13/14th Emléktábla Workshop

Extremal combinatorics / Matroid theory 08.13 - 08.18.2023.

Preliminary Schedule for the Extremal Combinatorics group:

Sunday

Arrival at the hotel from 14:00.
16:30 Short introduction/presentation of problems
18:00 Dinner

Monday-Thursday

7:30-8:30 Breakfast
9:00-17:00 Working in groups of 3-5
12:20-14:00 Lunch
16:45 Presentations of daily progress
18:00 Dinner

Friday

Check-out until 14:00.

List of Participants

Péter Ágoston, ELTE
János Barát, University of Pannónia
Pál Bärnkopf, ELTE
Zoltán Blázsik, Rényi Institute /Szeged
Márton Borbényi, ELTE
Endre Csóka, Rényi
Márk Di-Giovanni, Budapest
Panna Tímea Fekete, ELTE
Dániel Gerbner, Rényi Institute
Andrzej Grzesik, Jagellonian University
Ervin Győri, Rényi Institute
Anna Halfpap, University of Montana
Barnabás Janzer, Warwick
Oliver Janzer, Cambridge
Attila Jung, ELTE
Balázs Keszegh, Rényi Institute
Bartłomiej Kielak, Jagiellonian University
Benedek Kovács, ELTE
Van Magnan, University of Montana
Ryan Martin, Iowa State University
Abhishek Methuku, ETH Zürich
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Zoltán Lóránt Nagy, ELTE
Cory Palmer, University of Montana
Magdalena Prorok, AGH University Kraków
Dömötör Pálvölgyi, ELTE-MTA
Balázs Patkós, Rényi Institute
Nika Salia
Ádám Schweitzer, Berlin
Casey Tompkins, Rényi Institute
Máté Vizer, Rényi Institute
Ryan Wood, University of Montana
Russ Woodroofe, University of Primorska

The maximum number of edges in a K_6-minor-free graph of girth 5

by János Barát

We use n for the number of vertices of G, and m for the number of edges. The following is a fundamental question in extremal graph theory:
How many edges can an n-vertex K_t-minor-free graph have and what do the extremal graphs look like?

We would like to add a girth condition.
What is the maximum number of edges in a K_t-minor-free graph with n vertices and girth g?

We obtained some results in this direction. In particular for $t = 4, 5$ and $g = 5$, see [1, 2]. Aigner-Horev and Krakovski [3] proved that any K_6-minor-free graph of girth 6 has at most $3n - 6$ edges. We are still mostly interested in the case $g = 5$.

Problem 1. *What is the maximum number of edges in a K_6-minor-free graph with n vertices and girth 5?*

This is probably a difficult problem. We have a conjecture. If it is true, then it would have consequences to list coloring. There is a recent paper on ArXiv, which could be relevant: [4].

References

[1] János Barát. Extremal K_4-minor-free graphs without short cycles. *Periodica Mathematica Hungarica*, 86: 108–114, 2023.

[2] János Barát. On the number of edges in a K_5-minor-free graph of given girth. *Discrete Mathematics* to appear

[3] Elad Aigner-Horev, Roi Krakovski. Extremal results regarding K_6-minors in graphs of girth at least 5. *Journal of Combinatorics*, 2(3): 463–479, 2011.

[4] Maria Chudnovsky, Alex Scott, Paul Seymour, Sophie Spirkl Bipartite graphs with no K_6 minor.
`https://arxiv.org/pdf/2204.10119.pdf`

r-regular graphs are not uniquely hamiltonian

by Pál Bärnkopf

A graph G is said to be uniquely hamiltonian if it contains precisely one Hamiltonian cycle.

Problem 1. *[1] If G is a finite r-regular, simple graph, where $r > 2$, then G is not uniquely hamiltonian.*

This conjecture has been proved for all odd values of r [2] and for all even values of $r > 23$. [3]

References

[1] John Sheehan: The multiplicity of Hamiltonian circuits in a graph. Recent advances in graph theory (Proc. Second Czechoslovak Sympos., Prague, 1974), pp. 477-480. Academia, Prague, 1975

[2] A.G. Thomason. Hamiltonian cycles and uniquely edge colourable graphs. Annals of Discrete Mathematics, 3:259–268, 1978.

[3] P. Haxell, B. Seamone, and J. Verstraete. Independent dominating sets and hamiltonian cycles. Journal of Graph Theory, 54:233–244, 2007.

Jones' conjecture

by Pál Bärnkopf

For a graph G, let $cp(G)$ denote the cardinality of a maximum cycle packing (collection of vertex disjoint cycles) and let $cc(G)$ denote the cardinality of a minimum feedback vertex set (set of vertices X so that $G - X$ is acyclic).

Problem 1. *[1] For every planar graph G, $cc(G) \leq 2cp(G)$.*

The conjecture holds for subcubic planar graphs. [2]

References

[1] Ton Kloks, Chuan-Min Lee, and Jiping Liu. New algorithms for k-face cover, k-feedback vertex set, and k-disjoint cycles on plane and planar graphs. In International Workshop on Graph-Theoretic Concepts in Computer Science, pages 282–295. Springer, 2002.

[2] Bonamy, M., Dross, F., Masařík, T., Nadara, W., Pilipczuk, M., & Pilipczuk, M. Jones' Conjecture in subcubic graphs. arXiv preprint arXiv:1912.01570, 2019.

Saturation numbers for a disjoint union of graphs

by Zoltán L. Blázsik

We will consider only finite, undirected, simple graphs. Let H be a fixed graph. A graph G is called H-free if it has no subgraph isomorphic to H. It is called H-saturated if it is H-free but for any edge e from the complement of G, the graph $G+e$ does contain a copy of H. The saturation number of H denoted by $\mathrm{sat}(H,n)$ is the minimum number of edges in an H-saturated graph on n vertices.

