Profile polytopes of some classes of families

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20th December 2011

Abstract

The profile vector of a family $\mathcal{F}$ of subsets of an n-element set is $(f_0, f_1, \ldots, f_n)$ where f_i denotes the number of the i-element members of $\mathcal{F}$. In this paper we determine the extreme points of the set of profile vectors for some classes of families, including complement-free k-Sperner families and self-complementary k-Sperner families. Using these results we determine the maximum cardinality of intersecting k-Sperner families.

1 Introduction and preliminaries

Let us start with basic notation.

Let $[n] = \{1, \ldots, n\}$ be the underlying set. If $F \subseteq [n]$ then $\overline{F}$ denotes the complement of F. Let $\mathcal{F}$ be a family of subsets of $[n]$ ($\mathcal{F} \subseteq 2^{[n]}$). Then $\mathrm{co}(\mathcal{F}) = \{X \subseteq [n] : \overline{X} \in \mathcal{F}\}$ and $\mathcal{F}_i$ denotes the subfamily of the i-element subsets in $\mathcal{F}$: $\mathcal{F}_i = \{F : F \in \mathcal{F}, |F| = i\}$. Its size $|\mathcal{F}_i|$ is denoted by f_i. The vector $\mathbf{p}(\mathcal{F}) = (f_0, f_1, \ldots, f_n)$ in the $(n+1)$-dimensional Euclidian space $\mathbb{R}^{n+1}$ is called the *profile* of $\mathcal{F}$. The vector $\mathbf{p}_0(\mathcal{F}) = (f_1, f_2, \ldots, f_{n-1})$ is called the *reduced profile* of $\mathcal{F}$.

A *chain* is a family $\mathcal{C} = \{C_1, \ldots, C_i\}$ with some integer i such that $C_1 \subset C_2 \subset \ldots \subset C_i$. A *full chain* is a chain of length $n+1$, i.e. a family $\mathcal{C} = \{C_0, C_1, \ldots, C_n\}$ such that $C_0 \subset C_1 \subset \ldots \subset C_n$. A family $\mathcal{F}$ is *intersecting* if there exist no $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cap F_2 = \emptyset$. $\mathcal{F}$ is *co-intersecting* if there exist no $F_1, F_2 \in \mathcal{F}$ such that $F_1 \cup F_2 = [n]$, i.e. $\overline{F_1} \cap \overline{F_2} = \emptyset$. A family is *Sperner* (or *antichain*) if it does not contain any chain of length 2, and *k-Sperner*, if it does not contain any chain of length $k+1$, or equivalently if the intersection of the family and a chain contains at most k members.

If Λ is a finite set in $\mathbb{R}^d$, its *convex hull* $\mathrm{conv}(\Lambda)$ is the set of all convex combinations of the elements of Λ. A point of Λ is an *extreme point* if it is not a convex combination of other points of Λ. It is easy to see that the convex hull of a set is equal to the convex hull of the extreme points of the set.

Let $\mathbf{A}$ be a class of families of subsets of $[n]$. We denote by $\Lambda(\mathbf{A})$ the set of profiles of the families belonging to $\mathbf{A}$:

$$\Lambda(\mathbf{A}) = \{\mathbf{p}(\mathcal{F}) : \mathcal{F} \in \mathbf{A}\},$$

and similarly we denote by $\Lambda_0(\mathbf{A})$ the set of reduced profiles. $\Gamma(\mathbf{A})$ denotes the set of the extreme points of $\Lambda(\mathbf{A})$. We simply call them the extreme points of $\mathbf{A}$. Similarly $\Gamma_0(\mathbf{A})$ denotes the set of the extreme points of $\Lambda_0(\mathbf{A})$. The profile polytope of $\mathbf{A}$ is $\mathrm{conv}(\Lambda(\mathbf{A}))$.

We call an extreme point $\mathbf{v}$ of a set Λ *essential* if there is no other point $\mathbf{u} \in \Lambda$ with $\mathbf{v} \leq \mathbf{u}$ (it denotes $v_i \leq u_i$ for every i). $\Gamma^*(\mathbf{A})$ and $\Gamma_0^*(\mathbf{A})$ denote the sets of essential extreme points of the sets of profiles and reduced profiles, respectively. We say that a set of vectors $\Gamma = \{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *dominates* a set Λ of vectors if for any $\mathbf{v} \in \Lambda$ there are constants $\lambda_1, \ldots \lambda_m \geq 0, \ \sum_{i=1}^m \lambda_i \leq 1$ satisfying $\mathbf{v} \leq \sum_{i=1}^m \lambda_i \mathbf{v}_i$. We say that $\mathbf{A}$ is *hereditary* if $\mathcal{F} \subseteq \mathcal{F}' \in \mathbf{A}$ implies $\mathcal{F} \in \mathbf{A}$. We will use the following proposition ([5]).

Proposition 1.1. *If* $\mathbf{A}$ *is hereditary, then*

(i) any element of $\Gamma(\mathbf{A})$ *can be obtained by changing some coordinates of an element of* $\Gamma^*(\mathbf{A})$ *to zero.*

(ii) If $\Gamma \subset \Lambda(\mathbf{A})$ *dominates* $\Lambda(\mathbf{A})$ *then* $\Gamma^*(\mathbf{A}) \subset \Gamma$.

For the reduced profiles the analogous statement is true.

Let us give some motivations for studying profile polytopes. Suppose we are given a weight function $w : \{0, \ldots, n\} \to \mathbb{R}$, and the weight of a family $\mathcal{F}$ is defined to be $\sum_{F \in \mathcal{F}} w(|F|)$, which is equal to $\sum_{i=0}^n w(i) f_i$. Usually we are interested in the maximum of the weight of the families in a class $\mathbf{A}$. Several well-known results in extremal set theory can be formulated this way.

We want to maximize the sum, i.e. find a family $\mathcal{F}_0 \in \mathbf{A}$ and an inequality $\sum_{i=0}^n w(i) f_i = w(\mathcal{F}) \leq w(\mathcal{F}_0) = c$. This is a linear inequality, and it is always maximized in an extreme point (if the weight function is positive, it is maximized in an essential extreme point). We usually want to find the maximum weight, but conversely, it can help us to determine the extreme points. Basic linear programming gives the following lemma.

Lemma 1.2. *Let S be a set of profile vectors.*

i) Suppose that for every weight the maximum is given by an element of S. Then S contains all the extreme points.

ii) Suppose that for every positive weight the maximum is given by an element of S. Then S contains all the essential extreme points.

The profile polytopes were introduced by P.L. Erdős, P. Frankl and G.O.H. Katona in [4]. Later they applied the circle method to profile polytopes in [5]. This method will be an important tool in this paper.

The circle method was introduced by G.O.H. Katona [8]. Let the elements of the set $[n]$ be placed around a circle such that $i+1$ is next to i for all $i = 1, 2, \ldots, n-1$ and 1 is next to n in clockwise direction: we will also say that $i+1$ is to the right from i. We consider these numbers mod n. Elements next to each other will be called *consecutive*. A set of consecutive elements will be called an *interval*. Denote the interval of elements between a and b by $[a, b]$ (endpoints included): this is the set of elements $a, a+1, \ldots, b$. The family of all intervals on the circle will be denoted by $\mathcal{H}$.

