# Chain intersecting families

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**Abstract** Let $\mathcal{F}$ be a family of subsets of an $n$-element set. $\mathcal{F}$ is called *(p,q)-chain intersecting* if it does not contain chains $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ with $A_p \cap B_q = \emptyset$. The maximum size of these families is determined in this paper. Similarly to the $p = q = 1$ special case (intersecting families) this depends on the notion of *r-complementing-chain-pair-free* families, where $r = p + q - 1$. A family $\mathcal{F}$ is called *r-complementing-chain-pair-free* if there is no chain $\mathcal{L} \subseteq \mathcal{F}$ of length $r$ such that the complement of every set in $\mathcal{L}$ also belongs to $\mathcal{F}$. The maximum size of such families is also determined here and optimal constructions are characterized.

**Key words.** Extremal family, Disjoint chains, Chain-intersecting family, Complementing-chain-pair-free family

## 1. Introduction and preliminaries

In the present paper the following problem posed by Gyula O.H. Katona will be solved in a more general form.

**Problem:** Given a natural number $n$ and a family of sets $\mathcal{F}$ on an $n$ element universe in which there are no three sets $A, B$ and $C$ satisfying $A \subsetneq B$ and $B \cap C = \emptyset$. How many sets can such an $\mathcal{F}$ contain at most?

This problem was solved by the first author [1] and (independently, in a little more restricted form) by the second author [6]. We solve, among others, the following generalization of the above problem:

**Problem:** Given the natural numbers $n$, $p$ and $q$ and a family of sets $\mathcal{F}$ on the $n$ element universal set in which there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ such that $A_p \cap B_q = \emptyset$. How many sets can such an $\mathcal{F}$ contain at most?

The following notations will be used.

**Notations:** For a natural number $n$ let $[n] = \{1, 2, \ldots, n\}$. The power set of a set $V$ will be denoted by $2^V$. The complement of a subset $A$ of $V$ will be denoted by $\overline{A}$ (the universal set will always be clear from the context). The collection of all $k$ element subsets of the set $V$ will be denoted by $\binom{V}{k}$. For a set family $\mathcal{F} \subseteq 2^V$ let $\mathrm{co}(\mathcal{F}) = \{X \subseteq V : \overline{X} \in \mathcal{F}\}$.

A set family $\mathcal{F}$ will be called **intersecting** if $A \cap B \neq \emptyset$ for $A, B \in \mathcal{F}$. A family is called **complementing pair free** if it does not contain a pair of complementing sets.

In the whole article $n, p, q$ and $r$ will denote natural numbers with $1 \leq p, q, r \leq n$. Our universal set in this paper will always be the set $[n] = \{1, 2, \dots, n\}$. We need the following definitions.

**Definition 1.** *A family $\mathcal{F} \subseteq 2^{[n]}$ is called $(p,q)$-**chain-intersecting** if there are no sets $A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \dots \subsetneq B_q$ in $\mathcal{F}$ such that $A_p \cap B_q = \emptyset$ (the tops of two chains of sizes $p$ and $q$ in $\mathcal{F}$ always intersect).*

Observe, that the $p = q = 1$ case of this definition gives intersecting families: in this paper we will generalize the following theorem on the maximum cardinality of intersecting families (see [4]).

**Theorem 1.** *The largest cardinality of an intersecting family on the $n$ element universe is $2^{n-1}$.*

*Proof* An intersecting family certainly cannot contain complementing sets: so for each complementing pair we can include at most one of them. It is easy to find intersecting families achieving this bound, the family of all sets containing a common element is an example. □

This proof is included here because complementing pair free families are used in it. We give the largest cardinality of a $(p,q)$-chain-intersecting family over the $n$ element set for any values of $n, p$ and $q$. We do this using the following generalization of complementing pair free families.

**Definition 2.** *A $r$-**complementing-chain-pair** is a family of form $\mathcal{L} \cup \mathrm{co}(\mathcal{L})$ where $\mathcal{L} \subseteq 2^n$ is a chain of length $r$. A family $\mathcal{F} \subseteq 2^{[n]}$ is called $r$-**complementing-chain-pair-free** if it does not contain an $r$-complementing-chain-pair.*

Observe that a $(p,q)$-chain-intersecting family $\mathcal{F}$ is also $(p+q-1)$-complementing-chain-pair-free: if there was a $(p+q-1)$-complementing-chain-pair $\mathcal{L} \cup \mathrm{co}(\mathcal{L})$ in $\mathcal{F}$ then the smallest $p$ members and the complements of the largest $q$ members of $\mathcal{L}$ would give a forbidden configuration.

The structure of this paper is the following: in Section 2 we introduce some $r$-complementing-chain-pair-free families and state the main theorem about their optimality. As a consequence, the optimal cardinality of $(p,q)$-chain-intersecting families is observed. In Section 3 two simple proofs of a special case of the main result is given: one is using an adaptation of the method of permutations, while the other uses a rephrasing of the problem. Section 4 contains a full proof of the main theorem stated in Section 2. This proof uses the method of cyclic permutations. The last section contains comments and open problems on related topics.

In the end of this section we give some more definitions and cite results that will be needed in the paper. The following theorem of Erdős, Ko and Rado ([4]) will be used:

**Theorem 2.** *If $k$ and $n$ are natural numbers with $k \leq n/2$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality for*

$$\mathcal{F} \sim \{X \subseteq [n] : \ |X| = k \text{ and } x \in X\}$$

*for some $x \in [n]$.* □

(See the definition of relation $\sim$ in Definition 3: it is used here to express the particularities of the $k = n/2$ case!)

