Combinatorics and Graph Theory II

Midterm 2, November 27, 2023, 10.15-11.45

Grading guide

It is a very rough grading guide. A statement, scored in the guide, does not get the points automatically, only if it is used properly. Solutions, different from the presented one, are also scored similarly.

1. Prove that there is a graph G with $ch(G) = 4$ and $\chi(G) = 3$.

We know that $ch(K_{n,n^n}) = n + 1$. 3 points

So $ch(K_{3,27}) = 4$. 2 points

The only problem is that $\chi(K_{3,27}) = 2$. So, add a triangle, disjoint from the $K_{3,27}$. For the triangle, $ch = \chi = 3$. 3 points

So, for the disjoint union G, we have $ch(G) = 4$ and $\chi(G) = 3$. 2 points

2. Prove that $R(3, 3, 3) \leq R(4, 6)$.

Let G be a complete graph of $R(4, 6)$ vertices. We have to prove that in any 3-coloring of its edges, there is a monochromatic triangle. 2 points

Take an arbitrary red-white-green coloring. From that we define another coloring, where red edges remain red, but white and green edges are colored blue. (so we merge the colors white and green). By the definition of $R(4, 6)$, we either have a complete subgraph of 4 vertices, all of whose edges are red, or a complete subgraph of 6 vertices, all of whose edges are blue. 3 points

In the first case we are done immediately: then there is also a a complete subgraph of 4 vertices, all of whose edges are red in the original coloring, so there is also a red triangle. 2 points

In the second case there is a complete subgraph of 6 vertices, all of whose edges are blue. So in the original coloring, there is a complete subgraph of 6 vertices, all of whose edges are white or green. But we know that in any two-coloring of the complete subgraph of 6 vertices there is a monochromatic triangle. So there is a white or green triangle in the original coloring. 3 points

3. The graph H has n vertices ($n \geq 3$) and if we remove ANY edge of it, the remaining graph does not contain a triangle. What is the maximum possible number of edges of such a graph (in terms of n)?

We distinguish two cases. 1. H does not contain a triangle. In this case by Mantel's theorem the maximum number of edges is $\lceil n/2 \rceil \cdot \lfloor n/2 \rfloor$. 4 points

2. H does contain a triangle. But then, by the assumption, H cannot have any other edge. Therefore, H has exactly 3 edges. 4 points

Clearly, any graph which does not contain a triangle, satisfies the conditions of the problem. Therefore, the maximum number of edges of such a graph is exactly $\lceil n/2 \rceil \cdot \lfloor n/2 \rfloor$. 2 points

4. Prove that for any r there is an $N(r)$ with the following property. For any r-coloring of the numbers $1, 2, \ldots, N(r)$, there are three numbers, a, b, c, of the same color such that $a + b = c$ and a is a multiple of 2023.

According to the Schur theorem, for any r there is a $K(r)$ with the property that for any r-coloring of the numbers $1, 2, \ldots, K(r)$, there are three numbers, a, b, c, of the same color such that $a + b = c$. 2 points

Color the numbers $1, 2, \ldots, 2023 \cdot K(r)$ with r colors. Consider only the multiples of 2023. The coloring of these numbers induce a coloring of the numbers $1, 2, \ldots, K(r)$: color number k in the new coloring to the color of $2023 \cdot k$ in the original coloring. 3 points

By Schur's theorem, in the new coloring there are three numbers, a, b, c, of the same color such that $a+b=c$. 2 points

But then the numbers $2023 \cdot a$, $2023 \cdot b$, $2023 \cdot c$ satisfy the conditions. They are monochromatic, multiple of 2023, and $2023 \cdot a + 2023 \cdot b = 2023 \cdot c$ 3 points

5. Suppose that $\mathcal{F} \subseteq 2^{[10]}$ is a simple set system (one subset can appear at most once), any two of the sets have a nonempty intersection, but three different sets always have an empty intersection. Prove that $|\mathcal{F}| \leq 5$.

It follows from the conditions, that for any two sets $A, B \in \mathcal{F}$, there is an element $i \in [10]$ such that $i \in A \cap B$. 3 points

Moreover, for any two different pairs, (A, B) and (A', B') the common elements, i and i', are different since three sets have an empty intersection. 3 points

Therefore, $10 \geq \binom{|\mathcal{F}|}{2}$, and the it follows that $|\mathcal{F}| \leq 5$. 4 points

6. Suppose that $\mathcal{F} \subseteq 2^{[n]}$ is a simple set system (one subset can appear at most once). We know that if $A, B \in \mathcal{F}$ and $A \subset B$, then $|A| = 1$. Prove that

$$|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor} + n.$$

Divide $\mathcal{F}$ into two parts, $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$. Let $\mathcal{F}_1$ contain the 1-element subsets of $\mathcal{F}$, and let $\mathcal{F}_2$ contain the other sets. 3 points

Clearly, $|\mathcal{F}_1| \leq n$. 2 points

On the other hand, in $\mathcal{F}_2$ there are no two sets A and B with $A \subset B$, so it is a Sperner system. Therefore, by Sperner's theorem, $|\mathcal{F}_2| \leq \binom{n}{\lfloor n/2 \rfloor}$. 3 points

Summing up,

$$|\mathcal{F}| = |\mathcal{F}_1| + |\mathcal{F}_2| \leq \binom{n}{\lfloor n/2 \rfloor} + n.$$

2 points