l-chain profile vectors

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Abstract

The l-chain profile vector of a set system $\mathcal{F}$ on an underlying set X of size n is defined to be the vector of length $\binom{n+1}{l}$ in which the αth component ($\alpha = (\alpha_1, \alpha_2, ..., \alpha_l) \quad 0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$) is the number of l-chains in $\mathcal{F}$ with the smallest set having size α_1, the second smallest α_2, and so on. We modify the method of P.L. Erdős, P. Frankl and G.O.H. Katona to determine the l-chain profile polytope of some sets of families including k-Sperner, complement free, complement free k-Sperner families.

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1 Introduction

Basic problems in extremal set theory consider a set $\mathbb{A}$ of families of subsets of an n-element set all having some fixed property (Sperner, intersecting, complement free, etc.) and ask for the family with largest size. That is we sum over all sets in the family, adding one for each set and we ask for the family with the largest sum (largest *weight*). A natural generalization of this problem if in the summation we allow *weights* different from 1, but depending only on the size of the subset in question. For example, in the famous LYM-inequality a k-element set has weight $\frac{1}{\binom{n}{k}}$.

The tool that helps dealing with this generalized problem is the so-called *profile vector* of a family $\mathcal{F}$. It is a vector $f(\mathcal{F}) \in \mathbb{R}^{n+1}$ whose components are defined by

$$f(\mathcal{F})_i = |\{F \in \mathcal{F} : |F| = i\}|.$$

Given a set of families $\mathbb{A}$, let $\mu(\mathbb{A})$ denote the set of the profile vectors of families in $\mathbb{A}$ (i.e. $\{f(\mathcal{F}) : \mathcal{F} \in \mathbb{A}\}$). Now determining the family in $\mathbb{A}$ with the largest weight (for a weight-function w as above, where we denote the weight of an i-element set by w_i) is equivalent to finding the family of which the profile maximizes

$$w \cdot f(\mathcal{F}) = \sum_{i=0}^{n} w_i f(\mathcal{F})_i$$

among profiles in $\mu(\mathbb{A})$.

A basic fact of linear programming says that this maximum is attained at some *extreme point* of the *convex hull* of $\mu(\mathbb{A})$, which we call the *profile polytope* of $\mathbb{A}$ and we denote it by $\langle\mu(\mathbb{A})\rangle$. And if all w_is are non-negative, then the maximum is taken at an *essential extreme point* (i.e. an extreme point that is maximal with respect to the component-wise ordering). Therefore to establish "all linear inequalities" concerning a set $\mathbb{A}$ of families, one has to determine its profile polytope and its extreme points. The first result in this area (implicitly, without using the notion of the profile polytope) is due to Katona [8], the systematic investigation of profile polytopes was started by P.L. Erdős, P. Frankl, G.O.H. Katona in [4], [5]. A survey on this topic can be found in the book of Konrad Engel [3].

However, there are problems dealing with other kinds of weight functions, and problems not dealing with sets of some families, but subfamilies of families. A family of sets is called k-Sperner (and Sperner or *antichain* if $k = 1$) if it does not contain a chain of length $k+1$ (a sequence of sets of length $k+1$ in which every set contains the previous one). By the dual version of Dilworth' theorem we know that being k-Sperner is equivalent to being the union of at most k antichains. A natural question is the following: let $l \leq k$ be two integers, how many l-chains can be contained in a family without a $k+1$-chain (i.e. in a k-Sperner family). Note, that this problem is analogous to the celebrated theorem of Turán [13], generalized by Sauer [12] (see also [2]) stating that for all integers $2 \leq r \leq s$ a graph on n vertices without a clique of size $s+1$ can contain at most as many cliques of size r as does the s-partite Turán graph on n vertices (the complete s-partite graph with equipartite partition).

To deal with the above problem we introduce the notion of *l-chain profile vector* of a family $\mathcal{F}$ on an n-element underlying set. This has $\binom{n+1}{l}$ components, and the αth component f_α, where $\alpha = (\alpha_1, ..., \alpha_l)$ with $0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$, denotes the number of l-chains in $\mathcal{F}$ in which the smallest set has size α_1, the second smallest has size α_2, and so on. Note that for $l = 1$ this is just the original notion of the profile vector.