Let α be a cyclic permutation. If $F \subset [n]$, then $\alpha(F) = \{\alpha(i) : i \in F\}$. For a family $\mathcal{F}$ let $\mathcal{F}_\alpha$ denote the family of the intervals in $\mathcal{F}$, i.e. $\mathcal{F}_\alpha = \alpha(\mathcal{F}) \cap \mathcal{H}$, where $\alpha(\mathcal{F}) = \{\alpha(F) : F \in \mathcal{F}\}$. Similarly, for a class of families $\mathbf{A}$ let $\mathbf{A}_\alpha = \{\mathcal{F}_\alpha : \mathcal{F} \in \mathbf{A}\}$.

If $\mathbf{v} = (v_0, v_1, \ldots, v_n)$ then let

$$T(\mathbf{v}) = \left(v_0, v_1\binom{n}{1}/n, v_2\binom{n}{2}/n, \ldots, v_{n-1}\binom{n}{n-1}/n, v_n\right).$$

Theorem 1.3 ([5]). *If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are the extreme points of $\Lambda(\mathbf{A}_\alpha)$ for every given cyclic permutation α then*

$$\Lambda(\mathbf{A}) \subseteq \mathrm{conv}\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)\}.$$

This theorem is really useful if $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m) \in \Lambda(\mathbf{A})$ holds. (This can be easily checked.) Then $T(\mathbf{v}_1), \ldots, T(\mathbf{v}_m)$ are the extreme points of $\mathbf{A}$.

Definition 1.4. *Let $\mathcal{L} = \{L_0, L_1, \ldots, L_n\}$ be a chain, i.e. $L_0 \subset L_1 \subset \cdots \subset L_n$, where $|L_i| = i$. Then $\mathcal{K} = \mathcal{L} \cup \mathrm{co}(\mathcal{L})$ is a* ***complement chain-pair****, or briefly* ***chain-pair****.*

Definition 1.5. *$\mathcal{F}$ is a k-**antichainpair** family if $|\mathcal{F} \cap \mathcal{K}| \le k$ for every complement chain-pair $\mathcal{K}$.*

One can easily see that an l-Sperner family is $2l$-antichainpair. The notion of k-antichainpairs does not seem to be very interesting on its own, but we will able to use it to approach some other, more natural problems. In Section 2 we will study k-antichainpair families and determine their extreme points. In Section 3 the extreme points of some other classes of families are determined, using the k-antichainpair families. As a corollary, we will get the following statement.

Theorem 1.6. *Let $\mathcal{F}$ be an intersecting k-Sperner family of maximum cardinality. Then*

$$|\mathcal{F}| = \begin{cases} \displaystyle\sum_{i=\frac{n+1}{2}}^{\frac{n+1}{2}+k-1} \binom{n}{i} & \text{if } n \text{ is odd} \\ \binom{n-1}{n/2-1} + \displaystyle\sum_{i=n/2+1}^{n/2+k-1} \binom{n}{i} + \binom{n-1}{n/2+k} & \text{if } n \text{ is even.} \end{cases}$$

Note that the case $k = 1$ was proved by Milner [9].

2 The k-antichainpair families

For sake of completeness, we repeat some well-known definitions.

Let $\mathcal{F}$ be a family of k-element sets. Then the *shadow* of $\mathcal{F}$ is $\Delta\mathcal{F} = \{A \subseteq [n]\colon |A| = k-1, \text{ there exists } F \in \mathcal{F} \text{ such that } A \subseteq F\}$. The *shade* of $\mathcal{F}$ is $\nabla\mathcal{F} = \{A \subseteq [n]\colon |A| = k+1, \text{ there exists } F \in \mathcal{F} \text{ such that } F \subseteq A\}$. If $\mathcal{F}$ is a family of intervals, then these notions can be analogously defined but considering only intervals. More precisely let $\nabla_{\text{int}}\mathcal{F} := \{A \in \mathcal{H}\colon |A| = k+1, \text{ there exists } F \in \mathcal{F} \text{ such that } F \subseteq A\}$ and $\Delta_{\text{int}}\mathcal{F} = \{A \in \mathcal{H}\colon |A| = k-1, \text{ there exists } F \in \mathcal{F} \text{ such that } A \subseteq F\}$. We note here an important property of ∇_{int}: $|\nabla_{\text{int}}\mathcal{F}| \geq |\mathcal{F}|$, and equality holds if and only if $\mathcal{F}$ is empty or the full level.

Let us introduce our most important notations. Suppose $A, B \subseteq \{0, \dots, n\}$ are disjoint sets. Then

$$(\mathbf{x}_{A,B})_i = \begin{cases} \binom{n}{i} & \text{if } i \in A \\ \binom{n-1}{i-1} & \text{if } i \leq n/2 \text{ and } i \in B \\ \binom{n-1}{i} & \text{if } i > n/2 \text{ and } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

Here $\binom{n-1}{-1} := \binom{n-1}{n} := 1$. We also define

$$(\mathbf{u}_{A,B})_i = \begin{cases} 1 & \text{if } i = 0, n \text{ and } i \in A \cup B \\ n & \text{if } i \neq 0, n \text{ and } i \in A \\ i & \text{if } i \neq 0,\ i \leq n/2 \text{ and } i \in B \\ n-i & \text{if } i \neq n,\ i > n/2 \text{ and } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

The main example for the family with profile vector $\mathbf{x}_{A,B}$ is the following: $\mathcal{F}_{A,B} = \mathcal{F}_A \cup \mathcal{F}'_B$ where $\mathcal{F}_A = \{F \subset [n] : |F| = i \text{ for some } i \in A\}$ and

$\mathcal{F}'_B = \{F \subset [n] : 1 \in F$ and $|F| = j$ for a $j \in B, j \leq n/2\} \cup \{F \subset [n] : 1 \notin F$ and $|F| = j$ for a $j \in B, j > n/2\}$. Let $\mathcal{G}_{A,B} = \mathcal{F}_{A,B} \cap \mathcal{H}$, where $\mathcal{H}$ is the family of intervals, defined in Section 1. The profile vector of $\mathcal{G}_{A,B}$ is $\mathbf{u}_{A,B}$.

Note that $i \in A$ means $\mathcal{F}_{A,B}$ contains the full level i. On the other hand $i \in B$ means $\mathcal{F}_{A,B}$ contains a part of level i. One could call it "half level", as its size is at most half of the size of the full level, but it can actually be much smaller. However, as we will see, in some sense it acts as a half level. One can easily see that $\mathcal{F}_{A,B}$ is a $2|A| + |B|$-antichainpair family.

It is easy to see, that the 1-antichainpair families are the intersecting and co-intersecting Sperner families. The profile polytope of the class of these families has been determined by Konrad Engel and Péter Erdős([3]).

Theorem 2.1. *([3]) The extreme points of the profile-polytope of intersecting and co-intersecting families are the vectors $\mathbf{x}_{A,B}$ where $A = \emptyset$, $|B| \leq 1$ and $0, n \notin B$.*

Theorem 2.2 is a generalization of this result, and it will help us to determine the extreme points of the profile polytope of some other classes of families.