In 1964, Erdős, Hajnal and Moon [1] introduced the concept of saturation numbers and determined the saturation number of a complete graph. Since then, saturation numbers have been extensively studied, e.g. Kászonyi and Tuza [2] gave a general upper bound (linear function of n) on $\mathrm{sat}(H,n)$. However, the function $\mathrm{sat}(H,n)$ is not monotone with respect to H nor n. It still remains unknown, in general whether $\lim_{n\to\infty}\frac{\mathrm{sat}(H,n)}{n}$ exists. In 1988, Tuza [3] conjectured that this limit does exist, in which case it is denoted by $\mathrm{satlim}(H)$.

A general lower bound was given by Cameron and Puleo [4] on $\mathrm{sat}(H,n)$. They defined a weight of an edge as $\mathrm{wt}(uv) = |N(u)\cap N(v)| + \max d(u), d(v)$ and the weight of a graph H as $\mathrm{wt}(H) = \min_{uv\in E(H)}\mathrm{wt}(uv)$, and then proved that $\mathrm{sat}(H,n) \geq \frac{\mathrm{wt}(H)-1}{2}n + O(1)$. This immediately implies $\mathrm{satlim}(H) \geq \frac{\mathrm{wt}(H)-1}{2}$, if it exists. They also introduced a new notion, a graph H is sat-sharp if $\mathrm{satlim}(H) = \frac{\mathrm{wt}(H)-1}{2}$. Note that H is sat-sharp if and only if for every $n \geq |V(H)|$, there exists an H-saturated graph G of order n such that $|E(G)| \leq \frac{\mathrm{wt}(H)-1}{2}n + o(n)$.

Cameron and Puleo proved that the class of sat-sharp graphs is closed under the addition of isolated and dominating vertices, and showed that the disjoint union of cliques is sat-sharp as well. They conjectured the following in general:

Problem 1. *If H_1 and H_2 are sat-sharp graphs, then their disjoint union $H_1 + H_2$ is sat-sharp.*

Or in other words, the class of sat-sharp graphs is closed under taking disjoint unions. Zhang [5] further suggested that the following strengthening might be true as well.

Problem 2. *If H_1 is sat-sharp and H_2 is arbitrary such that $\mathrm{wt}(H_1) \leq \mathrm{wt}(H_2)$ then the disjoint union $H_1 + H_2$ is sat-sharp.*

References

[1] P. Erdős, A. Hajnal, J.W. Moon. A problem in graph theory. Amer. Math. Monthly, 71:1107–1110, 1964.

[2] L. Kászonyi, Zs.Tuza. Saturated graphs with minimal number of edges. Journal of graph theory, 10(2):203–210, 1986.

[3] Zs. Tuza. Extremal problems on saturated graphs and hypergraphs. Ars Combin.,25(B):105–113, 1988.

[4] A. Cameron, G.J. Puleo. A lower bound on the saturation number, and graphs for which it is sharp., Discrete Math., 345(7):112867, 2022.

[5] C. Zhang. Problem proposal at GRWC 2021 `https://drive.google.com/file/d/1KVB2ZF2swKcSSDgfnKIQ0GCVuU7YwdV3/view`

Extreme cases of matrix sampling

by Panna Fekete

Given an $n \times n$ matrix $\mathcal{A}$, its cut norm is defined as:

$$\|\mathcal{A}\|_\square = \frac{1}{n^2} \max_{S,T \subseteq [n]} \left| \sum_{i \in S, j \in T} \mathcal{A}_{ij} \right|.$$

Let $k \geq 2$ be an integer, $X := (X_1, X_2, \ldots, X_k) \in \{1, 2, \ldots, n\}^k$ be a random k-vector consisting of i.i.d. random variables with uniform distribution and let $\mathcal{A}_X := \mathcal{A}[X]$ denote the $k \times k$ matrix with values given by $(\mathcal{A}(X_i, X_j))_{i \neq j \in [k]}$, and 0 on the main diagonal.

The following holds:

Theorem 1 (see [1]). *Let $\mathcal{A}$ be a $[0,1]$-valued $n \times n$ matrix. Then with probability at least $1 - 2e^{-\sqrt{k}/8}$*

$$|\|\mathcal{A}_X\|_\square - \|A\|_\square| \leq O\left(k^{-1/4}\right).$$

Problem 1. *What are the "bad" matrices with large deviation, and how sharp is the exponent?*

All definitions can be extended to kernels and unbounded kernels $U \in L^1([0,1]^2)$. (See [2]). Also the above theorem extends to both cases, to kernels it was proven by Borgs, Chayes, Lovász, Sós, Vesztergombi ([3]).

Problem 2. *Given $m \geq 3$, for what $U \in L^m_{sym}([0,1]^2)$ and $\phi > (m-3)/4m$ can the following inequality fail with constant (in k) probability?*

$$\|\mathcal{U}_X\|_\square - \|U\|_\square \leq O\left(\sqrt{\ln k}\, k^{-\frac{1}{4} + \frac{1}{4m} - \phi}\right)$$

Problem 3. *Given $m \geq 3$, for what $U \in L^m_{sym}([0,1]^2)$ and $\gamma < 1/m$ can the following inequality fail with constant (in k) probability?*

$$\|\mathcal{U}_X\|_\square - \|U\|_\square \geq -O\left(k^{-1/2+\gamma}\sqrt{\ln k}\right)$$

These questions were raised during the work supported by ERC Synergy Grant No. 810115.