The rest of the paper is organized as follows. In Section 2 we provide some more definitions and make some easy remarks on the connection between the original notion of profile vector and the newly introduced notion of l-chain profile. In Section 3 we prove our main tool in determining l-chain profile polytopes, and in Section 4 we use this tool (the reduction method) to get some results on some sets of families.

2 Definitions and remarks

In this section we give some further definitions and describe some basic connections between the extreme points in the l-chain case and the extreme points in the original (1-chain) case.

Notation. For $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l), 0 \leq \alpha_1 < \alpha_2 < ... < \alpha_l \leq n$ we define the following multinomial coefficient:

$$\binom{n}{\alpha} = \prod_{i=1}^{l} \binom{n-\alpha_{i-1}}{\alpha_i - \alpha_{i-1}} = \frac{n!}{\alpha_1!(\alpha_2-\alpha_1)!...(\alpha_l-\alpha_{l-1})!(n-\alpha_l)!}$$

where $\alpha_0 = 0$ and $0! = 1$ as usual. Note that $\binom{n}{\alpha}$ is the number of l-chains that can be formed from subsets of an n-element set in such a way that the smallest set has size α_1, the second smallest has size α_2 and so on.

Definition: Given an underlying set X and a family $\mathcal{F}$ of its subsets, the upset of $\mathcal{F}$ is $\mathcal{U}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \subseteq G\}$ and the downset of $\mathcal{F}$ is $\mathcal{D}(\mathcal{F}) = \{G \subseteq X : \exists F \in \mathcal{F} \text{ such that } F \supseteq G\}$.

Definition: A set $\mathbb{A}$ of families is upward (downward) closed if $\mathcal{F} \in \mathbb{A}$ implies $\mathcal{U}(\mathcal{F}) \in \mathbb{A}$ ($\mathcal{D}(\mathcal{F}) \in \mathbb{A}$).

Examples: Clearly the set of t-intersecting (t-co-intersecting) families is upward (downward) closed. (A family $\mathcal{F}$ is said to be t-intersecting if for any two $F_1, F_2 \in \mathcal{F}$ $|F_1 \cap F_2| \geq t$, and a family $\mathcal{G}$ is said to be t-co-intersecting if $\overline{\mathcal{G}} = \{\overline{G} : G \in \mathcal{G}\}$ is t-intersecting or equivalently if for any two $G_1, G_2 \in \mathcal{G}$ $|\overline{G_1 \cup G_2}| \geq t$.)

Definition: Let $\mu_l(\mathbb{A})$ denote the set of all l-chain profile vectors of families in $\mathbb{A}$, $\langle \mu_l(\mathbb{A}) \rangle$ its convex hull, $\mathcal{E}_l(\mathbb{A})$ the extreme points of $\langle \mu_l(\mathbb{A}) \rangle$ and $E_l(\mathbb{A})$ the families from $\mathbb{A}$ with l-chain profile in $\mathcal{E}_l(\mathbb{A})$. Let furthermore $\mathcal{E}_l^*(\mathbb{A})$ denote the essential extreme points and $E_l^*(\mathbb{A})$ the corresponding families.

Theorem 1 *For any upward or downward closed set of families $\mathbb{A} \subseteq 2^{2^X}$ and for any $l \geq 1$*

$$\mathcal{E}_l^*(\mathbb{A}) \subseteq \mu_l(E_1^*(\mathbb{A})).$$

Note that equality does not always hold as the set of intersecting families, the family $\mathcal{F} = \{F \subseteq X : |F| > |X|/2\}$ and any $l > |X|/2$ shows.

Proof: The proof is the same for downward and upward closed sets of families, so we assume that $\mathbb{A}$ is upward closed.

Let $E_1^*(\mathbb{A}) = \{\mathcal{F}_1, \mathcal{F}_2, .., \mathcal{F}_m\}$ and let f^i the profile of $\mathcal{F}_i$, $f^{i,l}$ the l-chain profile of $\mathcal{F}^i$ and $f_\alpha^{i,l}$ its αth component.