Theorem 2.2. *The essential extreme points of the k-antichainpair families are the vectors $\mathbf{x}_{A,B}$ where $2|A| + |B| = k$ and $0, n \notin A$, $|B \setminus \{0, n\}| \leq 1$.*

Note that it helps understand the statement if we consider the elements of B as half levels. In that case a less precise form of the statement would be the following. The extremal families are those which consist of $k/2$ levels, where $\emptyset$ counts as a half level (and not a full level) on its own, similarly $[n]$, and there is at most one half level besides them.

Let us introduce the following notations. $\Gamma_k = \{\mathbf{u}_{A,B} : 2|A| + |B| = k, 0, n \notin A, |B \setminus \{0, n\}| \leq 1\}$. $\Lambda_k = \{\mathbf{u}_{A,\emptyset} : \ |A| \leq k\}$ and $\Lambda'_k = \{\mathbf{u}_{A,\emptyset} : \ |A| = k\}$.

We will need the following lemma (our main lemma):

Lemma 2.3. *The set of essential extreme points of the profile-polytope of k-antichainpair families on the circle is Γ_k.*

Before the proof some other lemmas are needed:

Lemma 2.4. *Let $\mathcal{G}$ be k-antichainpair family on the circle. Then $|\mathcal{G}| \leq k/2$.*

Proof. Let $\mathcal{A}_i = \{[x, i] : x \in [n], x \neq i + 1\}$ be the family of intervals, which 'end' in i, and $\mathcal{B}_i = \{[i, y] : y \in [n], y \neq i - 1\}$. Then $\mathcal{A}_i \cup \mathcal{B}_{i+1}$ is a subfamily of a chain-pair for every i. Thus $|\mathcal{G} \cap (\mathcal{A}_i \cup \mathcal{B}_{i+1})| \leq k$. If we count the elements of $\mathcal{G}$ in all $\mathcal{A}_i$s and $\mathcal{B}_j$s, we consider every interval two times. On the other hand we get at most kn. ■

Lemma 2.5. *Let $\mathcal{G}$ be $2l+1$-antichainpair without whole levels such that the cardinalities of the intervals are at least $\lceil \frac{n-l}{2} \rceil$ and at most $\lceil \frac{n+l}{2} \rceil$. Then $|\mathcal{G}| \leq \lfloor n/2 \rfloor + l(n-1) + \lfloor l/2 \rfloor$.*

Proof. There are at most $l+1$ nonempty levels. Hence a forbidden configuration (what would violate the $2l+1$-antichainpair property) consists of a chain of length $l+1$, and all the complements. Thus if $G, \overline{G} \notin \mathcal{G}$ and $\lceil \frac{n-l}{2} \rceil \leq |G| \leq \lceil \frac{n+l}{2} \rceil$, we can add G to the family $\mathcal{G}$ without violating the $2l+1$-antichainpair property. It also means that if $G \in \mathcal{G}$ and $\overline{G} \notin \mathcal{G}$, then $\mathcal{G} \setminus \{G\} \cup \{\overline{G}\}$ is also $2l+1$-antichainpair.

Replace all intervals of size less than $n/2$ by their complements, if the complement is not in $\mathcal{G}$. After that, if $n+l$ is odd, replace all intervals of size $\lceil \frac{n+l}{2} \rceil$ by their complements (note that those could not in $\mathcal{G}$ as their size is smaller than $\lceil \frac{n-l}{2} \rceil$ in this case). Now we are given a $2l+1$-antichainpair $\mathcal{G}'$ with $|\mathcal{G}'| = |\mathcal{G}|$. There are $\lceil l/2 \rceil$ nonempty levels above $n/2$ in $\mathcal{G}'$. If any of those levels are not whole levels, we can add the missing intervals to the family $\mathcal{G}$, because their complements are not in $\mathcal{G}'$. Note that there are no whole levels below $n/2$ in $\mathcal{G}'$ (because there are no whole levels at all in $\mathcal{G}$).

Let $\mathcal{A}$ be the family of intervals of size $\lfloor n/2 \rfloor$, such that there is a chain of length $l+1$ in $\mathcal{G}'$ containing this interval. Suppose $A, A' \in \mathcal{A}$ and $A \cap A' = \emptyset$. There are chains $B_1 \subset \dots B_x \subset A$ and $C_1 \subset \dots C_x \subset A'$ in $\mathcal{G}'$ which contain members of every possible size (x is either $\lfloor l/2 \rfloor$ or $\lceil l/2 \rceil$, depending on the parity of n and l). Then $B_1, \dots B_x, A, \overline{A'}, \overline{C}_x \dots \overline{C}_y$ and $C_1, \dots C_x, A', \overline{A}, \overline{B}_x \dots \overline{B}_y$ constitute a forbidden configuration, where $y = 1$ if $n+l$ is even and $y = 2$ otherwise. Note that if n is even then $A = \overline{A'}$.

This leads to a contradiction, hence $\mathcal{A}$ is intersecting, so $|\mathcal{A}| \leq \lfloor n/2 \rfloor$.

Clearly $\mathcal{G}' \setminus \mathcal{A}$ is an l-Sperner family, hence it is the union of l Sperner families. It is an easy exercise to see that a Sperner family contains at most n intervals and it contains exactly n intervals only if there is a j such that the family contains all j element intervals. $\mathcal{G}' \setminus \mathcal{A}$ can contain all j element intervals only if $j > n/2$, hence at most $\lfloor l/2 \rfloor$ times. Hence $|\mathcal{G}' \setminus \mathcal{A}| \leq l(n-1) + \lfloor l/2 \rfloor$, so $|\mathcal{G}| = |\mathcal{G}'| \leq \lfloor n/2 \rfloor + l(n-1) + \lfloor l/2 \rfloor$. $\square$

Lemma 2.6. *Let $\mathcal{G} \subseteq \mathcal{H}$ be a family on the circle, such that $\emptyset, [n] \notin \mathcal{G}$ and $|\mathcal{G}| \leq in$ $(1 \leq i \leq n-1)$. Then $\mathbf{p}(\mathcal{G}) \in \mathrm{conv}(\Lambda_i)$.*

Proof. Clearly this class of families is hereditary, and if we change some coordinates of an $\mathbf{u}_{A,\emptyset} \in \Lambda'_i$ to 0, the vector remains in Λ_i. So it is enough to prove that the essential extreme points are in Λ'_i (in that case all the extreme points are in Λ_i, and $\mathbf{p}(\mathcal{G})$ is their convex combination).

We use the following approach: A positive weight function is always maximized in at least one of the essential extreme points. Moreover, for every essential extreme point there is a non-negative weight function such that it is the unique maximum. Hence it is enough to prove that for every non-negative weight function w the maximum is given by a profile vector $\mathbf{u}_{A,\emptyset}$, such that $|A| = i$, i. e. there is a set $A \subset \{0, \dots, n\}$ and a family $\mathcal{G}'$ such that $|A| = i$, $\mathbf{p}(\mathcal{G}') = \mathbf{u}_{A,\emptyset}$ and $w(\mathcal{G}) \leq w(\mathcal{G}')$.

Let $w(j_1) \geq w(j_2) \geq \cdots \geq w(j_{n-1})$ be the order of the numbers $1, \dots n-1$ with respect to w. Then the weight of at most in intervals is maximum if they are all the j_1 element intervals, all the j_2 element intervals, and so on, while there are no more than in intervals. Clearly, this is the union of i complete levels, denoted by $\mathcal{G}'$. ■

Note that using this lemma together with Theorem 1.3 and some simple observations one could determine the extreme points of the profile polytope of the i-Sperner families, and also of those $2l$-antichainpairs which contain neither $\emptyset$ nor $[n]$.

