Almost intersecting families of sets

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Abstract

Let us write $\mathcal{D}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \cap G = \emptyset\}$ for a set G and a family $\mathcal{F}$. Then a family $\mathcal{F}$ of sets is said to be $(\leq l)$-almost intersecting (l-almost intersecting) if for any $F \in \mathcal{F}$ we have $|\mathcal{D}_{\mathcal{F}}(F)| \leq l$ ($|\mathcal{D}_{\mathcal{F}}(F)| = l$). In this paper we investigate the problem of finding the maximum size of an $(\leq l)$-almost intersecting (l-almost intersecting) family $\mathcal{F}$.

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1 Introduction

We will use standard notation: $[n]$ denotes the set of the first n positive integers $\{1, 2, ..., n\}$ and for any set X we write $\binom{X}{k}$ for the family of all k element subsets of X and 2^X for the power set of X. For a family $\mathcal{F} \subseteq 2^X$ we write $\overline{\mathcal{F}} = \{\overline{F} : F \in \mathcal{F}\}$. We will say that a family $\mathcal{F}$ is *intersecting* if $F, G \in \mathcal{F}$ implies $F \cap G \neq \emptyset$. Moreover, $\mathcal{F}$ is *trivially intersecting* if $\bigcap_{F \in \mathcal{F}} F \neq \emptyset$ and a family $\mathcal{F}$ is called *Sperner* if there exist no $F, G \in \mathcal{F}$ with $F \subsetneq G$.

One of the basic results about extremal set families is due to Erdős, Ko and Rado [7] and states that if $2k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then the size of $\mathcal{F}$ is at

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most $\binom{n-1}{k-1}$, furthermore if $2k < n$, then equality holds if and only if $\mathcal{F}$ is a trivially intersecting family. The non-uniform version (i.e. when sets in $\mathcal{F}$ need not to have equal size) of the above theorem is also due to Erdős, Ko and Rado. However, it is rather an easy exercise to prove that any intersecting family $\mathcal{G} \subseteq 2^{[n]}$ can be extended to an intersecting family $\mathcal{G}'$ of size 2^{n-1} and there exists no intersecting family of larger size.

These theorems attracted the attention of many researchers. Several generalizations have been proved, the intersecting condition has been relaxed or strengthened in many ways. One relaxation is to allow some fixed number of disjoint pairs formed by members of the family $\mathcal{F}$ [1, 5, 8]. In this paper we consider families $\mathcal{F}$ where for any $F \in \mathcal{F}$ there are at most a fixed number of sets disjoint from F. More precisely, for any set G and family $\mathcal{F}$ let $\mathcal{D}_{\mathcal{F}}(G) = \{F \in \mathcal{F} : F \cap G = \emptyset\}$ be the subfamily of the sets disjoint from G. We say that a family $\mathcal{F}$ is $(\leq l)$-*almost intersecting* if for every set $F \in \mathcal{F}$ we have $|\mathcal{D}_{\mathcal{F}}(F)| \leq l$, and we say that a family $\mathcal{F}$ is l-*almost intersecting* if for every set $F \in \mathcal{F}$ we have $|\mathcal{D}_{\mathcal{F}}(F)| = l$. We will address the problem of finding the largest size of an $(\leq l)$-almost intersecting (l-almost intersecting) family $\mathcal{F} \subset 2^{[n]}$ both in the uniform and in the non-uniform case. Clearly, $l = 0$ gives back the original problem of Erdős, Ko and Rado.

The rest of the paper is organized as follows: in Section 2 we consider k-uniform l-almost intersecting families. Among others we prove the following conjecture for $k = 2$.

Conjecture 1.1. *For any k there exists $l_0 = l_0(k)$ such that if $l \geq l_0$ and $\mathcal{F}$ is a k-uniform l-almost intersecting family, then $|\mathcal{F}| \leq (l+1)\binom{2k-2}{k-1}$.*

The following construction shows that if true, Conjecture 1.1 is sharp: $\{F \cup \{i\} : F \in \binom{[2k-2]}{k-1}, i \in \{2k-1, 2k, ..., 2k+l-1\}\}$.

In Section 3 we consider non-uniform l-almost intersecting families and solve the problem completely if l is 1 or 2.

In Section 4 we prove that for any fixed k and l if $n \geq n_0(k,l)$, then the largest k-uniform $(\leq l)$-almost intersecting family is a trivially intersecting family. Determining the smallest possible n_0 remains open except for the case $l = 1$ for which we prove the minimum of $n_0(k,1)$ is $2k+2$.

Section 5 deals with non-uniform $(\leq l)$-almost intersecting families. We settle the problem for $l = 1, 2$. For larger l we conjecture the following.

Conjecture 1.2. *For any positive integer $l \geq 2$ there exists $n_0 = n_0(l)$ such that if $n \geq n_0$ and $\mathcal{F} \subset 2^{[n]}$ is an $(\leq l)$-almost intersecting family, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathcal{F}$ is the family of sets of size at least $n/2$ and (if n is odd) the sets of size $\lfloor n/2 \rfloor$ not containing a fixed element of $[n]$.

2 Restrictive case - Uniform families

In this section we investigate k-uniform l-almost intersecting families. The following notion and Theorem 2.1 will play a very important role in our proofs. The collection of pairs of sets $(A_i, B_i)_{i=1}^m$ are said to form a *cross-intersecting* family if for any $1 \leq i, j \leq m$ we have $A_i \cap B_j = \emptyset$ if and only if $i = j$.

The main theorem about cross-intersecting families of pairs is the following result.

Theorem 2.1 (Bollobás [3]). *If the pairs $(A_i, B_i)_{i=1}^m$ form a cross-intersecting family, then the following inequality holds:*

$$\sum_{i=1}^{m} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1,$$

in particular if $|A_i| \leq k$ and $|B_i| \leq l$ for all $1 \leq i \leq m$, then $m \leq \binom{k+l}{k}$ and equality holds if and only if the pairs are all possible partitions into sets of size k and l of some $(k+l)$-set X.

The following easy corollary settles Conjecture 1.1 when $l = 1$.

Corollary 2.2. *If $\mathcal{F}$ is a k-uniform 1-almost intersecting family, then $|\mathcal{F}| \leq \binom{2k}{k}$ and equality holds if and only if $\mathcal{F} = \binom{X}{k}$ with $|X| = 2k$.*

Proof. For any k-uniform 1-almost intersecting family $\mathcal{F} = \{F_1, F_2, ..., F_m\}$, let $A_i = F_i$ and let B_i denote the only set in $\mathcal{F}$ which is disjoint from F_i. Then the pairs $(A_i, B_i)_{i=1}^m$ form a cross-intersecting family and we are done by Theorem 2.1. □

The next lemma shows that for any positive integers k and l there exists an upper bound on the size of a k-uniform l-almost intersecting family which is independent of the size of the ground set.