References

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Some non-degenerate generalized Turán problems

by Dániel Gerbner

Given graphs H and F and a positive integer n, the generalized Turán number $ex(n, H, F)$ is the largest number of copies of H in n-vertex F-free graphs. Recently several papers have dealt with the case $F = K_r$ and $\chi(H) < r$. We say that H is r-Turán-good if (for n large enough) the $(r-1)$-partite Turán graph is extremal (i.e., has the most copies of H among K_r-free graphs), and H is weakly r-Turán-good if a complete $(r-1)$-partite graph is extremal.

Morrison, Nir, Norin, Rzażewski, and Wesolek [1] showed that every H is r-Turán-good if $r \geq r_0(H)$. Their threshold for r_0 is $300|V(H)|^9$ and they mention that they did not optimize this bound. They suggest that it may be much smaller, but proving even a quadratic bound would need new ideas.

Clearly we have that if $r \geq r_1(H)$, then H is weakly r-Turán-good for some $r_1(H) \leq r_0(H)$. We know some graphs H with $r_1(H) > \chi(H) + 1$ from [2], but this is the only known lower bound on these thresholds.

Problem 1. *Improve the bounds on $r_0(H)$ and $r_1(H)$.*

One can view the result of Morrison, Nir, Norin, Rzażewski, and Wesolek as stating that we have the same extremal graph for $ex(n, K_r)$ and $ex(n, H, K_r)$ if r is large enough.

Problem 2. *Is it true that for any graph H, if $\chi(F)$ is large enough, then there is a graph that is extremal for both $ex(n, F)$ and $ex(n, H, F)$? Is it true for some graph $H \neq K_2$?*

References

[1] Morrison, Nir, Norin, Rzażewski, and Wesolek, Every graph is eventually Turán-good, Journal of Combinatorial Theory, Series B Volume 162, September 2023, Pages 231–243.

[2] Grzesik, Győri, Salia and Tompkins, Subgraph Densities in K_r-Free Graphs, Electronic J. Comb., 30(1), 2023, P1.51.

The number of cycles in tournaments

by Andrzej Grzesik

For an integer $k \geq 3$ what is the largest number of directed k-cycles in a tournament on n vertices? The solution is known for $k = 3$ from 1940s and for $k = 4$ from 1960s, but answers for some other lengths appeared only recently. It was shown in [1] that asymptotically the maximum for odd k is achieved in any regular tournament, while for even k not divisible by 4 in a random tournament. It is conjectured that for k divisible by 4, the maximum is obtained for odd n in a tournament on $\{0, 1, \ldots, n-1\}$, in which every vertex i is connected to all vertices from $\{i+1, i+2, \ldots, i+\frac{n-1}{2}\}$ considered modulo n.

Problem 1. *For any k divisible by 4 determine the maximum number of k-cycles in a tournament on n vertices.*

It is known for $k = 4$ and asymptotically for $k = 8$. The algebraic proof for $k = 8$ and k not divisible by 4 presented in [1] does not seem to be a good approach for k divisible by 4.

One may also consider a problem of maximization of other subgraphs. For instance, in [2] the asymptotic maximum number of directed paths of a given length was determined. They also asked for other orientations of a path.

Problem 2. *Determine the maximum number of copies of a given oriented graph H in a tournament on n vertices.*

References

[1] A. Grzesik, D. Král', L. M. Lovász, J. Volec. Cycles of a given length in tournaments, Journal of Combinatorial Theory, Series B, 158(1):117–145, 2023.

[2] A. Sah, M. Sawhney, Y. Zhao. Paths of given length in tournaments, arXiv: 2012.00262, 2020.

Rainbow Turán Problems

by Andrzej Grzesik

Consider a following rainbow version of the Turán problem. For a graph F and an integer k we consider k graphs $G_1, G_2, \ldots, G_k$ on the same set of n vertices and ask for the maximum possible number of edges in each graph avoiding appearance of a copy of F having at most one edge from each graph. In other words, for every $i \in \{1, 2, \ldots, k\}$ we color edges of G_i in color i and forbid all copies of F having non-repeated colors, so called rainbow copies.

For an example, if F is a triangle, then it follows from [5] that for $k \geq 4$ colors the best possible number of edges in each color without having a rainbow triangle is equal to $\frac{1}{4}n^2$. This is achieved in the balanced complete bipartite graph (the same in each color), as in Mantel's theorem. Surprisingly, for 3 colors the answer is different. In [1] it is proved that the optimal asymptotic bound is $\left(\frac{26-2\sqrt{7}}{81}\right) n^2 \approx 0.2557n^2$.

The question remains open for larger complete graphs. For instance, for K_4 from [5] we have an optimal bound for at least 8 colors, but for 6 and 7 colors there is even no conjectured extremal construction.

Problem 1 (Problem 1.4 in [1], Question 1.16 in [3])**.** *For every positive integer r, determine the smallest real number δ_r so that if $G_1, \ldots, G_{\binom{r}{2}}$ are graphs on a common set of n vertices with $|E(G_i)| \geq \delta_r n^2$ for every $1 \leq i \leq \binom{r}{2}$ then there exists a rainbow K_r, i.e., a set of r vertices and one edge from each G_i that together form a clique on this set of vertices.*

Similar questions can be asked for other graphs. In particular, for a path on 4 vertices the problem was solved in [2] with two types of possible extremal constructions. For larger paths in [4] it is conjectured that for k colors and a forbidden rainbow path on $k+1$ vertices the best construction is to take two disjoint cliques of size $n/2$, each consisting of colored edges from two different sets of colors.

Problem 2. *Solve the rainbow Turán problem for some particular forbidden rainbow graphs, for example paths on at least 5 vertices.*

References

[1] R. Aharoni, M. DeVos, S. González, A. Montejano, R. Šámal. A rainbow version of Mantel's Theorem, Advances in Combinatorics 2, 2020.

