A NEW FRACTAL DIMENSION: THE TOPOLOGICAL HAUSDORFF DIMENSION

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ABSTRACT. We introduce a new concept of dimension for metric spaces, the so called *topological Hausdorff dimension*. It is defined by a very natural combination of the definitions of the topological dimension and the Hausdorff dimension. The value of the topological Hausdorff dimension is always between the topological dimension and the Hausdorff dimension, in particular, this new dimension is a non-trivial lower estimate for the Hausdorff dimension.

We examine the basic properties of this new notion of dimension, compare it to other well-known notions, determine its value for some classical fractals such as the Sierpinski carpet, the von Koch snowflake curve, Kakeya sets, the trail of the Brownian motion, etc.

As our first application, we generalize the celebrated result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. They proved that certain curves show up in the limit set when passing a critical probability, and we prove that actually 'thick' families of curves show up, where roughly speaking the word thick means that the curves can be parametrized in a natural way by a set of large Hausdorff dimension. The proof of this is basically a lower estimate of the topological Hausdorff dimension of the limit set. For the sake of completeness, we also give an upper estimate and conclude that in the non-trivial cases the topological Hausdorff dimension is almost surely strictly below the Hausdorff dimension.

Finally, as our second application, we show that the topological Hausdorff dimension is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space.

CONTENTS

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1. Introduction

The term 'fractal' was introduced by Mandelbrot in his celebrated book [15]. He formally defined a subset of a Euclidean space to be a fractal if its topological dimension is strictly smaller than its Hausdorff dimension. This is just one example to illustrate the fundamental role these two notions of dimension play in the study of fractal sets. To mention another such example, let us recall that the topological dimension of a metric space X is the infimum of the Hausdorff dimensions of the metric spaces homeomorphic to X, see [10].

The main goal of this paper is to introduce a new concept of dimension, the so called *topological Hausdorff dimension*, that interpolates the two above mentioned dimensions in a very natural way. Let us recall the definition of the (small inductive) topological dimension (see e.g. [4, 10]).

Definition 1.1. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Our new dimension will be defined analogously, however, note that this second definition will not be inductive, and also that it can attain non-integer values as well. The Hausdorff dimension of a metric space X is denoted by $\dim_H X$, see e.g. [5] or [16]. In this paper we adopt the convention that $\dim_H \emptyset = -1$.

Definition 1.2. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

(Both notions of dimension can attain the value ∞ as well, actually we use the convention $\infty - 1 = \infty$, hence $d = \infty$ is a member of the above set.)

It was not this analogy that initiated the study of this new concept. Our original motivation was that this notion grew out naturally from our investigations of the following topic. In [12] B. Kirchheim considered generic (in the sense of Baire category) continuous functions f, defined on $[0,1]^d$. He proved that for every $y \in$

$\operatorname{int} f([0,1]^d)$ we have $\dim_H f^{-1}(y) = d-1$, that is, as one would expect, 'most' level sets are of Hausdorff dimension $d-1$. The next problem is about generalizations of this result to fractal sets in place of $[0,1]^d$.

Problem 1.3. *Describe the Hausdorff dimension of the level sets of the generic continuous function defined on a compact metric space.*

It has turned out that the topological Hausdorff dimension is the right concept to deal with this problem. We will essentially prove that the value $d-1$ in Kirchheim's result has to be replaced by $\dim_{tH} K - 1$, see the end of this introduction or Section 7 for the details.

We would also like to mention another potentially very interesting motivation of this new concept. Unlike most well-known notions of dimension, such as packing or box-counting dimensions, the topological Hausdorff dimension is smaller than the Hausdorff dimension. As it is often an important and difficult task to estimate the Hausdorff dimension from below, this gives another reason why to study the topological Hausdorff dimension.

It is also worth mentioning that there is another recent approach by M. Urbański [22] to combine the topological dimension and the Hausdorff dimension. However, his new concept, called the transfinite Hausdorff dimension is quite different in nature from ours, e.g. it takes ordinal numbers as values.

Moreover, the first listed author (using ideas of U. B. Darji and the third listed author) recently generalized the results of the paper for maps taking values in $\mathbb{R}^n$ instead of $\mathbb{R}$. The new concept of dimension needed to describe the Hausdorff dimension of the fibers of the generic continuous map is called the nth inductive topological Hausdorff dimension, see [1].

Next we say a few words about the main results and the organization of the paper.

In Section 3 we discuss some alternative definitions of the topological Hausdorff dimension yielding the same concept. Recall that the following classical theorem in fact describes an alternative recursive definition of the topological dimension.

Theorem 3.1. *If X is a non-empty separable metric space then*

$$\dim_t X = \min\{d : \exists A \subseteq X \textit{ such that } \dim_t A \le d-1 \textit{ and } \dim_t(X \setminus A) \le 0\}.$$

The next result shows that by replacing one instance of $\dim_t A$ by $\dim_H A$ we again obtain the notion of topological Hausdorff dimension.

Theorem 3.6. *If X is a non-empty separable metric space then*

$$\dim_{tH} X = \min\{d : \exists A \subseteq X \textit{ such that } \dim_H A \le d-1 \textit{ and } \dim_t(X \setminus A) \le 0\}.$$

As a corollary we also obtain that actually inf = min in our original definition of the topological Hausdorff dimension, which is not only interesting, but will also be used in one of the applications. We discuss the analogues of the other definitions of the topological dimension as well, such as the large inductive dimension and the Lebesgue covering dimension.

In Section 4 we investigate the basic properties of the topological Hausdorff dimension. Among others, we prove the following.

Theorem 4.4. $\dim_t X \leq \dim_{tH} X \leq \dim_H X$.

We also verify that $\dim_{tH} X$ satisfies some standard properties of a dimension, such as monotonicity, bi-Lipschitz invariance and countable stability for closed sets. We discuss the existence of G_δ hulls and F_σ subsets with the same topological Hausdorff dimension, as well. Moreover, we check that this concept is genuinely new in that $\dim_{tH} X$ cannot be expressed as a function of $\dim_t X$ and $\dim_H X$.

In Section 5 we compute $\dim_{tH} X$ for some classical fractals, like the Sierpiński triangle and carpet, the von Koch curve, etc. For example

Theorem 5.4. *Let T be the Sierpiński carpet. Then* $\dim_{tH} T = \frac{\log 6}{\log 3} = \frac{\log 2}{\log 3} + 1$.

(Note that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$ while the Hausdorff dimension of the triadic Cantor set equals $\frac{\log 2}{\log 3}$.)

We also consider Kakeya sets (see [5] or [16]). Unfortunately, our methods do not give any useful information concerning the Kakeya Conjecture.

Theorem 5.6. *For every $d \in \mathbb{N}^+$ there exist a compact Kakeya set of topological Hausdorff dimension 1 in $\mathbb{R}^d$.*

Following [13] by T. W. Körner we prove somewhat more, since we essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1.

We show that the range of the Brownian motion almost surely (i.e. with probability 1) has topological Hausdorff dimension 1 in every dimension except perhaps 2 and 3. These two cases remain the most intriguing open problems of the paper.

Problem 5.8. *Determine the almost sure topological Hausdorff dimension of the range of the d-dimensional Brownian motion for $d = 2$ or 3. Equivalently, determine the smallest $c \geq 0$ such that the range can be decomposed into a totally disconnected set and a set of Hausdorff dimension at most $c - 1$ almost surely.*

We also relate the planar case to a well-known open problem of W. Werner and solve the dual version of this problem, that is, the version in which the notion of Wiener measure is replaced by Baire category. In a similar vein, we also show that the range of the generic (in the sense of Baire category) continuous map $f \colon [0,1] \to \mathbb{R}^d$ is of topological Hausdorff dimension 1 for every d.

As our first application in Section 6 we generalize a result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. This limit set $M = M^{(p,n)}$ is a random Cantor set, which is constructed by dividing the unit square into $n \times n$ equal sub-squares and keeping each of them independently with probability p, and then repeating the same procedure recursively for every sub-square. (See Section 6 for more details.)

Theorem 6.1 (Chayes-Chayes-Durrett, [3])**.** *There exists a critical probability* $p_c = p_c^{(n)} \in (0,1)$ *such that if* $p < p_c$ *then* M *is totally disconnected almost surely, and if* $p > p_c$ *then* M *contains a nontrivial connected component with positive probability.*

It will be easy to see that this theorem is a special case of our next result.

Theorem 6.2. *For every* $d \in [0,2)$ *there exists a critical probability* $p_c^{(d)} = p_c^{(d,n)} \in (0,1)$ *such that if* $p < p_c^{(d)}$ *then* $\dim_{tH} M \le d$ *almost surely, and if* $p > p_c^{(d)}$ *then* $\dim_{tH} M > d$ *almost surely (provided* $M \neq \emptyset$*).*

Theorem 6.1 essentially says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, which roughly speaking means a 'Lipschitz copy' of $C \times [0,1]$ with $\dim_H C > d - 1$.

We also give a numerical upper bound for $\dim_{tH} M$ which implies the following.

Corollary 6.18. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

In Section 7 we answer Problem 1.3 as follows.

Corollary 7.14. *If* K *is a compact metric space with* $\dim_t K > 0$ *then* $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ *for the generic* $f \in C(K)$*.*

(If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, thus every non-empty level set is of Hausdorff dimension 0.)

If K is also sufficiently homogeneous, e.g. self-similar then we can actually say more.

Corollary 7.16. *If* K *is a self-similar compact metric space with* $\dim_t K > 0$ *then* $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ *for the generic* $f \in C(K)$ *and the generic* $y \in f(K)$*.*

It can actually also be shown that in the equation $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ (for the generic $f \in C(K)$) the supremum is attained. On the other hand, one cannot say more in a sense, since there is a K such that for the generic $f \in C(K)$ there is a *unique* $y \in \mathbb{R}$ for which $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. Moreover, in certain situations we can replace 'the generic $y \in f(K)$' with 'for every $y \in \operatorname{int} f(K)$' as in Kirchheim's theorem. The results of this last paragraph appeared elsewhere, see [2].

Finally, in Section 8 we list some open problems.

2. Preliminaries

Let (X,d) be a metric space. For $A, B \subseteq X$ let us define $\operatorname{dist}(A,B) = \inf\{d(x,y) : x \in A,\ y \in B\}$. Let $B(x,r)$ and $U(x,r)$ stand for the closed and open ball of radius r centered at x, respectively. Set $B(A,r) = \{x \in X : \operatorname{dist}(\{x\},A) \le r\}$ and $U(A,r) = \{x \in X : \operatorname{dist}(\{x\},A) < r\}$. We denote by $\operatorname{cl} A$, $\operatorname{int} A$ and ∂A the closure, interior and boundary of A, respectively. The diameter of a set A is denoted by $\operatorname{diam} A$. We use the convention $\operatorname{diam} \emptyset = 0$. For two metric spaces (X, d_X) and (Y, d_Y) a function $f \colon X \to Y$ is *Lipschitz* if there exists a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \le C \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The smallest such constant C is called the Lipschitz constant of f and denoted by $\operatorname{Lip}(f)$. A function $f \colon X \to Y$

is called *bi-Lipschitz* if f is a bijection and both f and f^{-1} are Lipschitz. For a metric space X and $s \geq 0$ the *s-dimensional Hausdorff measure* is defined as

$$\mathcal{H}^s(X) = \lim_{\delta \to 0+} \mathcal{H}^s_\delta(X), \text{ where}$$

$$\mathcal{H}^s_\delta(X) = \inf \left\{ \sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : X \subseteq \bigcup_{i=1}^{\infty} U_i, \ \forall i \operatorname{diam} U_i \leq \delta \right\}.$$

The *Hausdorff dimension of* X is defined as

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

It is not difficult to see using the regularity of $\mathcal{H}^s_\delta$ that every set is contained in a G_δ set of the same Hausdorff dimension. For more information on these concepts see [5] or [16].

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. By Baire's Category Theorem co-meager sets are dense. It is not difficult to show that a set is co-meager iff it contains a dense G_δ set. We say that the generic element $x \in X$ has property $\mathcal{P}$ if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The term 'typical' is also used instead of 'generic'. Our two main examples will be $X = C(K)$ endowed with the supremum metric (for some compact metric space K) and $X = \mathcal{K}$, that is, a certain subspace of the non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric (i.e. $d_H(K_1, K_2) = \min\{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\}$). See e.g. [11] for more on these concepts.

A topological space X is called *totally disconnected* if X is empty or every connected component of X is a singleton. If $\dim_t X \leq 0$ then X is clearly totally disconnected. If X is locally compact then $\dim_t X \leq 0$ iff X is totally disconnected, see [4, 1.4.5.]. If X is a σ-compact metric space then $\dim_t X \leq 0$ iff X is totally disconnected, see the countable stability of topological dimension zero for closed sets [4, 1.3.1.].

3. Equivalent definitions of the topological Hausdorff dimension

The goal of this section is to prove some equivalent definitions which will play an important role later. Perhaps the main point here is that while the original definition is of local nature, we can find an equivalent global definition.

Let us recall the following decomposition theorem for the topological dimension, see [4, 1.5.7.], which can be regarded as an equivalent definition.

Theorem 3.1. *If X is a non-empty separable metric space then*

$$\dim_t X = \min\{d : \exists A \subseteq X \textit{ such that } \dim_t A \leq d-1 \textit{ and } \dim_t(X \setminus A) \leq 0\}.$$

(Note that as above, ∞ is assumed to be a member of the above set, moreover we use the convention $\min\{\infty\} = \infty$.)

The main goal of this section is to prove Theorem 3.6, an analogous decomposition theorem for the topological Hausdorff dimension, which yields an equivalent definition of the topological Hausdorff dimension. As a by-product, we obtained that in the definition of $\dim_{tH} X$ the infimum is attained, see Corollary 3.8.

Remark 3.2. One can actually check that, as usual in dimension theory, the assumption of separability cannot be dropped.

Let X be a given non-empty separable metric space. For the sake of notational simplicity we use the following notation.

Notation 3.3.

$$P_{tH} = \{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\},$$
$$P_{dH} = \{d : \exists A \subseteq X \text{ such that } \dim_H A \leq d-1 \text{ and } \dim_t(X \setminus A) \leq 0\}.$$

We assume $\infty \in P_{tH}, P_{dH}$.

The following lemmas are the heart of the section.

Lemma 3.4. $P_{tH} = P_{dH}$.

Proof. First we prove $P_{tH} \subseteq P_{dH}$. Assume $d \in P_{tH}$ and $d < \infty$. Then there exists a countable basis $\mathcal{U}$ of X such that $\dim_H \partial U \leq d-1$ for all $U \in \mathcal{U}$. Let $A = \bigcup_{U \in \mathcal{U}} \partial U$, then the countable stability of the Hausdorff dimension yields $\dim_H A \leq d-1$, and the definition of A clearly implies $\dim_t(X \setminus A) \leq 0$. Hence $d \in P_{dH}$.

Now we prove $P_{dH} \subseteq P_{tH}$. Assume $d \in P_{dH}$ and $d < \infty$. Let us fix $x \in X$ and $r > 0$. To verify $d \in P_{tH}$ we need to find an open set $U \subseteq X$ such that $x \in U \subseteq U(x,r)$ and $\dim_H \partial U \leq d-1$. Since $d \in P_{dH}$, there is a set $A \subseteq X$ such that $\dim_H A \leq d-1$ and $\dim_t(X \setminus A) \leq 0$. As $X \setminus A$ is a separable subspace of X with topological dimension 0, by the separation theorem for topological dimension zero [4, 1.2.11.] there is a so-called partition between x and $X \setminus U(x,r)$ disjoint from $X \setminus A$. This means that there exist disjoint open sets $U, U' \subseteq X$ such that $x \in U$, $X \setminus U(x,r) \subseteq U'$ and $(X \setminus (U \cup U')) \cap (X \setminus A) = \emptyset$. In particular, $x \in U \subseteq U(x,r)$. Moreover, $\partial U \cap (X \setminus A) = \emptyset$, so $\partial U \subseteq A$, thus $\dim_H \partial U \leq \dim_H A \leq d-1$. Hence $d \in P_{tH}$. □

Lemma 3.5. $\inf P_{dH} \in P_{dH}$.

Proof. Let $d = \inf P_{dH}$, we may assume $d < \infty$. Set $d_n = d + 1/n$ for all $n \in \mathbb{N}^+$. As $d_n \in P_{dH}$, there exist sets $A_n \subseteq X$ such that $\dim_H A_n \leq d_n - 1$ and $\dim_t(X \setminus A_n) \leq 0$. We may assume that the sets A_n are G_δ, since we can take G_δ hulls with the same Hausdorff dimension. Let $A = \bigcap_{n=1}^{\infty} A_n$, then clearly $\dim_H A \leq d-1$. As $X \setminus A_n$ are F_σ sets such that $\dim_t(X \setminus A_n) \leq 0$ and $X \setminus A = \bigcup_{n=1}^{\infty}(X \setminus A_n)$, countable stability of topological dimension zero for F_σ sets [4, 1.3.3. Corollary] yields $\dim_t(X \setminus A) \leq 0$. Hence $d \in P_{dH}$. □

Now the main result of the section is an easy consequence of Lemma 3.4, Lemma 3.5 and the definition of the topological Hausdorff dimension.