Lemma 2.3. *For any $k, l \in \mathbb{N}$ where $l > 0$, if $\mathcal{F}$ is a k-uniform l-almost intersecting family, then we have $|\mathcal{F}| \leq l\binom{2kl}{kl}$.*

Proof. Let $\mathcal{F} = \{F_1, F_2, ..., F_m\}$ and let $I_j = \{i \in [m] : \mathcal{D}_{\mathcal{F}}(F_j) = \mathcal{D}_{\mathcal{F}}(F_i)\}$. Note that for any $j \leq m$ we have $|I_j| \leq l$ as otherwise any $G \in \mathcal{D}_{\mathcal{F}}(F_j)$ would be disjoint from at least $l+1$ sets of $\mathcal{F}$, a contradiction. Let $J \subseteq [m]$ be a set of indices so that $I_j \neq I_{j'}$ for any $j, j' \in J$ and $|J| \geq \frac{m}{l}$. For every $j \in J$ we define the sets $A_j := \cup_{i \in I_j} F_i$, $B_j := \cup_{G \in \mathcal{D}_{\mathcal{F}}(F_j)} G$. Clearly they form a cross-intersecting family and $|A_j|, |B_j| \leq kl$, hence we are done by Theorem 2.1. □

We finish this section by settling the case $k = 2$ of Conjecture 1.1. Instead of talking about 2-uniform families we state our result in the language of graphs.

Theorem 2.4. *If G is an l-almost intersecting graph with no isolated vertices, then G has at most $2l+2$ edges and the unique extremal graph is $K_{2,l+1}$ provided $l \neq 1, 3, 5, 6$. For $l = 1, 3, 6$ the unique extremal graphs are K_4, K_5, K_6 respectively, and for $l = 5$ there are two extremal graphs: $K_{2,6}$ and the complement of a matching on 6 vertices.*

Proof. If G is not connected, then there is a cut C_1, C_2 of $V(G)$ with $e(C_1, C_2) = 0$, $e(C_1), e(C_2) > 0$. But then by the l-almost intersecting property of G we obtain $e(C_1), e(C_2) \leq l$ and thus $e(G) \leq 2l$. Hence we can suppose G is connected. Let e be the number of edges and n the number of vertices. If there are exactly l edges disjoint from any fixed edge, then for any edge (u, v) we have $d(u) + d(v) - 1 = e - l$. Thus for any vertex all its neighbors have fixed degree. We distinguish two cases.

Case 1. G is d-regular for an integer d.

Then clearly $e = 2d + l - 1$ and $dn = 2e = 2(2d + l - 1)$, hence $(n - 4)d = 2l - 2$. Since d is the degree of all vertices in G, it is at most $n-1$. Hence $2l-2 = (n-4)d \geq (d-3)d = (d-\frac{3}{2})^2 - \frac{9}{4}$ and thus $d - \frac{3}{2} \leq \sqrt{2l + \frac{1}{4}}$. Then $e = 2d+l-1 \leq l+2\sqrt{2l + \frac{1}{4}}+2$, which is strictly less than $2l + 2$ if $l > 8$.

If $d = n - 1$, then $G = K_n$ and every edge is disjoint from $\binom{n-2}{2}$ other edges. As by the above, we must have $\binom{n-2}{2} = l \leq 8$, thus we obtain $n \leq 7$ and K_4, K_5 and K_6 do contain more edges than $K_{2,2}, K_{2,4}$ and $K_{2,7}$ respectively. Otherwise $d \leq n-2$ can be supposed, and by repeating the previous calculation we get $e \leq l + 2\sqrt{2l - 1} + 1$, which is strictly less than $2l + 2$ if $l > 5$. For $l = 5$ there is equality here, this gives the two extremal graphs.

$2l - 2 = d(d - 2)$ is impossible for $l < 5$, hence we can suppose $d \leq n - 3$. The simple calculation used in the previous paragraphs gives $e \leq l + 2\sqrt{2l - \frac{7}{4}}$ in this case, which is always strictly less than $2l + 2$.

Case 2. G is not regular and thus the degrees give a 2-coloring of G.

Let us denote the two color classes by A and B, their cardinality by a and b, and the degrees by d_A and d_B. Then $e = ad_A = bd_B = \frac{1}{2}(ad_A + bd_B) = l + d_A + d_B - 1$. Clearly $d_A \leq b$ and $d_B \leq a$, hence we get $ab - a - b \leq l - 1$ and $e \leq ab$.

Let $a \leq b$ (in fact $a = b$ happens only if G is regular, hence we can assume $a < b$). Then $a = 1$ means all the edges intersect and $a = 2$ gives at most $2(l+1)$ edges, with equality only in the case of $K_{2,l+1}$. Hence we can suppose $a \geq 3$. By $a + b = n$ we obtain $ab \geq 3(n - 3)$, thus $2n - 9 \leq ab - a - b \leq l - 1$ and $n \leq \frac{l}{2} + 5$. If $l > 4$ then it implies $e \leq ab < 2l + 2$. If $l \leq 4$ then the inequalities $a + b \leq 7$ and $b > a \geq 3$ give $a = 3$ and $b = 4$, but then $d_A = 4$ and $d_B = 3$ is necessary by the biregularity of G. Thus we must have $G = K_{3,4}$, hence $l = 6$, a contradiction. $\square$

3 Restrictive case - Non-uniform families

In this section we consider l-almost intersecting families that are not necessarily uniform. We start our investigations with a useful definition and a proposition valid for arbitrary l. For any family $\mathcal{F}$ of sets, the *comparability graph* $G(\mathcal{F})$ is the graph with vertex set $V(G) = \mathcal{F}$ and edge set $E(G) = \{(F_1, F_2) : (F_1 \subsetneq F_2) \vee (F_2 \subsetneq F_1)\}$.

Proposition 3.1. *If $\mathcal{F}$ is an l-almost intersecting family, then all connected components of $G(\mathcal{F})$ have size at most l.*

Proof. For any pair $F_1, F_2 \in \mathcal{F}$ of sets with $F_1 \subset F_2$, we have $\mathcal{D}_{\mathcal{F}}(F_1) = \mathcal{D}_{\mathcal{F}}(F_2)$ as all sets disjoint from F_2 are disjoint from F_1 as well and $|\mathcal{D}_{\mathcal{F}}(F)| = l$ for any $F \in \mathcal{F}$. We obtain that if F, F' lie in the same component of $G(\mathcal{F})$, then we have $\mathcal{D}_{\mathcal{F}}(F) = \mathcal{D}_{\mathcal{F}}(F')$. Therefore if a component C of $G(\mathcal{F})$ consisted of more than l vertices, then $|\mathcal{D}_{\mathcal{F}}(H)| > l$ would hold for any set $H \in \mathcal{D}_{\mathcal{F}}(F)$ with $F \in C$. $\square$

As a special case of Proposition 3.1 we obtain that an l-almost intersecting family does not contain an *l-fork* (a family of $l+1$ sets $F_0, F_1, ..., F_l$ with $F_0 \subsetneq F_i$ for all $1 \leq i \leq l$), hence the following theorem of De Bonis and Katona can be used (a weaker version was obtained earlier by Thanh [13]).

Theorem 3.2 (De Bonis, Katona [6])**.** *If a family $\mathcal{F} \subseteq 2^{[n]}$ does not contain an r-fork, then $|\mathcal{F}| \leq (1 + \frac{2r}{n} + O(\frac{1}{n^2}))\binom{n}{\lfloor n/2 \rfloor}$.*

Corollary 3.3. *If $\mathcal{F}$ is an l-almost intersecting family, then $|\mathcal{F}| \leq (1+\frac{2l}{n}+O(\frac{1}{n^2}))\binom{n}{n/2}$.*

As we will see, this bound is asymptotically tight if $l = 1$. If $l = 2$ the bound is off by a factor of 2 as shown by Theorem 3.14 and we conjecture that it is even further from the truth for larger values of l, but this is the best bound we have at the moment.

Now let us consider the case $l = 1$.