We have to prove that the l-chain profile f^l of any family $\mathcal{F}$ in $\mathbb{A}$ can be dominated by a convex combination of the $f^{i,l}$s. Denote the profile of $\mathcal{F}$ by f. Clearly we have

$$f^l_\alpha \leq f_{\alpha_1}\binom{n-\alpha_1}{\alpha^*},$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l), \alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_l - \alpha_1)$. Inequality holds with equality for the f^i_αs and the $f^{i,l}_\alpha$s. The fact that the f^is are the essential extreme points of $\langle \mu_1(\mathbb{A})\rangle$ means that for some convex combination $c_i, i = 1, ..., m$

$$f \leq \sum_{i=1}^m c_i f^i.$$

But then

$$f^l_\alpha \leq f_{\alpha_1}\binom{n-\alpha_1}{\alpha^*} \leq \binom{n-\alpha_1}{\alpha^*}\sum_{i=1}^m c_i f^i_{\alpha_1} = \sum_{i=1}^m c_i f^{i,l}_\alpha,$$

which completes the proof. □

Since the convex hull of the profile polytope of the set of intersecting families were determined by P.L. Erdős, P. Frankl and G.O.H. Katona in [5], Theorem 1 provides the essential extreme points of the convex hull of the l-chain profile polytopes.

Definition: For any family $\mathcal{F}$ on a base set X let $\mathrm{conv}(\mathcal{F}) = \{G \subseteq X : \exists F, F' \in \mathcal{F}(F \subseteq G \subseteq F')\}$ denote its *convex closure.* $\mathcal{F}$ is said to be convex if $\mathcal{F} = \mathrm{conv}(\mathcal{F})$.

Definition: A set of families $\mathbb{A}$ is said to be convex closed if $\mathcal{F} \in \mathbb{A}$ implies $\mathrm{conv}(\mathcal{F}) \in \mathbb{A}$.

Example: The basic example for a convex closed set is the set of intersecting and co-intersecting families.

Theorem 2 *For any convex closed set of families $\mathbb{A} \subseteq 2^{2^X}$ and for any $l \geq 2$*

$$\mathcal{E}^*_l(\mathbb{A}) \subseteq \mu_l(E^*_2(\mathbb{A})).$$

Proof: The proof is analogous to that of Theorem 1, the inequality needed is

$$f^l_\alpha \leq f^2_{\alpha_1,\alpha_l}\binom{\alpha_l - \alpha_1}{\alpha^*}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$, $\alpha^* = (\alpha_2 - \alpha_1, \alpha_3 - \alpha_1, ..., \alpha_{l-1} - \alpha_1)$ and for families with essential extreme profile inequality holds with equality. □

Unfortunately the extreme points of neither the 1-chain, nor the 2-chain profile polytope are known for the set of intersecting and co-intersecting families.

3 The reduction method

In this section we describe our main tool in determining the l-chain profile polytope of families of sets with some given property. We call this tool the reduction method. In fact, this is not a new one. Most of the proofs of results already obtained went this way, what we observed that the method works for the l-chain case as well, and - what seems to us more important - in some cases it is enough to reduce the original problem to a maximal chain or to a pair of complementary chains (for the precise definitions, see below) instead of some more complicated structures what previous proofs ([4], [5], etc.) did mostly. We have to mention that Sali in [11] uses arguments involving reduction to a maximal chain, but we are not aware of any proofs with reduction to a pair of complementary chains.

Definition: For any l let $T^l_{\mathbf{C}}$ denote the following operator acting on the $\binom{n+1}{l}$-dimensional $\mathbf{R}$-space whose coordinates are indexed by l-tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_l)$, where $0 \leq \alpha_1 < \alpha_2 < \ldots < \alpha_l \leq n$.

$$T^l_{\mathbf{C}} : e \mapsto T^l_{\mathbf{C}}(e) \qquad \text{where} \qquad T^l_{\mathbf{C}}(e)_\alpha = \binom{n}{\alpha} e_\alpha.$$

Definition: For a family $\mathcal{F}$ on a base set X and a maximal chain $\mathbf{C}$ in X let $\mathcal{F}(\mathbf{C}) = \{F \in \mathcal{F} \cap \mathbf{C}\}$ and for a set of families $\mathbb{A}$ let $\mathbb{A}(\mathbf{C}) = \{\mathcal{F}(\mathbf{C}) : \mathcal{F} \in \mathbb{A}\}$.

