Combinatorics and Graph Theory II

Midterm 1, October 11, 2023, 12.15-13.45

Grading guide

It is a very rough grading guide. A statement, scored in the guide, does not get the points automatically, only if it is used properly. Solutions, different from the presented one, are also scored similarly.

1. Give all simple plane graphs G (graphs drawn in the plane with no crossings) with $e(G) = f(G) + 1$.

By the Euler formula, $n - e + f = k + 1$. 3 points

Substituting $e - 1$ for f we get that $n - 1 = k + 1$, so $k = n - 2$. 3 points

There are three possibilities: G contains a triangle and $n - 3$ isolated points, or G contains two independent edges and $n - 4$ isolated points, or two edges with a common endpoint and $n - 3$ isolated vetices. 3 points

Each satisfy the conditions, in the first case $e = 3$, $f = 2$, in the second case $e = 2$, $f = 1$, in the third case $e = 2$, $f = 1$. 1 point

2. G is a graph, α is an edge of it. The graph $G \setminus \alpha$ (α removed from G) is planar. Show that $\chi(G) \leq 5$.

The graph $G \setminus \alpha$ is planar, so by the Four Color Theorem $\chi(G \setminus \alpha) \leq 4$. 5 points

Take a four coloring of $G \setminus \alpha$ and add the edge α. Its two endpoints might have the same color. Color one of its endpoints with a new, fifth color. This is a proper coloring. 5 points

3. Suppose that G is a bipartite, simple, connected plane graph. Its dual, G^*, is also a bipartite graph. Prove that G^* cannot be simple.

Since G is connected, it does not have isolated vertices. 1 point

If G had a vertex of degree 1, G^* would have a loop, which is impossible since G is bipartite. (Or: G had a vertex of degree 1, G^* would have a loop, so G^* is not simple are we are done.) 1 point

If G had a vertex of degree 2, G^* would have two parallel edges, so G^* is not simple are we are done. 1 point

If G had a vertex of degree 3, G^* would have a triangle, which is impossible since G is bipartite. 2 points

So, all degrees, $d_i \geq 4$. 2 points

Therefore, $2e = \sum_{i=1}^{n} d_i \geq 4n$, so $e \geq 2n$ which is impossible for a simple bipartite graph. This proves that G^* cannot be simple. 3 points

4. Let $n > 0$, The vertices of G are $v_1, v_2, \ldots, v_n$. v_i and v_j are connected by an edge if $|i - j| - 1$ is divisible by 3. For which values of n will G be perfect?

For $n \leq 4$ it is perfect, since all graphs of at most 4 vertices are perfect. 4 points

For $n = 5$ G is a cycle of length 5, which is not perfect. 3 points

For $n > 5$ the vertices $v_1, v_2, \ldots, v_5$ span a C_5, so again, G is not perfect. 3 points

5. Determine the list chromatic number of $K_{2,1000}$.

Let $v_1, v_2, \ldots, v_{1000}$ be the vertices in one class, x, y in the other class. Suppose that there is a list of colors of length 3 on each vertex. Color x and y arbitrarily from their lists. Now for each of $v_1, v_2, \ldots, v_{1000}$ there are two forbidden colors, the clolrs of x and y. So, we can find ecah of them a color from their respective lists, since each list contains 3 colors. This shows that $ch(K_{2,1000}) \leq 3$. 5 points

Now we we assign lists of length 2 to each vertex so that G cannot be colored from those lists. Let $L(x) = \{1, 2\}$, $L(y) = \{3, 4\}$, $L(v_1) = \{1, 3\}$, $L(v_2) = \{1, 4\}$, $L(v_3) = \{2, 3\}$, $L(v_4) = \{2, 4\}$. The lists of the other vertices are arbitrary. Suppose that we can color G, by symmetry we can assume that $c(x) = 1$, $c(y) = 3$. But then v_1 cannot be colored! Therefore, $ch(K_{2,1000}) \geq 3$, summarizing, $ch(K_{2,1000}) = 3$. 5 points

6. The vertices of G are $v_1, v_2, \ldots, v_n$. v_i and v_j are connected by an edge if $i > 2j$ or $j > 2i$. Prove that G is perfect.

Introduce a relation on the numbers $1, 2, \ldots, n$. $i \prec j$ if $j > 2i$. 2 points

This relation is irreflexive, $i \prec i$ never holds. 1 point

It is antisymmetric, $i \prec j$ and $j \prec i$ cannot hold at the same time. 1 point

And it is transitive: Suppose that $i \prec j$ and $j \prec k$. Then $j > 2i$ and $k > 2j$. Therefore, $k > 2j > 4i > 2i$ so $i \prec k$. 3 points

Therefore, it is a partial order. Take the comparability graph of $(\{1, 2, \ldots, n\}, \prec)$, it is exactly the graph G, since comparability graphs are perfect, G is perfect. 3 points