[2] S. Babiński, A. Grzesik. Graphs without a rainbow path of length 3, arXiv: 2211.02308, 2022.

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Edge-ordered graphs with almost linear extremal functions

by Oliver Janzer

Gerbner, Methuku, Nagy, Pálvölgyi, Tardos and Vizer [1] introduced the following problem. Given an edge-ordered graph H, at most how many edges can an n-vertex edge-ordered graph have without containing H as a subgraph (with the ordering of the edges respected)? Let us denote the answer by $ex_<(n,H)$.

For the Turán number of unordered graphs, there is an important dichotomy which says that if H is a forest, then $ex(n,H)=O(n)$ and otherwise $ex(n,H)=\Omega(n^c)$ for some $c>1$ (which depends on H). However, there are edge ordered forests H with $ex(n,H)=\omega(n)$.

Let us call a vertex in an edge-ordered graph H *close* if the edges adjacent to it are consecutive in the edge ordering of H. Now we say that an edge-ordered forest H has order chromatic number 2 if it has a proper 2-colouring in which all vertices in one colour class are close. Every edge ordered graph H which is not a forest with order chromatic number 2 has $ex_<(n,H)=\Omega(n^c)$ for some $c>1$. Gerbner et al. [1] conjectured that the converse is also true, that is, for every edge ordered forest H with order chromatic number 2 we have $ex_<(n,H)=n^{1+o(1)}$. This was recently proved by Kucheriya and Tardos.

Theorem 1 (Kucheriya and Tardos)**.** *For every edge-ordered forest H with order chromatic number 2, $ex_<(n,H)=n2^{O(\sqrt{\log n})}$.*

They conjectured that the following stronger bound should be true.

Problem 1 (Kucheriya and Tardos)**.** *For every edge-ordered forest H with order chromatic number 2, $ex_<(n,H)=n(\log n)^{O(1)}$.*

In fact, the largest known extremal number for an edge-ordered forest with order chromatic number 2 is $\Theta(n\log n)$, so they asked the following question.

Problem 2 (Kucheriya and Tardos)**.** *Is there an edge-ordered tree H with order chromatic number 2 such that $ex_<(n,H)=\omega(n\log n)$?*

References

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Turán-type problems for somewhat shorter 3-uniform tight cycles (Nina)

Nina Kamčev

Warning. The following conjectures might be just as difficult as finding the corresponding Turán densities, but I still think it's worth thinking about whether the approaches [2, 1] can be improved.

We will be considering tight 3-uniform cycles on m vertices, which we denote C_m. The hypergraph obtained from C_m by removing one edge is denoted by C_m^-. Recently, for sufficiently large constant m, the Turán densities for C_m and C_m^- have been determined in [2] and [1]. The following problems are the reason that both papers need large m.

Conjecture 1. *For any $c > 0$ and sufficiently large n, the following holds.*
(i) If H is a maximal C_5-free 3-graph on n vertices, then we can remove cn^3 edges from H to obtain a hypergraph with no tight cycles C_m with $3 \mid m$.
(ii) If H is a maximal C_5^--free 3-graph on n vertices, then we can remove cn^3 edges from H to obtain a hypergraph with no copies of C_m^- with $3 \nmid m$.

We expect the Conjecture to hold since in both cases, the conjectured extremal examples actually do not have any cycles C_m (or C_m^- respectively) whose length is not divisible by 3. (Note also that the assumption $3 \nmid m$ is necessary everywhere since 3-partite cycles have Turán density 0.)

A good 'toy case' to get some intuition on this problem are corresponding statements for odd cycles in (2-uniform) graphs.

References

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Thrackles of convex hulls

by Balázs Keszegh

Recently two different generalizations of linear thrackles to convex hulls were defined. Let $Conv(S)$ denote the convex hull of S.

Definition 1. *[2] Suppose that we are given a set P of n points and a family $\mathcal{S}$ of subsets of P whose convex hulls are all different and form the family $C(\mathcal{S}) = \{Conv(S) : S \in \mathcal{S}\}$. We say that $C(\mathcal{S})$ is a* thrackle of convex sets *if there exists a finite set $W \supseteq P$ such that the following holds:*

1. *$|C_i \cap C_j \cap W| = 1$ for any two different $C_1, C_2 \in C(\mathcal{S})$.*

In [2] they conjecture that such thrackles of convex sets have at most n convex hulls and prove various partial results.

Definition 2. *[1] Suppose that we are given a set P of n points in general position in the plane, and a family $\mathcal{S}$ of subsets of P whose convex hulls are all different and form the family $C(\mathcal{S}) = \{Conv(S) : S \in \mathcal{S}\}$. We say that $C(\mathcal{S})$ is a* convex hull thrackle *on P if the following hold:*

1. *$C_1 \not\subset C_2$ for any two different $C_1, C_2 \in C(\mathcal{S})$;*
2. *$C_1 \cap C_2 \neq \emptyset$ for any two different $C_1, C_2 \in C(\mathcal{S})$;*
3. *$C_1 \cap C_2 \cap C_3 \subset P$ for any three different $C_1, C_2, C_3 \in C(\mathcal{S})$.*

In [3] it was proved among others that such convex hull thrackles may have $n + 1$ convex hulls but always have at most $2n$ convex hulls.