Theorem 3.6. *If X is a non-empty separable metric space then*

$$\dim_{tH} X = \min\{d : \exists A \subseteq X \text{ such that } \dim_H A \leq d-1 \text{ and } \dim_t(X \setminus A) \leq 0\}.$$

Since every set is contained in a G_δ set of the same Hausdorff dimension, and a σ-compact metric space is totally disconnected iff it has topological dimension zero, the above theorem yields the following equivalent definition.

Theorem 3.7. *For a non-empty σ-compact metric space X*

$$\dim_{tH} X = \min\{d : \exists A \subseteq X \text{ such that } \dim_H A \leq d-1 \text{ and } X \setminus A \text{ is totally disconnected}\}.$$

Moreover, as a by-product of Lemma 3.4 and Lemma 3.5 we obtain that the infimum is attained in the original definition, which will play a role in one of the applications.

Corollary 3.8. *If X is a non-empty separable metric space then*

$$\dim_{tH} X = \min\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \le d-1 \text{ for every } U \in \mathcal{U}\}.$$

Remark 3.9. There are even more equivalent definitions of the topological dimension, and one can prove that the appropriate analogues result in the same notion as well, but since these statements will not be used in the sequel we only state the results here. For some details consult [1].

Moreover, Section 7 contains further equivalent definitions in the compact case in connection with the level sets of generic continuous functions.

The next classical theorem shows that the notion of large inductive dimension coincides with the other definitions of topological dimension for separable metric spaces.

Theorem 3.10. *For every non-empty separable metric space X*

$$\dim_t X = \min\{d : \forall F \subseteq X \text{ closed and } \forall V \text{ open with } F \subseteq V,\ \exists U \text{ open such that } F \subseteq U \subseteq V \text{ and } \dim_t \partial U \le d-1\}.$$

The natural analogue yields the same concept again.

Theorem 3.11. *For every non-empty separable metric space X*

$$\dim_{tH} X = \min\{d : \forall F \subseteq X \text{ closed and } \forall V \text{ open with } F \subseteq V,\ \exists U \text{ open such that } F \subseteq U \subseteq V \text{ and } \dim_H \partial U \le d-1\}.$$

Next we take up the Lebesgue covering dimension.

For a family $\mathcal{A}$ of sets and $m \in \mathbb{N}^+$ let $T_m(\mathcal{A})$ denote the set of points covered by at least m members of $\mathcal{A}$.

Theorem 3.12. *For every non-empty separable metric space X*

$$\dim_t X = \min\{d : \forall \text{ finite open cover } \mathcal{U} \text{ of } X\ \exists \text{ a finite open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ such that } T_{d+2}(\mathcal{V}) = \emptyset\}.$$

And as above

Theorem 3.13. *For every separable metric space X with $\dim_t X > 0$*

$$\dim_{tH} X = \min\{d : \forall \varepsilon > 0\ \forall \text{ finite open cover } \mathcal{U} \text{ of } X\ \exists \text{ a finite open refinement } \mathcal{V} \text{ of } \mathcal{U} \text{ such that } \mathcal{H}^{d-1+\varepsilon}_\infty(T_2(\mathcal{V})) \le \varepsilon\}.$$

4. Basic properties of the topological Hausdorff dimension

Let X be a metric space. Since $\dim_t X = -1 \iff X = \emptyset \iff \dim_H X = -1$, we easily obtain

Fact 4.1. $\dim_{tH} X = 0 \iff \dim_t X = 0$.

As $\dim_H X$ is either -1 or at least 0, we obtain

Fact 4.2. *The topological Hausdorff dimension of a non-empty space is either* 0 *or at least* 1.

These two facts easily yield

Corollary 4.3. *Every metric space with a non-trivial connected component has topological Hausdorff dimension at least one.*

The next theorem states that the topological Hausdorff dimension is between the topological and the Hausdorff dimension.

Theorem 4.4. *For every metric space X*

$$\dim_t X \leq \dim_{tH} X \leq \dim_H X.$$

Proof. We can clearly assume that X is non-empty. It is well-known that $\dim_t X \leq \dim_H X$ (see e.g. [10]), which easily implies $\dim_t X \leq \dim_{tH} X$ using the definitions. The second inequality is obvious if $\dim_H X = \infty$. If $\dim_H X < 1$ then $\dim_t X = 0$ (since $\dim_t X \leq \dim_H X$ and $\dim_t X$ only takes integer values) and by Fact 4.1 we obtain $\dim_{tH} X = 0$, hence the second inequality holds. Therefore we may assume that $1 \leq \dim_H X < \infty$. The following lemma is basically [16, Thm. 7.7]. It is only stated there in the special case $X = A \subseteq \mathbb{R}^n$, but the proof works verbatim for all metric spaces X.

Lemma 4.5. *Let X be a metric space and $f\colon X \to \mathbb{R}^m$ be Lipschitz. If $s \geq m$, then*

$$\int^\star \mathcal{H}^{s-m}\left(f^{-1}(y)\right) \mathrm{d}\mathcal{H}^m(y) \leq c(m)\,\mathrm{Lip}(f)^m \mathcal{H}^s(X), \tag{4.1}$$

where $\int^\star$ denotes the upper integral and $c(m)$ is a finite constant depending only on m.

Now we return to the proof of Theorem 4.4. We fix $x_0 \in X$ and define $f\colon X \to \mathbb{R}$ by $f(x) = d_X(x, x_0)$. Using the triangle inequality it is easy to see that f is Lipschitz with $\mathrm{Lip}(f) \leq 1$. We fix $n \in \mathbb{N}^+$ and apply Lemma 4.5 for f and $s = \dim_H X + \frac{1}{n} > 1 = m$. Hence

$$\int^\star \mathcal{H}^{s-1}(f^{-1}(y))\,\mathrm{d}\mathcal{H}^1(y) \leq c(1)\mathcal{H}^s(X) = 0.$$

Thus $\mathcal{H}^{s-1}(f^{-1}(y)) = \mathcal{H}^{\dim_H X + \frac{1}{n} - 1}(f^{-1}(y)) = 0$ holds for a.e. $y \in \mathbb{R}$. Since this is true for all $n \in \mathbb{N}^+$, we obtain that $\dim_H f^{-1}(y) \leq \dim_H X - 1$ for a.e. $y \in \mathbb{R}$. From the definition of f it follows that $\partial U(x_0, y) \subseteq f^{-1}(y)$. Hence there is a neighborhood basis of x_0 with boundaries of Hausdorff dimension at most $\dim_H X - 1$, and this is true for all $x_0 \in X$, so there is a basis with boundaries of Hausdorff dimension at most $\dim_H X - 1$. By the definition of the topological Hausdorff dimension this implies $\dim_{tH} X \leq \dim_H X$. □

There are some elementary properties one expects from a notion of dimension. Now we verify some of these for the topological Hausdorff dimension.

Extension of the classical dimension. Theorem 4.4 implies that the topological Hausdorff dimension of a countable set equals zero, moreover, for open subspaces of $\mathbb{R}^d$ and for smooth d-dimensional manifolds the topological Hausdorff dimension equals d.

Monotonicity. Let $X \subseteq Y$. If $\mathcal{U}$ is a basis in Y then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is a basis in X, and $\partial_X(U \cap X) \subseteq \partial_Y U$ holds for all $U \in \mathcal{U}$. This yields

Fact 4.6 (Monotonicity). *If $X \subseteq Y$ are metric spaces then* $\dim_{tH} X \le \dim_{tH} Y$.

Bi-Lipschitz invariance. First we prove that the topological Hausdorff dimension does not increase under Lipschitz homeomorphisms. An easy consequence of this that our dimension is bi-Lipschitz invariant, and does not increase under an injective Lipschitz map on a compact space. After obtaining corollaries of Theorem 4.7 we give some examples illustrating the necessity of certain conditions in this theorem and its corollaries.

Theorem 4.7. *Let X, Y be metric spaces. If $f\colon X \to Y$ is a Lipschitz homeomorphism then* $\dim_{tH} Y \le \dim_{tH} X$.

Proof. Since f is a homeomorphism, if $\mathcal{U}$ is a basis in X then $\mathcal{V} = \{f(U) : U \in \mathcal{U}\}$ is a basis in Y, and $\partial f(U) = f(\partial U)$ for all $U \in \mathcal{U}$. The Lipschitz property of f implies that $\dim_H \partial V = \dim_H \partial f(U) = \dim_H f(\partial U) \le \dim_H \partial U$ for all $V = f(U) \in \mathcal{V}$. Thus $\dim_{tH} Y \le \dim_{tH} X$. □

This immediately implies the following two statements.

Corollary 4.8 (Bi-Lipschitz invariance). *Let X, Y be metric spaces. If $f\colon X \to Y$ is bi-Lipschitz then* $\dim_{tH} X = \dim_{tH} Y$.

Corollary 4.9. *If K is a compact metric space, and $f\colon K \to Y$ is* one-to-one *Lipschitz then* $\dim_{tH} f(K) \le \dim_{tH} K$.

The following example shows that we cannot drop injectivity here. First we need a well-known lemma.

Lemma 4.10. *Let $M \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Then there exists a Lipschitz onto map $f\colon M \to [0,1]$.*

Proof. Let us choose a compact set $C \subseteq M$ of positive Lebesgue measure. Define $f\colon M \to [0,1]$ by

$$f(x) = \frac{\lambda\left((-\infty, x) \cap C\right)}{\lambda(C)},$$

where λ denotes the one-dimensional Lebesgue measure. Then it is not difficult to see that f is Lipschitz (with $\mathrm{Lip}(f) \le \frac{1}{\lambda(C)}$) and $f(C) = f(M) = [0,1]$. □

Example 4.11. Let $K \subseteq \mathbb{R}$ be a Cantor set (that is, a set homeomorphic to the middle-thirds Cantor set) of positive Lebesgue measure. By Fact 4.1, $\dim_{tH} K = \dim_t K = 0$. Using Lemma 4.10 there is a Lipschitz map $f\colon K \to [0,1]$ such that $f(K) = [0,1]$. By Theorem 4.4, $\dim_{tH}[0,1] = 1$, hence $\dim_{tH} K = 0 < 1 = \dim_{tH}[0,1] = \dim_{tH} f(K)$.

The next example shows that Corollary 4.9 does not hold without the assumption of compactness. We even have a separable metric counterexample.

Example 4.12. Let C be the middle-thirds Cantor set, and $f\colon C \times C \to [0,2]$ be defined by $f(x,y) = x + y$. It is well-known and easy to see that f is Lipschitz and $f(C \times C) = [0,2]$. Therefore one can select a subset $X \subseteq C \times C$ such that $f|_X$ is a bijection from X onto $[0,2]$. Then X is separable metric. Monotonicity and $\dim_t(C \times C) = 0$ imply $\dim_{tH} X \le \dim_{tH}(C \times C) = 0$. Therefore, f is one-to-one and Lipschitz on X but $\dim_{tH} X = 0 < 1 = \dim_{tH}[0,2] = \dim_{tH} f(X)$.

Our last example shows that the topological Hausdorff dimension is not invariant under homeomorphisms. Not even for compact metric spaces.

Example 4.13. Let $C_1, C_2 \subseteq \mathbb{R}$ be Cantor sets such that $\dim_H C_1 \neq \dim_H C_2$. We will see in Theorem 4.21 that $\dim_{tH}(C_i \times [0,1]) = \dim_H C_i + 1$ for $i = 1, 2$. Hence $C_1 \times [0,1]$ and $C_2 \times [0,1]$ are homeomorphic compact metric spaces whose topological Hausdorff dimensions disagree.

Stability and countable stability. As the following example shows, similarly to the case of topological dimension, stability does not hold for non-closed sets. That is, $X = \bigcup_{n=1}^{k} X_n$ does not imply $\dim_{tH} X = \max_{1\leq n\leq k} \dim_{tH} X_n$.

Example 4.14. Theorem 4.4 implies $\dim_{tH} \mathbb{R} = 1$, and Fact 4.1 yields $\dim_{tH} \mathbb{Q} = \dim_t \mathbb{Q} = 0$ and $\dim_{tH}(\mathbb{R} \setminus \mathbb{Q}) = \dim_t(\mathbb{R} \setminus \mathbb{Q}) = 0$. Thus $\dim_{tH} \mathbb{R} = 1 > 0 = \max\{\dim_{tH} \mathbb{Q}, \dim_{tH}(\mathbb{R} \setminus \mathbb{Q})\}$, and therefore stability fails.

As a corollary, we now show that as opposed to the case of Hausdorff (and packing) dimension, there is no reasonable family of measures inducing the topological Hausdorff dimension. Let us say that a 1-parameter family of measures $\{\mu^s\}_{s\geq 0}$ is *monotone* if $\mu^s(A) = 0$, $s < t$ implies $\mu^t(A) = 0$. The family of Hausdorff (or packing) measures certainly satisfies this criterion. It is not difficult to see that monotonicity implies that the induced notion of dimension, that is, $\dim A = \inf\{s : \mu^s(A) = 0\}$ is countably stable. Hence we obtain

Corollary 4.15. *There is no monotone 1-parameter family of measures $\{\mu^s\}_{s\geq 0}$ such that $\dim_{tH} A = \inf\{s : \mu^s(A) = 0\}$.*

However, just like in the case of topological dimension, even countable stability holds for *closed* sets.

Theorem 4.16 (Countable stability for closed sets)**.** *Let X be a separable metric space and $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n $(n \in \mathbb{N})$ are closed subsets of X. Then $\dim_{tH} X = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$.*

Proof. Monotonicity clearly implies $\dim_{tH} X \geq \sup_{n\in\mathbb{N}} \dim_{tH} X_n$.

For the opposite inequality let $d = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$, we may assume $d < \infty$. Theorem 3.6 yields that there are sets $A_n \subseteq X_n$ such that $\dim_H A_n \leq d - 1$ and $\dim_t(X_n \setminus A_n) \leq 0$. We may assume that the sets A_n are G_δ, since we can take G_δ hulls with the same Hausdorff dimension. Let $A = \bigcup_{n=0}^{\infty} A_n \subseteq X$, the countable stability of the Hausdorff dimension implies $\dim_H A \leq d - 1$. As the sets $X_n \setminus A_n$ are F_σ in X with $\dim_t(X_n \setminus A_n) \leq 0$ and $X \setminus A \subseteq \bigcup_{n=0}^{\infty}(X_n \setminus A_n)$, monotonicity and countable stability of topological dimension zero for F_σ sets [4, 1.3.3. Corollary] yield $\dim_t(X \setminus A) \leq 0$. Finally, $\dim_H A \leq d - 1$ and $\dim_t(X \setminus A) \leq 0$ together with Theorem 3.6 imply $\dim_{tH} X \leq d$, and the proof is complete. □

Corollary 4.17. *The same holds for F_σ sets, as well.*

Regularity. Next we investigate the existence of G_δ hulls and F_σ subsets with the same topological Hausdorff dimension.

Theorem 4.18 (Enlargement theorem for topological Hausdorff dimension)**.** *If X is a metric space and $Y \subseteq X$ is a separable subspace then there exists a G_δ set $G \subseteq X$ such that $Y \subseteq G$ and $\dim_{tH} G = \dim_{tH} Y$.*

Proof. We may assume $Y \neq \emptyset$. Theorem 3.6 implies that there exists $Z \subseteq Y$ such that $\dim_H Z = \dim_{tH} Y - 1$ and $\dim_t(Y \setminus Z) \leq 0$. Let $A \subseteq X$ be a G_δ set such that $Z \subseteq A$ and $\dim_H A = \dim_H Z = \dim_{tH} Y - 1$. By [4, 1.2.14.] there exists a G_δ set $B \subseteq X$ such that $Y \setminus Z \subseteq B$ and $\dim_t B \le 0$. Let $G = A \cup B$, then G is a G_δ subset of X with $Y \subseteq G$. Since $\dim_t(G \setminus A) \le \dim_t B \le 0$, Theorem 3.6 implies that $\dim_{tH} G \le \dim_{tH} A + 1 = \dim_{tH} Y$, and monotonicity yields $\dim_{tH} G = \dim_{tH} Y$. □

The following example shows that inner regularity does not hold even for G_δ subsets of Euclidean spaces.

Example 4.19. For every $d \in \mathbb{N}^+$ Mazurkiewicz [17] constructed a G_δ set $G \subseteq [0,1]^{d+1}$ such that $\dim_t G = d$ and G is totally disconnected (see [18, Theorem 3.9.3.] for a proof in English). Theorem 4.4 implies that $\dim_{tH} G \ge \dim_t G = d$. If $F \subseteq G$ is closed in $\mathbb{R}^{d+1}$ then F is compact and totally disconnected, thus $\dim_t F \le 0$, so Fact 4.1 implies $\dim_{tH} F \le 0$. Therefore countable stability of the topological Hausdorff dimension for closed sets yields that every F_σ subset of G has topological Hausdorff dimension at most 0.