Theorem 3.4. *If $\mathcal{F} \subseteq 2^{[n]}$ is a 1-almost intersecting family, then*

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{n/2} & \text{if } n \text{ is even} \\ 2\binom{n-1}{\lfloor n/2 \rfloor - 1} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathcal{F} = \binom{[n]}{n/2}$ provided n is even and $\mathcal{F} = \{F \in \binom{[n]}{\lfloor n/2 \rfloor} : x \in F\} \cup \{F \in \binom{[n]}{\lceil n/2 \rceil} : x \notin F\}$ for some fixed $x \in [n]$ provided n is odd.

Proof. Let $\mathcal{F}$ be a 1-almost intersecting family of maximum size. By Proposition 3.1 we know that $\mathcal{F}$ is Sperner and thus if n is even, we are done by Sperner's theorem [12].

Suppose n is odd. Clearly, $\mathcal{F}$ consists of disjoint pairs. Let us consider the subfamily $\mathcal{F}'$ that consists of the smaller set from each pair and let $\mathcal{F}'' = \mathcal{F} \setminus \mathcal{F}'$. Then $\mathcal{F}'$ is Sperner, intersecting and $2|F| \leq n$ for all $F \in \mathcal{F}'$. We use a theorem of Bollobás [4] that deals with families of this type. We only state the result for odd n.

Theorem 3.5 (Bollobás [4])**.** *If n is odd and $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting Sperner family with the property that $2|F| \leq n$ holds for all $F \in \mathcal{F}$, then*

$$|\mathcal{F}| \leq \binom{n-1}{\lfloor n/2 \rfloor - 1},$$

and equality holds if and only if $\mathcal{F} = \{F \in \binom{[n]}{\lfloor n/2 \rfloor} : x \in F\}$ for some fixed $x \in [n]$.

As $|\mathcal{F}'| = |\mathcal{F}''|$ it follows that $|\mathcal{F}| \leq 2\binom{n-1}{\lfloor n/2 \rfloor - 1}$ and $\mathcal{F}' = \{F \in \binom{[n]}{\lfloor n/2 \rfloor} : x \in F\}$ for some fixed $x \in [n]$. If $\mathcal{F}'' = \overline{\mathcal{F}'} = \{\overline{F}' : F' \in \mathcal{F}'\}$, then we are done. Otherwise there exists $F'' \in \mathcal{F}''$ with $|F''| \leq n/2$. Then the family $\mathcal{F}^* = (\mathcal{F}' \setminus \{F'\}) \cup \{F''\}$ (where F' is the unique set in $\mathcal{F}'$ which is disjoint from F'') satisfies the conditions of Theorem 3.5 and thus $\mathcal{F}^* = \{F \in \binom{[n]}{\lfloor n/2 \rfloor} : y \in F\}$ for some fixed $y \in [n]$ but both $x = y$ and $x \neq y$ is impossible. □

Let us continue with the case $l = 2$ by defining 2-almost intersecting families.

Construction 3.6. If $n = 2k+2$, then the following 2-almost intersecting family has size $2\binom{2k}{k} = (\frac{1}{2} + o(1))\binom{n}{n/2}$: let $\mathcal{G}_1, \mathcal{G}_2$ be a partition of $\binom{[2k]}{k}$ such that $G \in \mathcal{G}_1$ if and only if $\overline{G} \in \mathcal{G}_2$. Then $\mathcal{G} = \mathcal{G}_1 \cup \{G \cup \{2k+1\} : G \in \mathcal{G}_1\} \cup \mathcal{G}_2 \cup \{G \cup \{2k+2\} : G \in \mathcal{G}_2\}$ possesses the required properties.

Similarly, if $n = 2k+1$, then the following family is 2-almost intersecting: let $\mathcal{G}_1 = \{G \in \binom{[2k-1]}{k-1} : x \in G\}, \mathcal{G}_2 = \{G \in \binom{[2k-1]}{k} : x \notin G\}$ for some fixed $x \in [n]$ and define $\mathcal{G} = \mathcal{G}_1 \cup \{G \cup \{2k\} : G \in \mathcal{G}_1\} \cup \mathcal{G}_2 \cup \{G \cup \{2k+1\} : G \in \mathcal{G}_2\}$. Then $\mathcal{G}$ possesses the required property and has size $4\binom{2k-2}{k-2} = (\frac{1}{2} + o(1))\binom{n}{n/2}$.

In the remainder of this section we show that these constructions are best possible. Let $\mathcal{F} \subset 2^{[n]}$ be a 2-almost intersecting family and let us write $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2^U \cup \mathcal{F}_2^L$ where $\mathcal{F}_2^U = \{F \in \mathcal{F} : \exists F' \in \mathcal{F}, F' \subsetneq F\}, \mathcal{F}_2^L = \{F \in \mathcal{F} : \exists F' \in \mathcal{F}, F' \supsetneq F\}$ and $\mathcal{F}_1 = \mathcal{F} \setminus (\mathcal{F}_2^U \cup \mathcal{F}_2^L)$.

Proposition 3.7. *If $F \in \mathcal{F}_1$, then for any $G \supseteq \overline{F}$ we have $G \notin \mathcal{F}$.*

Proof. If such a G was in $\mathcal{F}$, then the two sets in $\mathcal{F}$ disjoint from G would be subsets of F, thus at least one of them should be a proper subset of F. That would contradict $F \in \mathcal{F}_1$. $\square$

Proposition 3.8. $\mathcal{F}_2^L \cap \mathcal{F}_2^U = \emptyset$.

Proof. The component of a set $F \in \mathcal{F}_2^L \cap \mathcal{F}_2^U$ in the comparability graph $G(\mathcal{F})$ would have size at least 3 contradicting Proposition 3.1. $\square$

Proposition 3.9. *For any $F \in \mathcal{F}_2^L$ ($F \in \mathcal{F}_2^U$) there exists exactly 1 set $F' \in \mathcal{F}_2^U$ ($F' \in \mathcal{F}_2^L$) with $F \subsetneq F'$ ($F \supsetneq F'$).*

Proof. This is the $l = 2$ special case of Proposition 3.1. $\square$

Corollary 3.10. *If $F \in \mathcal{F}_2^L$, then for any $G \supseteq \overline{F}$ we have $G \notin \mathcal{F}$.*

Proof. Any such G would contain the two sets in $\mathcal{F}$ that are disjoint from F' with $F \subsetneq F'$. $\square$

Proposition 3.11. *For any 2-almost intersecting family $\mathcal{F} \subset 2^{[n]}$ there exists another such family $\mathcal{G}$ with $|\mathcal{F}| = |\mathcal{G}|$ such that*
(i) *$G \in \mathcal{G}_2^U$ implies $\overline{G} \in \mathcal{G}_2^U$,*
(ii) *for any $G_1, G_2 \in \mathcal{G}$ with $G_1 \subsetneq G_2$ we have $|G_2 \setminus G_1| = 1$.*

Proof. If $\mathcal{F}_2^U \cup \mathcal{F}_2^L = \emptyset$ then $\mathcal{G} = \mathcal{F}$ satisfies **(i)** and **(ii)**. Otherwise let $F_1, F_2 \in \mathcal{F}$ with $F_1 \subsetneq F_2$. Then there exist 2 distinct sets $F', F'' \in \mathcal{F}$ which are disjoint from F_2 and therefore from F_1 and all other sets in $\mathcal{F}$ meet both F_1, F_2. Thus replacing F_1 with any set G satisfying $F_1 \subset G \subset F_2, |F_2 \setminus G| = 1$ will not violate the 2-almost intersecting property of the new family (as G is disjoint from F' and F'' and as a superset of F_1 meets all other sets of the family). By repeating this operation we can obtain a family satisfying **(ii)**.