A family $\mathcal{F} \subseteq 2^{[n]}$ will be called $k$-**Sperner** if it contains no chain $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k$ of $k+1$ different sets. For $k = 1$ this is the usual notion of Sperner families, for $k = 0$ the only 0-Sperner family is $\mathcal{F} = \emptyset$. We use the following theorem of [5] (which is a slight extension of a theorem of Erdős [3]):

**Theorem 3.** *Let $\mathcal{F}$ be a $k$-Sperner family of subsets of an $n$-element set. Then*

$$|\mathcal{F}| \leq \sum_{i=\lfloor (n-k+1)/2 \rfloor}^{\lfloor (n+k-1)/2 \rfloor} \binom{n}{i}$$

*holds with equality if and only if $\mathcal{F}$ is the family of all sets of sizes either in the interval $[\lfloor (n-k+1)/2 \rfloor, \lfloor (n+k-1)/2 \rfloor]$ or in the interval $[\lceil (n-k+1)/2 \rceil, \lceil (n+k-1)/2 \rceil]$.* $\square$

## 2. Complementing-chain-pair-free families

Let us make the following simple observation: if $A \in \mathcal{F} \subseteq 2^{[n]}$ and $\overline{A} \notin \mathcal{F}$ then the family $\mathcal{F}' = \mathcal{F} \setminus \{A\} \cup \overline{A}$ is an $r$-complementing-chain-pair-free family if and only if $\mathcal{F}$ was an $r$-complementing-chain-pair-free family. This follows simply from the fact that $\overline{A}$ cannot belong to a forbidden configuration in $\mathcal{F}'$. This motivates the following definitions.

**Definition 3.** *For families $\mathcal{F}, \mathcal{F}' \subseteq 2^{[n]}$ we say that $\mathcal{F}$ is* ***equivalent*** *to $\mathcal{F}'$ if $\mathcal{F}'$ can be obtained from $\mathcal{F}$ by replacing some sets by their complements. This is an equivalence relation denoted by $\mathcal{F} \sim \mathcal{F}'$. One member of each equivalence class will be distinguished by the following definition: a family $\mathcal{F}$ will be called* ***upwards-arranged****, if*

$$A \in \mathcal{F} \text{ and } \overline{A} \notin \mathcal{F} \implies |A| > |\overline{A}| \text{ or } (|A| = |\overline{A}| \text{ and } 1 \in A).$$

In the following sections we give the largest cardinality of an $r$-complementing-chain-pair-free family and show all the families achieving this bound. Define the following:

**Definition 4.** *For a positive integer $z$ the* ***upper*** *$z$* ***levels of*** *$2^{[n]}$ means the family of all sets of sizes $n, n-1, \ldots, n-z+1$ ( i.e. the upper $z$ levels in the lattice of $2^{[n]}$ ). The* ***upper*** *$z+1/2$* ***levels of*** *$2^{[n]}$ means the upper $z$ levels plus the sets of size $n-z$ containing a specific element, say 1: note that this is not half of the elements on that level, unless $z = n/2$. We introduce the following notations for these families:*

$$\tilde{\mathcal{F}}^z = \{X \subseteq [n] : |X| \geq n - z + 1\}$$

*denotes the upper $z$ levels,*

$$\tilde{\mathcal{F}}^{z+1/2} = \{X \subseteq [n] : |X| \geq n - z + 1 \text{ or } (|X| = n - z \text{ and } 1 \in X)\}$$

$$\tilde{\mathcal{F}}^{z+1/2}(x) = \{X \subseteq [n] : |X| \geq n - z + 1 \text{ or } (|X| = n - z \text{ and } x \in X)\}$$

*denotes the upper $z + 1/2$ levels with an arbitrary $x \in [n]$ .*

For the case $n$ even and $r = 3$ we have to introduce another family.

**Definition 5.** *For even $n$ let $\hat{\mathcal{F}}$ be the following family*

$$\hat{\mathcal{F}} = \{X \subseteq [n] : \ (|X| > n/2) \text{ or } (|X| = n/2 - 1) \text{ or } (|X| = n/2 \text{ and } 1 \in X)\}$$

The set families introduced above are the optimal $r$-complementing-chain-pair-free families (up to the relation $\sim$); with these notations our main theorem is the following (the proof is given in Section 4):

**Theorem 4.** *If $\mathcal{F}$ is an $r$-complementing-chain-pair-free family then $|\mathcal{F}| \leq |\tilde{\mathcal{F}}^{(n+r)/2}|$, with equality for*

- $\mathcal{F} \sim \tilde{\mathcal{F}}^{(n+r)/2}$ *if $n+r$ is even,*
- $\mathcal{F} \sim \tilde{\mathcal{F}}^{(n+r)/2}(x)$ *for some $x \in [n]$ if $n+r$ is odd and $r \neq 3$ and*
- $\mathcal{F} \sim \hat{\mathcal{F}}$ *or* $\mathcal{F} \sim \tilde{\mathcal{F}}^{(n+r)/2}(x)$ *for some $x \in [n]$ if $n+r$ is odd and $r = 3$.*

As a consequence we immediately get the following:

**Theorem 5.** *The largest cardinality of a $(p,q)$-chain-intersecting family is equal to the cardinality of the upper $(n+p+q-1)/2$ levels.*

*Proof* Since a $(p,q)$-chain-intersecting family $\mathcal{F}$ is also $(p+q-1)$-complementing-chain-pair-free family it has cardinality not more than that of the upper $(n+p+q-1)/2$ levels. But this bound is achieved: the upper $(n+p+q-1)/2$ levels form a $(p,q)$-chain-intersecting family. □

As one can see the largest cardinality of a $(p,q)$-chain-intersecting family depends only on the sum of $p$ and $q$. However, if one considers the optimal constructions in the case $p+q=4$ and $n$ is even there is some asymmetry: the family $\hat{\mathcal{F}}$ is an optimal $(2,2)$-chain-intersecting family but it is not $(3,1)$-chain-intersecting.

## 3. The method of permutations for a chain-pair

In this section we show a simple proof of Theorem 4 in the case when $n+r$ is even. The proof uses an adaptation of the method of permutations, first used by Lubell [8]. The 'uniqueness' part of the theorem is not proved, either. So we prove the following, weaker theorem:

**Theorem 6.** *If $n+r$ is even and $\mathcal{F}$ is an $r$-complementing-chain-pair-free family then the cardinality of $\mathcal{F}$ is not more than the cardinality of the upper $(n+r)/2$ levels.*

*Proof* For a permutation $\pi$ of $[n]$ we define the following family of subsets:

$$\mathcal{H}_\pi = \mathcal{A}_\pi \cup \mathrm{co}(\mathcal{A}_\pi)$$

where

$$\mathcal{A}_\pi = \{\emptyset, \{\pi(1)\}, \{\pi(1), \pi(2)\}, \ldots, \{\pi(1), \pi(2), \ldots, \pi(n-1)\}\}.$$

$\mathcal{H}_\pi$ will be called a *full chain pair* and let $\mathcal{H} := \mathcal{H}_{\mathrm{id}}$ (where id denotes the identity permutation). For a positive integer $z$ let $\mathcal{H}_\pi^z = \mathcal{H}_\pi \cap \tilde{\mathcal{F}}^z$ and $\mathcal{H}_\pi^{z+1/2} = \mathcal{H}_\pi^z \cup \{X\}$ where $X$ is one of the two $n-z$-element sets of $\mathcal{H}_\pi$ (so here we really have a 'half level', but there is no set family $\mathcal{F}$ satisfying $\forall \pi:\ \mathcal{H}_\pi^{z+1/2} = \mathcal{H}_\pi \cap \mathcal{F}$).

