Combinatorics and graph theory 2.

Recitation 12, December 4 2023.

Homogeneous linear recursions, generator functions

Fibonacci numbers: $F_0 = 0$, $F_1 = 1$ and for every $n > 1$, $F_{n+1} = F_n + F_{n-1}$. Then

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

1. Solve the recursion $a_0 = 1, a_1 = 0 \;\; a_n = 5a_{n-1} - 6a_{n-2}$.
2. Solve the recursion $a_0 = 3, a_1 = -3 \;\; a_n = -6a_{n-1} - 9a_{n-2}$.
3. Solve the recursion $a_0 = 3, a_1 = 6, a_2 = 0 \;\; a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$.
4. How many different ways can we go up a stairway of n stairs if each step is of 1 or 2 stairs?
5. How many ways can you cover a $2 \times n$ table with 1×2 and 2×2 dominoes?
6. Solve the following *non-homogeneous* linear resursion. $a_0 = 0, \;\; a_1 = 0, \;\; a_n = a_{n-1} + a_{n-2} + 1$.
7. Suppose that for some K, $a_n = 2Ka_{n-1} - K^2 a_{n-2}$.
   a. $a_0 = 1$, $a_1 = K$. Show that $a_n = K^n$.
   b. $a_0 = 0$, $a_1 = K$. Show that $a_n = nK^n$.
8. Give c_n with a linear recursion if $c_n = \frac{1}{2}\left(\frac{\sqrt{17}-3}{2}\right)^n + \frac{1}{3}\left(\frac{-\sqrt{17}-3}{2}\right)^n$.
9. Let $a_1 = 0$ and for $n \geq 1$, $a_{n+1} = \frac{n+1}{n} a_n + n^2 - 1$. Give a_n in closed form.
   The same for $a_1 = -1$ és $a_{n+1} = 2a_n + n + 1$.
10. Solve the recursion $a_0 = 1, \;\; a_n = 8a_{n-1} + 10^{n-1}$.
11. Let $g_0 = 1$ and $g_n = g_{n-1} + 2g_{n-2} + \ldots + (n-1)g_1 + ng_0$. Give g_n in closed form.
12. What is the generator function of the sequences
    $1, 1, 1, \ldots$;
    $1, 2, 4, 8, \ldots$;
    $1, 2, 3, 4, \ldots$;
    $1, 0, 1, 0, 1, \ldots$?
13. Express the sequence c_n with a_n and b_n if we have $C(x) = A(x)B(x)$ for their generator functions.
14. Let $g(n)$ be the number of non-selfcrossing walks of length n from the origin, where each step is by one unit to the East, West, or North.
    Give $g(n)$ in closed form.
15. For the sequence $a_0, a_1, \ldots$ we have $a_n = 4a_{n-1} - 4a_{n-2}$, $a_0 = 1$, $a_1 = x$. For what values of x does $\lim_{n\to\infty} a_n = -\infty$ hold?
16. Prove that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, and that $F_1 + \ldots + F_n = F_{n+2} - 1$. ($F_n$ is the n-th Fibonacci number.)
17. Prove that $F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$.