Suppose that in the above situation we have $F', F'' \neq \overline{F}_2$. Then either of these sets can be replaced with $\overline{F}_2$ without violating the the 2-almost intersecting property of the new family. By repeating this operation we can obtain a family satisfying **(i)**. $\square$

We will call a 2-almost intersecting family *good* if it satisfies **(i)** and **(ii)** of the above proposition. If $\mathcal{G}$ is good, then for any $G \in \mathcal{G}_2^U$ there exist $G_1, G_2 \in \mathcal{G}$ such that $G_1 \subsetneq G, G_2 \subsetneq \overline{G}, |G \setminus G_1| = |\overline{G} \setminus G_2| = 1$. Let x be the only element of $G \setminus G_1$ and y be the only element of $\overline{G} \setminus G_2$ and let $G^* = G \setminus \{x\} \cup \{y\} = G_1 \cup \{y\}$.

Proposition 3.12. *Let $\mathcal{G} \subseteq 2^{[n]}$ be a good 2-almost intersecting family. Then for any $G \in \mathcal{G}_2^U$ the sets G_1, G_2, G^* defined as above satisfy the following:*

(i) $G^* \notin \mathcal{G}$,
(ii) $\overline{G^*} \notin \mathcal{G}$,
(iii) *the only set in* $\mathcal{G} \cup \overline{\mathcal{G}}$ *contained in* G^* *is* G_1,
(iv) *the only set in* $\mathcal{G} \cup \overline{\mathcal{G}}$ *containing* G^* *is* $\overline{G}_2$,
(v) $\{G^* : G \in \mathcal{G}_2^U\}$ *is a Sperner family.*

Proof. The statements **(i)**,**(ii)** follow from Proposition 3.9 used for G_1 and G_2. A set $G' \in \mathcal{G}$ contradicting **(iii)** would be a third set in $\mathcal{G}$ which is disjoint from G_2, while for a set $G' \in \overline{\mathcal{G}}$ contradicting **(iii)** its complement $\overline{G'}$ would contradict Proposition 3.9 for G_2. The statement of **(iv)** follows similarly using G_1 in place of G_2.

Finally, **(v)** follows from the fact that $G'^* \subsetneq G^*$ would imply that G_2 and G_1' are disjoint and therefore there would be three sets in $\mathcal{G}$ which are disjoint from G_2. □

Lemma 3.13. *Let* $\mathcal{G} \subseteq 2^{[n]}$ *be a good 2-almost intersecting family. Then the following inequality holds:*

$$\sum_{G\in\mathcal{G}} \frac{2}{\binom{n}{|G|}} - \sum_{G\in\mathcal{G}_1} \frac{2}{(n-|G|)\binom{n}{|G|}} -$$

$$\sum_{G\in\mathcal{G}_2^L} \left(\frac{2}{(n-|G|)\binom{n}{|G|}} + \frac{1}{(n-|G|)(n-|G|-1)\binom{n}{|G|}} \right) -$$

$$\sum_{G\in\mathcal{G}_2^U} \left(\frac{1}{(n-|G|)\binom{n}{|G|}} + \frac{1}{|G|\binom{n}{|G|}} - \frac{1}{|G|(n-|G|)\binom{n}{|G|}} \right) \leq 1. \tag{1}$$

Proof. First note that $\emptyset \notin \mathcal{G}$ as there would be only one other set in $\mathcal{G}$, which could not be disjoint from two sets.

Let us consider the pairs $(G, \mathcal{C})$ with $G \in \mathcal{G}$, $\mathcal{C}$ is a maximal chain in $[n]$ and $G \in \mathcal{C}$. For any set G there are $|G|!(n-|G|)!$ chains containing G. Thus the number of pairs is exactly $\sum_{G\in\mathcal{G}} |G|!(n-|G|)!$. On the other hand by Proposition 3.8 every chain $\mathcal{C}$ may contain at most 2 sets from $\mathcal{G}$, thus the number of such pairs is $n! + c_2 - c_0$ where c_i is the number of chains containing i sets from $\mathcal{G}$. By Proposition 3.9 we have $c_2 = \sum_{G\in\mathcal{G}_2^L} |G|!(n-|G|-1)!$.

We would like to get a lower bound on c_0. Let us consider the set $\mathcal{S}_{\overline{G}}$ of chains that contain the complement $\overline{G}$ of a fixed set $G \in \mathcal{G}_1 \cup \mathcal{G}_2^L$. By Proposition 3.7 and Corollary 3.10, if $G \in \mathcal{G}_1 \cup \mathcal{G}_2^L$, there can be no $G' \in \mathcal{G}$ with $G' \supseteq \overline{G}$. A k-subset of $\overline{G}$ is contained in $|G|!k!(n-|G|-k)!$ chains that go through $\overline{G}$. Thus, if $G \in \mathcal{G}_1$, as the empty set cannot be in $\mathcal{G}$ and there are exactly 2 sets in $\mathcal{G}$ which are disjoint from G, we obtain that there are at least $(n-|G|-2)\cdot(n-|G|-1)!|G|!$ chains in $\mathcal{S}_{\overline{G}}$ that do not contain any set from $\mathcal{G}$. If $G \in \mathcal{G}_2^L$, then we know that the 2 sets in $\mathcal{G}$ contained in $\overline{G}$ have size $n-|G|-1$ and $n-|G|-2$ and thus the number of chains in $\mathcal{S}_{\overline{G}}$ that avoid $\mathcal{G}$ is $|G|!((n-|G|)!-(n-|G|-1)!-(n-|G|-2)!)$. By definition there do not exist 2 sets G, G' in $\mathcal{G}_1 \cup \mathcal{G}_2^L$ with $G \subsetneq G'$, thus we have $\mathcal{S}_{\overline{G}} \cap \mathcal{S}_{\overline{G'}} = \emptyset$ for any distinct $G, G' \in \mathcal{G}_1 \cup \mathcal{G}_2^L$.

Finally, let us consider sets $G \in \mathcal{G}_2^U$. By Proposition 3.12 (and using the definitions preceding the proposition) we get that any chain that contains G^* but contains neither G_1 nor $\overline{G}_2$ avoids $\mathcal{G} \cup \overline{\mathcal{G}}$. Therefore these chains are different from all chains in $\cup_{G \in \mathcal{G}_1 \cup \mathcal{G}_2^L} \mathcal{S}_{\overline{G}}$ and contain no sets from $\mathcal{G}$. The number of such chains for one fixed $G \in \mathcal{G}_2^U$ is $(|G|!-(|G|-1)!)((n-|G|)!-(n-|G|-1)!)$. Note that by Proposition 3.12 **(v)** we have $\mathcal{S}_{G'^*} \cap \mathcal{S}_{G''^*} = \emptyset$ for any $G', G'' \in \mathcal{G}_2^U$.