Problem 1. *Give improved bounds on the size of these two types of thrackles.*

References

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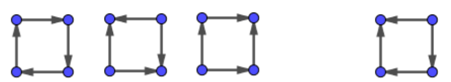
Orientations without even-type 4-cycles

by Benedek Kovács

Consider the following graph $G = (V, E)$, which is a well-known example for a graph on a fixed number of vertices avoiding a subgraph $K_{2,3}$ and having asymptotically the highest possible number of edges (see e.g. [1, Construction 3.15]):

- Let $\mathbb{F}_p^2$ denote the two-dimensional vector space over the p-element field $\mathbb{F}_p$. We say that two elements $(a, b) \in \mathbb{F}_p^2$ and $(a', b') \in \mathbb{F}_p^2$ are equivalent ($(a, b) \sim (a', b')$) if there exists a value $\lambda \in \{\pm 1\}$ such that $a' = \lambda a$ and $b' = \lambda b$ in $\mathbb{F}_p$. The vertices of the graph G are the two-element equivalence classes of $\mathbb{F}_p^2 \setminus \{0, 0\}$ under $\sim$ (so $|V| = \frac{p^2-1}{2}$).
- We join two vertices represented by (a, b) and (c, d) by an edge if $ac + bd = \pm 1$. Note that this definition is compatible with the equivalence classes.

In directed graphs, a 4-cycle (of the underlying simple graph) is called *even-type* or *odd-type*, depending on the parity of the number of edge-flips required to reach that 4-cycle from a cyclically oriented 4-cycle. The figure below has examples of even-type and odd-type 4-cycles (note that this list is exhaustive up to orientation-preserving isomorphisms):



We are interested in providing an orientation of G that avoids even-type 4-cycles, at least in an asymptotic sense. Note that in G, almost every pair of vertices have 2 common neighbours, so the total number of 4-cycles in G is $\frac{1}{4}n^2 + o(n^2)$ where $n = |V| = \frac{p^2-1}{2}$. If it is not possible to completely avoid even-type 4-cycles, it would still be interesting to see what their minimum number would be asymptotically.

Problem 1. *What is the biggest possible value $\lambda \in [0, 1]$ such that for each prime p there exists an orientation $\vec{G}$ of G such that the total number of odd-type 4-cycles is at least $\lambda \cdot \frac{1}{4}n^2 - o(n^2)$?*

Random orientations give $\lambda = \frac{1}{2}$, as 8 of the 16 possible orientations of a 4-cycle are odd-type. By representing each point with the lexicographically smaller of the two possible coordinate pairs in $[0, p-1]^2$, one can obtain orientations with $\lambda = \frac{2}{3}$, verified by a program for small cases p. Such an orientation would be to have an edge $(x, y) \to (x', y')$ iff $xx' + yy' \equiv +1 \mod p$.

Motivation: The problem is motivated by the following variant of the Zarankiewicz problem: in an $n \times n$ matrix A, we would like to put the highest possible number of 1's and 2's (and put 0's in the rest of the cells) such that $A_{i,j} = 1 \Leftrightarrow A_{j,i} = 2$, and for some d, each row has d 1's and d 2's, and for every pair of columns, there exists either exactly one row where they both have 1's, or exactly one row where they both have 2's (but never both).

Since Turán-type extremal constructions, like the graph G above (avoiding $K_{2,3}$), tend to be stable, it is likely that our problem about orientations is the same as an asymptotic version of the motivating question.

References

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1-close Sperner sequences

by Dániel T. Nagy

For a subset L of $[n]$, we say that a set system $\mathcal{F}$ is *L-close Sperner* if every pair $F, G \in \mathcal{F}$, $F \neq G$ satisfies

$$\min\{|F \setminus G|, |G \setminus F|\} \in L.$$

Theorem 1 (Nagy, Patkós [1]). *If the set system $\{F_1, F_2 \dots, F_m\} \subseteq 2^{[n]}$ is L-close Sperner for some $L \subseteq [n]$, then*

$$m \leq \sum_{h=0}^{|L|} \binom{n}{h}.$$

The subsets of $[n]$ can be represented by their characteristic vectors, $\{0,1\}$-sequences of length n. Instead of those one can consider sequences with entries from $\{0, 1, \dots q-1\}$. We say that a set of sequences $A \subseteq \{0, 1, \dots, q-1\}^n$ is $\{1\}$*-close Sperner* if for any distinct $a, b \in A$ we have

$$\min\{|\{i : a_i < b_i\}|, |\{i : a_i > b_i\}|\} = 1.$$

Problem 1. *Find the largest possible set of sequences $A \subseteq \{0, 1, \dots, q-1\}^n$ with the $\{1\}$-close Sperner property.*

Result 2 ([1]). *For any $\{1\}$-close Sperner $A \subseteq \{0, 1, \dots, q-1\}^n$, $|A| \leq O(n^{q-1})$.*

Proof. From every $\{0, 1, \dots, q-1\}$-sequence of length n in A, we will create a $\{0,1\}$-sequence of length $(q-1)n$. Replace every number with a $\{0,1\}$-sequence of length $q-1$ like this: the number x is transformed to a sequence of x 1s followed by $(q-1-x)$ 0s. We can think of the resulting sequences as the characteristic vectors of some sets $\{F_1, F_2 \dots, F_m\} \subseteq 2^{[(q-1)n]}$. The resulting set system will have the $\{1, 2, \dots, q-1\}$-close Sperner property. Theorem 1 implies

$$m \leq \sum_{h=0}^{q-1} \binom{(q-1)n}{h} = O(n^{q-1})$$

□

In [1] we gave a construction of size $(q-1)(n-1)$ and conjectured that $|A| \leq O(n)$ holds for any fixed q. I propose that we try to close the gap between $\Omega(n)$ and $O(n^{q-1})$ in Problem 1. After this we may consider $\{0, 1, \dots, q-1\}$-sequences with the L-close Sperner property for a set L different from $\{1\}$.