Products. Now we investigate products from the point of view of topological Hausdorff dimension. By product of two metric spaces we will always mean the l^2-product, that is,

$$d_{X\times Y}((x_1,y_1),(x_2,y_2)) = \sqrt{d_X^2(x_1,x_2) + d_Y^2(y_1,y_2)}.$$

First we recall a well-known statement, see [5, Chapters 3] and [5, Product formula 7.3] for the definition of the upper box-counting dimension and the proof, respectively. In fact, [5] works in Euclidean spaces only, but the proof goes through verbatim to general metric spaces.

Lemma 4.20. *Let X, Y be non-empty metric spaces such that $\dim_H Y = \overline{\dim}_B Y$, where $\overline{\dim}_B$ is the upper box-counting dimension. Then*

$$\dim_H(X \times Y) \le \dim_H X + \dim_H Y.$$

Now we prove our next theorem which provides a large class of sets for which the topological Hausdorff dimension and the Hausdorff dimension coincide.

Theorem 4.21. *Let X be a non-empty separable metric space. Then*

$$\dim_{tH}(X \times [0,1]) = \dim_H(X \times [0,1]) = \dim_H X + 1.$$

Proof. From Theorem 4.4 it follows that $\dim_{tH}(X \times [0,1]) \le \dim_H(X \times [0,1])$.

Applying Lemma 4.20 for $Y = [0,1]$ we deduce that

$$\dim_H(X \times [0,1]) \le \dim_H X + \dim_H [0,1] = \dim_H X + 1.$$

Finally, we prove that $\dim_H X + 1 \le \dim_{tH}(X \times [0,1])$. Let us define $\mathrm{pr}_X : X \times [0,1] \to X$ as $\mathrm{pr}_X(x,y) = x$ and let $Z = X \times [0,1]$. Theorem 3.6 implies that there is a set $A \subseteq Z$ such that $\dim_H A \le \dim_{tH} Z - 1$ and $\dim_t(Z \setminus A) \le 0$. Since $Z \setminus A$ is totally disconnected, A intersects $\{x\} \times [0,1]$ for all $x \in X$, thus $\mathrm{pr}_X(A) = X$. Projections do not increase the Hausdorff dimension, thus

$$\dim_{tH} Z - 1 \ge \dim_H A \ge \dim_H \mathrm{pr}_X(A) = \dim_H X.$$

Hence $\dim_{tH}(X \times [0,1]) \ge \dim_H X + 1$, and the proof is complete. □

Remark 4.22. We cannot drop separability here. Indeed, if X is an uncountable discrete metric space then it is not difficult to see that $\dim_{tH}(X \times [0,1]) = 1$ and $\dim_H(X \times [0,1]) = \dim_H X = \infty$.

Separability is a rather natural assumption throughout the paper. First, the Hausdorff dimension is only meaningful in this context (it is always infinite for non-separable spaces), secondly for the theory of topological dimension this is the most usual framework.

Corollary 4.23. *If X is a non-empty separable metric space then*

$$\dim_{tH}\left(X \times [0,1]^d\right) = \dim_H\left(X \times [0,1]^d\right) = \dim_H X + d.$$

The possible values of $(\dim_t X, \dim_{tH} X, \dim_H X)$. Now we provide a complete description of the possible values of the triple $(\dim_t X, \dim_{tH} X, \dim_H X)$. Moreover, all possible values can be realized by compact spaces as well.

Theorem 4.24. *For a triple $(d,s,t) \in [0,\infty]^3$ the following are equivalent.*

(i) There exists a compact metric space K such that $\dim_t K = d$, $\dim_{tH} K = s$, and $\dim_H K = t$.
(ii) There exists a separable metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.
(iii) There exists a metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.
(iv) $d = s = t = -1$, or $d = s = 0$, $t \in [0,\infty]$, or $d \in \mathbb{N}^+ \cup \{\infty\}$, $s,t \in [1,\infty]$, $d \leq s \leq t$.

Proof. The implications $(i) \implies (ii)$ and $(ii) \implies (iii)$ are obvious, and $(iii) \implies (iv)$ can easily be checked using Fact 4.1 and Theorem 4.4.

It remains to prove that $(iv) \implies (i)$. First, the empty set takes care of the case $d = s = t = -1$. Let now $d = s = 0$, $t \in [0,\infty]$. For $t \in [0,\infty]$ let K_t be a Cantor set with $\dim_H K_t = t$. Such sets are well-known to exist already in $[0,1]^n$ for large enough n in case $t < \infty$, whereas if C is the middle-thirds Cantor set then $C^{\mathbb{N}}$ is such a set for $t = \infty$. Then clearly $\dim_t K_t = \dim_{tH} K_t = 0$ and $\dim_H K_t = t$, so we are done with this case.

Finally, let $d \in \mathbb{N}^+ \cup \{\infty\}$, $s,t \in [1,\infty]$, $d \leq s \leq t$. We may assume $d < \infty$, otherwise the Hilbert cube provides a suitable example. (Indeed, clearly $\dim_t [0,1]^{\mathbb{N}} = \dim_{tH} [0,1]^{\mathbb{N}} = \dim_H [0,1]^{\mathbb{N}} = \infty$.) Define $K_{d,s,t} = (K_{s-d} \times [0,1]^d) \cup K_t$ (this can be understood as the disjoint sum of metric spaces, but we may also assume that all these spaces are in the Hilbert cube, so the union is well defined). Since $\dim_t(X \times Y) \leq \dim_t X + \dim_t Y$ for non-empty spaces (see e.g. [4]), we obtain $\dim_t(K_{s-d} \times [0,1]^d) = 0 + d = d$. Hence, by the stability of the topological dimension for closed sets, $\dim_t K_{d,s,t} = \max\left\{\dim_t\left(K_{s-d} \times [0,1]^d\right), \dim_t K_t\right\} = \max\{d, 0\} = d$. Using Corollary 4.23 and the stability of the topological Hausdorff dimension for closed sets we infer that $\dim_{tH} K_{d,s,t} = \max\left\{\dim_{tH}\left(K_{s-d} \times [0,1]^d\right), \dim_{tH} K_t\right\} = \max\{s-d+d, 0\} = s$. Again by Corollary 4.23 and by the stability of the Hausdorff dimension we obtain that $\dim_H K_{d,s,t} = \max\left\{\dim_H\left(K_{s-d} \times [0,1]^d\right), \dim_H K_t\right\} = \max\{s-d+d, t\} = \max\{s,t\} = t$. This completes the proof. □

The topological Hausdorff dimension is not a function of the topological and the Hausdorff dimension. As a particular case of the above theorem we obtain that there are compact metric spaces X and Y such that $\dim_t X = \dim_t Y$ and $\dim_H X = \dim_H Y$ but $\dim_{tH} X \neq \dim_{tH} Y$. This immediately implies the following, which shows that the topological Hausdorff dimension is indeed a genuinely new concept.

Corollary 4.25. $\dim_{tH} X$ *cannot be calculated from* $\dim_t X$ *and* $\dim_H X$*, even for compact metric spaces.*

5. Calculating the topological Hausdorff dimension

5.1. **Some classical fractals.** First we present certain natural examples of compact sets K with $\dim_t K = \dim_{tH} K < \dim_H K$. Let S be the Sierpiński triangle, then it is well-known that $\dim_t S = 1$ and $\dim_H S = \frac{\log 3}{\log 2}$.

Theorem 5.1. *Let S be the Sierpiński triangle. Then* $\dim_{tH}(S) = 1$.

Proof. Let $\varphi_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2,3)$ be the three similitudes with ratio $1/2$ for which $S = \bigcup_{i=1}^3 \varphi_i(S)$. Sets of the form $\varphi_{i_n} \circ \cdots \circ \varphi_{i_1}(S)$, $n \in \mathbb{N}$, $j \in \{1,\dots,n\}$, $i_j \in \{1,2,3\}$ are called the elementary pieces of S. It is not difficult to see that

$$\mathcal{U} = \{\operatorname{int}_S H : H \text{ is a finite union of elementary pieces of } S\}$$

is a basis of S such that $\#\partial_S U$ is finite for every $U \in \mathcal{U}$. Therefore $\dim_H \partial_S U \le 0$, and hence $\dim_{tH} S \le 1$. On the other hand, S contains a line segment, therefore $\dim_{tH} S \ge \dim_{tH}[0,1] = 1$ by monotonicity. □

Now we turn to the von Koch snowflake curve K. Recall that $\dim_t K = 1$ and $\dim_H K = \frac{\log 4}{\log 3}$.

Fact 5.2. *If K is homeomorphic to $[0,1]$ then* $\dim_{tH} K = 1$.

Proof. By Corollary 4.3 we obtain that $\dim_{tH} K \ge 1$. On the other hand, since K is homeomorphic to $[0,1]$, there is a basis in K such that $\#\partial U \le 2$ for every $U \in \mathcal{U}$. Thus $\dim_{tH} K \le 1$. □

Corollary 5.3. *Let K be the von Koch curve. Then* $\dim_{tH} K = 1$.

Next we take up a natural example of a compact set K with $\dim_t K < \dim_{tH} K < \dim_H K$. Let T be the Sierpiński carpet, then it is well-known that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$.

Theorem 5.4. *Let T be the Sierpiński carpet. Then* $\dim_{tH}(T) = \frac{\log 2}{\log 3} + 1 = \frac{\log 6}{\log 3}$.

Proof. Let C denote the middle-thirds Cantor set. Observe that $C \times [0,1] \subseteq T$. Then monotonicity and Theorem 4.21 yield $\dim_{tH} T \ge \dim_{tH}(C \times [0,1]) = \dim_H C + 1 = \frac{\log 2}{\log 3} + 1$.

Let us now prove the opposite inequality. For $n \in \mathbb{N}$ and $i = 1,\dots,3^n$ let $z_i^n = \frac{2i-1}{2(3^n)}$. Then clearly

$$\{z_i^n : n \in \mathbb{N},\, i \in \{1,\dots,3^n\}\}$$

is dense in $[0,1]$. Let L be a horizontal line defined by an equation of the form $y = z_i^n$ or a vertical line defined by $x = z_i^n$. It is easy to see that $L \cap T$ consists of finitely many sets geometrically similar to the middle-thirds Cantor set. Using these lines it is not difficult to construct a rectangular basis $\mathcal{U}$ of T such that $\dim_H \partial_T U = \frac{\log 2}{\log 3}$ for every $U \in \mathcal{U}$, and hence $\dim_{tH} T \le \frac{\log 2}{\log 3} + 1$. □

Finally we remark that, by Theorem 4.21, $K = C \times [0,1]$ (where C is the middle-thirds Cantor set) is a natural example of a compact set with $\dim_t K < \dim_{tH} K = \dim_H K$.

5.2. **Kakeya sets.**

Definition 5.5. A subset of $\mathbb{R}^d$ is called a *Kakeya set* if it contains a non-degenerate line segment in every direction (some authors call these sets *Besicovitch sets*).

According to a surprising classical result, Kakeya sets of Lebesgue measure zero exist. However, one of the most famous conjectures in analysis is the Kakeya Conjecture stating that every Kakeya set in $\mathbb{R}^d$ has Hausdorff dimension d. This is known to hold only in dimension at most 2 so far, and a solution already in $\mathbb{R}^3$ would have a huge impact on numerous areas of mathematics.

It would be tempting to attack the Kakeya Conjecture using $\dim_{tH} K \le \dim_H K$, but the following theorem, the main theorem of this subsection will show that unfortunately we cannot get anything non-trivial this way.

Theorem 5.6. *There exists a Kakeya set $K \subseteq \mathbb{R}^d$ of topological Hausdorff dimension 1 for every integer $d \ge 1$.*

This result is of course sharp, since if a set contains a line segment then its topological Hausdorff dimension is at least 1.

We will actually prove somewhat more, since we will essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1. This idea, as well as most of the others in this subsection are already present in [13] by T. W. Körner. However, he only works in the plane and his space slightly differs from ours. For the sake of completeness we provide the rather short proof details.

Let $(\mathcal{K}, d_H)$ be the set of compact subsets of $\mathbb{R}^{d-1} \times [0,1]$ endowed with the Hausdorff metric, that is, for each $K_1, K_2 \in \mathcal{K}$

$$d_H(K_1, K_2) = \min \{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\},$$

where $B(K, r) = \left\{x \in \mathbb{R}^{d-1} \times [0,1] : \operatorname{dist}(x, K) \le r\right\}$. It is well-known that $(\mathcal{K}, d_H)$ is a complete metric space, see e.g. [11].

Let

$$\Gamma = \{(x_1, \dots, x_{d-1}, 1) : 1/2 \le x_i \le 1, \quad i = 1, \dots, d-1\}$$

denote a subset of directions in $\mathbb{R}^d$. A closed line segment w connecting $\mathbb{R}^{d-1} \times \{0\}$ and $\mathbb{R}^{d-1} \times \{1\}$ is called a standard segment.

Let us denote by $\mathcal{F} \subseteq \mathcal{K}$ the system of those compact sets in $\mathbb{R}^{d-1} \times [0,1]$ in which for each $v \in \Gamma$ we can find a standard segment w parallel to v. First we show that $\mathcal{F}$ is closed in $\mathcal{K}$. Let us assume that $F_n \in \mathcal{F}$, $K \in \mathcal{K}$ and $F_n \to K$ with respect to d_H. We have to show that $K \in \mathcal{F}$. Let $v \in \Gamma$ be arbitrary. Since $F_n \in \mathcal{F}$, there exists a $w_n \subseteq F_n$ parallel to v for every n. It is easy to see that $\bigcup_{n \in \mathbb{N}} F_n$ is bounded, hence we can choose a subsequence n_k such that w_{n_k} is convergent with respect to d_H. But then clearly $w_{n_k} \to w$ for some standard segment $w \subseteq K$, and w is parallel to v. Hence $K \in \mathcal{F}$ indeed.

Therefore, $(\mathcal{F}, d_H)$ is a complete metric space and hence we can use Baire category arguments.

The next lemma is based on [13, Thm. 3.6.].

Lemma 5.7. *The generic set in $(\mathcal{F}, d_H)$ is of topological Hausdorff dimension 1.*

Proof. The rational cubes form a basis of $\mathbb{R}^d$, and their boundaries are covered by the rational hyperplanes orthogonal to one of the usual basis vectors of $\mathbb{R}^d$.

Therefore, it suffices to show that if S is a fixed hyperplane orthogonal to one of the usual basis vectors then $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\}$ is co-meager.

For $n \in \mathbb{N}^+$ define

$$\mathcal{F}_n = \left\{ F \in \mathcal{F} : \mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(F \cap S) < \frac{1}{n} \right\}.$$

In order to show that $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\} = \bigcap_{n \in \mathbb{N}^+} \mathcal{F}_n$ is co-meager, it is enough to prove that each $\mathcal{F}_n$ contains a dense open set.

For $p \in \mathbb{R}^d$, $v \in \Gamma$ and $0 < \alpha < \pi/2$ we denote by $C(p, v, \alpha)$ the following doubly infinite closed cone

$$C(p, v, \alpha) = \{x \in \mathbb{R}^d : \text{ the angle between the lines of } v \text{ and } x - p \text{ is at most } \alpha\}.$$

We denote by $V(C(p, v, \alpha))$ the set of those vectors $u = (u_1, ..., u_{d-1}, 1)$ for which there is a line in $\operatorname{int}(C(p, v, \alpha)) \cup \{p\}$ parallel to u. Then $V(C(p, v, \alpha))$ is relatively open in $\mathbb{R}^{d-1} \times \{1\}$.

The sets of the form $C'(p, v, \alpha) = C(p, v, \alpha) \cap \left(\mathbb{R}^{d-1} \times [0, 1]\right)$ will be called truncated cones, and the system of truncated cones will be denoted by $\mathcal{C}'$. A truncated cone $C'(p, v, \alpha)$ is S-compatible if either $C'(p, v, \alpha) \cap S = \{p\}$, or $C'(p, v, \alpha) \cap S = \emptyset$. The set of S-compatible truncated cones is denoted by $\mathcal{C}'_S$. Define $\mathcal{F}_S$ as the set of those $F \in \mathcal{F}$ that can be written as the union of finitely many S-compatible truncated cones and finitely many points in $\mathbb{R}^{d-1} \times [0, 1]$.

Next we check that $\mathcal{F}_S$ is dense in $\mathcal{F}$.

Suppose $F \in \mathcal{F}$ is arbitrary and $\varepsilon > 0$ is given. First choose finitely many points $\{y_i\}_{i=1}^t$ in F such that $F \subseteq B\left(\{y_i\}_{i=1}^t, \varepsilon\right)$. Let $v \in \Gamma$ be arbitrary, then there exists a standard segment $w_v \subseteq F$ parallel to v. By the choice of S and Γ, clearly $w_v \nsubseteq S$, hence we can choose p_v and α_v such that $C'(p_v, v, \alpha_v) \in \mathcal{C}'_S$ and $d_H(C'(p_v, v, \alpha_v), w_v) \leq \varepsilon$. Obviously $v \in V(C(p_v, v, \alpha_v))$, so $\{V(C(p_v, v, \alpha_v))\}_{v \in \Gamma}$ is an open cover of the compact set Γ. Therefore, there are $\{C'(p_{v_i}, v_i, \alpha_{v_i})\}_{i=1}^m$ in $\mathcal{C}'_S$ such that $\Gamma \subseteq \bigcup_{i=1}^m V(C(p_{v_i}, v_i, \alpha_{v_i}))$. Put $F' = \bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \cup \{y_1, \dots, y_t\}$, then $F' \in \mathcal{F}_S$. It is easy to see that $\bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \subseteq B(F, \varepsilon)$, and combining this with $\{y_i\}_{i=1}^t \subseteq F$ we obtain that $F' \subseteq B(F, \varepsilon)$. By the choice of $\{y_i\}_{i=1}^t$ we also have $F \subseteq B(F', \varepsilon)$. Thus $d_H(F, F') \leq \varepsilon$.