Lemma 2.7. *Let $\mathcal{G} \subseteq \mathcal{H}$ be a family on the circle, such that $\emptyset, [n] \notin \mathcal{G}$. Let $m = \min\{|G| : G \in \mathcal{G}\}$ and $M = \max\{|G| : G \in \mathcal{G}\}$. Suppose $m \leq n - M$ and $|\mathcal{G}| \leq in + m$. Then $\mathbf{p}(\mathcal{G})$ is in the convex hull of the vectors $\mathbf{u}_{A,B}$ where $|A| \leq i$ and $0, n \notin A \cup B$, $|B| \leq 1$.*

Proof. We follow the proof of Lemma 2.6, hence we give here only a sketch. There are only few differences. Again when we are given a non-negative weight function w, we want to construct a family $\mathcal{G}'$ such that its profile vector is $\mathbf{u}_{A,B}$ with $|A| = i$, $|B| = 1$ and $w(\mathcal{G}) \leq w(\mathcal{G}')$.

This time we order only the numbers between m and M with respect to w, hence the order is $w(j_1) \geq w(j_2) \geq \cdots \geq w(j_{M-m+1})$. Then the weight of at most $in + m$ intervals is maximum if they are all the j_1 element intervals, all the j_2 element intervals, and so on. It means that all the $j_1, j_2, \dots, j_i$ intervals are in the family of maximum weight, and m intervals of size j_{i+1}.

Let $\mathcal{G}'$ be the family of all $j_1, \dots, j_i$ element intervals and j_{i+1} intervals of size j_{i+1} if $j_{i+1} \leq n/2$, or $n - j_{i+1}$ intervals of size j_{i+1} if $j_{i+1} > n/2$. We assumed that $m \leq j_{i+1} \leq M \leq n - m$, hence there are at least m intervals of size j_{i+1} in $\mathcal{G}'$. It follows that $w(\mathcal{G}')$ is at least the above mentioned maximum weight, hence it is at least the weight of $\mathcal{G}$. The profile of $\mathcal{G}'$ is listed in the lemma. ■

Lemma 2.8. *Let $\mathcal{G}$ be a $2l+1$-antichainpair family ($0 < l$) on the circle such that $\emptyset, [n] \notin \mathcal{G}$. Let us suppose, that for all $0 < i < l$ the set of essential extreme points of the profile polytope of the $2l+1-2i$-antichainpair families is $\Gamma_{2l-2i+1}$. Suppose moreover that $\mathcal{G}$ can be decomposed in the following way: $\mathcal{G} = \mathcal{G}^1 \cup \mathcal{G}^2$ ($\mathcal{G}^1 \cap \mathcal{G}^2 = \emptyset$), where $\mathcal{G}^1$ is $2l+1-2i$-antichainpair, $|\mathcal{G}^2| \leq in$*

and there are no $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2|$. Then the profile of $\mathcal{G}$ is dominated by Γ_{2l+1}.

Proof. By the assumptions $\mathbf{p}(\mathcal{G}^1)$ is dominated by $\Gamma_{2l+1-2i}$. By Lemma 2.6 $\mathbf{p}(\mathcal{G}^2)$ is dominated by Λ_i.

It means there is at least one convex combination of some elements $\mathbf{u}_{C,\emptyset}$ of Λ'_i which dominates $\mathbf{p}(\mathcal{G}^2)$. We choose that one of those convex combinations, which is in some sense minimal: there are no vectors with coefficient zero (they could be deleted), and if $C' \subset C$ and the coefficient of $\mathbf{u}_{C,\emptyset}$ is not zero, then we cannot change $\mathbf{u}_{C,\emptyset}$ to $\mathbf{u}_{C',\emptyset}$ without violating the domination. This convex combination can be achieved by deleting and changing vectors.

In this convex combination each C is clearly zero in every coordinate where $\mathbf{p}(\mathcal{G}^2)$ is zero. Similarly we can choose a convex combination of vectors from $\Gamma_{2l+1-2i}$ which dominates $\mathbf{p}(\mathcal{G}^1)$ and each of them has zero in every coordinate where $\mathbf{p}(\mathcal{G}^1)$ is zero.

Hence the following is true: if $\mathbf{u}_{C,\emptyset}$ is a vector with non-zero coefficient in the convex combination which dominates $\mathbf{p}(\mathcal{G}^2)$ and $\mathbf{u}_{A,B}$ is a vector with non-zero coefficient from the convex combination which dominates $\mathbf{p}(\mathcal{G}^1)$, then $(A \cup B) \cap C = \emptyset$. Thus $\mathbf{u}_{A,B} + \mathbf{u}_{C,\emptyset} = \mathbf{u}_{A\cup C,B}$ is in Γ_{2l+1}. It is easy to see that the sum of $\mathbf{p}(\mathcal{G}^1)$ and $\mathbf{p}(\mathcal{G}^2)$ (which is $\mathbf{p}(\mathcal{G})$) is dominated by a convex combination of the sum of the $\mathbf{u}_{A,B}$s and $\mathbf{u}_{C,\emptyset}$s, and these sums are all in Γ_{2l+1}. ■

Proof of Lemma 2.3. If $\mathbf{u}_{A,B} \in \Gamma_k$, then the family $\mathcal{G}_{A,B}$ shows that $\mathbf{u}_{A,B}$ is a profile vector.

In order to prove that these are the extreme points, we use induction on k. As it was mentioned before, the case $k = 1$ is known (see [3]), the case $k = 0$ is trivial. If $\emptyset$ and/or $[n]$ are in the family, the other sets form a $(k-2)$- or $(k-1)$-antichainpair family, so by induction it is enough to prove the statement for the reduced profiles. Lemma 2.4 and Lemma 2.6 finishes the proof in case k is even, hence from now on we suppose $k = 2l + 1$.

Let $\mathcal{G}$ be a family, which does not contain $\emptyset$ and $[n]$. If there is a complete level (all i element intervals), which is a subfamily of $\mathcal{G}$, let $\mathcal{G}^2$ be this level, and $\mathcal{G}^1$ be the family of the other sets in $\mathcal{G}$. By Lemma 2.8 and the induction hypothesis we are done in this case. Hence we can assume that there is no complete level in $\mathcal{G}$.

Suppose indirectly that $\mathbf{p}(\mathcal{G})$ is not in $\mathrm{conv}(\Gamma_k)$. Let $m = \min\{|G| : G \in \mathcal{G}\}$ and $M = \max\{|G| : G \in \mathcal{G}\}$. We can assume that $m \le n - M$ (otherwise we can replace all elements by their complements, then we get a convex combination and there we can replace every coordinate i by coordinate $n-i$). If $|\mathcal{G}| \le nl + m$ then Lemma 2.7 finishes the proof. Hence we can also assume that $|\mathcal{G}| > nl + m$.

If $m \geq \frac{n-l}{2}$, then Lemma 2.5 implies $|\mathcal{G}| \leq nl+m$, which is a contradiction. Indeed, $|\mathcal{G}| \leq \lfloor n/2 \rfloor + l(n-1) + \lfloor l/2 \rfloor = ln + \lfloor n/2 \rfloor - \lceil l/2 \rceil \leq ln + \lceil \frac{n-l}{2} \rceil \leq nl + m$.

