Combinatorics and Graph Theory 2.

Recitation 1, September 6.

Perfect graphs

Things to know

$\omega(G)$: clique number, size of the biggest complete subgraph.

$\chi(G)$: chromatic number, the least number of colors needed to color G such that adjacent vertices receive different colors.

$\chi(G) \geq \omega(G)$.

G graph is perfect if for each induced subgraph G' of G (including G as well): $\chi(G') = \omega(G')$.

Interval graphs: Each vertex corresponds to an interval on the line. Two vertices are adjacent if and only if the corresponding intervals intersect each other.

Bipartite graphs, interval graphs, line graphs of bipartite graphs and the complementers of these are perfect graphs

Weak perfect graph theorem: (Lovász 72): G is perfect if and only if $\overline{G}$ is perfect.

Strong perfect graph theorem (Chudnovsky, Robertson, Seymour, Thomas, 2002): G is perfect if and only if G does not contain an odd cycle of length at least five or its complement as an induced subgraph.

1. (Shift graph) Let $m > 1$. Let the vertices of the shift graph S_m be all pairs (i, j), where $1 \leq i < j \leq m$. (We can imagine (i, j) as an interval.) Two vertices, i.e. (i, j) and (i', j') are adjacent if and only if either $i = j'$ or $j = i'$. In other words one of the two intervals starts where the other ends.

   a. Show that S_m does not contain a triangle.

   b. Show that $\chi(S_m) \geq \log_2 m$.

2. Show directly that complements of bipartite graphs are perfect.

3. Show directly that complements of interval graphs are perfect.

4. a. Give a perfect graph which is not an interval graph, but complement of an interval graph. b. Give a perfect graph which is not a complement of an interval graph, but an interval graph. c. Give a perfect graph which is not a complement of an interval graph and not an interval graph. d. Give a perfect graph which is a complement of an interval graph and an interval graph.

5. G is called as a circular-arc graph if its vertices correspond to intervals on a circle and two vertrices are adjacent if and only if the corresponding intervals intersect each other.

   a. Show a circular-arc graph which is not perfect.

   b. Prove that if G is a circular-arc graph, then $\chi(G) \leq 2\omega(G)$.

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6. Let G_n be a graph whose vertices are $1, 2, 3, \ldots, n$, and ij is an edge of G_n if and only if, i and j are coprimes. Determine $\chi(G_n)$ and $\omega(G_n)$. Is G_n perfect when n is large enough?

7. a. Let H be the disjoint union of the perfect graphs G and F. Show that H is also perfect.

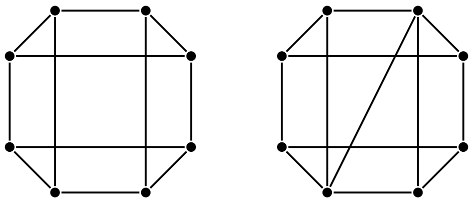
   b. Form H' from H such that we connect every vertex of G with every vertex of F. Show that H' is also perfect.

8. The vertices of the knight-graph G correspond to the tiles of the 8×8 chessboard. Two vertices are adjacent in G if the knight can move from one to the other in one move. (The knight moves one horizontally or vertically, and two in the other direction.) Show that G is perfect. What about other figures? (bishop, rook, queen, king)

9. Show that a graph G is perfect if and only if each induced subgraph G' of G contains an independent set A, such that A intersects each maximum clique of G'.

10. G is a splitgraph if its vertex set is the union of a clique and an independent set. Show that split graphs are perfect.

11. Are these graphs perfect?



12. A tree T is given and $F_1, \ldots, F_n$ are subtrees of T. We give a graph G over the set $\{F_1, \ldots, F_n\}$: F_i and F_j $(i \neq j)$ are adjacent if and only if they share a vertex. Show that G is perfect.

Homework

For every $n > 0$, K_n is the complete graph on n vertices, $K_{n,n}$ is the complete bipertite graph, it has n vertices in both classes and contains all n^2 edges between them. The line graph of G, denoted by $L(G)$, is the following:

1. $V(L(G)) = E(G)$
2. $\{e_1, e_2\} \in E(G) \iff e_1$ and e_2 share a common endpoint in G.

1. Determine all $n > 0$ values for which $L(K_n)$ is perfect.

2. Suppose that $n > 10$. a. Is $L(K_{n,n})$ perfect? b. Is $L(L(K_{n,n}))$ perfect?

3. Give a graph on 2023 vertices which is not perfect, but every nontrivial (that is, of less than 2023 vertices) induced subgraph of it is perfect.