Now using our dense set $\mathcal{F}_S$ we verify that $\mathcal{F}_n$ contains a dense open set $\mathcal{U}$. We construct for all $F_0 \in \mathcal{F}_S$ a ball in $\mathcal{F}_n$ centered at F_0. By the definition of S-compatibility $F_0 \cap S$ is finite. Hence we can easily choose a relatively open set $U_0 \subseteq S$ such that $F_0 \cap S \subseteq U_0$ and $\mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(U_0) < \frac{1}{n}$. Let us define

$$\mathcal{U} = \{F \in \mathcal{F} : F \cap S \subseteq U_0\}.$$

Clearly $F_0 \in \mathcal{U}$, $\mathcal{U} \subseteq \mathcal{F}_n$ and it is easy to see that $\mathcal{U}$ is open in $\mathcal{F}$. This completes the proof. □

From this we obtain the main theorem of the subsection as follows.

Proof of Theorem 5.6. By the above lemma we can choose $F_0 \in \mathcal{F}$ such that $\dim_{tH} F_0 = 1$. Then F_0 contains a line segment in every direction of Γ, hence we can choose finitely many isometric copies of it, $\{F_i\}_{i=1}^n$ such that the compact set $K = \bigcup_{i=0}^n F_i$ contains a line segment in every direction. By the Lipschitz invariance of the topological Hausdorff dimension $\dim_{tH} F_i = \dim_{tH} F_0$ for all i, and by the stability of the topological Hausdorff dimension for closed sets $\dim_{tH} K = 1$. □

5.3. **Brownian motion.** One of the most important stochastic processes is the Brownian motion (see e.g. [19]). Its range and graph also serve as important examples of fractal sets in geometric measure theory. Since the graph is always homeomorphic to $[0,\infty)$, Fact 5.2 and countable stability for closed sets yield that its topological Hausdorff dimension is 1. Hence we focus on the range only.

Each statement in this paragraph is to be understood to hold with probability 1 (almost surely). Clearly, in dimension 1 the range is a non-degenerate interval, so it has topological Hausdorff dimension 1. Moreover, if the dimension is at least 4 then the range has no multiple points ([19]), so it is homeomorphic to $[0,\infty)$, which in turn implies as above that the range has topological Hausdorff dimension 1 again.

Problem 5.8. *Let $d = 2$ or 3. Determine the almost sure topological Hausdorff dimension of the range of the d-dimensional Brownian motion. Equivalently, determine the smallest $c \geq 0$ such that the range can be decomposed into a totally disconnected set and a set of Hausdorff dimension at most $c - 1$ almost surely.*

Indeed, the two formulations of the problem are equivalent by Theorem 3.7. The following open problem of W. Werner [19, p. 384.] is closely related to Problem 5.8 in the case $d = 2$.

Notation 5.9. By a *curve* we mean a continuous map $\gamma\colon [0,1] \to \mathbb{R}^d$. Let us denote by $\operatorname{Ran}(\gamma)$ the *range* of γ. If $\gamma\colon [0,1] \to \mathbb{R}^2$ is a closed curve and $p \in \mathbb{R}^2 \setminus \operatorname{Ran}(\gamma)$ then let us denote by $WN(\gamma, p)$ the *winding number* of γ with respect to p, see [7] for the definition.

Problem 5.10 (W. Werner)**.** *Is it true almost surely for the planar Brownian motion $B\colon [0,\infty) \to \mathbb{R}^2$ that for every $x, y \in \mathbb{R}^2 \setminus \operatorname{Ran}(B)$ there exists a curve $\gamma\colon [0,1] \to \mathbb{R}^2$ such that $\gamma(0) = x$, $\gamma(1) = y$, and $\operatorname{Ran}(\gamma) \cap \operatorname{Ran}(B)$ is finite?*

Remark 5.11. An affirmative answer to Problem 5.10 would probably also solve Problem 5.8. More precisely, if there exists any function $b\colon (0,\infty) \to (0,\infty)$ with $\lim_{x\to 0+} b(x) = 0$ such that the above curve γ can be constructed in the disc $U(x, b(|x - y|))$ then one can build a basis of $\operatorname{Ran}(B)$ as follows. Let $x \in \operatorname{Ran}(B)$ and $\varepsilon > 0$ be given, and pick $\delta > 0$ such that $|x - y| < \delta$ implies $b(|x - y|) < \frac{\varepsilon}{2}$. Select a sequence of points $x_0, x_1, \dots, x_n = x_0$ on the circle of radius $\frac{\varepsilon}{2}$ centered at x such that $x_i \notin \operatorname{Ran}(B)$ and $|x_i - x_{i+1}| < \delta$ for every $i = 0, \dots, n-1$. Moreover, we can also assume that the argument of the vectors $x_i - x$ $(i = 0, \dots, n-1)$ is an increasing sequence in $[0, 2\pi)$. This implies that if for every $i = 0, \dots, n-1$ we construct a curve γ_i from x_i to x_{i+1} in $U(x_i, \frac{\varepsilon}{2})$ intersecting $\operatorname{Ran}(B)$ finitely many times, and glue these curves together to obtain a closed curve γ then the winding number $WN(\gamma, x) = 1$, and hence x is in a bounded component $U_{x,\varepsilon}$ of $\mathbb{R}^2 \setminus \operatorname{Ran}(\gamma)$ see [7, Proposition 3.16.]. Then it is easy to see that $\partial U_{x,\varepsilon} \subseteq \operatorname{Ran}(\gamma)$, hence $\partial U_{x,\varepsilon} \cap \operatorname{Ran}(B)$ is finite, and $x \in U_{x,\varepsilon} \subseteq U(x,\varepsilon)$. Therefore, the sets of the form $U_{x,\varepsilon} \cap \operatorname{Ran}(B)$ form a basis of $\operatorname{Ran}(B)$ such that the boundary of $U_{x,\varepsilon}$ relative to $\operatorname{Ran}(B)$ is finite for all x and ε, thus the topological Hausdorff dimension of $\operatorname{Ran}(B)$ is 1 almost surely.

While we can solve neither Problem 5.8 nor Problem 5.10, we are able to solve their Baire category duals. First we need some preparation.

Definition 5.12. Let us denote by $\mathcal{C}^d$ the space of curves from $[0,1]$ to $\mathbb{R}^d$ endowed with the supremum metric. As this is a complete metric space, we can use Baire category arguments.

A *standard line segment* in $\mathbb{R}^d$ is a closed non-degenerate line segment parallel to one of the standard basis vectors $e_1 = (1,0,\dots,0),\dots,e_d = (0,0,\dots,1)$.

The next theorem gives an affirmative answer to the category dual of Problem 5.8.

Theorem 5.13. *The range of the generic $f \in \mathcal{C}^d$ has topological Hausdorff dimension 1 for every $d \in \mathbb{N}^+$.*

Proof. If $d = 1$ then the statement is straightforward, thus we may assume $d > 1$. By Corollary 4.3 the range of every non-constant $f \in \mathcal{C}^d$ has topological Hausdorff dimension at least 1, thus we need to show that the generic $f \in \mathcal{C}^d$ has topological Hausdorff dimension at most 1. The rational cubes form a basis of $\mathbb{R}^d$, and their boundaries are covered by countably many hyperplanes, therefore it is enough to show that if S is a fixed hyperplane then $\{f \in \mathcal{C}^d : \dim_H (\mathrm{Ran}(f) \cap S) = 0\}$ is co-meager. By rotating our coordinate system we may suppose that S is not parallel to any vector in the standard basis $\{e_1,\dots,e_d\}$. For $n \in \mathbb{N}^+$ define

$$\mathcal{F}_n = \left\{ f \in \mathcal{C}^d : \mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}} (\mathrm{Ran}(f) \cap S) < \frac{1}{n} \right\}.$$

In order to show that $\{f \in \mathcal{C}^d : \dim_H (\mathrm{Ran}(f) \cap S) = 0\} = \bigcap_{n\in\mathbb{N}^+} \mathcal{F}_n$ is co-meager, it is enough to prove that each $\mathcal{F}_n$ is a dense open set. Let us fix $n \in \mathbb{N}^+$. The regularity of $\mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}$ implies that $\mathcal{F}_n$ is open. Let $\mathcal{G}$ be the set of curves $f \in \mathcal{C}^d$ such that $\mathrm{Ran}(f)$ is a union of finitely many standard line segments. It is easy to see that $\mathcal{G}$ is dense in $\mathcal{C}^d$. As $\mathrm{Ran}(f) \cap S$ is finite for every $f \in \mathcal{G}$, we have $\mathcal{G} \subseteq \mathcal{F}_n$. Thus $\mathcal{F}_n$ is dense in $\mathcal{C}^d$, and the proof is complete. □

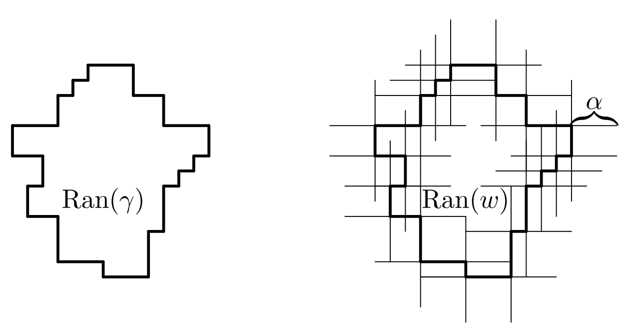


FIGURE 1. A standard polygon γ and the corresponding α-wire w

Definition 5.14. Let $C \subseteq \mathbb{R}^2$ and let $p, q \in \mathbb{R}^2 \setminus C$. We say that C *separates p and q* if these points belong to different connected components of $\mathbb{R}^2 \setminus C$. A metric space is called a *continuum* if it is compact and connected. A set $V \subseteq \mathbb{R}^2$ is a *standard open set* if it is a union of finitely many axis-parallel open squares.

We say that $f \in \mathcal{C}^2$ is a *standard curve* if there exist $0 = x_0 < \cdots < x_m = 1$ such that f maps the interval $I_k = [x_{k-1}, x_k]$ bijectively to the standard line segment $f(I_k)$, and $f(I_k)$ is orthogonal to $f(I_{k+1})$ for all $k \in \{1, \dots, m\}$, where we use the notation $I_{m+1} = I_1$. Notice that the points x_k are uniquely determined. We say that $f(x_k)$ and I_k are the *turning points* and *edge intervals* of f, respectively. A simple closed standard curve $\gamma \in \mathcal{C}^2$ is a *standard polygon*.

Let $\alpha > 0$. Let us define $w \in \mathcal{C}^2$ to be an *α-wire* if there exists a standard polygon $\gamma \in \mathcal{C}^2$ with edge intervals $I_1, \dots, I_m$ such that the lengths of $\gamma(I_k)$ are at most α, and if E_k denotes the segment we obtain from the edge $\gamma(I_k)$ by expanding it by length α in both directions, then $w(I_k) = E_k$ for all $k \in \{1, \dots, m\}$, see Figure 1. We say that the E_k are the *edges* of the α-wire w, and E_k and E_{k+1} are *adjacent* if $k \in \{1, \dots, m\}$, where $E_{m+1} = E_1$. Of course, w passes through points of $w(I_k) \setminus \gamma(I_k)$ more than once.

The above definitions easily imply the following facts.

Fact 5.15. *The standard curves form a dense set in $\mathcal{C}^2$.*

Fact 5.16. *Let $E, E' \subseteq \mathbb{R}^2$ be two adjacent edges of an α-wire. Assume that $f \in \mathcal{C}^2$ and $I, I' \subseteq [0,1]$ are disjoint closed intervals such that $f(I) = E$ and $f(I') = E'$. Then for all $g \in U(f, \alpha/2)$ we obtain that $g(I) \cap g(I') \neq \emptyset$.*

The next theorem answers the category dual of Problem 5.10 in the negative.

Theorem 5.17. *For the generic $f \in \mathcal{C}^2$ if $p, q \in \mathbb{R}^2 \setminus \operatorname{Ran}(f)$ and $\operatorname{Ran}(f)$ separates p and q then for every continuum $C \subseteq \mathbb{R}^2$ with $p, q \in C$ the intersection $C \cap \operatorname{Ran}(f)$ has cardinality continuum.*

Before proving Theorem 5.17 we need some lemmas.

Lemma 5.18. *Let $f \in \mathcal{C}^2$ be a standard curve and let $p, q \in \mathbb{R}^2 \setminus \operatorname{Ran}(f)$. If $V \subseteq \mathbb{R}^2$ is a standard open set such that $V \cap \operatorname{Ran}(f)$ separates p and q then there is a standard polygon $\gamma \in \mathcal{C}^2$ such that $\operatorname{Ran}(\gamma) \subseteq V \cap \operatorname{Ran}(f)$ separates p and q.*

Proof. We say that an open line segment S is a *marginal segment* if $S \subseteq V \cap \operatorname{Ran}(f)$ and one of its endpoints is in ∂V and the other endpoint is either in ∂V, or is a turning point, or is a point attained by f more than once. Let $S_1, \dots, S_n \subseteq V \cap \operatorname{Ran}(f)$ be the marginal segments. Clearly, they are pairwise disjoint. Set $K = (V \cap \operatorname{Ran}(f)) \setminus (\bigcup_{i=1}^n S_i)$. Then $K \subseteq V \cap \operatorname{Ran}(f)$ is compact.

First we prove that K separates p and q. Assume to the contrary that this is not the case, then there is a standard curve $g \in \mathcal{C}^2$ with $g(0) = p$ and $g(1) = q$ such that $K \cap \operatorname{Ran}(g) = \emptyset$. Let y_1 be an endpoint of S_1 in ∂V, if g meets S_1 then we can modify g in a small neighborhood of S_1 such that the modified curve passes through y_1 and avoids $S_1 \cup K$. Continuing this procedure we obtain a curve $\widetilde{g} \in \mathcal{C}^2$ with $\widetilde{g}(0) = p$ and $\widetilde{g}(1) = q$ such that $(V \cap \operatorname{Ran}(f)) \cap \operatorname{Ran}(\widetilde{g}) = \emptyset$, but this contradicts the fact that $V \cap \operatorname{Ran}(f)$ separates p and q.

Now elementary considerations show that there is a standard polygon $\gamma \in \mathcal{C}^2$ such that $\operatorname{Ran}(\gamma) \subseteq K$. For the sake of completeness we mention that [20, Thm. 14.3.] implies that K contains a component C separating p and q. Then C is a locally connected continuum, so [23, (2.41)] yields that there exists a simple closed curve $\gamma \in \mathcal{C}^2$ such that $\operatorname{Ran}(\gamma) \subseteq C$ separates p and q, and after reparametrization γ will be a standard polygon. □

Lemma 5.19. *Let $p, q \in \mathbb{R}^2$ and $\varepsilon > 0$. Assume that $f \in \mathcal{C}^2$ is a standard curve and $V \subseteq \mathbb{R}^2$ is a standard open set such that $V \cap \operatorname{Ran}(f)$ separates p and q. Then there exist a standard curve $f_0 \in \mathcal{C}^2$, standard open sets $V_0, V_1 \subseteq \mathbb{R}^2$ and $\delta > 0$ such that*

(i) $V_0, V_1 \subseteq V$ *and* $\operatorname{dist}(V_0, V_1) > 0$,
(ii) $V_j \cap \operatorname{Ran}(g)$ *separates* p *and* q *if* $j \in \{0,1\}$ *and* $g \in U(f_0, \delta)$,
(iii) $U(f_0, \delta) \subseteq U(f, \varepsilon)$,
(iv) $f_0 = f$ *on* $[0,1] \setminus f^{-1}(V)$.

Proof. It is enough to construct a not necessarily standard $f_0 \in \mathcal{C}^2$ with the above properties, since by Fact 5.15 we can replace it with a standard one.

By Lemma 5.18 there exists a standard polygon $\gamma \in \mathcal{C}^2$ such that $\operatorname{Ran}(\gamma) \subseteq V \cap \operatorname{Ran}(f)$ separates p and q. Let $\Gamma = \operatorname{Ran}(\gamma)$, we may assume by decreasing ε if necessary that $U(\Gamma, \varepsilon) \subseteq V$ and $p, q \notin U(\Gamma, \varepsilon)$. Let us suppose that q is in the bounded component of $\mathbb{R}^2 \setminus \Gamma$. Then the winding numbers $WN(\gamma, p) = 0$ and $WN(\gamma, q) = \pm 1$, see [7, Proposition 3.16.] and [7, Proposition 5.20.]. It is easy to see that we can fix a small enough $\alpha \in (0, \varepsilon/9)$ such that there exist non-intersecting α-wires $w_0, w_1 \in U(\gamma, \varepsilon/3)$, see Figure 2.

