2 test sets?*

Theorem 4.1 shows that the answer is affirmative for $a \geq 1$ instead of $a > 0$.

## References

[1] K. Falconer: *Fractal Geometry: Mathematical Foundations and Applications.* Second Edition. John Wiley & Sons, 2003.
[2] C. Cutler, The density theorem and Hausdorff inequality for packing measure in general metric spaces, *Illinois J. Math.* **39** (1995), 676-694.
[3] A. S. Kechris: *Classical descriptive set theory.* Springer-Verlag, 1995.
[4] G. Leoni: *A first course in Sobolev spaces.* Graduate Studies in Mathematics, 105. American Mathematical Society, Providence, RI, 2009.
[5] J. Matoušek: *Using the Borsuk-Ulam theorem. Lectures on topological methods in combinatorics and geometry*, Universitext. Springer-Verlag, Berlin, 2003.
[6] P. Mattila: *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.

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