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Embedding r-degenerate bipartite graphs in regular graphs - a step towards Erdős' conjecture

by Zoltán Lóránt Nagy

A graph F is **r-degenerate** if it contains no subgraph of minimum degree at least $r+1$. Erdős conjectured that if F is an r-degenerate bipartite graph, then the Turán number of F has order of magnitude $ex(n;F) = O(n^{2-1/r})$. This holds for those graphs F where every vertex in one of the partition class of F has degree at most r, showed by Füredi and pointed out later in the famous short proof of Fox and Sudakov in [2] using the tool called **dependent random choice**. It has been refined in [1] by Alon, Krivelevich and Sudakov to prove a general bound for the family of r-degenerate bipartite graphs as well.

Theorem 1. *$ex(n;F) = O(n^{2-\frac{1}{4r}})$ for r-degenerate bipartite graphs.*

The aim is to improve the exponent.

The approach is, generally speaking, to obtain an embedding result for such graphs to graphs where almost (but not all) r-sets have large common neighbourhood. In fact, the proof of Theorem 1 uses a refined version of the dependent random choice (you can consider it as a black box) and then gain a bipartite subgraph $G' = G(A', B')$ where **all** r-tuples of A' and of B' have many common neighbours. Then they use a simple greedy embedding $F \to G'$.

Problem 1 (Embedding F into large almost highly regular graphs)**.** *Choose your favourite r and a r-degenerate bipartite graph F. Determine or bound the function $h(r,v,n)$ for which F can be embedded to all $G' = G(A', B')$ graphs where the r-tuples of A' and of B' have at least v common neighbours for $v := v(F)$, except for at most $h(r,v,n)$ many of them, where n is the number of vertices of G'.*

A randomized embedding algorithm seems to be a suitable approach. For some specific graphs F, some ideas and results are proved in [3, 4]. The main question would be the decide whether $h(r,v,n)$ can be as large as $\delta\binom{n}{r}$ for some constant δ, which can have a dependence on v; since one might apply the dep. random choice to obtain a gadget G' with the properties above from sparser graph G.

A related problem is the following.

Problem 2 (Labelling vertices to cover all F subgraphs)**.** *Take a positive integer r, an r-degenerate bipartite graph F and a large host graph G on n vertices where all r-sets of the partition classes of G have large common neighbourhood. How many points should one erase from G to guarantee that F is not a subgraph of the remaining graph?*

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Turán numbers for hypergraph paths and trees

by Cory Palmer

The goal of these problems is to generalize the Erdős-Gallai theorem and Erdős-Sós conjecture to r-graphs. For $a+b=r$, the (a,b)-*blowup of the* ℓ-*edge path* P_ℓ is the r-graph $P_\ell(a,b)$ obtained from the graph path P_ℓ by alternating blowing up its vertices to a-sets and b-sets. More precisely, take $\ell+1$ pairwise disjoint sets $A_0, A_1, \ldots, A_\ell$ with $|A_i| = a$ for i even and $|A_i| = b$ for i odd and define the r-edges of $P_\ell(a,b)$ as the sets of the form $A_{i-1} \cup A_i$.

The following is from a recent paper of Füredi et al. [2]:

Theorem 1. *If (1) ℓ is odd, or (2) ℓ is even and $a > b$, or (3) $(\ell, a, b) = (4, 1, 2)$, then*

$$\mathrm{ex}_r(n, P_\ell(a,b)) = \left\lfloor \frac{\ell-1}{2} \right\rfloor \binom{n}{r-1} + o(n^{r-1}).$$

If $a \neq b$, $a, b \geq 2$ and $\ell = 2k-1$ odd, then the extremal family $\Psi_{k-1}(n,r)$ of all r-edges that meet a fixed set of $k-1$ (universal) vertices is unique.

This is essentially 3/4 of the cases. The remaining case is:

Conjecture 1. *If $r \geq 3$, $\ell \geq 2$ and $a \leq b$, then $\mathrm{ex}_r(n, P_{2\ell}(a,b))$ is attained by $\Psi_{\ell-1}(n,r)$.*

Let T be a tree with classes of size s and t. An (a,b)-blowup of T is defined analogously by blowing up vertices in the class of size s to a-sets and vertices in the class of size t to b-sets.

Theorem 2. *Fix $r \geq 3$, $s, t \geq 2$, $a+b=r$ and $b < a < r$. Let T with (a,b)-blowup $T(a,b)$. Then*

$$\mathrm{ex}_r(n, T(a,b)) \leq (t-1)\binom{n}{r-1} + o(n^{r-1})$$

and this bound is asymptotically sharp when $t \leq s$.

Conjecture 2. *When $t > s$, then*

$$\mathrm{ex}_r(n, T(a,b)) \leq (s-1)\binom{n}{r-1} + o(n^{r-1}).$$

It is also natural to try and extend these theorems for trees to forests.

Problem 3. *Determine*

$$\lim_{n\to\infty} \frac{\mathrm{ex}_r(n, F(a,b))}{\binom{n}{r-1}}.$$

The key method in [2] is the Δ-system method; particularly Füredi's intersection semilattice lemma [1]. A good overview can be found in slides by Kostochka. More problems for particular blowups of trees can be found in a very new paper of Füredi and Kostochka [3].

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Number Balancing

by Dömötör Pálvölgyi

The following conjecture came up in a recent joint work with Gábor Damásdi, Nóra Frankl and János Pach (see `https://doi.org/10.5817/CZ.MUNI.EUROCOMB23-044`).