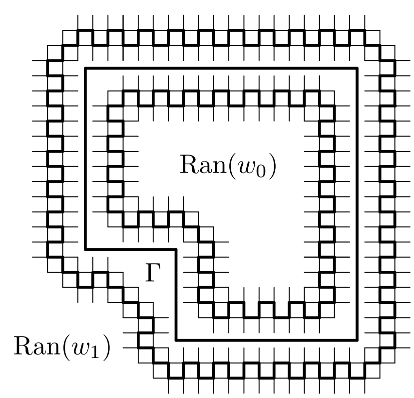


FIGURE 2. Illustration to Lemma 5.19

Let $W_j = \operatorname{Ran}(w_j)$ for $j \in \{0,1\}$, then clearly $W_0 \cap W_1 = \emptyset$ and $W_0 \cup W_1 \subseteq U(\Gamma, \varepsilon/3)$. Let us denote the edges of w_0 and w_1 by $E_1, \dots, E_{n_0}$ and $E_{n_0+1}, \dots, E_{n_0+n_1}$, respectively. Set $n = n_0 + n_1$. Since $W_0 \cup W_1 \subseteq U(\Gamma, \varepsilon/3) \subseteq U(\operatorname{Ran}(f), \varepsilon/3)$ and $\operatorname{diam} E_k \leq 3\alpha < \varepsilon/3$, there exist distinct points $z_1, \dots, z_n \in [0,1]$ such that $E_k \subseteq U(f(z_k), 2\varepsilon/3)$ for all $k \in \{1, \dots, n\}$. By the continuity of f there are pairwise disjoint closed non-degenerate intervals $I_1, \dots, I_n \subseteq [0,1]$ such that $E_k \subseteq U(f(x), 2\varepsilon/3)$ for all $x \in I_k$ and $k \in \{1, \dots, n\}$. Notice that each $x \in \bigcup_{k=1}^n I_k$ satisfies $f(x) \in U(W_0 \cup W_1, 2\varepsilon/3) \subseteq U(\Gamma, \varepsilon) \subseteq V$, thus $\bigcup_{k=1}^n I_k \subseteq f^{-1}(V)$. Now for all $k \in \{1, \dots, n\}$ define $f_0|_{I_k}$ such that $f_0(I_k) = E_k$ and let $f_0 = f$ on $[0,1] \setminus f^{-1}(V)$. It is easy to check that $|f(x) - f_0(x)| < 2\varepsilon/3$ for every $x \in \bigcup_{k=1}^n I_k \cup \big([0,1] \setminus f^{-1}(V)\big)$. Then Tietze's Extension Theorem implies that we can extend f_0 continuously to $[0,1]$ such that $f_0 \in B(f, 2\varepsilon/3)$. Property (iv) follows from the definition of f_0.

Now let $\delta = \min\{\alpha/2, \operatorname{dist}(W_0, W_1)/5\} > 0$. As $f_0 \in B(f, 2\varepsilon/3)$ and $\delta \le \alpha/2 < \varepsilon/3$, we obtain $U(f_0, \delta) \subseteq U(f, \varepsilon)$, thus property (iii) holds.

Let us pick standard open sets V_0 and V_1 such that $U(W_j, \delta) \subseteq V_j \subseteq U(W_j, 2\delta)$ for $j \in \{0, 1\}$, then

$$V_j \subseteq U(W_j, 2\delta) \subseteq U(\Gamma, 2\delta + \varepsilon/3) \subseteq U(\Gamma, \varepsilon) \subseteq V,$$

and

$$\operatorname{dist}(V_0, V_1) \ge \operatorname{dist}(W_0, W_1) - 4\delta \ge \delta > 0,$$

hence property (i) holds.

Finally, let us fix $g \in U(f_0, \delta)$, we need to prove for property (ii) that $V_0 \cap \operatorname{Ran}(g)$ separates p and q, and of course the same argument works for $V_1 \cap \operatorname{Ran}(g)$. Let us define the compact set $K = \bigcup_{k=1}^{n_0} g(I_k)$. Then $\bigcup_{k=1}^{n_0} f_0(I_k) = W_0$ yields $K \subseteq U(W_0, \delta) \subseteq V_0$, thus $K \subseteq V_0 \cap \operatorname{Ran}(g)$. Therefore it is enough to prove that K separates p and q. Consider $0 = x_0 < x_1 < \dots < x_{n_0} = 1$ such that $w_0([x_{k-1}, x_k]) = E_k$ for every $k \in \{1, \dots, n_0\}$. Then $g \in U(f_0, \delta) \subseteq U(f_0, \alpha/2)$ and Fact 5.16 imply that $g(I_k) \cap g(I_{k+1}) \neq \emptyset$ for all $k \in \{1, \dots, n_0 - 1\}$ and $g(I_{n_0}) \cap g(I_1) \neq \emptyset$. Thus we can choose $u_k, v_k \in I_k$ for all $k \in \{1, \dots, n_0\}$ such that $g(v_k) = g(u_{k+1})$ and let $h_k \colon [x_{k-1}, x_k] \to [u_k, v_k]$ be a homeomorphism with $h_k(x_{k-1}) = u_k$ and $h_k(x_k) = v_k$, where we use the notations $u_{n_0+1} = u_1$ and $[u, v] = [\min\{u, v\}, \max\{u, v\}]$. Let us define $\varphi \colon [0, 1] \to K$ such that $\varphi|_{[x_{k-1}, x_k]} = g \circ h_k$ for all $k \in \{1, \dots, n_0\}$. Then φ is a closed curve with $\operatorname{Ran}(\varphi) \subseteq K$, so it is enough to prove that $\operatorname{Ran}(\varphi)$ separates p and q. The function $WN(\varphi, \cdot)$ is constant on the components of $\mathbb{R}^2 \setminus \operatorname{Ran}(\varphi)$ by [7, Proposition 3.16.], thus it is sufficient to prove that $WN(\varphi, p) \neq WN(\varphi, q)$. From $g \in U(f_0, \delta)$, $f_0(I_k) = E_k$ and $\operatorname{diam} E_k < \varepsilon/3$ it follows that $\varphi \in U(w_0, \delta + \varepsilon/3) \subseteq U(w_0, 2\varepsilon/3)$. Thus $w_0 \in U(\gamma, \varepsilon/3)$ yields $\varphi \in U(\gamma, \varepsilon)$. Hence φ and γ are homotopic in $U(\Gamma, \varepsilon) \subseteq \mathbb{R}^2 \setminus \{p, q\}$, so $WN(\varphi, p) = WN(\gamma, p) = 0$ and $WN(\varphi, q) = WN(\gamma, q) = \pm 1$, see [7, Corollary 3.8.]. The proof is complete. $\square$

Lemma 5.20. *Assume $p, q \in \mathbb{R}^2$ and $\varepsilon > 0$. Let $f \in \mathcal{C}^2$ be a standard curve and let $V_1, \dots, V_m \subseteq \mathbb{R}^2$ be pairwise disjoint standard open sets such that $V_k \cap \operatorname{Ran}(f)$ separates p and q for all $k \in \{1, \dots, m\}$. Then there exist a non-empty open set $\mathcal{V} \subseteq U(f, \varepsilon)$ and standard open sets $V_{kj} \subseteq \mathbb{R}^2$ such that for all $k \in \{1, \dots, m\}$ and $j \in \{0, 1\}$*

(i) $V_{kj} \subseteq V_k$ and $\operatorname{dist}(V_{k0}, V_{k1}) > 0$,
(ii) $V_{kj} \cap \operatorname{Ran}(g)$ separates p and q if $g \in \mathcal{V}$.

Proof. Let $g_0 = f$ and $\delta_0 = \varepsilon$. Now for $k \in \{0, \dots, m-1\}$ we construct by induction a standard curve $g_k \in \mathcal{C}^2$, a $\delta_k > 0$, and for all $i \in \{1, \dots, k\}$ and $j \in \{0, 1\}$ standard open sets V_{ij} such that for all $i \in \{1, \dots, k\}$ and $j \in \{0, 1\}$

(1) $V_{ij} \subseteq V_i$ and $\operatorname{dist}(V_{i0}, V_{i1}) > 0$,
(2) $V_{ij} \cap \operatorname{Ran}(g)$ separates p and q if $g \in U(g_k, \delta_k)$,
(3) $U(g_k, \delta_k) \subseteq U(f, \varepsilon)$,
(4) $g_k = f$ on $[0, 1] \setminus f^{-1}(V_1 \cup \dots \cup V_k)$.

The case $k = 0$ is done, since (3) and (4) obviously hold, while (1) and (2) hold vacuously, since there are no V_{ij}'s.

Let us now take up the inductive step. Since $g_k = f$ on $f^{-1}(V_{k+1}) \subseteq [0, 1] \setminus f^{-1}(V_1 \cup \dots \cup V_k)$, we obtain that $V_{k+1} \cap \operatorname{Ran}(g_k)$ separates p and q. Lemma 5.19

applied to the standard curve g_k, standard open set V_{k+1} and $\delta_k > 0$ yields that we can satisfy properties (1)-(4) for $k+1$.

Finally, the non-empty open set $\mathcal{V} = U(g_m, \delta_m) \subseteq U(f, \varepsilon)$ and the constructed standard open sets V_{ij} satisfy properties (i)-(ii). □

Now we are ready to prove Theorem 5.17.

Proof of Theorem 5.17. Let $\mathcal{F}$ be the set of functions $f \in \mathcal{C}^2$ such that if $\mathrm{Ran}(f)$ separates a pair of points p and q then for every continuum $C \subseteq \mathbb{R}^2$ with $p, q \in C$ the intersection $C \cap \mathrm{Ran}(f)$ has cardinality continuum. We need to show that $\mathcal{F}$ is co-meager in $\mathcal{C}^2$. For fixed $p, q \in \mathbb{R}^2$ let $\mathcal{F}_{p,q}$ be the set of functions $f \in \mathcal{C}^2$ such that if $\mathrm{Ran}(f)$ separates p and q then for every continuum $C \subseteq \mathbb{R}^2$ with $p, q \in C$ the intersection $C \cap \mathrm{Ran}(f)$ has cardinality continuum. As

$$\mathcal{F} = \bigcap_{p,q \in \mathbb{Q}\times\mathbb{Q}} \mathcal{F}_{p,q},$$

it is enough to show that the sets $\mathcal{F}_{p,q}$ are co-meager in $\mathcal{C}^2$. Let $p, q \in \mathbb{R}^2$, $p \neq q$ be arbitrarily fixed. In order to prove that $\mathcal{F}_{p,q}$ is co-meager, we play the Banach-Mazur game in the metric space $\mathcal{C}^2$: First player I chooses a non-empty open set $\mathcal{U}_1 \subseteq \mathcal{C}^2$, then player II chooses a non-empty open set $\mathcal{V}_1 \subseteq \mathcal{U}_1$, player I continues with a non-empty open set $\mathcal{U}_2 \subseteq \mathcal{V}_1$, and so on. By definition player II wins this game if $\bigcap_{n=1}^{\infty} \mathcal{V}_n \subseteq \mathcal{F}_{p,q}$. It is well-known that player II has a winning strategy iff $\mathcal{F}_{p,q}$ is co-meager in $\mathcal{C}^2$, see [21, Thm. 1] or [11, (8.33)]. Thus we need to prove that player II has a winning strategy.

Now we describe the strategy of player II. If there is an $f \in \mathcal{U}_1$ such that p and q are in the same component of $\mathbb{R}^2 \setminus \mathrm{Ran}(f)$ then there is an $\varepsilon > 0$ such that all functions in $U(f, \varepsilon) \subseteq \mathcal{U}_1$ have this property. Thus $U(f, \varepsilon) \subseteq \mathcal{F}_{p,q}$, and the strategy of Player II is the following. Let $\mathcal{V}_1 = U(f, \varepsilon)$, and the other moves of Player II are arbitrary. Then clearly $\bigcap_{n=1}^{\infty} \mathcal{V}_n \subseteq \mathcal{F}_{p,q}$, so Player II wins the game.

If $\mathrm{Ran}(f)$ separates p and q for all $f \in \mathcal{U}_1$ then the strategy of Player II is the following. The functions f with the property $p, q \notin \mathrm{Ran}(f)$ form a dense open subset in $\mathcal{U}_1$, so by Fact 5.15 there is a standard curve $f_1 \in \mathcal{U}_1$ such that $\mathrm{Ran}(f_1)$ separates p and q. Let $V \subseteq \mathbb{R}^2$ be a standard open set with $\mathrm{Ran}(f_1) \subseteq V$. Let $\varepsilon_1 > 0$ be so small that $U(f_1, \varepsilon_1) \subseteq \mathcal{U}_1$. Applying Lemma 5.20 for ε_1, f_1 and V implies that there is a non-empty open set $\mathcal{V}_1 \subseteq U(f_1, \varepsilon_1)$ and standard open sets $V_0, V_1 \subseteq V$ such that $\mathrm{dist}(V_0, V_1) > 0$ and $V_j \cap \mathrm{Ran}(g)$ separates p and q for every $j \in \{0, 1\}$ and $g \in \mathcal{V}_1$. Next we suppose that $n \in \mathbb{N}^+$ and after the nth move of Player II the non-empty open set $\mathcal{V}_n$ and standard open sets $V_{j_1 \dots j_i} \subseteq \mathbb{R}^2$ $(i \in \{1, \dots, n\},\ j_1, \dots, j_i \in \{0, 1\})$ are already defined such that for all $j_1, \dots, j_n \in \{0, 1\}$

(i) $V_{j_1 \dots j_n} \subseteq V_{j_1 \dots j_{n-1}}$ and $\mathrm{dist}(V_{j_1 \dots j_{n-1} 0}, V_{j_1 \dots j_{n-1} 1}) > 0$,
(ii) $V_{j_1 \dots j_n} \cap \mathrm{Ran}(g)$ separates p and q for all $g \in \mathcal{V}_n$.

Suppose that Player I continues with $\mathcal{U}_{n+1} \subseteq \mathcal{V}_n$. By Fact 5.15 Player II can choose a standard curve $f_{n+1} \in \mathcal{U}_{n+1}$. Then $f_{n+1} \in \mathcal{V}_n$ implies that (ii) holds for f_{n+1}. Let $\varepsilon_{n+1} > 0$ be so small that $U(f_{n+1}, \varepsilon_{n+1}) \subseteq \mathcal{U}_{n+1}$. Applying Lemma 5.20 for ε_{n+1}, f_{n+1} and the open sets $V_{j_1 \dots j_n}$ we obtain a non-empty open set $\mathcal{V}_{n+1} \subseteq U(f_{n+1}, \varepsilon_{n+1}) \subseteq \mathcal{U}_{n+1}$ and standard open sets $V_{j_1 \dots j_{n+1}} \subseteq \mathbb{R}^2$ witnessing that properties (i)-(ii) hold for $n+1$.

Finally, we need to prove that player II wins the game with the above strategy if $\mathrm{Ran}(f)$ separates p and q for all $f \in \mathcal{U}_1$. Let $f \in \bigcap_{n=1}^{\infty} \mathcal{V}_n$, we need to prove that $f \in \mathcal{F}_{p,q}$. Let $C \subseteq \mathbb{R}^2$ be a continuum with $p, q \in C$, we need to show that $C \cap \operatorname{Ran}(f)$ has cardinality continuum. Property (ii) implies that we can choose $c_{j_1 \dots j_n} \in C \cap (V_{j_1 \dots j_n} \cap \operatorname{Ran}(f))$ for every $n \in \mathbb{N}^+$ and $j_1, \dots, j_n \in \{0,1\}$. If $\underline{j} = (j_1, j_2, \dots) \in \{0,1\}^{\mathbb{N}}$ then let $c_{\underline{j}}$ be an arbitrary limit point of the set $\{c_{j_1 \dots j_n} : n \in \mathbb{N}^+\}$. Then the compactness of C and $\operatorname{Ran}(f)$ implies that $c_{\underline{j}} \in C \cap \operatorname{Ran}(f)$ for every $\underline{j} \in \{0,1\}^{\mathbb{N}}$. Thus it is enough to prove that if $\underline{j}, \underline{k} \in \{0,1\}^{\mathbb{N}}$, $\underline{j} \neq \underline{k}$ then $c_{\underline{j}} \neq c_{\underline{k}}$. Assume that $n \in \mathbb{N}^+$ is the minimal number such that the nth coordinate of $\underline{j}$ and $\underline{k}$ differ. As the sets $V_{j_1 \dots j_i}$ form a nested sequence by property (i), we obtain $c_{\underline{j}} \in \operatorname{cl} V_{j_1 \dots j_n}$ and $c_{\underline{k}} \in \operatorname{cl} V_{k_1 \dots k_n}$, and property (i) also yields $\operatorname{cl} V_{j_1 \dots j_n} \cap \operatorname{cl} V_{k_1 \dots k_n} = \emptyset$. Therefore $c_{\underline{j}} \neq c_{\underline{k}}$, and the proof is complete. □

6. Application I: Mandelbrot's fractal percolation process

In this section we take up one of the most important random fractals, the limit set M of the fractal percolation process defined by Mandelbrot in [14].