Adding the above observations together we obtain

$$\begin{aligned}\sum_{G \in \mathcal{G}} |G|!(n-|G|)! \leq n! &+ \sum_{G \in \mathcal{G}_2^L} |G|!(n-|G|-1)! \\ &- \sum_{G \in \mathcal{G}_1} (n-|G|-2) \cdot |G|!(n-|G|-1)! \\ &- \sum_{G \in \mathcal{G}_2^L} |G|!\left((n-|G|)!-(n-|G|-1)!-(n-|G|-2)!\right) \\ &- \sum_{G \in \mathcal{G}_2^U} (|G|!-(|G|-1)!)((n-|G|)!-(n-|G|-1)!). \qquad (2)\end{aligned}$$

Rearranging and dividing by $n!$ yield the statement of the lemma. $\square$

Clearly, the main term of the LHS of (1) is the first term $\sum_{G \in \mathcal{G}} \frac{2}{\binom{n}{|G|}}$ and all other terms are negligible compared to this, thus Construction 3.6 is asymptotically best possible. We need a little more work to prove the exact bound.

Theorem 3.14. *Let $\mathcal{F} \subseteq 2^{[n]}$ be a 2-almost intersecting family, then*

$$|\mathcal{F}| \leq \begin{cases} 2\binom{n-2}{\frac{n-2}{2}} & \text{if } n \text{ is even} \\ 4\binom{n-3}{\lfloor n/2 \rfloor - 2} & \text{if } n \text{ is odd,} \end{cases}$$

and this bound is best possible as shown by Construction 3.6.

Proof. Without loss of generality we may assume that $\mathcal{F}$ is good. Then we know that sets in $\mathcal{F}_2^U \cup \mathcal{F}_2^L$ come in pairs with sizes differing by 1. Let us consider the summands in (1). Let a_k denote the sum of all summands from sets in $\mathcal{F}_1$ of size k and let b_k denote the sum of all summands from pairs in $\mathcal{F}_2^U \cup \mathcal{F}_2^L$ such that the smaller set in the pair has size k. Clearly, if $m = \min\{a_k, \frac{b_k}{2} : 1 \leq k \leq n-1\}$, then $|\mathcal{F}| \leq \frac{1}{m}$. For convenience we rather work with

$$a'_k = n!a_k = 2k!(n-k)! - 2k!(n-k-1)!$$

and

$$\begin{aligned} b'_k = n!b_k &= 2k!(n-k)! + 2(k+1)!(n-k-1)! - 2k!(n-k-1)! - k!(n-k-2)! \\ &- (k+1)!(n-k-2)! - k!(n-k-1)! + k!(n-k-2)! \end{aligned}$$

First we claim that if n is even, the minimum of a'_k over k is $a'_{n/2}$ and if n is odd, it is $a'_{\lfloor n/2 \rfloor} = a'_{\lceil n/2 \rceil}$. Indeed, let

$$A_k = a'_{k+1} - a'_k = 2k!(n-k-2)![(k+1)(n-k-2) - (n-k-1)^2]$$

and observe that the expression in the brackets is quadratic in k. Furthermore if n is odd, one of its roots is between $n-3$ and $n-2$ and the other is $\lfloor n/2 \rfloor$. If n is even, it has a root between $n/2 - 1$ and $n/2$. This proves our claim.

Next we claim that the same holds for each b'_k, i.e. if n is even, the minimum of b'_k over k is $b'_{n/2}$ and if n is odd, it is $b'_{\lfloor n/2 \rfloor} = b'_{\lceil n/2 \rceil}$. This can be shown by a similar (but a bit more tedious) calculation involving

$$\begin{aligned} B_k = b'_{k+1} - b'_k &= k!(n-k-3)! \\ &\times [2(k+1)(k+2)(n-k-2) - 2(n-k)(n-k-1)(n-k-2) \\ &- (k+1)(3n-2k-4) + (3n-2k-2)(n-k-2)]. \end{aligned}$$

Finally, by substituting, $a_{\lfloor n/2 \rfloor} > \frac{1}{2} b_{\lfloor n/2 \rfloor}$ and the theorem follows as the size of the families in Construction 3.6 is exactly $\frac{2}{b_{\lfloor n/2 \rfloor}}$. □

The case when $l > 2$ remains unsolved. We finish this section with a question which is related to the $l = 2$ case. What is the maximum size of a family that satisfies the Sperner property in addition to being 2-almost intersecting (or l-intersecting for some $l \geq 2$)? Is the following construction optimal or asymptotically optimal?

Construction 3.15. Let $\mathcal{F}$ be an optimal 1-almost intersecting family on $[n-l-1]$ as in Theorem 3.4. Then $\mathcal{G} = \mathcal{F} \times \binom{[n-l,n]}{1} = \{F \cup \{x\} : F \in \mathcal{F}, x \in [n-l, n]\}$ is a Sperner and l-almost intersecting family of size $(\frac{l+1}{2^{l+1}} + o(1))\binom{n}{\lfloor n/2 \rfloor}$.

4 The less restrictive case - Uniform families

In this section we consider k-uniform $(\leq l)$-almost intersecting families. First, we prove that if n is large enough, then this relaxation of the intersecting property does not allow us to obtain a larger family, than what the Erdős-Ko-Rado result states.

Proposition 4.1. *For any $k, l \in \mathbb{N}$ there exists $n_0 = n_0(k, l)$ such that if $n \geq n_0$ and $\mathcal{F} \subset \binom{[n]}{k}$ is an $(\leq l)$-almost intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality if and only if $\mathcal{F}$ is the family of all k-sets containing a fixed element of $[n]$.*

Proof. If $\mathcal{F}$ is intersecting, then we are done by the Erdős-Ko-Rado theorem. If $F_1, F_2 \in \mathcal{F}$ are disjoint, then any $F \in \mathcal{F} \setminus (\mathcal{D}_{\mathcal{F}}(F_1) \cup \mathcal{D}_{\mathcal{F}}(F_2))$ should meet both F_1 and F_2 and thus $|\mathcal{F}| \leq k^2\binom{n-2}{k-2} + 2l$ which is smaller than $\binom{n-1}{k-1}$ if n is large enough. $\square$

The argument of Proposition 4.1 gives $n_0(k,l) \leq O(k^3 + kl)$. Using a theorem of Hilton and Milner [9] that states that a non-trivially intersecting family has size at most $\binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$ one can obtain a bound $n_0(k,l) = O(k^2 l)$ which is better than the previous bound if $l = o(k)$. Finding the smallest possible $n_0(k,l)$ seems to be an interesting problem.

Obviously, if $l \geq \binom{m}{k}$, then $\binom{[k+m]}{k}$ is $(\leq l)$-almost intersecting, furthermore if $l \geq \binom{n-k-1}{k-1}$, then the trivially intersecting family $\{F \in \binom{[n]}{k} : 1 \in F\}$ is not maximal. The following theorem states that $2k+2$ is a good choice for $n_0(k,1)$ provided $k \geq 3$. Note that the case $k = 2$ is trivial as if there exist $F_1, F_2 \in \mathcal{F}$ with $F_1 \cap F_2 = \emptyset$, then all other sets in $\mathcal{F}$ must intersect both F_1 and F_2 and thus be subsets of $F_1 \cup F_2$ and therefore $|\mathcal{F}| \leq 6$. Comparing this to the size of the trivially intersecting family gives $n_0(2,1) = 7$.