From now on we suppose $m < \frac{n-l}{2}$. We change $\mathcal{G}$ to get $\mathcal{G}'$. The skeleton of the algorithm is the following: we replace the intervals of minimum size by either their complements or their shade. If some new intervals have already been in the family, we repeat this procedure using them instead of the intervals of minimum size. This algorithm ends in at most k steps. After that the family is still k-antichainpair and the minimum size of intervals has increased. We might repeat this whole procedure several times.

Let $\mathcal{G}_0 = \mathcal{G}$, let m_i and M_i be the size of minimal, resp. the maximal elements of $\mathcal{G}_i$. We get $\mathcal{G}_{i+1}$ by the following steps. At first we have some initial steps.

Step 0a. If $m_i > n - M_i$, then replace all M_i-element sets by their complements. After that, using again the same letters for the maximum and the minimum size, $m_i \leq n - M_i$. If there are no $n - m_i$ elements, then let $\mathcal{G}_i^1$ denote the resulting family. Otherwise, we have a Step 0b:

Step 0b. If there is a pair $G, \overline{G} \in \mathcal{G}_i$ such that $|G| = m_i$, then delete all the m_i-element sets whose complements are not in $\mathcal{G}_i$, and add their complements, we denote the new family by $\mathcal{G}_i^1$. Otherwise replace all $n-m_i$-element sets by their complements, and let $\mathcal{G}_i^1$ be the resulting family.

Let $\mathcal{D}_i^1$ be the family of m_i-element sets in $\mathcal{G}_i^1$. Note that after these steps the family $\mathcal{G}_i^1$ is k-antichainpair and $m_i \leq n - M_i$.

Step 1. Let $\mathcal{G}_i^2 = (\mathcal{G}_i^1 \cup \nabla_{\text{int}} \mathcal{D}_i^1) \setminus \mathcal{D}_i^1$, and $\mathcal{D}_i^2 = \mathcal{G}_i^1 \cap \nabla_{\text{int}} \mathcal{D}_i^1$. What happens is that $\mathcal{D}_i^1$ is replaced by its shade. Some of the new members are already in $\mathcal{G}_i^1$, we count them twice: once in $\mathcal{G}_i^2$ and once in $\mathcal{D}_i^2$.

We say the algorithm meets the level $m+1$ in this step, since the size of the new members is $m+1$. Note that since $\mathcal{D}_i^1$ is not empty and not the full level, we know $|\nabla_{\text{int}} \mathcal{D}_i^1| \geq |\mathcal{D}_i^1|$, which implies $|\mathcal{G}_i^2| + |\mathcal{D}_i^2| > |\mathcal{G}_i^1|$.

Step 2j. Let $\mathcal{G}_i^{2j+1} = \mathcal{G}_i^{2j} \cup \text{co}(\mathcal{D}_i^{2j})$ and $\mathcal{D}_i^{2j+1} = \text{co}(\mathcal{G}_i^{2j}) \cap \mathcal{D}_i^{2j}$. We say the algorithm meets the level $n-m-j$ in this step, since the size of the new members is $n-m-j$.

Step 2j+1 (j>0). Let $\mathcal{G}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cup \nabla_{\text{int}} \mathcal{D}_i^{2j+1}$ and $\mathcal{D}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cap \nabla_{\text{int}} \mathcal{D}_i^{2j+1}$. We say the algorithm meets the level $m+j+1$ in this step, since the size of the new members is $m+j+1$.

Note that if a set D is in $\mathcal{D}_i^j$, then a subset or the complement of it is in $\mathcal{D}_i^{j-1}$. Moreover, we can easily find a chain-pair containing D and intersecting every $\mathcal{D}_i^t$ with $t < j$. This implies that if $\mathcal{G}_i$ is k-antichainpair, then $\mathcal{D}_i^k = \emptyset$. Let us finish the process immediately when $\mathcal{D}_i^j = \emptyset$ for some j and let $\mathcal{G}_i^* = \mathcal{G}_i^j$. If $\mathcal{G}_i$ is $2l'+1$-antichainpair, then there are at most $2l'+1$

steps, so the cardinality of a new interval is either at most $m_i + l'$ or at least $n-(m_i+l'-1)$. An important observation: there are no m_i-element intervals in $\mathcal{G}_i^*$.

Last step. For every j, if there are n j-element intervals in $\mathcal{G}_i^*$, delete all of them (delete the whole levels). This way we get $\mathcal{G}_{i+1}$.

Let q_i be the number of deleted levels before we get $\mathcal{G}_i$. We iterate this algorithm while $m_i < \frac{n-l-2q_i}{2}$.

Claim 1. If $\mathcal{G}_i$ is $(2l'+1)$-antichainpair and $m_i < \frac{n-l'}{2}$, then $\mathcal{G}_i^*$ is $(2l'+1)$-antichainpair, too.

Proof of Claim 1. The rather technical proof is based on the following simple observations. Suppose there is a forbidden configuration in $\mathcal{G}_i^*$. It contains a set $G \in \mathcal{G}_i^* \setminus \mathcal{G}_i$, then that is a member of $\mathcal{D}_i^j$ for some j, and then there is a chain-pair $\mathcal{K}$ containing D and intersecting every $\mathcal{D}_i^t$ with $t < j$. As we will see, we can replace G and a part of the forbidden configuration with those intersections of the $\mathcal{D}_i^j$s and $\mathcal{K}$ such that we get another forbidden configuration with less new elements.

Suppose indirectly, that there are $A_1 \subset \cdots \subset A_x$ and $B_1 \subset \cdots \subset B_y$ chains in $\mathcal{G}_i^*$, such that $x + y = 2l'+$ and there is a chain $\mathcal{C} = \{C_0, \ldots, C_n\}$, such that every A_j and every $\overline{B}_j$ is an element of $\mathcal{C}$. For example A_z is not in $\mathcal{G}_i$. Let the size of A_z be $m_i + l^*$, and $|B_{w-1}| < n - m_i - l^* \leq |B_w|$.

Our algorithm gives A_z. Let us suppose it is in the Step 2j+1, then there are intervals $D_{m_i} \subset D_{m_i+1} \subset, \ldots, \subset D_{m_i+l^*-1}$ in $\mathcal{G}_i$, such that their complements are in $\mathcal{G}_i$ too, $|D_t| = t$ for every t and $D_{m_i+l^*-1} \subset A_z$. (In this case $l^* = j$.) Clearly, $D_{m_i}, \ldots, D_{m_i+l^*-1}, C_{m_i+l^*}, \ldots, C_n$ form a part of a chain $\mathcal{C}'$. Therefore

$$\mathcal{G}_i \cap (\mathcal{C}' \cup co(\mathcal{C}')) \supseteq \{D_{m_i}, \ldots, D_{m_i+l^*-1}, A_{z+1}, \ldots, A_x,$$

$$B_1, \ldots, B_{w-1}, \overline{D}_{m_i+l^*-1}, \ldots \overline{D}_{m_i+1}\}.$$

These are at least $2l'+$ intervals. The cardinalities of $A_1, \ldots A_z$ are at least $m_i + 1$ and at most $m_i + l^*$, and we replace these intervals by $D_{m_i}, \ldots, D_{m_i+l^*-1}$, so we get l^* intervals in place of at most l^* intervals. Similarly, the cardinalities of $B_w, \ldots, B_y$ are at least $n - m_i - l^*$ and at most $n - m$ or $n - m - 1$, depending on Step 0b. We replace these intervals by $\overline{D}_{m_i+l^*-1}, \ldots \overline{D}_{m_i+1}$ and maybe $\overline{D}_{m_i}$, depending on Step 0b. We get $l^* + 1$ or l^* intervals in place of at most $l^* + 1$ or l^* intervals.