His original motivation was that this model captures certain features of turbulence, but then this random set turned out to be very interesting in its own right. For example, M serves as a very powerful tool for calculating Hausdorff dimension. Indeed, J. Hawkes has shown ([9]) that for a fixed Borel set B in the unit square (or analogously in higher dimensions) $M \cap B = \emptyset$ almost surely iff $\dim_H M + \dim_H B < 2$, and this formula can be used in certain applications to determine $\dim_H B$. Moreover, it can be shown that the range of the Brownian motion is so called *intersection-equivalent* to a percolation fractal (roughly speaking, they intersect the same sets with positive probability), and this can be used to deduce numerous dimension related results about the Brownian motion, see the works of Y. Peres, e.g. in [19].

Let us now formally describe the fractal percolation process. Let $p \in (0,1)$ and $n \geq 2$, $n \in \mathbb{N}$ be fixed. Set $M_0 = M_0^{(p,n)} = [0,1]^2$. We divide the unit square into n^2 equal closed sub-squares of side-length $1/n$ in the natural way. We keep each sub-square independently with probability p (and erase it with probability $1-p$), and denote by $M_1 = M_1^{(p,n)}$ the union of the kept sub-squares. Then each square in M_1 is divided into n^2 squares of side-length $1/n^2$, and we keep each of them independently (and also independently of the earlier choices) with probability p, etc. After k steps let $M_k = M_k^{(p,n)}$ be the union of the kept k^{th} level squares with side-length $1/n^k$. Let

$$M = M^{(p,n)} = \bigcap_{k=1}^{\infty} M_k. \tag{6.1}$$

The process we have just described is called *Mandelbrot's fractal percolation process*, and M is called its *limit set*.

Percolation fractals are not only interesting from the point of view of turbulence and fractal geometry, but they are also closely related to the (usual, graph-theoretic) percolation theory. In case of the fractal percolation the role of the clusters is played by the connected components. Our starting point will be the following celebrated theorem.

Theorem 6.1 (Chayes-Chayes-Durrett, [3])**.** *There exists a critical probability $p_c = p_c^{(n)} \in (0,1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.*

They actually prove more, the most powerful version states that in the supercritical case (i.e. when $p > p_c$) there is actually a unique unbounded component if the process is extended to the whole plane, but we will only concentrate on the most surprising fact that the critical probability is strictly between 0 and 1.

The main goal of the present section will be to prove the following generalization of the above theorem.

Theorem 6.2. *For every $d \in [0,2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0,1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \leq d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).*

In order to see that we actually obtain a generalization, just note that a compact space is totally disconnected iff $\dim_t M = 0$ ([4]), also that $\dim_{tH} M = 0$ iff $\dim_t M = 0$, and use $d = 0$. Theorem 6.1 basically says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, where the word thick is related to large Hausdorff dimension.

In the rest of this section first we do some preparations in the first subsection, then we prove the main theorem (Theorem 6.2) in the next subsection, and finally give an upper bound for $\dim_{tH} M$ and conclude that $\dim_{tH} M < \dim_H M$ almost surely in the non-trivial cases.

6.1. **Preparation.** For the proofs of the statements in the next two remarks see e.g. [3].

Remark 6.3. It is well-known from the theory of branching processes that $M = \emptyset$ almost surely iff $p \leq \frac{1}{n^2}$, so we may assume in the following that $p > \frac{1}{n^2}$.

If $\frac{1}{n^2} < p \leq \frac{1}{\sqrt{n}}$ then $\dim_t M = 0$ almost surely. Hence Fact 4.1 implies that $\dim_{tH} M = 0$ almost surely. (In fact, the same holds even for $p < p_c$, see Theorem 6.1.)

Remark 6.4. As for the Hausdorff dimension, for $p > \frac{1}{n^2}$ we have

$$\dim_H M = 2 + \frac{\log p}{\log n}$$

almost surely, provided $M \neq \emptyset$.

We will also need the 1-dimensional analogue of the process (intervals instead of squares). Here $M^{(1D)} = \emptyset$ almost surely iff $p \leq \frac{1}{n}$, and for $p > \frac{1}{n}$ we have

$$\dim_H M^{(1D)} = 1 + \frac{\log p}{\log n}$$

almost surely, provided $M^{(1D)} \neq \emptyset$.

Now we check that the almost sure topological Hausdorff dimension of M also exists.

Lemma 6.5. *For every $p > \frac{1}{n^2}$ and $n \geq 2$, $n \in \mathbb{N}$ there exists a number $d = d^{(p,n)} \in [0,2]$ such that*

$$\dim_{tH} M = d$$

almost surely, provided $M \neq \emptyset$.

Proof. Let N be the random number of squares in M_1. Let us set $q = P(M = \emptyset)$. Then $q < 1$ by Remark 6.3, and [5, Thm. 15.2] gives that q is the least positive root of the polynomial

$$f(t) = -t + \sum_{k=0}^{n^2} P(N = k)t^k.$$

Let us fix an arbitrary $x \in [0, \infty)$. As $q > 0$, we obtain $P(\dim_{tH} M \le x) > 0$. First we show that $P(\dim_{tH} M \le x)$ is a root of f.

If $N > 0$ then let $M_1 = \{Q_1, \dots, Q_N\}$, where the Q_i's are the first level subsquares. For every i and k let $M_k^{Q_i}$ be the union of those squares in M_k that are in Q_i, and let $M^{Q_i} = \bigcap_k M_k^{Q_i}$. (Note that this is not the same as $M \cap Q_i$, since in this latter set there may be points on the boundary of Q_i 'coming from squares outside of Q_i'.) Then M^{Q_i} has the same distribution as a similar copy of M (this is called statistical self-similarity), and hence for every i

$$P\left(\dim_{tH} M^{Q_i} \le x\right) = P\left(\dim_{tH} M \le x\right).$$

Using the stability of the topological Hausdorff dimension for closed sets and the fact that the M^{Q_i}'s are independent and have the same distribution under the condition $N = k > 0$, this implies

$$\begin{aligned} P\left(\dim_{tH} M \le x \,|\, N = k\right) &= P\left(\dim_{tH} M^{Q_i} \le x \text{ for each } 1 \le i \le k \,|\, N = k\right) \\ &= \left(P\left(\dim_{tH} M^{Q_1} \le x\right)\right)^k \\ &= \left(P\left(\dim_{tH} M \le x\right)\right)^k. \end{aligned}$$

For $N = 0$ we have $P\left(\dim_{tH} M \le x \,|\, N = 0\right) = 1$, therefore we obtain

$$\begin{aligned} P(\dim_{tH} M \le x) &= \sum_{k=0}^{n^2} P(N = k) P(\dim_{tH} M \le x \,|\, N = k) \\ &= \sum_{k=0}^{n^2} P(N = k)\left(P\left(\dim_{tH} M \le x\right)\right)^k, \end{aligned}$$

and thus $P(\dim_{tH} M \le x)$ is indeed a root of f for every $x \in [0, \infty)$.

As mentioned above, $q \ne 1$ and q is also a root of f. Moreover, 1 is obviously also a root, and it is easy to see that f is strictly convex on $[0, \infty)$, hence there are at most two nonnegative roots. Therefore q and 1 are the only nonnegative roots, thus $P(\dim_{tH} M \le x) = q$ or 1 for every $x \in [0, \infty)$.

If $P(\dim_{tH} M \le 0) = 1$ then we are done, so we may assume that $P(\dim_{tH} M \le 0) = q$. Then the distribution function $F(x) = P(\dim_{tH} M \le x \,|\, M \ne \emptyset) = (P(\dim_{tH} M \le x) - q)/(1 - q)$ only attains the values 0 and 1, and clearly $F(0) = 0$ and $F(2) = 1$. Thus there is a value d where it 'jumps' from 0 to 1. This concludes the proof. □

6.2. **Proof of Theorem 6.2; the lower estimate of** $\dim_{tH} M$**.** Set

$$p_c^{(d,n)} = \sup\left\{p : \dim_{tH} M^{(p,n)} \le d \text{ almost surely}\right\}.$$

First we need some lemmas. The following one is analogous to [8, p. 387].

Lemma 6.6. *For every $d \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 2$*

$$p_c^{(d,n)} < 1 \quad \iff \quad p_c^{(d,n^2)} < 1.$$

Proof. Clearly, it is enough to show that

$$p_c^{(d,n)} \left(1 - \left(1 - p_c^{(d,n)}\right)^{\frac{1}{n^2}}\right) \leq p_c^{(d,n^2)} \leq p_c^{(d,n)}. \tag{6.2}$$

We say that the random construction X is dominated by the random construction Y if they can be realized on the same probability space such that $X \subseteq Y$ almost surely.

Let us first prove the second inequality in (6.2). It clearly suffices to show that

$$\dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely} \implies \dim_{tH} M^{(p,n)} \leq d \text{ almost surely.}$$

But this is rather straightforward, since $M_{2k}^{(p,n)}$ is easily seen to be dominated by $M_k^{(p,n^2)}$ for every k, hence $M^{(p,n)}$ is dominated by $M^{(p,n^2)}$.

Let us now prove the first inequality in (6.2). Set $\varphi(x) = 1 - (1-x)^{1/n^2}$. We need to show that

$$0 < p < p_c^{(d,n)} \varphi(p_c^{(d,n)}) \implies \dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely.} \tag{6.3}$$

Since $x\varphi(x)$ is an increasing homeomorphism of the unit interval, $p = q\varphi(q)$ for some $q \in (0,1)$. Then clearly $q < p_c^{(d,n)}$, so $\dim_{tH} M^{(q,n)} \leq d$ almost surely. Therefore, in order to prove (6.3) it suffices to check that

$$M^{(p,n^2)} \text{ is dominated by } M^{(q,n)}. \tag{6.4}$$

First we check that

$$M_k^{(\varphi(q),n^2)} \text{ is dominated by } M_k^{(q,n)} \text{ for every } k, \tag{6.5}$$

and consequently $M^{(\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$. Indeed, in the second case we erase a sub-square of side length $\frac{1}{n}$ with probability $1-q$ and keep it with probability q, while in the first case we *completely* erase a sub-square of side length $\frac{1}{n}$ with the same probability $(1-\varphi(q))^{n^2} = 1-q$ and hence keep *at least a subset of it* with probability q.

But this will easily imply (6.4), which will complete the proof. Indeed, after each step of the processes $M^{(\varphi(q),n^2)}$ and $M^{(q,n)}$ let us perform the following procedures. For $M^{(\varphi(q),n^2)}$ let us keep every existing square independently with probability q and erase it with probability $1-q$ (we do not do any subdivisions in this case). For $M^{(q,n)}$ let us take one more step of the construction of $M^{(q,n)}$. Using (6.5) this easily implies that $M_k^{(q\varphi(q),n^2)}$ is dominated by $M_{2k}^{(q,n)}$ for every k, hence $M^{(q\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$, but $q\varphi(q) = p$, and hence (6.4) holds. □

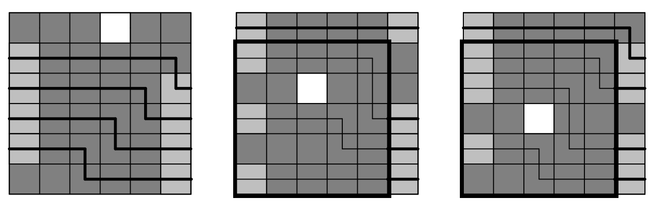
From now on let N be a fixed (large) positive integer to be chosen later. Recall that a square of level k is a set of the form $\left[\frac{i}{N^k}, \frac{i+1}{N^k}\right] \times \left[\frac{j}{N^k}, \frac{j+1}{N^k}\right] \subseteq [0,1]^2$.

Definition 6.7. A *walk of level k* is a sequence $(S_1, \dots, S_l)$ of non-overlapping squares of level k such that S_r and S_{r+1} are abutting for every $r = 1, \dots, l-1$, moreover $S_1 \cap (\{0\} \times [0,1]) \neq \emptyset$ and $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$.

In particular, the only walk of level 0 is $([0,1]^2)$.

Definition 6.8. We say that $(S_1, \dots, S_l)$ is a *turning walk (of level 1)* if it satisfies the properties of a walk of level 1 except that instead of $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$ we require that $S_l \cap ([0,1] \times \{1\}) \neq \emptyset$.

Lemma 6.9. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{0\} \times [0,1]$, and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{1\} \times [0,1]$. Moreover, let F^* be a square of level 1 such that the row of F^* does not intersect $\mathcal{S} \cup \mathcal{T}$. Then there exist $N-2$ non-overlapping walks of level 1 not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*



Case 1. Case 2. Case 3.

FIGURE 3. Illustration to Lemma 6.9

Proof. The proof is by induction on N. The case $N = 2$ is obvious.
Case 1. *F^* is in the top or bottom row.*

By simply ignoring this row it is straightforward how to construct the walks in the remaining rows.
Case 2. *F^* is not in the top or bottom row, and both top corners or both bottom corners are in $\mathcal{S} \cup \mathcal{T}$.*

Without loss of generality we may suppose that both top corners are in $\mathcal{S} \cup \mathcal{T}$. Let the straight walk connecting these two corners be one of the walks to be constructed. Then let us shift the remaining members of $\mathcal{T}$ to the left by one square, and either we can apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner of the original $N \times N$ many squares, or F^* is not among these $(N-1) \times (N-1)$ many squares and then the argument is even easier. Then one can see how to get the required walks.
Case 3. *Neither Case 1 nor Case 2 holds.*

Since there are only two squares missing on both sides, and F^* cannot be the top or bottom row, we infer that both $\mathcal{S}$ and $\mathcal{T}$ contain at least one corner. Since Case 2 does not hold, we obtain that both the top left and the bottom right corners or both the bottom left and the top right corners are in $\mathcal{S} \cup \mathcal{T}$. Without loss of generality we may suppose that both the top left and the bottom right corners are in $\mathcal{S} \cup \mathcal{T}$. By reflecting the picture about the center of the unit square if necessary, we may assume that F^* is not in the rightmost column. We now construct the first walk. Let it run straight from the top left corner to the top right corner, and then continue downwards until it first reaches a member of $\mathcal{T}$. Then, as above, we can similarly apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner, and we are done. $\square$

Lemma 6.10. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{0\} \times [0,1]$, and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level 1 intersecting $[0,1] \times \{1\}$ (the sets of starting and terminal squares). Moreover, let F^* be a square of level 1 (the forbidden square) such that the row of F^* does not intersect $\mathcal{S}$ and the column of F^* does not intersect $\mathcal{T}$. Then there exist $N-2$ non-overlapping turning walks not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*

Proof. Obvious, just take the simplest 'L-shaped' walks. □

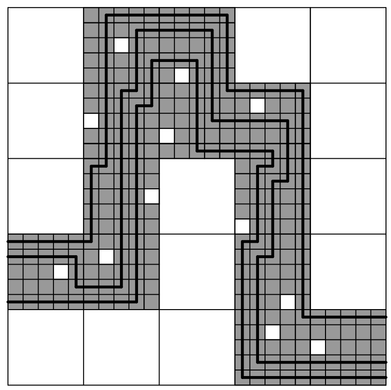


FIGURE 4. Illustration to Lemma 6.11

The last two lemmas will almost immediately imply the following.

Lemma 6.11. *Let $(S_1, \dots, S_l)$ be a walk of level k, and $\mathcal{F}$ a system of squares of level $k+1$ such that each S_r contains at most 1 member of $\mathcal{F}$. Then $(S_1, \dots, S_l)$ contains $N-2$ non-overlapping sub-walks of level $k+1$ avoiding $\mathcal{F}$.*

Proof. We may assume that each S_r contains exactly 1 member of $\mathcal{F}$. Let us denote the member of $\mathcal{F}$ in S_r by F_r^*. The sub-walks will be constructed separately in each S_r, using an appropriately rotated or reflected version of either Lemma 6.10 or Lemma 6.9. It suffices to construct $\mathcal{S}_r$ and $\mathcal{T}_r$ for every r (compatible with F_r^*) so that for every member of $\mathcal{T}_r$ there is an abutting member of $\mathcal{S}_{r+1}$. (Of course we also have to make sure that every member of $\mathcal{S}_1$ intersects $\{0\} \times [0,1]$ and every member of $\mathcal{T}_l$ intersects $\{1\} \times [0,1]$.) For example, the construction of $\mathcal{T}_r$ for $r < l$ is as follows. The squares S_r and S_{r+1} share a common edge E. Assume for simplicity that E is horizontal. Then $\mathcal{T}_r$ will consist of those sub-squares of S_r of level $k+1$ that intersect E and whose column differs from that of F_r^* and F_{r+1}^*. If these two columns happen to coincide then we can arbitrarily erase one more square. The remaining constructions are similar and the details are left to the reader. □

Definition 6.12. We say that a square in M_k is 1-*full* if it contains at least $N^2 - 1$ many sub-squares from M_{k+1}. We say that it is m-*full*, if it contains at least $N^2 - 1$ many $m-1$-full sub-squares from M_{k+1}. We call M *full* if M_0 is m-full for every $m \in \mathbb{N}^+$.