Denote by $d_i$ the number of sets of size $i$ in $\mathcal{H}_\pi$: that is $d_0 = d_n = 1$ and $d_1 = d_2 = \cdots = d_{n-1} = 2$.

Let $\mathcal{F}$ be any $r$-complementing-chain-pair-free family. We will evaluate the following double sum in two ways

$$\sum_{\pi} \sum_{F \in \mathcal{H}_\pi \cap \mathcal{F}} \frac{1}{d_{|F|}} \binom{n}{|F|} = \sum_{F \in \mathcal{F}} \sum_{\pi : F \in \mathcal{H}_\pi} \frac{1}{d_{|F|}} \binom{n}{|F|}. \tag{1}$$

The right hand side can be counted exactly: for a subset $F$ the number of permutations $\pi$ with $F \in \mathcal{H}_\pi$ is $d_{|F|}|F|!(n-|F|)!$. So the sum above is $|\mathcal{F}|n!$.

Now let us fix a permutation $\pi$ and give an upper bound on $\sum_{F\in\mathcal{H}_\pi\cap\mathcal{F}} \frac{1}{d_{|F|}}\binom{n}{|F|}$. Since $\mathcal{F}$ is an $r$-complementing-chain-pair-free family, there can be at most $r-1$ sets $F \in \mathcal{H}_\pi \cap \mathcal{F}$ with $\overline{F}$ also in $\mathcal{H}_\pi \cap \mathcal{F}$: clearly the middle $r-1$ levels of the sublattice $\mathcal{H}_\pi$ of $2^{[n]}$ give the largest $\frac{1}{d_{|F|}}\binom{n}{|F|}$ value (for any $i$ between 1 and $n-1$ we have $\frac{1}{d_i}\binom{n}{i} > \frac{1}{d_n}\binom{n}{n} = 1$, so it is really not worth choosing the empty set or $[n]$ instead of sets 'in the middle'). For every other member $F \in \mathcal{H}_\pi$ at most one of $F$ and $\overline{F}$ can belong to $\mathcal{F}$, but these give the same value of $\frac{1}{d_{|F|}}\binom{n}{|F|}$, so assume that the larger of them is in $\mathcal{F}$. So

$$\sum_{F\in\mathcal{H}_\pi\cap\mathcal{F}} \frac{1}{d_{|F|}}\binom{n}{|F|} \leq \sum_{F\in\mathcal{H}_\pi^{(n+r)/2}} \frac{1}{d_{|F|}}\binom{n}{|F|}. \tag{2}$$

The upper bounds are achieved for any $\pi$ if $\mathcal{F}$ is the family of the upper $(n+r)/2$ levels where $n+r$ is even, in which case the sum is just the cardinality of the upper $(n+r)/2$ levels. This gives the following

$$\sum_{F\in\mathcal{H}_\pi\cap\mathcal{F}} \frac{1}{d_{|F|}}\binom{n}{|F|} \leq \sum_{F\in\mathcal{H}_\pi\cap\tilde{\mathcal{F}}^{(n+r)/2}} \frac{1}{d_{|F|}}\binom{n}{|F|} = |\tilde{\mathcal{F}}^{(n+r)/2}|. \tag{3}$$

So we really obtained that $|\mathcal{F}| \leq |\tilde{\mathcal{F}}^{(n+r)/2}|$ as stated above. □

It should be noted that if $n+r$ is odd then the argument above gives only a bound on the size of $\mathcal{F}$ that is not tight if $r > 1$.

In the end of this section an even simpler proof of Theorem 6 is given by rephrasing the problem (though we still want to include the proof above for later reference). We can write any family $\mathcal{F} \subseteq 2^{[n]}$ in the form of a disjoint union $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$ where $\mathcal{F}_1$ consists exactly of those members $X$ of $\mathcal{F}$, for which $\overline{X}$ belongs to $\mathcal{F}$, too. Obviously $\mathcal{F}_1$ is closed under complementation. Observe that

$$\mathcal{F} \text{ is } r\text{-complementing-chain-pair-free} \iff \mathcal{F}_1 \text{ is } (r-1)\text{–Sperner}.$$

The proof of Theorem 6 is based on this observation.

*Proof (Proof of Theorem 6)* In the above decomposition of the optimal $r$-complementing-chain-pair-free family $\mathcal{F}$ the subfamily $\mathcal{F}_2$ will obviously contain exactly one of $X$ or $\overline{X}$ for every $X \notin \mathcal{F}_1$. So we have:

$$\begin{gathered}\mathcal{F} \text{ is an optimal } r\text{-complementing-chain-pair-free family} \iff \\ \mathcal{F}_1 \text{ is optimal among families that are} \\ (r-1)\text{–Sperner and closed under complementation} \\ (\text{and } |\mathcal{F}_2 \cap \{X, \overline{X}\}| = 1 \text{ for each } X \notin \mathcal{F}_1).\end{gathered}$$

If $n+r$ is even then – according to Theorem 3 – the unique optimal $r-1$-Sperner family is closed under complementation, so this has to be $\mathcal{F}_1$; the theorem is proved and we also obtained that the optimal $r$-complementing-chain-pair-free family is unique (up to the relation $\sim$). □

The observations given above together with Theorem 4 (to be proved in the next section) will give us the following result:

**Theorem 7.** *The maximum size of a family that is $k$–Sperner and closed under complementation is*

- $\sum_{i=(n-k+1)/2}^{(n+k-1)/2}\binom{n}{i}$ *if $n+k-1$ is even and*
- $\sum_{i=(n-k+2)/2}^{(n+k-2)/2}\binom{n}{i}+2\binom{n}{(n-k)/2-1}$ *if $n+k-1$ is odd.*

*The optimal families correspond to the optimal $(k+1)$-complementing-chain-pair-free families.* □

## 4. The full proof of the main theorem

In our proof we will use **the method of cyclic permutations** developed by Gyula O.H. Katona in [7]. First of all let us introduce some terminology.