Theorem 4.2. *If $k \geq 3$ and $n \geq 2k+2$ and $\mathcal{F} \subset \binom{[n]}{k}$ is an (≤ 1)-almost intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$ with equality if and only if $\mathcal{F}$ is the family of all k-sets containing a fixed element of $[n]$.*

Proof. We will use Katona's cycle method [10]. We call a subset S of $[n]$ an *interval* in a cyclic permutation π of $[n]$ if $S = \{\pi(i), \pi(i+1), ..., \pi(i+|S|-1)\}$ for some i and addition is modulo n. We say that two sets S, T are *separated* in a cyclic permutation π if there are disjoint intervals S', T' in π with $S \subseteq S', T \subseteq T'$ and $S' \cup T' = [n]$. Let $\mathcal{F}_0 = \{F \in \mathcal{F} : \mathcal{D}_{\mathcal{F}}(F) = 0\}, \mathcal{F}_1 = \{F \in \mathcal{F} : \mathcal{D}_{\mathcal{F}}(F) = 1\}$. We define two types of *objects* and give them weights:

1. for (F, π) with $F \in \mathcal{F}_0$ and F an interval in π, give the weight $\frac{1}{k}$;
2. for $(\{F, F'\}, \pi)$ with $F, F' \in \mathcal{F}_1$ and F, F' separated in π, give the weight $\frac{2}{n}$.

Lemma 4.3. *If $2k \leq n$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is a (≤ 1)-almost intersecting family, then for any cyclic permutation π of $[n]$ the sum of the weights of all objects having π as second coordinate is at most 1.*

Proof. For simplicity we identify objects with their first coordinates. We say that for any cyclic permutation π there is a *milestone* between any consecutive elements $\pi(i), \pi(i+1)$. For any object with second coordinate π we define two *separating milestones*: if the object is of type 1, then the milestones lying between the interval and its complement are the separating milestones. If the object is of type 2, then there exist intervals S', T' showing this fact and we take the milestones between S' and T'.

We claim that any milestone can belong to at most one object. Otherwise if both objects are of type 1, then to be different they must lie on the two sides of the milestone and as $2k \leq n$ they do not intersect. If one of the objects F is of type 1 and the other $\{S, T\}$ is of type 2, then $F \subseteq S'$ or $F \subseteq T'$ and thus either $F \cap T = \emptyset$ or $F \cap S = \emptyset$. Finally, if both objects $\{S_1, T_1\}, \{S_2, T_2\}$ are of type 2, then $S_1 \subseteq S_2'$ or $S_2 \subseteq S_1'$ and thus $S_1 \cap T_2 = \emptyset$ or $S_2 \cap T_1 = \emptyset$.

As each object has 2 separating milestones, the lemma immediately follows if all objects are of type 2. Let us assume that there is at least one object F of type 1 and we say that F *crosses* the milestones between any two of its elements. Then for any other object O, the interval F crosses at least one of the separating milestones of O. This is clear if O is of type 1 while if $O = \{S, T\}$, then not crossing any of the milestones would mean $F \subseteq S'$ or $F \subseteq T'$ and thus $F \cap T = \emptyset$ or $F \cap S = \emptyset$. But clearly an interval of size k may cross at most $k-1$ intervals, thus we obtain that in this case there are at most k objects with second coordinate π and as $\frac{1}{k} \geq \frac{2}{n}$ the lemma follows. $\square$

Let α denote the number of cyclic permutation in which the disjoint k-sets S and T are separated. Clearly, α does not depend on the actual choice of S and T. With this notation and Lemma 4.3 we obtain the following inequality

$$|\mathcal{F}_0| k!(n-k)! \frac{1}{k} + |\mathcal{F}_1| \alpha \frac{1}{n} \leq (n-1)!$$

Thus we will be done if we can prove that

$$k!(n-k)! \frac{1}{k} < \alpha \frac{1}{n}. \tag{3}$$

We calculate α in the following way: consider the family $\mathcal{G} = \binom{[2k]}{k}$ and count the objects $(\{G_1, G_2\}, \pi)$ with $G_1, G_2 \in \mathcal{G}$ and separated in π. On the one hand, this is $\frac{1}{2}\binom{2k}{k}\alpha$. On the other hand, this is $k(n-1)!$ as any cyclic permutation π of $[n]$ contains exactly k separated pairs from $\mathcal{G}$. Thus we obtain

$$\alpha = \frac{2k(n-1)!}{\binom{2k}{k}}.$$

After substituting this into (3) we need only to verify

$$k!(n-k)! \frac{1}{k} < \frac{2k(n-1)!}{n\binom{2k}{k}}$$

which is equivalent to

$$\frac{(2k)!}{k!k^2} < \frac{2(n-1)!}{(n-k)!n}.$$

This holds if $2k+2 \leq n$ provided $k \geq 5$. For $k < 5$ the above inequality holds if $2k+3 \leq n$. The cases $k = 3$, $n = 8$ and $k = 4$, $n = 10$ can be verified by changing the weight of all objects $(\{F, F'\}, \pi)$ to $1/k$ when at least one of F and F' is an interval in π. One can easily check that Lemma 4.3 holds with the modified weights. We leave the details to the reader. $\square$

Note that since the family $\binom{[2k]}{k}$ is 1-almost intersecting and has size $\binom{2k}{k} > \binom{2k+1-1}{k-1}$, $n_0(k,1) = 2k+2$ is best possible. We finish this section with an easy double counting proof that settles the case of $n = 2k+1$.

Proposition 4.4. *If $\mathcal{F} \subseteq \binom{[2k+1]}{k}$ is an (≤ 1)-almost intersecting family, then $|\mathcal{F}| \leq \binom{2k}{k}$ and equality holds if and only if $\mathcal{F}$ is the family all k-sets not containing a fixed element of $[2k+1]$.*

Proof. Let us double count the pairs (F, G) with $F \in \mathcal{F}$, $G \notin \mathcal{F}$. On one hand this is at least $k|\mathcal{F}|$ as for any $F \in \mathcal{F}$ out of the $k+1$ many k-sets disjoint from F at most 1 can be in $\mathcal{F}$. On the other hand the number of such pairs is at most $(\binom{2k+1}{k} - |\mathcal{F}|)(k+1)$. We obtain $(\binom{2k+1}{k} - |\mathcal{F}|)(k+1) \geq k|\mathcal{F}|$ and by rearranging we get the stated bound on $|\mathcal{F}|$.

To characterize the case of equality note that all lower and upper bounds in the previous argument hold with equality if and only if $\mathcal{F}$ is a 1-almost intersecting family. Thus we are done by Corollary 2.2. $\square$

5 The less restrictive case - Non-uniform families

In this section we consider the problem of finding the maximum size of an $(\leq l)$-almost intersecting family $\mathcal{F} \subset 2^{[n]}$. Theorem 5.3 will settle the case of $l = 1$, we prove Conjecture 1.2 for the case $l = 2$.

Remark 5.1. A family where all supersets of any member of the family belong to the family as well is called an *upset*. For any $(\leq l)$-almost intersecting family $\mathcal{F}$ there is another family $\mathcal{F}'$ of the same size which is an upset. Indeed, if for two sets F, G we have $F \subsetneq G$, $F \in \mathcal{F}$, $G \notin \mathcal{F}$, then $\mathcal{F} \setminus \{F\} \cup \{G\}$ is $(\leq l)$-almost intersecting provided $\mathcal{F}$ is as well. Thus it is enough to prove the upper bound in Conjecture 1.2 for upsets, furthermore uniqueness for upsets implies uniqueness for arbitrary families. Indeed, consider an $(\leq l)$-almost intersecting family $\mathcal{F}$ of maximum size. By applying repeatedly the above operation, we obtain a family $\mathcal{F}'$ such that

$|\mathcal{F}| = |\mathcal{F}'|$ and $\mathcal{F}' \setminus \{F\} \cup \{G\} = \mathcal{F}''$, where $F \subsetneq G, F \in \mathcal{F}', G \notin \mathcal{F}'$ and $\mathcal{F}''$ is the unique family described in Conjecture 1.2. but then $\mathcal{D}_{\mathcal{F}'}(F) \geq \lfloor \frac{1}{2}(n-1) \rfloor$ which contradicts the $(\leq l)$-almost intersecting property of $\mathcal{F}'$.