The following lemma was the key realization in [3].

Lemma 6.13. *There exists a* $p^{(N)} < 1$ *such that for every* $p > p^{(N)}$ *we have* $P\left(M^{(p,N)} \text{ is full}\right) > 0$.

See [3] or [5, Prop. 15.5] for the proof.

Definition 6.14. Let $L \leq N$ be positive integers. A compact set $K \subseteq [0,1]$ is called (L,N)-regular if it is of the form $K = \bigcap_{i\in\mathbb{N}} K_i$, where $K_0 = [0,1]$, and K_{k+1} is obtained by dividing every interval I in K_k into N many non-overlapping closed intervals of length $1/N^{k+1}$, and choosing L many of them for each I.

The following fact is well-known, see e.g. the more general [5, Thm. 9.3].

Fact 6.15. *An* (L,N)*-regular compact set has Hausdorff dimension* $\frac{\log L}{\log N}$.

Next we prove the main result of the present subsection.

Proof of Theorem 6.2. Let $d \in [0,2)$ be arbitrary. First we verify that, for sufficiently large N, if $M = M^{(p,N)}$ is full then $\dim_{tH} M > d$. The strategy is as follows. We define a collection $\mathcal{G}$ of disjoint connected subsets of M such that if a set intersects each member of $\mathcal{G}$ then its Hausdorff dimension is larger than $d-1$. Then we show that for every countable open basis $\mathcal{U}$ of M the union of the boundaries, $\bigcup_{U\in\mathcal{U}} \partial_M U$ intersects each member of $\mathcal{G}$, which clearly implies $\dim_{tH} M > d$.

Let us fix an integer N such that

$$N \geq 6 \text{ and } \frac{\log(N-2)}{\log N} > d-1, \tag{6.6}$$

and let us assume that M is full. Using Lemma 6.13 at each step we can choose $N-2$ non-overlapping walks of level 1 in M_1, then $N-2$ non-overlapping walks of level 2 in M_2 in each of the above walks, etc. Let us denote the obtained system at step k by

$$\mathcal{G}_k = \left\{\Gamma_{i_1,\dots,i_k} : (i_1,\dots,i_k) \in \{1,\dots,N-2\}^k\right\},$$

where $\Gamma_{i_1,\dots,i_k}$ is the union of the squares of the corresponding walk. (Set $\mathcal{G}_0 = \{\Gamma_\emptyset\} = \{[0,1]^2\}$.) Let us also put

$$C_k = \left\{y \in [0,1] : (0,y) \in \bigcup \mathcal{G}_k\right\}$$

and define

$$C = \bigcap_{k\in\mathbb{N}} C_k.$$

Then clearly C is an $(N-2,N)$-regular compact set, therefore Fact 6.15 yields that $\dim_H C = \frac{\log(N-2)}{\log N} > d-1$. We will also need that $\dim_H(C\setminus\mathbb{Q}) > d-1$, but this is clear since $\dim_H C > 0$ and hence $\dim_H(C\setminus\mathbb{Q}) = \dim_H C$.

For every $y \in C\setminus\mathbb{Q}$ and every $k \in \mathbb{N}$ there is a unique $(i_1,\dots,i_k)$ such that $(0,y) \in \Gamma_{i_1,\dots,i_k}$. (For y's of the form $\frac{i}{N^l}$ there may be two such $(i_1,\dots,i_k)$'s, and we would like to avoid this complication.) Put $\Gamma_k(y) = \Gamma_{i_1,\dots,i_k}$ and $\Gamma(y) = \bigcap_{k=1}^\infty \Gamma_k(y)$. Since $\Gamma(y)$ is a decreasing intersection of compact connected sets, it is itself connected ([4]). (Actually, it is a continuous curve, but we will not need this here.) It is also easy to see that it intersects $\{0\}\times[0,1]$ and $\{1\}\times[0,1]$.

We can now define

$$\mathcal{G} = \{\Gamma(y) : y \in C \setminus \mathbb{Q}\}.$$

Next we prove that $\mathcal{G}$ consists of disjoint sets. Let $y, y' \in C \setminus \mathbb{Q}$ be distinct. Pick $l \in \mathbb{N}$ so large such that $|y - y'| > \frac{6}{N^l}$. Then there are at least 5 intervals of level l between y and y'. Since we always chose $N-2$ intervals out of N along the construction, there can be at most 4 consecutive non-selected intervals, therefore there is a $\Gamma_{i_1,\dots,i_l}$ separating y and y'. But then this also separates $\Gamma_l(y)$ and $\Gamma_l(y')$, hence $\Gamma(y)$ and $\Gamma(y')$ are disjoint.

Now we check that for every $y \in C \setminus \mathbb{Q}$ and every countable open basis $\mathcal{U}$ of M the set $\bigcup_{U \in \mathcal{U}} \partial_M U$ intersects $\Gamma(y)$. Let $z_0 \in \Gamma(y)$ and $U_0 \in \mathcal{U}$ such that $z_0 \in U_0$ and $\Gamma(y) \nsubseteq U_0$. Then $\partial_M U_0$ must intersect $\Gamma(y)$, since otherwise $\Gamma(y) = (\Gamma(y) \cap U_0) \cup (\Gamma(y) \cap \operatorname{int}_M(M \setminus U_0))$, hence a connected set would be the union of two non-empty disjoint relatively open sets, a contradiction.

Thus, as explained in the first paragraph of the proof, it is sufficient to prove that if a set Z intersects every $\Gamma(y)$ then $\dim_H Z > d-1$. This is easily seen to hold if we can construct an onto Lipschitz map

$$\varphi\colon \bigcup \mathcal{G} \to C \setminus \mathbb{Q}$$

that is constant on every member of $\mathcal{G}$, since Lipschitz maps do not increase Hausdorff dimension, and $\dim_H(C \setminus \mathbb{Q}) > d-1$. Define

$$\varphi(z) = y \text{ if } z \in \Gamma(y),$$

which is well-defined by the disjointness of the members of $\mathcal{G}$.

Let us now prove that this map is Lipschitz. Let $y, y' \in C \setminus \mathbb{Q}$, $z \in \Gamma(y)$, and $z' \in \Gamma(y')$. Choose $l \in \mathbb{N}^+$ such that $\frac{1}{N^l} < |y - y'| \leq \frac{1}{N^{l-1}}$. Then using $N \geq 6$ we obtain $|y - y'| > \frac{6}{N^{l+1}}$, thus, as above, there is a walk of level $l+1$ separating z and z'. Therefore $|z - z'| \geq \frac{1}{N^{l+1}}$, and hence

$$|\varphi(z) - \varphi(z')| = |y - y'| \leq \frac{1}{N^{l-1}} = N^2 \frac{1}{N^{l+1}} \leq N^2 |z - z'|,$$

therefore φ is Lipschitz with Lipschitz constant at most N^2.

To finish the proof, let n be given as in Theorem 6.2 and pick $k \in \mathbb{N}$ so large that $N = n^{2^k}$ satisfies (6.6). If $p > p^{(N)}$ then using Lemma 6.13 we deduce that

$$P\left(\dim_{tH} M^{(p,N)} > d\right) \geq P\left(M^{(p,N)} \text{ is full }\right) > 0,$$

which implies $p_c^{(d,N)} < 1$. Iterating k times Lemma 6.6 we infer $p_c^{(d,n)} < 1$.

Now, if $p > p_c^{(d,n)}$ then

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \neq \emptyset\right) \geq P(\dim_{tH} M^{(p,n)} > d) > 0.$$

Combining this with Lemma 6.5 we deduce that

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \neq \emptyset\right) = 1,$$

which completes the proof of the theorem. □

Remark 6.16. It is well-known and not difficult to see that $\lim\limits_{p \to 1} P(M^{(p,n)} = \emptyset) = 0$. Using this it is an easy consequence of the previous theorem that for every integer

$n > 1$, $d < 2$ and $\varepsilon > 0$ there exists a $\delta = \delta^{(n,d,\varepsilon)} > 0$ such that for all $p > 1 - \delta$

$$P\left(\dim_{tH} M^{(p,n)} > d\right) > 1 - \varepsilon.$$

6.3. **The upper estimate of** $\dim_{tH} M$**.** The argument of this subsection will rely on some ideas from [3].

Theorem 6.17. *If* $p > \frac{1}{\sqrt{n}}$ *then almost surely*

$$\dim_{tH} M \leq 2 + 2\frac{\log p}{\log n}.$$

Proof. A segment is called a *basic segment* if it is of the form $\left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \{\frac{j}{n^k}\}$ or $\{\frac{j}{n^k}\} \times \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right]$, where $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

It suffices to show that for every basic segment S and for every $\varepsilon > 0$ there exists (almost surely, a random) arc $\gamma \subseteq [0,1]^2$ connecting the endpoints of S in the ε-neighborhood of S such that $\dim_H(M \cap \gamma) \leq 1 + 2\frac{\log p}{\log n}$. Indeed, we can almost surely construct the analogous arcs for all basic segments, and hence obtain a basis of M consisting of 'approximate squares' whose boundaries are of Hausdorff dimension at most $1 + 2\frac{\log p}{\log n}$, therefore $\dim_{tH} M \leq 2 + 2\frac{\log p}{\log n}$ almost surely.

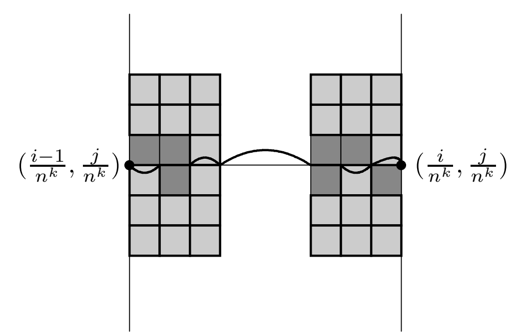


FIGURE 5. Construction of the arc γ connecting the endpoints of S

Let us now construct such an arc γ for S and $\varepsilon > 0$. We may assume that S is horizontal, hence it is of the form $S = \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \{\frac{j}{n^k}\}$ for some $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

We divide S into n subsegments of length $\frac{1}{n^{k+1}}$, and we call a subsegment $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times \{\frac{j}{n^k}\}$ *bad* if both the adjacent squares $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k} - \frac{1}{n^{k+1}}, \frac{j}{n^k}]$ and $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k}, \frac{j}{n^k} + \frac{1}{n^{k+1}}]$ are in M_{k+1}. Otherwise we say that the subsegment is *good*. Let B_1 denote the union of the bad segments. Then inside every bad segment we repeat the same procedure, and obtain B_2 and so on. It is easy to see that this process is (a scaled copy of) the 1-dimensional fractal percolation with p replaced by p^2. Let $B = \bigcap_l B_l$ be its limit set. Then by Remark 6.4 (note that $p^2 > \frac{1}{n}$) we obtain $\dim_H B = 1 + \frac{\log p^2}{\log n} = 1 + 2\frac{\log p}{\log n}$ or $B = \emptyset$ almost surely. So it suffices to construct a γ connecting the endpoints of S in the ε-neighborhood of S such that $\gamma \cap M = B$ (except perhaps some endpoints, but all the endpoints form a countable set, hence a set of Hausdorff dimension 0).

But this is easily done. Indeed, for every good subsegment I let γ_I be an arc connecting the endpoints of I in a small neighborhood of I such that γ is disjoint from M apart from the endpoints (this is possible, since either the top or the bottom square was erased from M). Then $\gamma = \left(\bigcup_{I \text{ is good}} \gamma_I\right) \cup B$ works. □

Using Remarks 6.3 and 6.4 this easily implies

Corollary 6.18. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

Remark 6.19. Calculating the exact value of $\dim_{tH} M$ seems to be difficult, since it would provide the value of the critical probability p_c of Chayes, Chayes and Durrett (where the phase transition occurs, see above), and this is a long-standing open problem.

7. Application II: The Hausdorff dimension of the level sets of the generic continuous function

Now we return to Problem 1.3. The main goal is to find analogues to Kirchheim's theorem, that is, to determine the Hausdorff dimension of the level sets of the generic continuous function defined on a compact metric space K.

Let us first note that the case $\dim_t K = 0$, that is, when there is a basis consisting of clopen sets is trivial because of the following well-known and easy fact. For a short proof see [2, Lemma 2.6].

Fact 7.1. *If K is a compact metric space with $\dim_t K = 0$ then the generic continuous function is one-to-one on K.*

Corollary 7.2. *If K is a compact metric space with $\dim_t K = 0$ then every non-empty level set of the generic continuous function is of Hausdorff dimension 0.*

Hence from now on we can restrict our attention to the case of positive topological dimension.

In the first part of this section we prove Theorem 7.12 and Corollary 7.14, our main theorems concerning level sets of the generic function defined on an arbitrary compact metric space, then we use this to derive conclusions about homogeneous and self-similar spaces in Theorem 7.15 and Corollary 7.16.

7.1. Arbitrary compact metric spaces.

The goal of this subsection is to prove Theorem 7.12. In order to do this we will need two equivalent definitions of the topological Hausdorff dimension.

Let us fix a compact metric space K with $\dim_t K > 0$, and let $C(K)$ denote the space of continuous real-valued functions equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments.

Definition 7.3. Define

$$P_l = \{d \geq 1 : \exists G \subseteq K \text{ such that } \dim_H G \leq d-1 \text{ and the generic } f \in C(K) \text{ is one-to-one on } K \setminus G\}.$$

Definition 7.4. We say that a continuous function f is *d-level narrow*, if there exists a dense set $S_f \subseteq \mathbb{R}$ such that $\dim_H f^{-1}(y) \leq d-1$ for every $y \in S_f$. Let $\mathcal{N}_d$ be the set of d-level narrow functions. Define

$$P_n = \{d : \mathcal{N}_d \text{ is somewhere dense in } C(K)\}.$$

Now we repeat the definition of the topological Hausdorff dimension.

Definition 7.5. Let $\dim_{tH} K = \inf P_{tH}$, where

$$P_{tH} = \{d : K \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \le d-1 \text{ for every } U \in \mathcal{U}\}.$$

We assume that by definition $\infty \in P_n, P_l, P_{tH}$.

Now we show the following theorem.

Theorem 7.6. *If K is a compact metric space with $\dim_t K > 0$ then*

$$P_{tH} = P_l = P_n.$$

Theorem 7.6 and Corollary 3.8 immediately yield two new equivalent definitions for the topological Hausdorff dimension.

Theorem 7.7. *If K is a compact metric space with $\dim_t K > 0$ then*

$$\dim_{tH} K = \min P_l = \min P_n.$$

Before proving Theorem 7.6 we need a lemma.

Lemma 7.8. *Let $K_1 \subseteq K_2$ be compact metric spaces and*

$$R\colon C(K_2) \to C(K_1), \quad R(f) = f|_{K_1}.$$

If $\mathcal{F} \subseteq C(K_1)$ is co-meager then so is $R^{-1}(\mathcal{F}) \subseteq C(K_2)$.

Proof. The map R is clearly continuous. Using the Tietze Extension Theorem it is not difficult to see that it is also open. We may assume that $\mathcal{F}$ is a dense G_δ set in $C(K_1)$. The continuity of R implies that $R^{-1}(\mathcal{F})$ is also G_δ, thus it is enough to prove that $R^{-1}(\mathcal{F})$ is dense in $C(K_2)$. Let $\mathcal{U} \subseteq C(K_2)$ be non-empty open, then $R(\mathcal{U}) \subseteq C(K_1)$ is also non-empty open, hence $R(\mathcal{U}) \cap \mathcal{F} \neq \emptyset$, and therefore $\mathcal{U} \cap R^{-1}(\mathcal{F}) \neq \emptyset$. □

Next we prove Theorem 7.6. The proof will consist of three lemmas.

Lemma 7.9. $P_{tH} \subseteq P_l$.

Proof. Assume $d \in P_{tH}$ and $d < \infty$. Let $\mathcal{U}$ be a countable basis of K such that $\dim_H \partial U \le d-1$ for all $U \in \mathcal{U}$. Now the assumption $\dim_t K \ge 1$ and Theorem 4.4 yield $d \ge \dim_{tH} K \ge 1$. Let $F = \bigcup_{U\in\mathcal{U}} \partial U$, the countable stability of Hausdorff dimension implies $\dim_H F \le d-1$. Then there exists a G_δ set $G \subseteq K$ such that $F \subseteq G$ and $\dim_H G = \dim_H F \le d-1$. The above definitions clearly imply $\dim_t(K \setminus G) \le \dim_t(K \setminus F) \le 0$.