Hence it is a forbidden configuration (in $\mathcal{G}_i \cup \mathcal{G}_i^*$), and less new elements are in it. We repeat this procedure until there are no new elements, which is a contradiction.

If after some repeats all the remaining intervals from $\mathcal{G}_i^* \setminus \mathcal{G}_i$ were given in the Step 2j, the procedure is similar.

Sketch: We can replace $A_z, \dots, A_y$ and $B_1, \dots, B_{w-1}$ by a part of a chain-pair, which is given by our algorithm. Then we get a forbidden configuration with less new elements. $\square$

Let us continue with some simple observations.

Observation 1. $\min\{|G| : G \in \mathcal{G}_i^*\} = m_i + 1$.

Observation 2. $\min\{|G| : G \in \mathcal{G}_{i+1}\} > m_i$.

Observation 3. $M_{i+1} \leq M_i$.

Note that $m_i < \frac{n-l-2q_i}{2}$ is used here. Also note that M_i can decrease only in Step 0a.

Claim 2. Let $j > 0$. Then $|\mathcal{G}_i^{2j+2}| + |\mathcal{D}_{2j+2}| > |\mathcal{G}_i^{2j+1}| + |\mathcal{D}_{2j+1}|$.

Proof of Claim 2. By definition $\mathcal{G}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cup \nabla_{\text{int}} \mathcal{D}_i^{2j+1} \setminus \mathcal{D}_i^{2j+1}$ and $\mathcal{D}_i^{2j+2} = \mathcal{G}_i^{2j+1} \cap \nabla_{\text{int}} \mathcal{D}_i^{2+1}$. Hence $|\mathcal{G}_i^{2j+2}| + |\mathcal{D}_i^{2j+2}| = |\mathcal{G}_i^{2j+1}| + |\nabla_{\text{int}} \mathcal{D}_i^{2j+1}|$. We have to show that $|\nabla_{\text{int}} \mathcal{D}_i^{2j+1}| > |\mathcal{D}_i^{2j+1}|$. As it was mentioned, the operator ∇_{int} increases the size of a non-empty family except the case that the family is a whole level. Suppose indirectly that is the case, $\mathcal{D}_i^{2j+1}$ is a whole level. By definition $\mathcal{D}_i^{2j+1} = \text{co}(\mathcal{G}_i^{2j}) \cap \mathcal{D}_i^{2j}$, hence in this case $\mathcal{D}_i^{2j}$ is a whole level too and $\mathcal{G}_i^{2j-1}$ contains its complement level.

A pair of complement levels $p, n-p$ with $p < n-p$ can be involved at most twice in the algorithm. At first the level p is met by the algorithm in an odd step, and the complement level in the next step. Then the level $n-p$ can be met again, this time in an odd step and then the level p in the next step.

In our case it means that $\mathcal{D}_i^{2j}$ is the level $n-p$, and the level p becomes full when it is first met (in an odd step). Moreover, the level $n-p$ becomes full in the next step. But it would mean that the level p cannot remain full, which is a contradiction. $\square$

Observation 4. $|\mathcal{G}_i^*| > |\mathcal{G}_i|$.

Claim 3. $|\mathcal{G}_i| \geq |\mathcal{G}| + m_i - m - q_i n$.

Proof of Claim 3. We use induction on i. The case $i = 0$ is trivial. It is enough to prove, that $|\mathcal{G}_i| \geq |\mathcal{G}_j| + m_i - m_j - (q_i - q_j)n$ for a $j < i$.

If there is a number $j < i$ and an interval $G \in \mathcal{G}_i \setminus \mathcal{G}_j$, then let j be the biggest such number. G is an interval of size at least m_i, or a complement of an interval of size at least m_i, hence there are at least $2(m_i - m_j)$ steps in the $j+1$st iteration. There are at least $m_i - m_j$ odd steps, and Claim 2 shows in these steps the size is increased by at least 1. All the decreasing is $(q_i n - q_j n)$ so the change of the size between $\mathcal{G}_j$ and $\mathcal{G}_i$ is at least $m_i - m_j - (q_i n - q_j n)$, and the proof is done.

If $\mathcal{G}_i \not\subseteq \mathcal{G}$, then there is a $G \in \mathcal{G}_i$ which is not in $\mathcal{G} = \mathcal{G}_0$, hence $j = 0$ finishes the proof.

If $\mathcal{G}_i \subseteq \mathcal{G}$, then all the new intervals have been deleted, some of them as a member of a whole level, others as intervals with minimum size. $\mathcal{G}^1 := \mathcal{G}_i$ and

$\mathcal{G}^2 := \mathcal{G} \setminus \mathcal{G}_i$. Clearly $|\mathcal{G}^1| \geq \mathcal{G} - q_i n$, since the size can decrease only when whole levels are deleted, and the entire decreasing is $q_i n$. Thus $|\mathcal{G}^2| \leq q_i n$. If there are no $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2|$, we can apply Lemma 2.8, and we proved Lemma 3, which is a contradiction (we supposed indirectly, that the lemma is not true). If there are $G_1 \in \mathcal{G}^1$ and $G_2 \in \mathcal{G}^2$ such that $|G_1| = |G_2| = a$, then $a \geq m_i$. Then G_2 could not be deleted as an interval on minimum size, only as a member of a full level. Hence all the a-element intervals were deleted at least once during the algorithm, i.e. they are members of $\mathcal{G}^*_{j-1} \setminus \mathcal{G}_j$ for a $j < i$. Then $G_1 \in \mathcal{G}_i \setminus \mathcal{G}_j$, and this finishes the proof. □

It is important to see, that this claim is not true in general, it follows from the indirect assumption.

As it was mentioned, we iterate this algorithm while $m_i < \frac{n-l-2q_i}{2}$. By Observations 2 and 3 it cannot go forever, finally we get a family violating this property, i.e. a $2l'+1$-antichainpair, such that the cardinalities of the intervals are at least $\lceil \frac{n-l'}{2} \rceil$ and at most $\lceil \frac{n+l'}{2} \rceil$. We denote this family by $\mathcal{G}'$. There are no whole levels in $\mathcal{G}'$. By Lemma 2.5 we know $|\mathcal{G}'| \leq \lfloor n/2 \rfloor + l'(n-1) + \lfloor l'/2 \rfloor$.