We start with a lemma that we will use in proving Theorem 5.3 and Theorem 5.5 and might be useful in verifying Conjecture 1.2.

Lemma 5.2. *Let l be a positive integer. If an $(\leq l)$-almost intersecting family $\mathcal{F}$ is an upset such that the size m of a minimum set in $\mathcal{F}$ is at most $\frac{n-l}{2}$, then there exists another $(\leq l)$-almost intersecting family $\mathcal{F}'$ with $|\mathcal{F}| \leq |\mathcal{F}'|$ such that the size of a minimum set in $\mathcal{F}'$ is $m+1$. Furthermore, if $m < \frac{n-l}{2}$, then $|\mathcal{F}| < |\mathcal{F}'|$.*

Proof. Let us write $\mathcal{F}_i = \{F \in \mathcal{F} : |F| = i\}$ and define the bipartite graph $G(\mathcal{V}_1, \mathcal{V}_2, E)$ where $\mathcal{V}_1 = \mathcal{F}_m$, $\mathcal{V}_2 = \{G \in \binom{[n]}{n-m-1} \setminus \mathcal{F}_{n-m-1} : \exists F \in \mathcal{F}_m, F \cap G = \emptyset\}$, $E = \{(F,G) : F \cap G = \emptyset\}$. Clearly, for any $G \in \mathcal{V}_2$ we have $d(G) \leq m+1$. Also, for any $F \in \mathcal{F}_m = \mathcal{V}_1$ we have $d(F) \geq n-m-l+1$ as there are $n-m$ sets of size $n-m-1$ which are disjoint from F and at most $l-1$ of them belong to $\mathcal{F}$ by the $(\leq l)$-almost intersecting property since $\mathcal{F}$ being an upset guarantees that $\overline{F} \in \mathcal{F}$ provided $\mathcal{D}_{\mathcal{F}}(F) \neq \emptyset$. It follows that $|\mathcal{V}_1| \leq |\mathcal{V}_2|$, furthermore if $m < \frac{n-l}{2}$, then $|\mathcal{V}_1| < |\mathcal{V}_2|$.

Let us define $\mathcal{F}' = (\mathcal{F} \setminus \mathcal{V}_1) \cup \mathcal{V}_2$. By the above, we have $|\mathcal{F}| \leq |\mathcal{F}'|$ and if $m < \frac{n-l}{2}$, then $|\mathcal{F}| < |\mathcal{F}'|$. We still have to show that $\mathcal{F}'$ is $(\leq l)$-almost intersecting. Since all sets in $\mathcal{V}_2$ have size $n-m-1$ and the minimum sets in $\mathcal{F}'$ have size $m+1$ for any set $G \in \mathcal{V}_2$ we have $|\mathcal{D}_{\mathcal{F}'}(G)| = 1$. Also, for any set $F \in \mathcal{F} \cap \mathcal{F}'$ if $|F| > m+1$, then $\mathcal{D}_{\mathcal{F}'}(F) \subseteq \mathcal{D}_{\mathcal{F}}(F)$ and thus $|\mathcal{D}_{\mathcal{F}'}(F)| \leq |\mathcal{D}_{\mathcal{F}}(F)| \leq l$. Finally, let us consider a set $F \in \mathcal{F} \cap \mathcal{F}'$ with $|F| = m+1$. If there was no set $F' \subset F$ in $\mathcal{F}_m$, then $\mathcal{D}_{\mathcal{F}}(F) = \mathcal{D}_{\mathcal{F}'}(F)$, while if there was, then the only set in $\mathcal{D}_{\mathcal{F}'}(F) \setminus \mathcal{D}_{\mathcal{F}}(F)$ is $\overline{F}$, therefore $|\mathcal{D}_{\mathcal{F}'}(F)| \leq |\mathcal{D}_{\mathcal{F}}(F)| + 1$. To finish the proof of the lemma observe that as $\overline{F}' \in \mathcal{F}$ we have $|\mathcal{D}_{\mathcal{F}}(F)| < |\mathcal{D}_{\mathcal{F}}(F')| \leq l$. $\square$

The next theorem considers the case $l = 1$.

Theorem 5.3. *If $n \geq 2$ and $\mathcal{F} \subset 2^{[n]}$ is an (≤ 1)-almost intersecting family, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathcal{F}$ is the family of sets of size at least $n/2$ and (if n is odd) the sets of size $\lfloor n/2 \rfloor$ containing a fixed element of $[n]$.

Proof. Let $\mathcal{F}$ be an (≤ 1)-almost intersecting family of maximum size. Remark 5.1 shows that we may assume that $\mathcal{F}$ is an upset and by Lemma 5.2 we may assume that the size of a minimum set in $\mathcal{F}$ is at least $\lceil \frac{n-1}{2} \rceil$. If n is even, this means $\mathcal{F} \subseteq \{F \subseteq [n] : |F| \geq n/2\}$ and we are done.

If $n = 2k+1$ is odd, then Lemma 5.2 gives $\mathcal{F} \subseteq \{F \subseteq [2k+1] : |F| \geq k\}$. We claim that $\binom{[2k+1]}{k+1} \subseteq \mathcal{F}$. Indeed, if $G \in \binom{[2k+1]}{k+1}$, then $\mathcal{D}_{\mathcal{F}}(G) \leq 1$, thus the only reason for which G could not be in $\mathcal{F}$ is $\overline{G} \in \mathcal{F}$ and there exists $G' \in \mathcal{F}$ such that $G' \subsetneq G$. But then G would belong to $\mathcal{F}$ as $\mathcal{F}$ is an upset, a contradiction.

As $\binom{[2k+1]}{k+1} \subseteq \mathcal{F}$, we know that for every $F \in \mathcal{F}_k$ we have $\overline{F} \in \mathcal{F}$ and thus $\mathcal{F}_k = \{F \in \mathcal{F} : |F| = k\}$ must form an intersecting family. Thus we are done by the Erdős-Ko-Rado Theorem. □

Theorem 5.3 can also be derived from a result by Bernáth and Gerbner [2]. We define a family $\mathcal{F}$ to be (p,q)-*chain intersecting* if $A_1 \subsetneq A_2 \subsetneq ... \subsetneq A_p$, $B_1 \subsetneq B_2 \subsetneq ... \subsetneq B_q$ with $A_i, B_j \in \mathcal{F}$ implies $A_p \cap B_q \neq \emptyset$.

Theorem 5.4 (Bernáth, Gerbner [2]). *If $\mathcal{F} \subseteq 2^{[n]}$ is (p,q)-chain intersecting, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=(n-p-q+3)/2}^{n} \binom{n}{i} & \text{if } \ n-p-q \text{ is odd} \\ \binom{n-1}{\lfloor (n-p-q+3)/2 \rfloor - 1} + \sum_{i=\lceil (n-p-q+3)/2 \rceil}^{n} \binom{n}{i} & \text{if } \ n-p-q \text{ is even.} \end{cases}$$

Theorem 5.3 follows from Theorem 5.4 by letting $p=1$, $q=2$ as the (1,2)-chain intersecting property is equivalent to the condition that $\mathcal{D}_{\mathcal{F}}(F)$ is Sperner for all $F \in \mathcal{F}$. Bernáth and Gerbner also deal with the case of equality. We do not state their complete result for sake of brevity.