Theorem 3 *For any set of families $\mathbb{A} \subseteq 2^{2^X}$ if the extreme points $e_1, e_2, ..., e_m$ of $\langle \mu_l(\mathbb{A}(\mathbf{C}))\rangle$ do not depend on the choice of $\mathbf{C}$, then*

$$\langle \mu_l(\mathbb{A})\rangle \subseteq \langle \{T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathcal{C}}(e_m)\}\rangle.$$

Proof: The modification of the argument in [5] works. Let $\mathcal{F}$ be an element of $\mathbb{A}$ with l-profile $f = (..., f_\alpha, ...)$. For $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ with $|F_i| = \alpha_i, i = 1, ..., l$ let $\underline{w}(\mathbf{F})$ be the vector of length $\binom{n+1}{l}$ with $1/n!$ in the αth component and 0 everywhere else (where n is the size of the base set). Consider the sum $\sum \underline{w}(\mathbf{F})$ for all pairs $(\mathbf{C}, \mathbf{F})$, where $\mathbf{C}$ is a maximal chain on X and $\mathbf{F} \subset \mathcal{F} \cap \mathbf{C}$ an l-chain. For a fixed $\mathbf{C}$ we have

$$\sum_{\mathbf{F} \in \mathcal{F}(\mathbf{C})} \underline{w}(\mathbf{F}) = \frac{1}{n!}(\text{profile of } \mathcal{F}(\mathbf{C})).$$

Here the profile of $\mathcal{F}(\mathbf{C})$ is a convex linear combination $\sum_{i=1}^m \lambda_i(\mathbf{C}) e_i$ of the e_is. Therefore

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}} \sum_{\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{C}} \frac{1}{n!} \sum_{i=1}^m \lambda_i(\mathbf{C}) e_i = \sum_{i=1}^m \frac{1}{n!} (\sum_{\mathbf{C}} \lambda_i(\mathbf{C})) e_i \tag{1}$$

holds where $\sum_{\mathbf{C}} \frac{1}{n!} \sum_{i=1}^m \lambda_i(\mathbf{C}) = 1$. Thus $\sum \underline{w}(\mathbf{F})$ is a convex linear combination of the e_is.

Summing in the other way around, we have

$$\sum_{\mathbf{C},\mathbf{F}} \underline{w}(\mathbf{F}) = \sum_{\mathbf{F}} \sum_{\mathbf{C}} \underline{w}(\mathcal{F}) =$$

$$\sum_{\mathbf{F}} (0, 0, ..., \frac{|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!}{n!}, ..., 0) = (..., \frac{f_\alpha}{\binom{n}{\alpha}}, ...), \quad (2)$$

since for a fixed $\mathbf{F} = \{F_1 \subset F_2 \subset ... \subset F_l\}$ there are exactly $|F_1|!(|F_2| - |F_1|)!...(|F_l| - |F_{l-1}|)!(n - |F_l|)!$ chains containing $\mathbf{F}$. So (1) and (2) give that this last vector is a convex linear combination of the e_is, which implies that f is the linear combination of $T^l_{\mathbf{C}}(e_1), ..., T^l_{\mathbf{C}}(e_m)$. $\square$

The structure of maximal chains are too simple, so using only them is not enough to determine the l-chain profile polytope of more difficult sets of families. But the proof of Theorem 3 works if we replace the chain by a pair of complementary maximal chains (i.e. for $i = 1, 2$ $\mathbf{C}^i = \{C^i_0, C^i_1, ..., C^i_n\}$ with $C^i_j = X \setminus C^{3-i}_{n-j} = \overline{C}^{3-i}_{n-j}$ for all $j = 0, 1, ..., n$). In the proof one has to write (instead of $\frac{1}{n!}$) $\frac{2}{(n!)}$ in the definition of $\underline{w}(\mathbf{F})$, and modify the definition of the T-operator to

$$(T^l_{\mathbf{C}_1,\mathbf{C}_2}(e))_\alpha = \frac{1}{d_\alpha}\binom{n}{\alpha},$$

where d_α is the number of α-type l-chains in the pair of complementing chains.

4 Applications

In this section we determine the profile polytope of some sets of families using the reduction method. In the first subsection the problem will be reduced to the case of the maximal chain while in the second subsection we will consider reduction to a pair of complement chains. Using the results obtained by the latter we will give examples when the extreme families of the l-profile polytope can really depend on l.