Obviously $\mathcal{G}' = \mathcal{G}_i$ for some i. Thus $|\mathcal{G}'| \geq |\mathcal{G}| + m_i - m - q_i n$. Clearly $q_i = l - l'$ and $m_i = \lceil \frac{n-l'}{2} \rceil$. Hence

$$|\mathcal{G}| \leq |\mathcal{G}'| - \lceil \frac{n-l'}{2} \rceil + m + nl - l'n \leq \lfloor n/2 \rfloor + l'(n-1) + \lfloor l'/2 \rfloor - \lceil \frac{n-l'}{2} \rceil + m + nl - l'n$$

$$\leq m + nl + \lfloor n/2 \rfloor - l' + \lfloor l'/2 \rfloor - \lceil \frac{n-l'}{2} \rceil \leq m + nl,$$

which is a contradiction. ■

Proof of Theorem 2.2. One can easily see, that it is enough to prove the theorem for reduced profiles. If $\mathbf{x}_{A,B}$ is one of the listed vectors, $\mathcal{F}_{A,B}$ is k-antichainpair, hence these vectors are profile vectors. Now we can apply Lemma 2.3 and Theorem 1.3, and we are done.■

3 Corollaries

In this section we determine the extreme points for some other classes of families.

Theorem 3.1. *The essential extreme points of the complement-free k-antichainpair families are $\mathbf{x}_{A,B}$ where $2|A| + |B| = k$, $0, n/2, n \notin A$, $|B \setminus \{0, n/2, n\}| \leq 1$, and $i \in A \cup B$ implies $n - i \notin A \cup B$ except for $i = n/2$.*

Proof. It is easy to see, that these are essential extreme points. Let w be a positive weight function and $\mathcal{F}$ be an optimal family for this weight. By

Lemma 1.2 it is enough to show a complement-free k-antichainpair family with the same weight such that its profile is listed in the theorem.

For any A, only one of A or $\overline{A}$ can belong to $\mathcal{F}$. It is the one which has greater weight, as replacing A by $\overline{A}$ would increase the weight otherwise, without violating the property. If $w(i) < w(n-i)$ then clearly $\mathcal{F}$ does not contain i element sets, if $w(i) = w(n-i)$, then we choose one of them, i, and replace all $n-i$ element sets of $\mathcal{F}$ to its complement. It does not change the weight, and $\mathcal{F}$ remains complement-free k-antichainpair.

Let $w'(i) = 0$, if $w(i) < w(n-i)$ or if $w(i) = w(n-i)$ and $i < n-i$. Otherwise let $w'(i) = w(i)$. Then the optimal complement-free k-antichainpair family for this weight will be also optimal for w, hence from now on we will deal only with w'. Let $\mathcal{K}_0$ be an optimal k-antichainpair for the weight w', and delete all the sets with weight 0. Then we get a family $\mathcal{K}_1$, which is also optimal for this weight, and almost complement-free: if A and $\overline{A}$ both are in $\mathcal{K}_1$, then $|A| = n/2$.

If n is odd, we are done: $\mathcal{K}_1$ is maximal for w', and one can easily see, that it is maximal for w, and its profile is listed in the theorem.

If n is even, we can assume, that all the $n/2$-element sets of $\mathcal{F}$ contain 1.

Case 1. $\mathcal{K}_1$ does not contain all the $n/2$ element sets. Then it contains at most $\binom{n-1}{n/2-1}$ members of size $n/2$. Its profile vector is $\mathbf{x}_{A,B}$. We can assume by Theorem 2.2 that this is the family $\mathcal{F}_{A,B}$ from Section 2 which was the main example for families with this profile vector. Then it is complement-free, and we are done.

Case 2. The optimal (for w') $k{+}1$-antichainpair, $\mathcal{K}_2$ contains $n/2$ element sets. Then let $\mathcal{A} = \{A \subset [n] : |A| = n/2,\ 1 \notin A\}$. We can assume by Theorem 2.2 that $\mathcal{K}_2$ contains $\mathcal{A}$. Let $\mathcal{K}_3 = \mathcal{K}_2 \setminus \mathcal{A}$, then it is a complement-free k-antichainpair and its profile vector is listed in the theorem. Let $\mathcal{F}' = \mathcal{F} \cup \mathcal{A}$. Then $w'(\mathcal{F}) = w'(\mathcal{F}') - \binom{n}{n/2}w'(n/2) \leq w'(\mathcal{K}_2) - \binom{n}{n/2}w'(n/2) = w'(\mathcal{K}_2 \setminus \mathcal{A})$, hence $\mathcal{K}_2 \setminus \mathcal{A}$ is (also) an optimal family, which has profile listed in the theorem.

Case 3. The optimal for w' k-antichainpair $\mathcal{K}_1$ contains all $n/2$ element sets, and the optimal $k{+}1$-antichainpair $\mathcal{K}_2$ does not contain any $n/2$ element sets. Let the profile vector of $\mathcal{K}_1$ (resp. $\mathcal{K}_2$) be $x_{A,B}$ ($x_{C,D}$).

Case 3.1. $|C| \geq |A|$. We know that $n/2 \in A \setminus C$, hence there is a $j \in C \setminus A$, where $j \neq 0, n/2, n$. If $\binom{n}{n/2}w'(n/2) \leq \binom{n}{j}w'(j)$, then we can replace the $n/2$ element sets in $\mathcal{K}_1$ by the j element sets, we found a family which is optimal for w' and does not contain all the $n/2$-element sets, and then we can apply Case 1. If $\binom{n}{n/2}w'(n/2) \geq \binom{n}{j}w'(j)$, then we can replace the j element sets in $\mathcal{K}_2$ by the $n/2$ element sets, and we can apply Case 2.

Case 3.2. $|C| < |A|$. $|A| \leq \lfloor k/2 \rfloor$, hence $|C| \leq \lfloor k/2 \rfloor - 1$. On the other hand $2|C| + |D| = k + 1$ and $|D| \leq 3$. It is possible only if $|A| = \lfloor k/2 \rfloor$ and $|C| = \lfloor k/2 \rfloor - 1$. Moreover $2|A| + |B| = k$, hence $|D| - |B| = 3$. It means $|D| = 3$ and $|B| = 0$. Then $\emptyset$ and $[n]$ are in $\mathcal{K}_2$ and not in $\mathcal{K}_1$. The family of $n/2$-element sets is not in $\mathcal{K}_2$, hence the weight w' of all the $n/2$-element sets is not more than the weight of $\emptyset$ and $[n]$ (otherwise we could exchange them). Then we can replace the $n/2$ element sets in $\mathcal{K}_1$ by $\emptyset$ and $[n]$, and we can apply Case 1. ■

Definition 3.2. *A family $\mathcal{F} \subseteq 2^{[n]}$ is called r-**complement-chain-pair-free** if there is no chain $A_1 \subsetneq A_2 \subsetneq \cdots \subsetneq A_r$ in $2^{[n]}$ such that all sets A_i and $\overline{A_i}$ belong to $\mathcal{F}$.*

The maximum size of families satisfying this property is known ([1]).

Theorem 3.3. *The essential extreme points of the profile-polytope of the class of r-complement-chain-pair-free families are $\mathbf{x}_{A,B}$, where $2|A| + |B| = n + r$, $i \notin A$ implies $n - i \in A$, except for $i = n/2$. In addition $0, n \notin B$, $|B \setminus \{n/2\}| \leq 1$.*

Proof. We determine the reduced essential extreme points, the rest of the proof is trivial. We will use Lemma 1.2. Let w be a positive weight function and $\mathcal{F}$ be an optimal r-complement-chain-pair-free family. We can assume that $\mathcal{F}$ is maximal, i.e. if $\mathcal{F} \cup \{F\}$ is r-complement-chain-pair-free, then $F \in \mathcal{F}$. It follows easily, that if $A \notin \mathcal{F}$, then $\overline{A} \in \mathcal{F}$. Moreover, replacing A by $\overline{A}$ does not violate the property. So $\mathcal{F}$ contains at least one element of every pair of complements, of course the one which has greater weight. If $w(i) = w(n-i)$, then choose one, for example i, and from all pairs $A, \overline{A}$ if $|A| = i$ and $A \notin \mathcal{F}$, then replace $\overline{A}$ by A in $\mathcal{F}$.