Conjecture 1 (Damásdi–Frankl–Pach–Pálvölgyi)**.** *If a sequence of reals* $-1 < x_1, \dots, x_k < 1$ *satisfies*

$$x_{i+1} = \begin{cases} 2x_i, & \text{if } 2|x_i| < 1 \\ 2x_i - 2, & \text{if } 2x_i > 1 \\ 2 + 2x_i, & \text{if } 2x_i < -1 \end{cases}$$

for $i = 1, \dots, k$*, where* $x_{k+1} = x_1$*, then there is a permutation* π *of* $\{1, \dots, k\}$ *such that* $0 \leq \sum_{i=1}^{j} x_{\pi(i)} < 1$ *for every* j*.*

This conjecture is similar to Steinitz's theorem, and to other vector balancing problems. Indeed, it can be proved for any x_i's satisfying the conditions of the conjecture, that $\sum_{i=1}^{k} x_i = 0$. We note that if the x_i's are any sequence satisfying $\sum_{i=1}^{k} x_i = 0$ and $|x_i| < 1/2$ for every i, then one can easily find a permutation for which $0 \leq \sum_{i=1}^{j} x_{\pi(i)} < 1$ for every j. But without this bound, we have to exploit that $x_{i+1} = 2x_i$, as otherwise there would be counterexamples, e.g., $0.6, 0.6, 0.6, -0.9, -0.9$. Could it be that the conjecture is true because we always have many i's such that $|x_i| < 1/2$, and these can be used somehow to take care of the other x_i's?

Intersection of Maximum Cliques

by Dömötör Pálvölgyi

The following conjecture came up in a recent joint work with Attila Jung, Balázs Keszegh and Yelena Yuditsky.

Conjecture 1 (Jung–Keszegh–Pálvölgyi–Yuditsky)**.** *If the size of the largest clique in a k-uniform hypergraph on n vertices is $\omega > (k-1)n/k$, then there is a vertex that is contained in all ω-sized cliques.*

The $k = 2$ case of this conjecture is a classic result of Hajnal [1]; the solution is just two lines, but not so simple to find. The $k = 3$ case is already wide open.

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Cycles on Every Ground Set

Nika Salia

Problem 1 (Salia [6])**.** *Let G be a bipartite graph, with partite sets A and B. Let for every subset A' of A, $|A| \geq 2$, the number of vertices incident with at least two vertices of A' is at least $|A'|$ in G. Then for every subset A' of A, $|A'| \geq 2$, there is a cycle $C_{A'}$ in G such that $V(C_{A'}) \cap A = A'$.*

This problem was inspired by the following problem.

Problem 2 (Kostochka, Lavrov, Luo, Zirlin [4])**.** *Let G be a $2-$connected bipartite graph, with partite sets A and B. Let for every subset A' of A, $|A| \geq 3$, the number of vertices incident with at least two vertices of A' is at least $|A'|$ in G. Then for every subset A' of A, $|A'| \geq 3$, there is a cycle $C_{A'}$ in G such that $V(C_{A'}) \cap A = A'$.*

Note that Problem 2 implies the former problem. While the latter problem was inspired by the following problems and longstanding conjecture.

For positive integers n, m, and δ with $\delta \leq m$, let $G(n, m, \delta)$ denote the set of all bipartite graphs with a partition (X, Y) such that $|X| = n \geq 2, |Y| = m$ and for every $x \in X$, $d(x) \geq \delta$. In 1981, Jackson [1] proved that if $\delta \geq \max\{n, \frac{m+2}{2}\}$, then every graph $G \in G(n, m, \delta)$ contains a cycle of length $2n$, i.e., a cycle that covers X. This result is sharp. Jackson also conjectured that if $G \in G(n, m, \delta)$ is 2-connected, then the upper bound on m can be weakened.

Theorem 1 (Conjecture of Jackson [1, 2] solved in [7])**.** *Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+5}{3}\}$, then every 2-connected graph $G \in G(n, m, \delta)$ contains a cycle of length $2n$.*

The restriction $\delta \geq \frac{m+5}{3}$ cannot be weakened.

Theorem 2 (Kostochka, Lavrov, Luo, Zirlin [5])**.** *Let m, n, δ be integers. If $\delta \geq \max\{n, \frac{m+10}{4}\}$, then every 3-connected graph $G \in G(n, m, \delta)$ contains a cycle of length $2n$.*

Theorem 2 is a natural 3-connected strengthening of Conjecture 1 for 2-connected graphs. Consider the following family of k-connected graphs.

Definition 3. *Let k be a positive integer, and let $n_1 \geq \ldots \geq n_{k+1} \geq 1$ be such that $n_1 + \ldots + n_{k+1} = n$. Let $G_k(n_1, \ldots, n_{k+1}; \delta) \in G(n, (k+1)(\delta - k) + k, \delta)$ be the bipartite graph obtained from $K_{\delta-k,n_1} \cup \ldots \cup K_{\delta-k,n_{k+1}}$ by adding k vertices $a_1, \ldots, a_k$ that are each adjacent to every vertex in the parts of size $n_1, \ldots, n_{k+1}$. Let $\mathcal{G}_k(n, \delta)$ be the collection of the graphs $G_k(n_1, \ldots, n_{k+1}; \delta)$ for all suitable choices of $n_1, \ldots, n_{k+1}$.*

Problem 3. *Let m, n, k, δ be integers. Suppose $k \geq 4$, $\delta \geq n$ and $m \leq (k+1)(\delta - k) + k - 1$. Is it true that every k-connected graph $G \in G(n, m, \delta)$ contains a cycle of length $2n$? Moreover, if $k \geq 3$, are the graphs in the family $\mathcal{G}_k(n, \delta)$ the only extremal examples with $m = (k+1)(\delta - k) + k$?*

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Clumsy packing of 1 by m dominos

Casey Tompkins

The classical packing problem asks for the maximum number of some type of geometric object that can be placed in a given space. Sands [3] and independently Gyárfás, Lehel, and Tuza [2] considered a dual version of this problem in the setting of domino tilings. In particular they determined the smallest fraction of an n by n chessboard with 1 by 2 dominos (in such a way that each domino coincides with two squares) such that no further domino can be added. The answer turns out to be $2/3$ and it is simple to find such a packing.