4.1 Reduction to the chain

Theorem 4 *For all $l \geq 1$ the extreme points of the convex hull of the l-chain profile vectors of convex families are the following:*

the all zero vector

$$\mathbf{0} = (0, ..., 0)$$

and for all $0 \leq i \leq j \leq n$ the vectors $v_{i,j}$

$$(v_{i,j})_\alpha = \begin{cases} \binom{n}{\alpha} \text{ if } & i \leq \alpha_1 < \alpha_l \leq j \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Proof: The vector $v_{i,j}$ is the l-profile of the family $\mathcal{F}_{i,j} = \{F \subseteq [n] : i \leq |F| \leq j\}$, which is convex.

On a chain any convex family must consist of some consecutive subsets of the chain. The theorem follows now from Theorem 3. □

Note that the set of convex families is not hereditary, therefore the extreme points (for the original profile vectors) need not be the ones obtained from the essential extreme points (in this case there is only one such, the profile of $2^{[n]}$) by changing some of the non zero components to zero - and as Theorem 4 shows, they are not those vectors, indeed.

Theorem 5 *For any $l \leq k$ the extreme points of the l-chain profile polytope of k-Sperner families are the following:*
the all zero vector

$$\mathbf{0} = (0, ..., 0, ...0)$$

and for all $l \leq z \leq k$ and $\beta = \{\beta_1, ..., \beta_z\}$ with $0 \leq \beta_1 < ... < \beta_z \leq n$ the vectors v_β

$$(v_\beta)_\alpha = \begin{cases} \binom{n}{\alpha} & \text{if } \alpha \subseteq \beta \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

The case $l = 1$ is a result of P.L. Erdős, P. Frankl and G.O.H. Katona [5].

Proof: It is trivial to see that these vectors are l-chain profiles of the corresponding levels, and they are convex linearly independent.

A k-Sperner family on a maximal chain consists of at most k sets, therefore its l-chain profile vector have ones in those components $\alpha = (\alpha_1, ..., \alpha_l)$ for which there is an element in the family with size α_i for all $i = 1, ..., l$. All these vectors are convex independent. Therefore they form the convex hull of the profile polytope on the chain, and Theorem 3 implies now Theorem 5. □

Applying Theorem 5 for the constant 1 weight function one gets

Corollary *For any $l \leq k$ if a family $\mathcal{F}$ on an n-element base set X does not contain a chain of length $k + 1$, then the number of l-chains in $\mathcal{F}$ is at most*

$$\max_{\beta \subset [0,n]; |\beta| = k} \sum_{\alpha \subseteq \beta; |\alpha| = l} \binom{n}{\alpha}.$$

Remarks.

- In the case $l = k$, even the very simple argument of [8] works. First we need a LYM-type inequality. To get this we double-count the pairs $(\mathbf{C}, \mathbf{F})$ where $\mathbf{C}$ is a maximal chain and $\mathbf{F}$ is an l-chain contained in $\mathbf{C}$. If we decompose the k-Sperner family into k antichains, then all sets of an **F** come from different antichains, and any **C** can contain at most k sets from our family, so by a standard calculation we obtain

$$\sum_\alpha \frac{f_\alpha}{\binom{n}{\alpha}} \leq \binom{k}{l}. \tag{5}$$

If $l = k$, then the RHS is 1, and we can finish the proof as follows

$$\sum_\alpha w_\alpha f_\alpha = \sum_\alpha \frac{f_\alpha w_\alpha}{\binom{n}{\alpha}}\binom{n}{\alpha} \leq \max_\alpha\{w_\alpha\binom{n}{\alpha}\}\sum_\alpha \frac{f_\alpha}{\binom{n}{\alpha}} \leq \max_\alpha\{w_\alpha\binom{n}{\alpha}\}, \tag{6}$$

where w_α is any non-negative weight function and the last inequality in (6) uses (5).