Let $\mathcal{F}_0$ be a subfamily of $\mathcal{F}$ which contains exactly one of A and $\overline{A}$ for every complement pair. It can be chosen such a way that there are only whole levels in $\mathcal{F}_0$, except for the level $n/2$.

Let $\mathcal{F}_1 = \mathcal{F} \setminus \mathcal{F}_0$, it is a complement-free r-antichainpair. Let $w'(i) = 0$ if all the i element sets are in $\mathcal{F}_0$, and $w'(i) = w(i)$ otherwise. Clearly the weight of $\mathcal{F}_1$ does not change. Theorem 3.1 says, what vectors $\mathbf{x}_{A,B}$ are maximal for w'; clearly we can suppose that if $w'(i) = 0$ then $i \notin A \cup B$. It is easy to see, that $\mathbf{p}(\mathcal{F}_0) + \mathbf{x}_{A,B}$ is listed in the theorem.■

The extreme points of the profile-polytope of the k-Sperner and the complement-free Sperner families are known ([5] and [3]). Moreover, the extreme points of the complement-free k-Sperner families are also determined ([6]) in the case n is odd.

Theorem 3.4. *The essential extreme points of the profile-polytope of the complement-free k-Sperner families are $\mathbf{x}_{A,B}$, where $2|A| + |B| = 2k$, $0, n \notin B$, $i \in A \cup B$ implies $n - i \notin A \cup B$ except for $i = n/2$, $n/2 \notin A$ and $|B \setminus \{n/2\}| \leq 1$.*

Proof. Clearly, $\mathcal{F}_{A,B}$ is complement-free k-Sperner with such profile. One can easily see, that it is enough to prove the theorem for reduced profiles. The complement-free k-Sperner families are complement-free $2k$-antichainpairs, which by Theorem 3.1 gives the statement. ■

One can easily see that Theorem 1.6 is a simple corollary of the previous theorem. Indeed, an intersecting family is complement-free, thus we can apply it, and simple calculation gives the result using the weight $w = 1$.

We call a family $\mathcal{F}$ *self-complementary* if $F \in \mathcal{F}$ implies $\overline{F} \in \mathcal{F}$. The maximal size of self-complementary k-Sperner families is determined in [1], the extreme points of the self-complementary Sperner families are also known ([3]).

Theorem 3.5. *The extreme points of the profile-polytope of the self-complementary k-Sperner families are $\mathbf{x}_{A,B}$, where $2|A| + |B| \leq 2k$, $0, n/2, n \notin B$, $i \in A$ implies $n - i \in A$, $i \in B$ implies $n - i \in B$ and either $|B| = 2$ and $|A| = k - 1$ or $|B| = 0$.*

Proof. Let Γ be the set of the vectors $\mathbf{x}_{A,B}$, where $2|A| + |B| \leq 2k$, $0, n/2, n \notin B$, $i \in A$ implies $n - i \in A$, $i \in B$ implies $n - i \in B$ and $|B| \leq 2$. Clearly these are the same properties as those in the theorem, except for the last one. Hence Γ contains more vectors, but they are in the convex hull of the vectors listed in the theorem. We show the following, equivalent statement: for any weight w the weight of a vector in Γ cannot exceed the weight of every vector listed in the theorem.

Let w be a weight function and define $w'(i) = w'(n-i) = \frac{w(i)+w(n-i)}{2}$. Clearly the weight of a self-complementary family does not change. The maximal weight in Γ is $w(\mathbf{x}_{A,B}) = w'(\mathbf{x}_{A,B}) = w'(\mathcal{F}_{A,B})$, and we can suppose $\mathcal{F}_{A,B}$ does not contain any sets of negative weight w' (it might contain some sets F such that $w(F) < 0$). If $\mathbf{x}_{A,B}$ is not listed in the theorem, than $|B| = 2$ and $|A| < k - 1$. But than we could add the other sets from the levels contained in B without violating the property or decreasing the weight.

Thus it is enough to prove that all the profile vectors of self-complementary k-Sperner families are in the convex hull of Γ.

Let $\mathcal{F}$ be the maximal for w' self-complementary k-Sperner family. $\mathcal{F}$ contains pairs $F, \overline{F}$. We define a partition of $\mathcal{F}$: for all the pairs $F, \overline{F}$ if $|F| < |\overline{F}|$ or $|F| = |\overline{F}|$ and $1 \in F$ then $F \in \mathcal{F}^1$ and $\overline{F} \in \mathcal{F}^2$. $\mathcal{F}^1$ and $\mathcal{F}^2$ are complement-free k-antichainpairs, such that all members of $\mathcal{F}^1$ (resp. $\mathcal{F}^2$) are of size at most (at least) $n/2$. Let $w''(i) = w'(i)$ if $i \leq n/2$ and 0 otherwise, and $w'''(i) = w''(n-i)$. Clearly $w''(\mathcal{F}^1) = w'(\mathcal{F}^1)$ and $w'''(\mathcal{F}^2) = w'(\mathcal{F}^2)$. Theorem 3.1 gives the maximal for w'' complement-free k-antichainpair (here the weight can be negative, so we have to use (i) of Proposition 1.1). It is easy to see, that the complement of this family is the maximal for w''' complement-free k-antichainpair. The union of this complement-free k antichainpair and its complement family is a self-complementary k-Sperner, and the profile vector of it is in Γ. Its weight is at least the weight of $\mathcal{F}$, hence we can apply Lemma 1.2 to finish the proof. ■

References

[1] A. Bernáth, D. Gerbner: Chain intersecting families, *Graphs and Combinatorics*, **23** (2007), 353-366.

[2] K. Engel: Sperner Theory, Encyclopedia of Mathematics and its Applications, 65. Cambridge University Press, Cambridge (1997)

[3] K. Engel, P.L. Erdős: Sperner families satisfying additional conditions and their convex hulls, *Graphs Combin.* **5** (1989), 47-56.

[4] P.L. Erdős, P. Frankl, G.O.H. Katona: Intersecting Sperner families and their convex hulls, *Combinatorica*, **4** (1984), 21-34.

[5] P.L. Erdős, P. Frankl, G.O.H. Katona: Extremal hypergraph problems and convex hulls,*Combinatorica*, **5** (1985), 11-26.

[6] D. Gerbner, B. Patkós: l-chain profile vectors, *SIAM Journal on Discrete Mathematics* **22** (2008), 185-193.

[7] P. Frankl, G.O.H. Katona: Polytopes determined by hypergraph classes, *European J. Combin.* **6 3** (1985), 233–243.

[8] G.O.H. Katona: A Simple proof of the Erdős-Chao Ko-Rado theorem, *J. Comb. Theory (B)*, **13** (1972), 183-184.

[9] E.C. Milner: A combinatorial theorem on systems of sets, *J. London Math. Soc.*, **43** (1968), 204-206.