This can be thought of as a "saturation" version of the classical packing problem. Such packings have also been termed "clumsy packings". Goddard [1] extended the above result to packings of other "ominos". In particular for 1 by m dominos he showed that at least $2/(m+1)$ fraction of the board must be covered and conjectured that the correct answer is $2m/(m^2+1)$. I propose we try to solve the case when $m = 3$.

Problem 1 (Goddard)**.** *Determine the smallest clumsey packing of an n by n chessboard by 1 by 3 dominos. Goddard conjectures the answer is* $3/5$

W. Goddard / Discrete Mathematics 137 (1995) 361–365

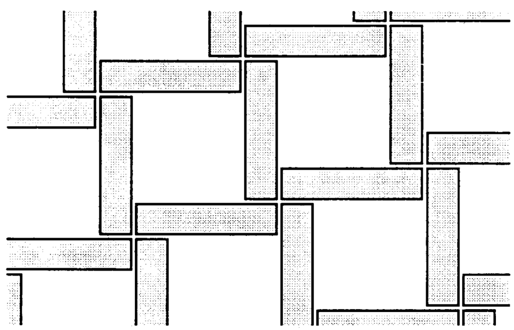

Fig. 3. The conjectured mistiling for long dominoes.

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Nonunimodal forests

by Russ Woodroofe

The *independence polynomial* of a graph G is the generating function for the number of independent sets of G, indexed by size. Thus, the coefficient α_i of x^i is the number of independent sets in G with i vertices. A polynomial is *unimodal* if its coefficients go up, then come down. That is, if there is some k (a *peak* of unimodality) so that $\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_k \geq \alpha_{k+1} \geq \cdots$. A polynomial is *nicely log-concave* if the coefficients are all nonnegative, the nonzero coefficients occur on an interval (so, no "internal zeros"), and if

$$\frac{\alpha_k}{\alpha_{k-1}} \geq \frac{\alpha_{k+1}}{\alpha_k} \tag{0.1}$$

whenever $a_{k-1}, a_k \neq 0$.

It is obvious that a nicely log-concave polynomial is unimodal. The nicely log-concave property is more robust than that of unimodality, however. For example, the product of two nicely log-concave polynomials has the same property, while the product of a nicely log-concave polynomial with a unimodal polynomial is unimodal. Unimodal and log-concave sequences are well-studied in combinatorics [2, 3, 7].

Alavi, Malde, Schwenk, and Erdős [1] asked whether the independence polynomial of every tree (or perhaps every forest) is unimodal. Since the independence polynomial of a graph with multiple connected components is exactly the product of the independence polynomials of its components, it is natural to ask whether perhaps every tree has nicely log-concave independence polynomial [5].

It is easy to computationally verify this condition for all trees with a small number (say, fewer than 20) of vertices. Indeed, the Nauty software [6] will efficiently produce a list of trees, and your favorite computer algebra system can quickly compute independence polynomials on small trees, particularly when we may remove a vertex of high degree. (Note, however, that the approach of the arXiv preprint [9] appears to be strictly worse than the naive approach.)

However, a recent paper of Kadrawi, Levit, Yosef, and Mizrachi [4] found by computer search two trees on 26 vertices whose independence polynomials fail the log-concavity inequality 0.1. One is shown in Figure 1 (diagram created with Gap.app [8]).

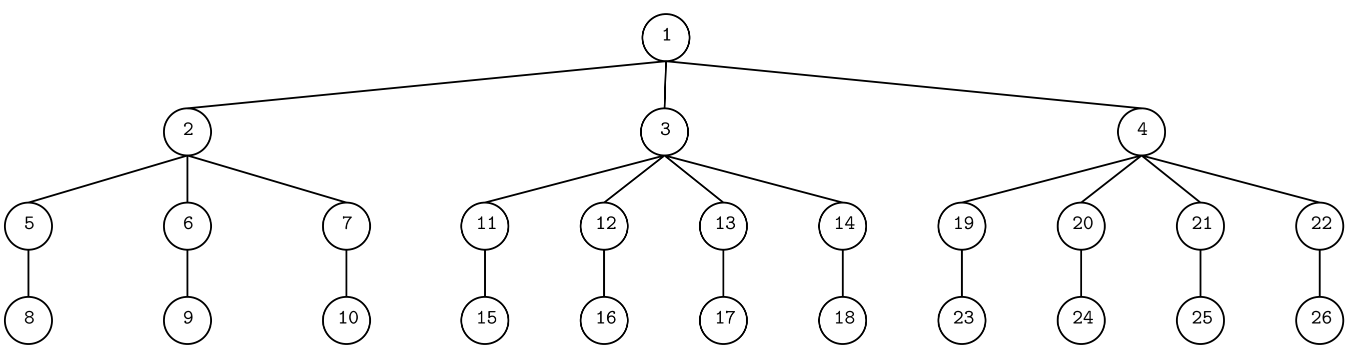

Figure 1: A tree whose independence polynomial fails log-concavity

Problem 1. *Construct a forest whose independence polynomial is not unimodal.*

A technique to do so might be to construct for use as connected components a large number of trees whose independence polynomials have the following properties:

1. they fail log-concavity, and
2. they are unimodal, but with peaks at widely differing indices.

The product of unimodal polynomials that are not log-concave is not guaranteed to be unimodal. The hope would be that widely differing peaks might tend to give a polynomial in the product that is not unimodal, hence the desired forest.

Problem 2. *Construct many trees whose independence polynomial fails log-concavity. Assuming all examples constructed are unimodal, where can peaks be found?*

Of course, it would also be of great interest to find a tree whose independence polynomial is not unimodal, but this is likely a considerably harder problem.

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