Let the elements of the set $[n]$ be placed around a circle ('a round table') such that $i+1$ is next to $i$ for all $i=1,2,\dots,n-1$ and 1 is next to $n$ in the clockwise direction: we will also say that $i+1$ is to the left of $i$, just as if they were persons around the table. We will simply write $i+1, i+2, \dots$ and mean that if we get a number greater than $n$ then we subtract $n$ (almost like modulo $n$ addition, but we do not want to change the underlying set to $\{0,1,\dots,n-1\}$). Elements next to each other will be called **consecutive**. A set of consecutive elements will be called an **interval** ($\emptyset$ and $[n]$ are also considered to be intervals). Denote the interval of elements between $a$ and $b$ by $[a,b]$ (endpoints included): this is the set of elements $a, a+1, \dots, b$. The family of all intervals on the circle will be denoted by $\mathcal{I}$, the family of intervals of length $k$ will be denoted by $\mathcal{I}_k$. The intersection of the set of intervals with the upper $z$ (or $z+1/2$) levels of $2^{[n]}$ will be called the **upper $z$ (or $z+1/2$) levels on the circle**. For convenience we introduce the following notations:

$$\tilde{\mathcal{G}}^z=(\tilde{\mathcal{F}}^z\cap\mathcal{I})\setminus\{[n]\},\ \tilde{\mathcal{G}}^{z+1/2}=(\tilde{\mathcal{F}}^{z+1/2}\cap\mathcal{I})\setminus\{[n]\},$$

$$\tilde{\mathcal{G}}^{z+1/2}(x)=(\tilde{\mathcal{F}}^{z+1/2}(x)\cap\mathcal{I})\setminus\{[n]\},\hat{\mathcal{G}}=(\hat{\mathcal{F}}\cap\mathcal{I})\setminus\{[n]\}.$$

Since the complement of an interval is again an interval, the equivalence relation $\sim$ is also well defined among the families on the circle. Fortunately, if $\mathcal{F}\sim\mathcal{F}'$ then $\sum_{F\in\mathcal{F}'}\binom{n}{|F|}=\sum_{F\in\mathcal{F}}\binom{n}{|F|}$ which will be useful in our lemma.

We simply say that the $r$-complementing-chain-pair-free family $\mathcal{G}$ is *optimal* if $\sum_{G\in\mathcal{G}'}\binom{n}{|G|}\le\sum_{G\in\mathcal{G}}\binom{n}{|G|}$ for any $r$-complementing-chain-pair-free family $\mathcal{G}'$ (optimality in this sense will only be used on the circle).

Essentially the method of cyclic permutations is an adaptation of the method of permutations on the circle, so we will need the analogous result (Lemma 2) on the circle.

**Lemma 1.** *If $\mathcal{G}$ is an optimal and upwards-arranged $r$-complementing-chain-pair-free family of intervals on the circle then $m:=\min\{|G|:G\in\mathcal{G}\}\ge(n-r+1)/2$.*

*Proof* Suppose, for a contradiction, that $m<(n-r+1)/2$. For an $m$ element set $G=[a,b]$ in $\mathcal{G}$ consider the following sequence of intervals:

$$\mathcal{S}_G=[a,b],\overline{[a,b]},[a,b+1],\overline{[a,b+1]},\dots,[a,b+r-1],\overline{[a,b+r-1]}.$$

Observe that $\mathcal{S}_G\not\subseteq\mathcal{G}$, because $\mathcal{G}$ is an $r$-complementing-chain-pair-free family. Denote the first member of $\mathcal{S}_G\setminus\mathcal{G}$ by $A_G$. We claim that $A_{G_1}\ne A_{G_2}$ for different $m$ element sets $G_1=[a_1,b_1]$ and $G_2=[a_2,b_2]$. This is true since a set in $\mathcal{S}_{G_1}\cap\mathcal{S}_{G_2}$ can only be of the form $[a_1, b_1 + k] = \overline{[a_2, b_2 + l]}$ (possibly with $G_1$ and $G_2$ exchanged) with $a_1 = b_2 + l + 1$ and $a_2 = b_1 + k + 1$, thus $l = n - k - 2m$. But this set cannot be equal to both $A_{G_1}$ and $A_{G_2}$, since this would mean that the first $2k$ members of $\mathcal{S}_{G_1}$ and the first $2l+1$ members of $\mathcal{S}_{G_2}$ all belong to $\mathcal{G}$, but these together give a chain of length $k + l = n - 2m > r - 1$ belonging to $\mathcal{G}$ with all complements, contradicting the $r$-complementing-chain-pair-free property.

We will want to perform the following operation:

$(*)$ Exchange every $m$ element set $G$ in $\mathcal{G}$ for $A_G$.

Then we want to prove that the family obtained is again $r$-complementing-chain-pair-free. However, the proof becomes technically a little bit simpler if – before performing operation $(*)$ – we first apply the following preparatory operation: for all $m$ element sets $G = [a, b]$ which have $A_G = \overline{[a, b + l]}$ for some $l \geq 0$ (note that $l > 0$ since $\mathcal{G}$ is upwards-arranged), replace $\overline{A_G}$ in $\mathcal{G}$ with $A_G$: we obtain another optimal family (denoted again by $\mathcal{G}$) which might not be upwards-arranged, but all $m$ element sets $G = [a, b]$ will have $A_G$ in the form $[a, b + k]$ for some $k > 0$. When performing this preparatory operation we do not introduce any new $m$ element sets (since the only $m$ element set in $\mathcal{S}_G$ was $G$, according to the hypothesis $m < (n - r + 1)/2$ and every $m$ element set remains in $\mathcal{G}$. Let us introduce the following notation: for an $m$ element set $G = [a, b]$ in $\mathcal{G}$ consider the following chain:

$$\mathcal{L}_G = [a, b], [a, b + 1], [a, b + 2], \ldots, [a, b + r - 1].$$

So $A_G$ is the smallest member of $\mathcal{L}_G \setminus \mathcal{G}$.