If $l = k$, then (5) gives the maximum number of k-chains that a k-Sperner family can contain. This is $\binom{n}{\alpha}$ where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_k)$ and the numbers $\alpha_1, \alpha_2 - \alpha_1, ...\alpha_k - \alpha_{k-1}$ differ by at most one. If $k+1$ divides n, then we get the uniqueness of the extremal system (take all $F \subseteq X$ with $|F| = \alpha_i$ for some $i = 1, ..., k$) automatically. If $k+1$ does not divide n, then we can lift (5) up to an AZ-type identity (for more details see [10], and for the original AZ-identity see the paper of Ahlswede and Zhang [1]) which will assure the uniqueness.

- With the notation of Section 3 Theorem 5 implies (if $\mathbb{S}_k$ denotes the set of k-Sperner families) $E_1(\mathbb{S}_k) = E_l(\mathbb{S}_k)$. But the bordering faces of the convex hulls $\langle\mu_1(\mathbb{S}_k)\rangle$ and $\langle\mu_l(\mathbb{S}_k)\rangle$ are "not the same". If $l = 1$ the convex hull determined by the faces given by the inequalities $0 \leq f_i \leq \binom{n}{i}$ and the LYM-inequality $\sum_i f_i/\binom{n}{i} \leq k$ (see [5]), while if $l > 1$ the hyperplanes given by $0 \leq f_\alpha \leq \binom{n}{\alpha}$ and the LYM-type inequality of (5) are bordering faces, but there are some additional ones, which can be seen by the following observation. Choosing $\binom{k}{l}$ αs in such a way that their union has size strictly larger than k and putting $f_\alpha = \binom{n}{\alpha}$ for these αs and 0 for the others, we obtain an essential extreme point of the polytope determined by the above inequalities, and which is not an l-chain profile of any k-Sperner families.

- In [9] the second author defined the distance of $\mathcal{F}$-free families. For two $\mathcal{F}$-free families $\mathcal{H}_1$ and $\mathcal{H}_2$ this is the number of copies of $\mathcal{F}$ in $\mathcal{H}_1 \cup \mathcal{H}_2$ (the edge (vertex) set of $\mathcal{H}_1 \cup \mathcal{H}_2$ is the union of the edge (vertex) sets of $\mathcal{H}_1$ and $\mathcal{H}_2$). For a collection $\mathbb{C}$ of forbidden families, the $\mathbb{C}$-free distance of two $\mathbb{C}$-free families is the sum of $\mathcal{F}$-free distances taken over all $\mathcal{F} \in \mathbb{C}$. If in the collection $\mathbb{C}$ we enumerate all kind of $k+1$-chains, then the $\mathbb{C}$-free families are the k-Sperner systems. The union of two such systems is a $2k$-Sperner family and of course all $2k$-Sperner families can be decomposed into two k-Sperner systems. So the maximum distance that two k-Sperner families can have is the maximum number of $k+1$-chains that a $2k$-Sperner family can contain, which by the Corollary of Theorem 5 is

$$\max_{\beta \subset [0,n]; |\beta| = 2k} \sum_{\alpha \subseteq \beta; |\alpha| = k+1} \binom{n}{\alpha}.$$

4.2 Reduction to a pair of complement chains

Theorem 6 *Let $n = 2m + 1$ and $k \le m + 1$. Then the extreme points of the 1-chain profile polytope (i.e. the ordinary profile polytope) of the set of complement-free k-Sperner families are the following vectors (indexed with a z-element ($z \le k$) subset α of $\{0, 1, 2, ..., n\}$ where $\alpha_i \in \alpha$ implies $n - \alpha_i \notin \alpha$)*

$$v_\alpha = (0, ..., 0, \binom{n}{\alpha_1}, 0, ..., 0, \binom{n}{\alpha_2}, 0, ..., 0, ..., 0, \binom{n}{\alpha_z}, 0, ..., 0).$$

Proof: By the version of Theorem 3 just mentioned in the first paragraph of this section, it is enough to prove the following

Lemma 1 *If $n = 2m + 1$ and $k \le m + 1$, then the extreme points of the profile polytope of complement-free k-Sperner families on a pair of maximal complement chains are the vectors with at most k non-zero components, where all the non-zero components are 2 (except for the first or the last component, if one of them is non-zero, it equals 1), and if the ith component is non-zero, then the $n - i$th component is zero.*

Proof of Lemma 1: If the non-zero components of such a vector are $\alpha_1, \alpha_2, ..., \alpha_z$ (satisfying the condition of the lemma), then the sets in the two chains with cardinality α_i for some $i = 1, ..., z$ form a complement-free k-Sperner family with the vector as profile.