As $K \setminus G$ is F_σ, we can choose compact sets K_n such that $K \setminus G = \bigcup_{n=1}^\infty K_n$ and $K_n \subseteq K_{n+1}$ for all $n \in \mathbb{N}^+$. Let $\mathcal{F}_n = \{f \in C(K_n) : f \text{ is one-to-one}\}$ and let us define $R_n\colon C(K) \to C(K_n)$ as $R_n(f) = f|_{K_n}$ for all $n \in \mathbb{N}^+$. Since $\dim_t K_n \le \dim_t(K\setminus G) \le 0$, Fact 7.1 implies that the sets $\mathcal{F}_n \subseteq C(K_n)$ are co-meager. Lemma 7.8 yields that $R_n^{-1}(\mathcal{F}_n) \subseteq C(K)$ are co-meager, too. As a countable intersection of co-meager sets $\mathcal{F} = \bigcap_{n=1}^\infty R_n^{-1}(\mathcal{F}_n) \subseteq C(K)$ is also co-meager. Clearly, every $f \in \mathcal{F}$ is one-to-one on all K_n, so $K_n \subseteq K_{n+1}$ $(n \in \mathbb{N}^+)$ yields that f is one-to-one on $\bigcup_{n=1}^\infty K_n = K \setminus G$. Hence $d \in P_l$. □

Lemma 7.10. $P_l \subseteq P_n$.

Proof. Assume $d \in P_l$ and $d < \infty$. By the definition of P_l, there exists $G \subseteq K$ such that $\dim_H G \leq d-1$ and for the generic $f \in C(K)$ for all $y \in \mathbb{R}$ we have $\#(f^{-1}(y) \setminus G) \leq 1$. Then $\dim_H G \leq d-1$ and $d \geq 1$ yield $\dim_H f^{-1}(y) \leq d-1$, so $\mathcal{N}_d$ is co-meager, thus (everywhere) dense. Hence $d \in P_n$. □

Lemma 7.11. $P_n \subseteq P_{tH}$.

Proof. Assume $d \in P_n$ and $d < \infty$. Let us fix $x_0 \in K$ and $r > 0$. To verify $d \in P_{tH}$ we need to find an open set U such that $x_0 \in U \subseteq U(x_0, r)$ and $\dim_H \partial U \leq d-1$. We may assume $\partial U(x_0, r) \neq \emptyset$, otherwise we are done.

By $d \in P_n$ we obtain that $\mathcal{N}_d$ is dense in a ball $B(f_0, 6\varepsilon)$, $\varepsilon > 0$. By decreasing r if necessary, we may assume that $\operatorname{diam} f_0(U(x_0,r)) \leq 3\varepsilon$. Then Tietze's Extension Theorem provides an $f \in B(f_0, 6\varepsilon)$ such that $f(x_0) = f_0(x_0)$ and $f|_{\partial U(x_0,r)}(x) = f_0(x_0) + 3\varepsilon$ for every $x \in \partial U(x_0, r)$. Since $\mathcal{N}_d$ is dense in $B(f_0, 6\varepsilon)$, we can choose $g \in \mathcal{N}_d$ such that $||f - g|| \leq \varepsilon$. By the construction of g it follows that $g(x_0) < \min\{g(\partial U(x_0, r))\}$. Hence in the dense set S_g (see Definition 7.4) there is an $s \in S_g$ such that

$$g(x_0) < s < \min \{ g (\partial U(x_0, r))\} . \tag{7.1}$$

Let

$$U = g^{-1}((-\infty, s)) \cap U(x_0, r),$$

then clearly $x_0 \in U \subseteq U(x_0, r)$. By (7.1) we have $\partial g^{-1}((-\infty, s)) \cap \partial U(x_0, r) = \emptyset$, therefore $\partial U \subseteq \partial g^{-1}((-\infty, s)) \subseteq g^{-1}(s)$. Using $s \in S_g$ we infer that $\dim_H \partial U \leq \dim_H g^{-1}(s) \leq d-1$. □

This concludes the proof of Theorem 7.6.

Now we are ready to describe the Hausdorff dimension of the level sets of generic continuous functions.

As already mentioned above, if $\dim_t K = 0$ then every level set of a generic continuous function on K consists of at most one point.

Theorem 7.12. *Let K be a compact metric space with $\dim_t K > 0$. Then for the generic $f \in C(K)$*

(i) $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every* $y \in \mathbb{R}$,
(ii) for every $d < \dim_{tH} K$ *there exists a non-degenerate interval* $I_{f,d}$ *such that* $\dim_H f^{-1}(y) \geq d-1$ *for every* $y \in I_{f,d}$.

This theorem actually readily follows from the following more precise, but slightly technical version.

Theorem 7.13. *Let K be a compact metric space with $\dim_t K > 0$. Then there exists a G_δ set $G \subseteq K$ with $\dim_H G = \dim_{tH} K - 1$ such that for the generic $f \in C(K)$*

(i) f is one-to-one on $K \setminus G$, hence $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$ *for every* $y \in \mathbb{R}$,
(ii) for every $d < \dim_{tH} K$ *there exists a non-degenerate interval* $I_{f,d}$ *such that* $\dim_H f^{-1}(y) \geq d-1$ *for every* $y \in I_{f,d}$.

Proof. Let us first prove (i). Theorem 7.7 implies $\dim_{tH} K = \min P_l$. Thus there exists a set $G \subseteq K$ such that $\dim_H G = \dim_{tH} K - 1$ and the generic $f \in C(K)$ is one-to-one on $K \setminus G$. By taking a G_δ hull of the same Hausdorff dimension we can assume that G is G_δ. As $\dim_{tH} K = \min P_l \geq 1$, we have $\dim_H G = \dim_{tH} K - 1 \geq$

0. Hence $\dim_H f^{-1}(y) \le \dim_H G = \dim_{tH} K - 1$ for the generic $f \in C(K)$ and for all $y \in \mathbb{R}$. Thus (i) holds.

Let us now prove (ii). Let us choose a sequence $d_k \nearrow \dim_{tH} K$. Theorem 7.7 yields $d_k < \dim_{tH} K = \min P_n$ for every $k \in \mathbb{N}^+$. Hence $\mathcal{N}_{d_k}$ is nowhere dense by the definition of P_n. It follows from the definition of $\mathcal{N}_d$ that for every $f \in C(K) \setminus \mathcal{N}_{d_k}$ there exists a non-degenerate interval I_{f,d_k} such that $\dim_H f^{-1}(y) \ge d_k - 1$ for every $y \in I_{f,d_k}$. But then (ii) holds for every $f \in C(K) \setminus (\bigcup_{k \in \mathbb{N}^+} \mathcal{N}_{d_k})$, and this latter set is clearly co-meager, which concludes the proof of the theorem. □

This immediately implies

Corollary 7.14. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

$$\sup \left\{ \dim_H f^{-1}(y) : y \in \mathbb{R} \right\} = \dim_{tH} K - 1.$$

7.2. **Homogeneous and self-similar compact metric spaces.** In this subsection we show that if the compact metric space is sufficiently homogeneous, e.g. self-similar (see [5] or [16]) then we can say much more.

Theorem 7.15. *Let K be a compact metric space with $\dim_t K > 0$ such that $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Before turning to the proof of this theorem we formulate a corollary. Recall that K is *self-similar* if there are injective contractive similitudes $\varphi_1, \dots, \varphi_k : K \to K$ such that $K = \bigcup_{i=1}^k \varphi_i(K)$. The sets of the form $\varphi_{i_1} \circ \varphi_{i_2} \circ \dots \circ \varphi_{i_m}(K)$ are called the *elementary pieces* of K. It is easy to see that every ball in K contains an elementary piece. Moreover, by Corollary 4.8 the topological Hausdorff dimension of every elementary piece is $\dim_{tH} K$. Hence, using monotonicity as well, we obtain that if K is self-similar then $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. This yields the following.

Corollary 7.16. *Let K be a self-similar compact metric space with $\dim_t K > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Proof of Theorem 7.15. Theorem 7.12 implies that for the generic $f \in C(K)$ for *every* $y \in \mathbb{R}$ we have $\dim_H f^{-1}(y) \le \dim_{tH} K - 1$, so we only have to prove the opposite inequality.

Let us consider a sequence $0 < d_k \nearrow \dim_{tH} K$. For $f \in C(K)$ and $k \in \mathbb{N}^+$ let

$$L_{f,k} = \left\{ y \in f(K) : \dim_H f^{-1}(y) \ge d_k - 1 \right\}.$$

First we show that it suffices to construct for every $k \in \mathbb{N}^+$ a co-meager set $\mathcal{F}_k \subseteq C(K)$ such that for every $f \in \mathcal{F}_k$ the set $L_{f,k}$ is co-meager in $f(K)$. Indeed, then the set $\mathcal{F} = \bigcap_{k \in \mathbb{N}^+} \mathcal{F}_k \subseteq C(K)$ is co-meager, and for every $f \in \mathcal{F}$ the set $L_f = \bigcap_{k \in \mathbb{N}^+} L_{f,k} \subseteq f(K)$ is also co-meager. Since for every $y \in L_f$ clearly $\dim_H f^{-1}(y) \ge \dim_{tH} K - 1$, this finishes the proof.

Let us now construct such an $\mathcal{F}_k$ for a fixed $k \in \mathbb{N}^+$. Let $\{B_n\}_{n \in \mathbb{N}}$ be a countable basis of K consisting of closed balls, and for all $n \in \mathbb{N}$ let $R_n : C(K) \to C(B_n)$ be defined as

$$R_n(f) = f|_{B_n}.$$

Let us also define

$$\mathcal{B}_n = \left\{ f \in C(B_n) : \exists I_{f,n} \text{ s. t. } \forall y \in I_{f,n} \ \dim_H f^{-1}(y) \geq d_k - 1 \right\},$$

(where $I_{f,n}$ is understood to be a non-degenerate interval). Finally, let us define

$$\mathcal{F}_k = \bigcap_{n \in \mathbb{N}} R_n^{-1}(\mathcal{B}_n).$$

First we show that $\mathcal{F}_k$ is co-meager. By our assumption $\dim_{tH} B_n = \dim_{tH} K > d_k$ (which also implies $\dim_t B_n > 0$ by Fact 4.1, since $d_k > 0$), thus Theorem 7.12 yields that $\mathcal{B}_n$ is co-meager in $C(B_n)$. Lemma 7.8 implies that $R_n^{-1}(\mathcal{B}_n)$ is co-meager in $C(K)$ for all $n \in \mathbb{N}$, thus $\mathcal{F}_k$ is also co-meager.

It remains to show that for every $f \in \mathcal{F}_k$ the set $L_{f,k}$ is co-meager in $f(K)$. Let us fix $f \in \mathcal{F}_k$. We will actually show that $L_{f,k}$ contains an open set in $\mathbb{R}$ which is a dense subset of $f(K)$. So let $U \subseteq \mathbb{R}$ be an open set in $\mathbb{R}$ such that $f(K) \cap U \neq \emptyset$. It is enough to prove that $L_{f,k} \cap U$ contains an interval. Since the B_n's form a basis, the continuity of f implies that there exists an $n \in \mathbb{N}$ such that $f(B_n) \subseteq U$. It is easy to see using the definition of $\mathcal{F}_k$ that $f|_{B_n} \in \mathcal{B}_n$, so there exists a non-degenerate interval $I_{f|_{B_n},k}$ such that for all $y \in I_{f|_{B_n},k}$ we have

$$\dim_H f^{-1}(y) \geq \dim_H \left(f|_{B_n}\right)^{-1}(y) \geq d_k - 1.$$

Thus $I_{f|_{B_n},k} \subseteq L_{f,k}$. Then $d_k - 1 > -1$ implies $\left(f|_{B_n}\right)^{-1}(y) \neq \emptyset$ for every $y \in I_{f|_{B_n},k}$, thus $I_{f|_{B_n},k} \subseteq f(B_n)$. But it follows from $f(B_n) \subseteq U$ that $I_{f|_{B_n},k} \subseteq U$. Hence $I_{f|_{B_n},k} \subseteq L_{f,k} \cap U$ and this completes the proof. □

8. Open Problems

First let us recall the most interesting open problem.

Problem 5.8. *Determine the almost sure topological Hausdorff dimension of the range of the d-dimensional Brownian motion for $d = 2$ and $d = 3$. Equivalently, determine the smallest $c \geq 0$ such that the range can be decomposed into a totally disconnected set and a set of Hausdorff dimension at most $c - 1$ almost surely.*

Now we collect a few more open problems. The first one concerns a certain Darboux property.

Problem 8.1. *Let $B \subseteq \mathbb{R}^d$ be a Borel set and $1 \leq c < \dim_{tH} B$ be arbitrary. Does there exist a Borel set $B' \subseteq B$ with $\dim_{tH} B' = c$?*

The following problem is motivated by the proof of Theorem 6.2. The idea is to look for some structural reason behind large topological Hausdorff dimension.

Problem 8.2. *Is it true that a compact metric space K satisfies $\dim_{tH} K \geq c$ iff it contains a family of* disjoint *non-degenerate continua such that each set meeting all members of this family is of Hausdorff dimension at least $c - 1$?*

The following remark shows that by dropping disjointness the problem becomes rather simple.

Remark 8.3. If K is a non-empty compact metric space and $\mathcal{S}$ is the collection of subsets of K intersecting every non-degenerate continuum then

$$\dim_{tH} K = \min\{\dim_H S + 1 : S \in \mathcal{S}\}.$$

Indeed, first let $S \in \mathcal{S}$ be arbitrary, and we prove that $\dim_{tH} K \leq \dim_H S + 1$. We may assume that S is G_δ, since we can take G_δ hulls with the same Hausdorff dimension. Then $K \setminus S$ is σ-compact. A compact subset of $K \setminus S$ does not contain non-degenerate continua by definition, so it is totally disconnected, thus it has topological dimension at most zero. Therefore the countable stability of topological dimension zero for closed sets [4, 1.3.1.] yields that $\dim_t(K \setminus S) \leq 0$. Hence Theorem 3.6 implies $\dim_{tH} K \leq \dim_H S + 1$.

Now we prove that there exists $S \in \mathcal{S}$ with $\dim_{tH} K = \dim_H S + 1$. Theorem 3.6 yields that there is a set $S \subseteq K$ with $\dim_H S = \dim_{tH} K - 1$ and $\dim_t(K \setminus S) \leq 0$. Then $K \setminus S$ cannot contain any non-degenerate continuum, so $S \in \mathcal{S}$.

Notice that the above remark does not apply even to G_δ subspaces of Euclidean spaces, see Example 4.19.

And finally,

Problem 8.4. *What is the right notion to describe the packing, lower box, or upper box dimension of the level sets of the generic continuous function $f \in C(K)$?*

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References

[1] R. Balka, Inductive topological Hausdorff dimensions and fibers of generic continuous functions, *Monatsh. Math.* **174** (2014), no. 1, 1–28.
[2] R. Balka, Z. Buczolich, M. Elekes, Topological Hausdorff dimension and level sets of generic continuous functions on fractals, *Chaos Solitons Fractals* **45** (2012), no. 12, 1579–1589.
[3] J. T. Chayes, L. Chayes, R. Durrett, Connectivity properties of Mandelbrot's percolation process, *Probab. Th. Rel. Fields* **77** (1988), 307–324.
[4] R. Engelking, *Dimension Theory*, North-Holland Publishing Company, 1978.
[5] K. Falconer, *Fractal geometry: mathematical foundations and applications*, Second Edition, John Wiley & Sons, 2003.
[6] K. Falconer and G. Grimmett, The critical point of fractal percolation in three and more dimensions, *J. Phys. A: Math. Gen.* **24** (1991), 491–494.
[7] W. Fulton, *Algebraic Topology. A First Course.* Springer-Verlag, 1995.
[8] G. Grimmett, *Percolation*, Second Edition, Springer-Verlag, 1999.
[9] J. Hawkes, Trees generated by a simple branching process, *J. London Math. Soc.* **24** (1981), 373–384.
[10] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948.
[11] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.
[12] B. Kirchheim, Hausdorff measure and level sets of typical continuous mappings in Euclidean spaces, *Trans. Amer. Math. Soc.* **347** (1995), no. 5, 1763–1777.
[13] T. W. Körner, Besicovitch via Baire, *Studia Math.* **158** (2003), no. 1, 65–78.
[14] B. Mandelbrot, Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier, *J. of Fluid Mechanics* **62** (1974), 331–358.
[15] B. Mandelbrot, *Les objets fractals. Forme, hasard et dimension.* Nouvelle Bibliothèque Scientifique, Flammarion, Editeur, Paris, 1975.
[16] P. Mattila, *Geometry of sets and measures in Euclidean spaces*, Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.
[17] S. Mazurkiewicz, Sur les problèmes κ et λ de Urysohn, *Fund. Math.* **10** (1927), 311–319.
[18] J. van Mill, *The infinite-dimensional topology of function spaces*, North-Holland Publishing Company, 2001.
[19] P. Mörters and Y. Peres, *Brownian motion*, With an appendix by Oded Schramm and Wendelin Werner, Cambridge University Press, 2010.

[20] M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge University Press, 1954.
[21] J. C. Oxtoby, The Banach-Mazur game and Banach's category theorem, Contributions to the theory of games, *Ann. of Math. Stud.* **39** (1957), Princeton University Press, 159–163.
[22] M. Urbański, Transfinite Hausdorff dimension, *Topology Appl.* **156** (2009), no. 17, 2762–2771.
[23] G. T. Whyburn, *Topological analysis*, Princeton University Press, 1958.

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