Now we can perform operation $(*)$: for every $m$ element set $G$ in $\mathcal{G}$ we replace $G$ by $A_G$. The family obtained will be denoted by $\mathcal{G}'$. Note that $|\mathcal{G}| = |\mathcal{G}'|$, $m' = \min\{|G| : G \in \mathcal{G}'\} > m$ and

$$\sum_{G \in \mathcal{G}} \binom{n}{|G|} < \sum_{G \in \mathcal{G}'} \binom{n}{|G|} \tag{4}$$

since $\binom{n}{|G|} < \binom{n}{|A_G|}$ for every $m$ element set $G \in \mathcal{G}$.

We will now prove that $\mathcal{G}'$ is also an $r$-complementing-chain-pair-free family and so (4) contradicts the optimality of $\mathcal{G}$. Suppose the contrary and consider a chain $\mathcal{L} = \{A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_r\}$ such that $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{G}'$. Of course $\mathcal{L} \cup \mathrm{co}(\mathcal{L})$ cannot be contained in $\mathcal{G}$: suppose $\mathcal{L} \not\subseteq \mathcal{G}$ (otherwise exchange $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$). For every $A \in (\mathcal{L} \cup \mathrm{co}(\mathcal{L})) \setminus \mathcal{G}$ there was an $m$ element set $G \in \mathcal{G}$ that got substituted by $A$ (that is $A = A_G$).

We show in two steps that there was a $r$-complementing-chain-pair in $\mathcal{G}$, which is a contradiction.

The first step is the following: let $k = \max\{i : A_i \notin \mathcal{G}\}$. Let $l = |A_k|$. Observe that $k \leq l - m$. Then there was an $m$ element set $G$ in $\mathcal{G}$ that got replaced with $A_k$ which means that $A_k$ is the first member of $\mathcal{L}_G$ that was not in $\mathcal{G}$. So there are $l - m \geq k$ members of $\mathcal{L}_G$ belonging to $\mathcal{G}$ along with their complements before $A_k$. Then we can replace the first $k$ elements of $\mathcal{L}$ by the first $k$ members of $\mathcal{L}_G$ and obtain a chain $\mathcal{L}'$.

If $\mathrm{co}(\mathcal{L}')$ still contains members of $\mathcal{G}' \setminus \mathcal{G}$ then a second step is needed; otherwise we are done ($\mathcal{L}' \cup \mathrm{co}(\mathcal{L}') \subseteq \mathcal{G}$ gives the contradiction). In this second step replace $\mathcal{L}'$ by $\mathrm{co}(\mathcal{L}')$ and repeat the preceding procedure: find the largest member $A = A_G$ of $\mathcal{L}' \setminus \mathcal{G}$ and exchange the beginning of $\mathcal{L}'$ for the beginning of $\mathcal{L}_G$ to obtain a chain $\mathcal{L}''$ of the same length with $\mathcal{L}'' \cup \mathrm{co}(\mathcal{L}'') \subseteq \mathcal{G}$, which is a contradiction. $\square$

We note that the operation we used in the preceding proof works in other situations, too. Let us give the skeleton of it:

$(*)$ In a set family $\mathcal{F}$ replace every set $G$ of minimum size with the first member of $\mathcal{S}_G \setminus \mathcal{F}$.

This operation depends on the definition of $\mathcal{S}_G$. Let us give two examples:

1. If $\mathcal{F}$ is a $(p,q)$-chain-intersecting family on the circle with $\mathcal{S}_G$ defined as above and $m < (n-(p+q-1)+1)/2$ then this operation also preserves the $(p,q)$-chain-intersecting property of the family $\mathcal{F}$: with this observation one can work out what the optimal $(p,q)$-chain-intersecting families are.
2. If $\mathcal{F}$ is an $r$-complementing-chain-pair-free family (not necessarily on the circle) and $\mathcal{S}_G$ is defined by means of a **symmetric chain partition** (a partition of $2^{[n]}$ into chains, each of form $\mathcal{L} = \{L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_h\}$ with $|L_1| + |L_h| = n$ and $|L_j| - |L_{j-1}| = 1$ $(\forall j \in \{2,\dots,h\})$; such a partition exists, see for example [2]) and $m < (n-r+1)/2$ then similar arguments show that the operation preserves the $r$-complementing-chain-pair-free property. This method gives a full proof of Theorem 4 for the case when $n+r$ is even, but it cannot handle the difficulties when it is odd.

**Lemma 2.** *Suppose that we are given an $r$-complementing-chain-pair-free family $\mathcal{G}$ on the circle not containing $\emptyset$ and $[n]$. Then*

$$\sum_{G\in\mathcal{G}} \binom{n}{|G|} \le \sum_{G\in\tilde{\mathcal{G}}^{(n+r)/2}} \binom{n}{|G|}. \tag{5}$$

*If equality holds then*

- $\mathcal{G} \sim \tilde{\mathcal{G}}^{(n+r)/2}$ *if $n+r$ is even*
- $\mathcal{G} \sim \tilde{\mathcal{G}}^{(n+r)/2}(x)$ *for some $x \in [n]$ if $n+r$ is odd and $r \neq 3$*
- $\mathcal{G} \sim \hat{\mathcal{G}}$ *or* $\mathcal{G} \sim \tilde{\mathcal{G}}^{(n+r)/2}(x)$ *for some $x \in [n]$ if $n$ is even and $r = 3$.*

*Proof* Lemma 1 implies the result if $n+r$ is even. It remains to consider the case when $n+r$ is odd and the minimum cardinality of a set in $\mathcal{G}$ is $m = (n-r+1)/2$. In the proof we assume that $\mathcal{G}$ is an optimal and upwards-arranged family.

Note that all sets of size $> n-m = (n+r-1)/2$ have to be in $\mathcal{G}$, because their complements are not present: so only sets of size between $m$ and $n-m$ are of interest (the 'middle $r$ levels' which are well defined here).