Now let $\mathcal{F}$ be a complement-free k-Sperner family on a pair of complement chains $\mathbf{C}_1, \mathbf{C}_2$ with profile vector f. Let α be the set of indices of the non-zero components of f. Partition α into three subsets. Let CL (complete levels) denote the indices α_i with $f_{\alpha_i} = 2$ and furthermore CL contains 0 (n) provided $f_0 = 1$ ($f_n = 1$). Let furthermore CP (complementing pairs) denote the indices $\alpha_i \in \alpha$ with $n - \alpha_i \in \alpha$, and let $R = \alpha \setminus (CL \cup CP)$. Note that $CP \cap CL = \emptyset$, for otherwise $\mathcal{F}$ would not be complement-free. Now form two subsets α^1, α^2 of α in the following way. Put all indices in CL into both α^1 and α^2. For all pairs of indices $i, n - i$ in CP (note that these are really pairs, for n is odd) put one of the indices into α^1 and the other into α^2. Finally, choose α^1 or α^2 for all indices of R in such a way, that $|\alpha^1| \le k$ and $|\alpha^2| \le k$ hold. (This is possible, for $\mathcal{F}$ is k-Sperner, therefore $|\alpha| \le 2k$.) Now let f^i, $i = 1, 2$ the following vectors.

$$f^i_j = \begin{cases} 2 \quad (1) \text{ if } \; j \neq 0, n \quad (j = 0, n) \quad j \in \alpha^i \\ 0 \qquad\qquad\qquad \text{otherwise.} \end{cases} \tag{7}$$

By the facts that both f^is are of the form of the statement of the lemma and $f = \frac{1}{2} f^1 + \frac{1}{2} f^2$, furthermore f^1 and f^2 are both profile vectors of k-Sperner families on $\mathbf{C}_1, \mathbf{C}_2$, the proof is completed. $\Box^{\text{lemma}}$ $\Box^{\text{theorem}}$

The case of complement-free families is very analogous (and even simpler), therefore we just sketch the proof.

Theorem 7 *The extreme points of the convex hull of the 1-chain profile vectors of complement-free families are the vectors corresponding to the families consisting of*

(i) *a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I, if n is odd,*

(ii) *a set I of levels with the property that the ith and the $n-i$th levels cannot be both in I and possibly half of the sets with size $n/2$ one from each pair of complementary sets, if n is even.*

Proof: It is easy to see that it is enough to solve the problem reduced to a pair of maximal complement chains. There the statement holds, since there a complement-free family can contain at most two sets out of the four with size i or $n-i$, and the vectors $(1,1),(0,1),(1,0)$ are convex combinations of the vectors $(2,0),(0,2),(0,0)$. □

Theorem 1 (and 2) states that for a certain class of sets of families all candidates for the families with essential extreme l-chain profiles are among the families with essential extreme 1-chain (2-chain) profile. Theorem 5 states, that for k-Sperner families the above statement is true for all extreme profiles (not only for essential extreme profiles). It seems natural to conjecture (with the notation of Section 2) that for all set of families $\mathbb{A}$ and $l>1$ $E_l(\mathbb{A}) \subseteq E_1(\mathbb{A})$ and/or $E_l^*(\mathbb{A}) \subseteq E_1^*(\mathbb{A})$. But this is false. Here we present two counterexamples.

The first example is based on Theorem 6. Note that the families corresponding to the extreme points cannot contain sets of size i and $n-i$ at the same time. Hence all 2-chain profiles of those families have 0 in their components indexed with the sets $\{i, n-i\}$, and therefore all their convex combinations have 0 in those components. But a pair of subsets in inclusion with size i and $n-i$ is of course a complement-free k-Sperner family (if $k \geq 2$), and its profile is not in the convex hull of the above-mentioned vectors.

The second example is absolutely analogous to the first one. According to Theorem 7 in the extremal families of the set of complement-free systems there are no pairs of sets in inclusion with size i and $n-i$ (so the corresponding component is 0 in any convex combinations), but there are complement-free families with such pairs.

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