A PROOF OF BURNSIDE'S FORMULA

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August 11, 2017

Theorem. *For any positive integer n, we have*

$$\log n! = \left(n + \frac{1}{2}\right)\log\left(n + \frac{1}{2}\right) - \left(n + \frac{1}{2}\right) + \frac{1}{2}\log(2\pi) - \sum_{j=1}^{\infty}\frac{\zeta(2j, n+1)}{2^{2j}2j(2j+1)},$$

where $\zeta(s,a) = \sum_{m=0}^{\infty}(m+a)^{-s}$ is the Hurwitz zeta function.

Proof. If k is a positive integer, then

$$\int_{k-1/2}^{k+1/2}\log t\,\mathrm{d}t = \int_{k}^{k+1/2}\log t\,\mathrm{d}t + \int_{k-1/2}^{k}\log t\,\mathrm{d}t = \int_0^{1/2}\log(k+t)\mathrm{d}t + \int_0^{1/2}\log(k-t)\mathrm{d}t$$
$$= \int_0^{1/2}\log(k^2-t^2)\mathrm{d}t = \log k + \int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t.$$

Employing this expression, we find

$$\log n! = \sum_{k=1}^{n}\log k = \sum_{k=1}^{n}\int_{k-1/2}^{k+1/2}\log t\,\mathrm{d}t - \sum_{k=1}^{n}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t$$
$$= \int_{1/2}^{n+1/2}\log t\,\mathrm{d}t - \sum_{k=1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t + \sum_{k=n+1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t$$
$$= \int_{1/2}^{n+1/2}\log t\,\mathrm{d}t - \int_0^{1/2}\log\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t + \sum_{k=n+1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t.$$

By the well-known product formula for the sine function,

$$\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2}\right) = \frac{\sin(\pi t)}{\pi t}$$

and hence,

$$\int_0^{1/2}\log\prod_{k=1}^{\infty}\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t = \int_0^{1/2}\log\frac{\sin(\pi t)}{\pi t}\mathrm{d}t = \int_0^{1/2}\log\sin(\pi t)\mathrm{d}t - \int_0^{1/2}\log(\pi t)\mathrm{d}t$$
$$= \int_0^{1/2}\log\sin(\pi t)\mathrm{d}t + \frac{1}{2} - \frac{1}{2}\log\left(\frac{\pi}{2}\right).$$

Applying symmetry and the double angle formula, we obtain

$$\int_0^{1/2}\log\sin(\pi t)\mathrm{d}t = \frac{1}{2}\int_0^{1}\log\sin(\pi t)\mathrm{d}t = \int_0^{1/2}\log\sin(2\pi t)\mathrm{d}t$$
$$= \frac{1}{2}\log 2 + \int_0^{1/2}\log\sin(\pi t)\mathrm{d}t + \int_0^{1/2}\log\cos(\pi t)\mathrm{d}t = \frac{1}{2}\log 2 + 2\int_0^{1/2}\log\sin(\pi t)\mathrm{d}t.$$

Re-arranging this equation, we find

$$\int_0^{1/2}\log\sin(\pi t)\mathrm{d}t = -\frac{1}{2}\log 2.$$

Collecting all the partial results, we have

$$\begin{aligned}\log n! &= \int_{1/2}^{n+1/2} \log t \mathrm{d}t + \frac{1}{2}\log\pi - \frac{1}{2} + \sum_{k=n+1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t \\ &= \left(n+\frac{1}{2}\right)\log\left(n+\frac{1}{2}\right) - \left(n+\frac{1}{2}\right) + \frac{1}{2}\log(2\pi) + \sum_{k=n+1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t.\end{aligned}$$

Using the power series expansion of the logarithm, we deduce

$$\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t = -\int_0^{1/2}\sum_{j=1}^{\infty}\frac{1}{j}\frac{t^{2j}}{k^{2j}}\mathrm{d}t = -\sum_{j=1}^{\infty}\frac{1}{j}\frac{1}{k^{2j}}\int_0^{1/2}t^{2j}\mathrm{d}t = -\sum_{j=1}^{\infty}\frac{1}{2^{2j}2j(2j+1)}\frac{1}{k^{2j}}.$$

Consequently,

$$\begin{aligned}\sum_{k=n+1}^{\infty}\int_0^{1/2}\log\left(1-\frac{t^2}{k^2}\right)\mathrm{d}t &= -\sum_{k=n+1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{2^{2j}2j(2j+1)}\frac{1}{k^{2j}} = -\sum_{j=1}^{\infty}\frac{1}{2^{2j}2j(2j+1)}\sum_{k=n+1}^{\infty}\frac{1}{k^{2j}} \\ &= -\sum_{j=1}^{\infty}\frac{1}{2^{2j}2j(2j+1)}\sum_{m=0}^{\infty}\frac{1}{(m+n+1)^{2j}} = -\sum_{j=1}^{\infty}\frac{\zeta(2j,n+1)}{2^{2j}2j(2j+1)},\end{aligned}$$

which completes the proof. ■