The lemma will be proved by induction on $r$. If $r=1$ then the result is obvious and well known: $\mathcal{G} \cap \mathcal{I}_{n/2}$ is an intersecting family on the circle, contains no complements: optimal constructions are families of all sets containing a common element or those that can be obtained from them by complementing some sets. If $\mathcal{G}$ is also upwards-arranged then $\mathcal{G} = \tilde{\mathcal{G}}^{(n+r)/2}$, as stated above.

If $r=2$ (so $m=(n-1)/2$) then $\mathcal{G}' = \mathcal{G} \cap (\mathcal{I}_{(n-1)/2} \cup \mathcal{I}_{(n+1)/2})$ is a 2-complementing-chain-pair-free family of maximum size among families in $\mathcal{I}_{(n-1)/2} \cup \mathcal{I}_{(n+1)/2}$ (all sets have equal weight, so maximizing the weight is the same as maximizing the size). So, for every $a \in [n]$ we have $|\mathcal{K}_a \cap \mathcal{G}| \le 3$, where

$$\mathcal{K}_a = \{[a,a+m-1], \overline{[a,a+m-1]}, [a,a+m], \overline{[a,a+m]}\}.$$

So if we consider

$$\sum_{a\in[n]} \sum_{G\in\mathcal{K}_a\cap\mathcal{G}'} 1 = \sum_{G\in\mathcal{G}'} \sum_{a: G\in\mathcal{K}_a} 1 \tag{6}$$

then we see that the right hand side is exactly $2|\mathcal{G}'|$ while the left hand side is at most $3n$ (this is similar to what we did in Section 3 in the permutation method but it gives a tight bound in this special case, though $n+r$ is odd, but this works only on the circle). So we really obtained that $|\mathcal{G}'| \leq 3n/2$. This implies $|\mathcal{G}'| \leq \lfloor 3n/2 \rfloor = |\tilde{\mathcal{G}}^{(n+r)/2} \cap \big(\mathcal{I}_{(n-1)/2} \cup \mathcal{I}_{(n+1)/2}\big)|$, which also gives

$$\sum_{G\in\mathcal{G}} \binom{n}{|G|} \leq \sum_{G\in\tilde{\mathcal{G}}^{(n+r)/2}} \binom{n}{|G|}. \tag{7}$$

If there were two disjoint sets of size $(n-1)/2$ then their complements would also be in $\mathcal{G}$ because of the upwards-arranged property, but this would contradict the 2-complementing-chain-pair-free property. So sets of size $(n-1)/2$ pairwise intersect each other, but then there are at most $\lfloor n/2 \rfloor$ of them and clearly the only intersecting families of intervals in $\mathcal{I}_{(n-1)/2}$ of cardinality $\lfloor n/2 \rfloor$ are those containing a common element $x$. So we really have

$$\mathcal{G}' = \{X \subseteq [n] : |X| = (n+1)/2 \text{ or } (|X| = (n-1)/2 \text{ and } x \in X)\}$$

for some $x \in [n]$, finishing the case $r = 2$.

Assume that $r \geq 3$. We claim that all (inclusionwise) minimal members of $\mathcal{G}$ are of size $m$ or $m+1$. Suppose indirectly that there is a minimal member $G = [a, a+m-1+l]$ of $\mathcal{G}$ with $l \geq 2$. This means that neither of the $m+1$ element sets $[a, a+m]$ and $[a+1, a+m+1]$ is in $\mathcal{G}$. But in this case their intersection, the $m$ element $[a+1, a+m]$ (which was not in $\mathcal{G}$) could be added to $\mathcal{G}$ while maintaining its $r$-complementing-chain-pair-free property. This is because a chain of length $r$ in $\mathcal{G} \cup \{[a+1, a+m]\}$ whose complements are also in $\mathcal{G}$ has to contain intervals from all of the middle $r$ levels, but a chain in $\mathcal{G} \cup \{[a+1, a+m]\}$ starting at $[a+1, a+m]$ will not contain any $m+1$ element interval. Hence we proved that all minimal sets in $\mathcal{G}$ are of size $m$ or $m+1$.

Let us denote by $\mathcal{A}$ the family of minimal members of $\mathcal{G}$, that is $\mathcal{A} = \{X \in \mathcal{G} : \nexists Y \in \mathcal{G} \text{ with } Y \subsetneq X\}$ and let $\mathcal{G}^1 = \mathcal{G} \setminus \mathcal{A}$. It is easy to see that $\mathcal{G}^1$ is also upwards-arranged if $r > 3$; in the case $r = 3$ this is not necessarily true, but this will cause no problem. We claim that $\mathcal{G}^1$ is an $(r-2)$-complementing-chain-pair-free family $\mathcal{G}^1$. This is proved by the following argument: suppose there is a $(r-2)$-complementing-chain-pair $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{G}^1$. Note that $\mathcal{L} \cup \mathrm{co}(\mathcal{L})$ must be contained in the middle $r-2$ levels. But then there would be members of $\mathcal{G}$ contained in the minimal members of $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$: the complements of these would be in $\mathcal{G}$, too, because of the upwards-arranged property if $m+1 < n/2$ which is true for $r > 3$ (if $r = 3$ the chains $\mathcal{L}$ and $\mathrm{co}(\mathcal{L})$ are of length 1: they have to form a complementing set pair of size $n/2 = m+1$ so the sets below them are of size $m$ and have complement in $\mathcal{G}$, too). So there would be a $r$-complementing-chain-pair in $\mathcal{G}$, a contradiction.

By induction $\sum_{G\in\mathcal{G}^1} \binom{n}{|G|} \leq \sum_{G\in\tilde{\mathcal{G}}^{(n+r-2)/2}} \binom{n}{|G|}$. The difference $\mathcal{A} = \mathcal{G} \setminus \mathcal{G}^1$ is a Sperner family with all sets of size $m$ or $m+1$. We use the following observation (originally due to Zoltán Füredi): a Sperner family on the circle has at most $n$ members and if it has $n$ members then all its elements are of equal size. If all members of $\mathcal{A}$ are of size $m$ then $\sum_{G\in\mathcal{A}} \binom{n}{|G|} = n\binom{n}{m}$. If less than $m$ members of $\mathcal{A}$ are of size $m$ then $\mathcal{G}$ was not optimal, since $\tilde{\mathcal{G}}^{(n+r)/2}$ is strictly better. So if there are sets of size $m+1$ in $\mathcal{A}$ as well then $\sum_{G\in\mathcal{A}} \binom{n}{|G|} \leq m\binom{n}{m} + (n-m-1)\binom{n}{m+1}$ with equality for

$$\mathcal{A} = \mathcal{A}(x) = \{X \in \mathcal{I} : (|X| = m \text{ and } x \in X) \text{ or } (|X| = m+1 \text{ and } x \notin X)\}$$

with an element $x \in [n]$ (only families of this form will be disjoint from an optimal $(r-2)$-complementing-chain-pair-free family and it is easy to see that $\mathcal{G}^1$ has to be such a family if $\mathcal{G}$ is an optimal $r$-complementing-chain-pair-free family).