The next theorem states that Conjecture 1.2 is true if $l = 2$.

Theorem 5.5. *If $n \geq 2$ and $\mathcal{F} \subset 2^{[n]}$ is an (≤ 2)-almost intersecting family, then*

$$|\mathcal{F}| \leq \begin{cases} \sum_{i=n/2}^{n} \binom{n}{i} & \text{if } \ n \text{ is even} \\ \binom{n-1}{\lfloor n/2 \rfloor} + \sum_{i=\lceil n/2 \rceil}^{n} \binom{n}{i} & \text{if } \ n \text{ is odd,} \end{cases}$$

and equality holds if and only if $\mathcal{F}$ is the family of sets of size at least $n/2$ and (if n is odd) the sets of size $\lfloor n/2 \rfloor$ not containing a fixed element of $[n]$.

Proof. Let us consider two cases according to the parity of n. If $n = 2k+1$ is odd, then by Remark 5.1 and Lemma 5.2 we can assume that $\mathcal{F}$ is an upset and all sets in $\mathcal{F}$ have size at least k. Therefore all sets F of size at least $k+2$ belong to $\mathcal{F}$ as $\mathcal{D}_{\mathcal{F}}(F) = \emptyset$. We claim that if $\mathcal{F}$ is maximal, then $\binom{[2k+1]}{k+1} \subseteq \mathcal{F}$. Indeed, if $|G| = k+1$, then $\mathcal{D}_{\mathcal{F}}(G) \leq 1$, thus the only reason for which G could not be in $\mathcal{F}$ is that $\overline{G} \in \mathcal{F}$ and $\mathcal{D}_{\mathcal{F}}(\overline{G}) > 2$. But then $G \in \mathcal{F}$ as $\mathcal{F}$ is an upset.

Now consider $\mathcal{F}_k = \mathcal{F} \cap \binom{[2k+1]}{k}$. Again, for all $F \in \mathcal{F}_k$ the set $\overline{F}$ belongs to $\mathcal{F}$, thus $\mathcal{F}_k$ is (≤ 1)-almost intersecting and then we are done by Proposition 4.4.

Suppose now that $n = 2k$ is even. Then by Remark 5.1 and Lemma 5.2 we can assume that $\mathcal{F}$ is an upset and all sets in $\mathcal{F}$ have size at least $k-1$. Furthermore Lemma 5.2 states that there is an upset $\mathcal{F}'$ with $|\mathcal{F}|$ many members each of size at least k. This implies the bound in the theorem. Suppose $\mathcal{F}_{k-1} \neq \emptyset$, then just as in the case of odd n, all sets of size at least $k+1$ belong to $\mathcal{F}$ and the proof of Lemma 5.2 shows that writing $\mathcal{B}_k = \{B \in \binom{[2k]}{k} \setminus \mathcal{F}_k\}$ we must have $|\mathcal{F}_{k-1}| = |\mathcal{B}_k|$ (as otherwise $(\mathcal{F} \setminus \mathcal{F}_{k-1}) \cup \mathcal{B}_k$ would be a larger (≤ 2)-almost intersecting family) and $\mathcal{F}_{k-1} = \Delta\overline{\mathcal{B}_k} = \{F \in \binom{[2k]}{k-1} : \exists B \in \mathcal{B}_k(F \subset \overline{B})\}$. The Lovász version [11] of the Kruskal-Katona shadow theorem states that if $\overline{\mathcal{B}_k} = m = \binom{x}{k}$ for some real number x, then $\Delta\overline{\mathcal{B}_k} \geq \binom{x}{k-1}$. As $\binom{x}{k} < \binom{x}{k-1}$ if $x < 2k-1$, we must have $|\mathcal{F}_{k-1}| = |\mathcal{B}_k| \geq \binom{2k-1}{k-1}$. Finally, note that $\mathcal{F}_{k-1}$ is intersecting as a pair $F, F' \in \mathcal{F}_{k-1}, F \cap F' = \emptyset$ would give, by the assumption that $\mathcal{F}$ is an upset, $|\mathcal{D}_{\mathcal{F}}(F)| \geq |\{G : F' \subseteq G \subseteq \overline{F}\}| = 4$. By the Erdős-Ko-Rado Theorem we obtain that $|\mathcal{F}_{k-1}| \leq \binom{2k-1}{k-2} < \binom{2k-1}{k-1}$. This contradiction finishes the proof of the theorem. □

Remark 5.6. For general l Lemma 5.2 gives that an $(\leq l)$-almost intersecting family $\mathcal{F}$ of maximum size is a subset of $\{F \in 2^{[n]} : |F| \geq \lfloor (n-l)/2 \rfloor + 1\}$. Theorem 5.4 can be used to give a better upper bound on the size of $\mathcal{F}$. To see this note that an $(\leq l)$-almost intersecting family which is an upset satisfies the $(1,p)$-chain intersecting property with $p = \lceil \log_2(l+1) \rceil$. Indeed, if not then there would exist a set $F \in \mathcal{F}$ and a chain $G_1 \subsetneq G_2 \subsetneq ... \subsetneq G_p$ in $\mathcal{F}$ such that $G_p \cap F = \emptyset$, but then all sets G with $G_1 \subseteq G \subseteq G_p$ would belong to $\mathcal{F}$ and thus $|\mathcal{D}_{\mathcal{F}}(F)| > l$ would hold.

References

[1] R. Ahlswede, Simple hypergraphs with maximal number of adjacent pairs of edges, J. Combinatorial Theory (B) **28** (1980), 164-167.

[2] A. Bernáth, D. Gerbner, Chain intersecting families. Graphs Combin. **23** (2007), 353–366.

[3] B. Bollobás, On generalized graphs, Acta Mathematica Academiae Scientarium Hungaricae **16** (1965) (34), 447–452.

[4] B. Bollobás, Sperner systems consisting of pairs of complementary subsets, Journal of Combinatorial Theory, Ser. A **15**, 363–366.

[5] B. Bollobás, I. Leader, Set Systems with few Disjoint Pairs, Combinatorica **23** (2003), 559–570.

[6] A. De Bonis, G.O.H. Katona, Largest families without an r-fork, Order **24**, 181–191.

[7] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford, **12** (1961), 313–318.

[8] P. Frankl, On the minimum number of disjoint pairs in a family of finite sets, J. Comb. Theory, Ser. A **22** (1977), 249–251.

[9] A.J.W. Hilton, E.C. Milner, Some intersection theorems for systems of finite sets. Quart. J. Math. Oxford Ser. (2) 18 (1967) 369–384.

[10] G.O.H. Katona, A simple proof of the Erdős-Chao Ko-Rado theorem, J. Combinatorial Theory Ser. B 13 (1972), 183–184.

[11] L. Lovász, *Combinatorial problems and exercises*, second edition, North-Holland Publishing Co., Amsterdam, 1993, 635 pp.

[12] E. Sperner, Ein Satz über Untermenge einer endlichen Menge, *Math Z.*, **27**(1928) 544-548.

[13] H.T. Thanh, An extremal problem with excluded subposets in the Boolean lattice. Order **15** 51–57.