It is easy to prove that $n\binom{n}{m} \le m\binom{n}{m} + (n-m-1)\binom{n}{m+1}$ if $m \le n/2-1$ which is true for $r \ge 3$ (with equality only for $r=3$). So

$$\sum_{G\in\mathcal{G}} \binom{n}{|G|} = \sum_{G\in\mathcal{G}^1} \binom{n}{|G|} + \sum_{G\in\mathcal{A}} \binom{n}{|G|} \tag{8}$$

$$\le \sum_{G\in\tilde{\mathcal{G}}^{(n+r-2)/2}} \binom{n}{|G|} + m\binom{n}{m} + (n-m-1)\binom{n}{m+1} \tag{9}$$

$$= \sum_{G\in\tilde{\mathcal{G}}^{(n+r)/2}} \binom{n}{|G|}. \tag{10}$$

Suppose we have equality here: then $\mathcal{G}^1$ has to be an optimal family for the $r-2$ case. If $r>3$ then $\mathcal{A} = \mathcal{A}(x)$; if $r=3$ then $\mathcal{A} = \{X \in \mathcal{I} : |X| = m\}$ or $\mathcal{A} = \mathcal{A}(x)$ (where $x \in [n]$). By induction

- If $r=3$ then $\mathcal{G}^1 \sim \tilde{\mathcal{G}}^{(n+r-2)/2}$ and
  - $\mathcal{A} = \{X \in \mathcal{I} : |X| = m\}$ which gives $\mathcal{G} = \hat{\mathcal{G}}$ (since $\mathcal{G}$ is upwards-arranged) or
  - $\mathcal{A} = \mathcal{A}(x)$ which is disjoint from $\mathcal{G}^1$ if $\mathcal{G}^1 = \tilde{\mathcal{G}}^{(n+r-2)/2}(x)$ giving $\mathcal{G} = \tilde{\mathcal{G}}^{(n+r)/2}(x)$.
- If $r>3$ and $r \ne 5$ then $\mathcal{G}^1 = \tilde{\mathcal{G}}^{(n+r-2)/2}(y)$ which is disjoint from $\mathcal{A} = \mathcal{A}(x)$ if $x=y$ giving $\mathcal{G} = \tilde{\mathcal{G}}^{(n+r)/2}(x)$.
- If $r=5$ then $\mathcal{A} = \mathcal{A}(x)$ and
  - $\mathcal{G}^1 = \tilde{\mathcal{G}}^{(n+r-2)/2}(y)$ which is disjoint from $\mathcal{A} = \mathcal{A}(x)$ if $x=y$ giving $\mathcal{G} = \tilde{\mathcal{G}}^{(n+r)/2}(x)$ or
  - $\mathcal{G}^1 = \hat{\mathcal{G}}$ which cannot be disjoint from $\mathcal{A}$ so gives no other solution.

□

Now we can prove our main theorem (Theorem 4) using the method of cyclic permutations.

*Proof (Proof of Theorem 4)* We prove by induction on $r$: the case $r=1$ is trivial. Suppose $n > r > 1$ (the case $r=n$ is again simple).

If $\mathcal{F}$ is an optimal $r$-complementing-chain-pair-free family then it contains at least one of the sets $\emptyset$ and $[n]$: suppose $[n] \in \mathcal{F}$.

If $\emptyset$ is also in $\mathcal{F}$ then $\mathcal{F}' = \mathcal{F} \setminus \{\emptyset\}$ is an $(r-1)$-complementing-chain-pair-free: if it contained a chain $\mathcal{L}'$ of length $r-1$ with $\mathcal{L}' \cup \mathrm{co}(\mathcal{L}') \subseteq \mathcal{F}'$ then $\mathcal{L} = \mathcal{L}' \cup \{[n]\}$ would give $\mathcal{L} \cup \mathrm{co}(\mathcal{L}) \subseteq \mathcal{F}$, a contradiction. So in this case $\mathcal{F}$ could not be optimal, since

$$|\mathcal{F}| \le |\tilde{\mathcal{F}}^{(n+r-1)/2}| + 1 < |\tilde{\mathcal{F}}^{(n+r)/2}|. \tag{11}$$

We can suppose that $\mathcal{F}$ is upwards-arranged. Hence $[n] \in \mathcal{F}$ and $\emptyset \notin \mathcal{F}$. Consider the family $\mathcal{F}' = \mathcal{F} \setminus \{[n]\}$. Evaluate the following double sum in two ways

$$\sum_{\pi} \sum_{F\in\mathcal{I}^\pi\cap\mathcal{F}'} \binom{n}{|F|} = \sum_{F\in\mathcal{F}'} \sum_{\pi: F\in\mathcal{I}^\pi} \binom{n}{|F|}, \tag{12}$$

where $\pi$ is an arbitrary cyclic permutation and $\mathcal{I}^\pi$ is the family of sets that form an interval in $\pi$.

The right hand side can be counted exactly: for a nonempty proper subset $F$ the number of cyclic permutations $\pi$ with $F \in \mathcal{I}^\pi$ is $|F|!(n-|F|)!$. So the sum above is $|\mathcal{F}'|n!$.

According to Lemma 2, for any fixed cyclic permutation $\pi$

$$\sum_{F\in\mathcal{I}^\pi\cap\mathcal{F}'}\binom{n}{|F|} \le \sum_{F\in\mathcal{I}^\pi\cap\mathcal{F}_0}\binom{n}{|F|} = m\binom{n}{m} + \sum_{k=m+1}^{n-1}\binom{n}{k} = n|\mathcal{F}_0|, \tag{13}$$

where $\mathcal{F}_0 = \tilde{\mathcal{F}}^{(n+r)/2}\setminus\{[n]\}$. So we obtain the inequality $|\mathcal{F}| \le |\tilde{\mathcal{F}}^{(n+r)/2}|$. This is satisfied with equality if and only if $\mathcal{F}'\cap\mathcal{I}^\pi$ is an optimal family on the circle for every cyclic permutation $\pi$ (of course $\mathcal{F}'\cap\mathcal{I}^\pi$ will be upwards-arranged for any $\pi$). By Lemma 1 this cannot be true if $m = \min\{|F| : F\in\mathcal{F}\} < (n-r+1)/2$ : so we are again done in the case $n+r$ even.

Suppose $n+r$ is odd and $m = \min\{|F| : F\in\mathcal{F}\} = (n-r+1)/2$ in the family $\mathcal{F}$. Consider first the case $r \ne 3$. All sets of size $> m$ must be in $\mathcal{F}$, otherwise a suitable cyclic permutation would give a family that is not $\tilde{\mathcal{G}}^{(n+r)/2}(x)$, so cannot be optimal on the circle. But in this case the family of sets of size $m$ in $\mathcal{F}$ must be intersecting because of the $r$-complementing-chain-pair-free property: so they form a maximum sized intersecting subfamily of $\binom{[n]}{m}$. By the Erdős-Ko-Rado theorem (Theorem 2), $\mathcal{F}\cap\binom{[n]}{m} = \{X\subseteq[n] : |X| = m \text{ and } x\in X\}$ for some $x\in[n]$.

Now suppose $r = 3$. Again, all sets of size $> n/2$ must be in $\mathcal{F}$. If there are 2 disjoint sets of size $n/2$ in $\mathcal{F}$ then all sets of size $n/2$ have to be in $\mathcal{F}$, otherwise we could easily find a cyclic permutation that gives a family which is not optimal on the circle (if $X, \overline{X}\in\mathcal{F}$ and $Z\notin\mathcal{F}$, each of size $n/2$ then we can find a permutation that takes all of them into intervals). So in this case $\mathcal{F} = \tilde{\mathcal{F}}^{(n+r)/2}$ (again, sets of size $m$ pairwise intersect here). If sets of size $n/2$ in $\mathcal{F}$ pairwise intersect then $\mathcal{F}$ can contain all sets of size $m$, so $\mathcal{F} = \hat{\mathcal{F}}$. $\square$

## 5. Comments and open problems

Related problems can be obtained from the following definitions:

**Definition 6.** *A family $\mathcal{F}\subseteq 2^{[n]}$ is called* ***strongly*** $(p,q)$***-chain-intersecting*** *if there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ in $\mathcal{F}$ such that $A_p\cap B_1 = \emptyset$ (the top of a chain of length p always intersects the bottom of a chain of length q in $\mathcal{F}$).*

**Definition 7.** *A family $\mathcal{F}\subseteq 2^{[n]}$ is called* ***totally*** $(p,q)$***-chain-intersecting*** *if there are no sets $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_p$ and $B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_q$ in $\mathcal{F}$ such that $A_1\cap B_1 = \emptyset$ (the bottoms of two chains of sizes p and q in $\mathcal{F}$ always intersect).*

Note, that the second definition is symmetric in $p$ and $q$, while the first one is not. Also note that totally $(p,q)$-chain-intersecting property of a family implies its strong $(p,q)$-chain-intersecting property and strong $(p,q)$-chain-intersecting property implies $(p,q)$-chain-intersecting property.

Another thing to notice is that if $\mathcal{F}$ is strongly $(p,q)$-chain-intersecting then it is also $p$-complementing-chain-pair-free if $p\ge q$. Consequently:

**Theorem 8.** *If $p\ge q$ then the largest cardinality of a strongly $(p,q)$-chain-intersecting family is equal to the cardinality of the upper $(n+p)/2$ levels.*

*Proof* Since a strongly $(p,q)$-chain-intersecting family $\mathcal{F}$ is also a $p$-complementing-chain-pair-free family if $p \geq q$, it has cardinality not more than that of the upper $(n+p)/2$ levels. But this bound is achieved: the upper $(n+p)/2$ levels form a strongly $(p,q)$-chain-intersecting family. □

*Conjecture 1.* If $\mathcal{F}$ is a strongly $(p,q)$-chain-intersecting and $p < q$, then $|\mathcal{F}| \leq \max(|\mathcal{R}_1|, |\mathcal{R}_2|)$, where $\mathcal{R}_1$ is the upper $(n+p)/2$ levels and $\mathcal{R}_2$ is the middle $q-1$ levels of $2^{[n]}$.

One can see, that the conjecture is true on the full chain pair $\mathcal{H}$, more precisely

$$\sum_{F \in \mathcal{H} \cap \mathcal{F}} \frac{1}{d_{|F|}} \binom{n}{|F|} \leq \max \left( \sum_{F \in \mathcal{H}^{(n+p)/2}} \frac{1}{d_{|F|}} \binom{n}{|F|}, \sum_{F \in \mathcal{R}_2 \cap \mathcal{H}} \frac{1}{d_{|F|}} \binom{n}{|F|} \right), \tag{14}$$

(see the notation in the proof of Theorem 6 on page 4). Note that $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly $(p,q)$-chain-intersecting.

The conjecture follows in the case $n+p$ is even or when the second quantity gives the maximum, with the same argument as in the proof of Theorem 6. If $n+p$ is odd and the first quantity is bigger then this gives only an upper bound, which is tight when $p=1$.

Turning to the totally $(p,q)$-chain-intersecting case, it is easy to see that in the full chain pair $\mathcal{H}$ the optimal totally $(p,q)$-chain-intersecting family is the following: $q-1$ sets from either chain, and the complete other chain, except $\emptyset$ and $[n]$ (where, wlog. $p \geq q$ is assumed). Unfortunately there is no family of sets $\mathcal{F}$ such that $\mathcal{H}_\pi \cap \mathcal{F}$ would be this optimal family for every permutation $\pi$. So it remains open to determine the size of the optimal totally $(p,q)$-chain-intersecting family.

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