Combinatorics and Graph Theory 2.

Recitation 2, September 13.

Comparability graphs, Dilworth theorem

Things to know

Let H be a set and $\preceq$ be a relation over its elements. Relation $\preceq$ is
reflexive, if for all $a \in H$: $a \preceq a$;
antisymmetric, if $a \preceq b, b \preceq a \longrightarrow a = b$;
transitive, if $a \preceq b, b \preceq c \longrightarrow a \preceq c$.

A reflexive, antisymmetric and transitive relation is called a **partial order**, and the pair $(H, \preceq)$ is called as a **partially ordered set (poset)**. If $a \preceq b$ and $a \neq b$, then we denote it by $a \prec b$.

If $a \preceq b$ or $b \preceq a$, then we say that a and b are **comparable,** otherwise a and b are **incomparable.**

Let $(H, \preceq)$ be a partially ordered set. The subset $\{a_1, a_2, \dots, a_k\} \subset H$ is called as a chain if $a_1 \prec a_2 \prec \cdots \prec a_k$, and it is called as an **antichain**, if for any i, j where $1 \leq i < j \leq k$ a_i and a_j are incomparable.

Dilworth's theorem: Let a be the size of the largest antichain in the partially ordered set $(H, \preceq)$. Then H can be partitioned into a chains but it cannot be partitioned into $a - 1$ chains.

Dual Dilworth's theorem: Let l be the size of the largest chain in the partially ordered set $(H, \preceq)$. Then H can be partitioned into l antichains but it cannot be partitioned into $l - 1$ antichains.

Denote the comparability graph which corresponds to the partially ordered set $(H, \preceq)$ by $G(H)$. The vertices of $G(H)$ correspond to the elements of H and two distinct vertices are adjacent if and only if the corresponding elements are comparable.

Lemma. Comparability graphs are perfect.

1. Let G be a perfect graph of n vertices. Show that $\omega(\overline{G})\omega(G) \geq n$.

2. Let $a_1, a_2, \dots, a_n$ be a sequence of numbers. Construct graph G on vertices $v_1, v_2, \dots, v_n$ from it as follows. For every $i > j$, v_i and v_j are connected in G if and only if $a_i > a_j$. Prove that G is perfect.

3. (Erdős-Szekeres Theorem)

   Let $A = a_1, a_2, \dots, a_m$ be a sequence of distinct numbers, $m = kl + 1$, $k, l > 0$.

   a. Prove that A contains an increasing subsequence of length $l + 1$ or a decreasing subsequence of length $k + 1$.

   b. Prove that the statement does not always hold for $m = kl$.

4. a. We have some cardboard boxes that correspond to the vertices of graph G. Two vertices are connected if and only if none of the two corresponding boxes fits into the other one. Show that G is perfect.

   b. Does that statement hold with flexible boxes or bags instead of cardboard boxes?

5. For given n circles in the plane we constrict the following graph G. The vertices correspond to the circles, two vertices are connected if and only if one of the circles is inside the other one. Show that G is perfect.

6. We have a finite set F. The vertices of G correspond to some subsets of F. Tho distinct vertices are connected in G if and only if one of the subsets contains the other one. Show that G is perfect.

7. We have a triangle ABC and n points in it. Show that we can find $\sqrt[3]{n}$ of the points such that every line determined by these points intersects the same two sides of ABC.

8. We have n points in the plane. Prove that we can choose $\sqrt{n}$ of them such that either any two determines a line which encloses an angle at most $\pi/6$ with the x-axis, or any two determines a line which encloses an angle at least $\pi/6$ with the x-axis.

9. Let $(H, \prec)$ be a poset. An element x is *maximal* (*minimal*) if there is no y with $x \prec y$ ($y \prec x$).

   a. Prove that the set of maximal (resp. minimal) elements is an antichain.

   b. Suppose that the maximal and minimal elements *together* form and antichain. Prove that all elements of H form an antichain.

10. Let $(H, \prec)$ be a poset, L a maximal chain whose maximal element is x, minimal element is y. Let $A = \{z_1, \ldots, z_a\}$ be a maximal antichain in H.

    Finally, let

$$H^+ = \{\ h \in H \mid \exists z \in A : z \preceq h\},$$

and

$$H^- = \{\ h \in H \mid \exists z \in A : h \preceq z\}.$$

   a. Show that $H^+ \cap H^- = A$.

   b. Show that $H^+ \cup H^- = H$.

   c. Show that $x \in H^+$, $y \in H^-$.

11. Let $G(V, E)$ be a graph such that $E = E_1 \cup E_2$, $G_1(V, E_1)$ and $G_2(V, E_2)$ are perfect, $|V| = 65$. Prove that G contains a clique of size 5 or an independent set of size 5.

12. Given 50 unit intervals on a line. Prove that either there is a point on the line contained in at least 8 intervals or there are 8 pairwise disjoint intervals.

13. Let $G(V, E)$ be a graph such that $E = E_1 \cup E_2$, $G_1(V, E_1)$ is perfect, $G_2(V, E_2)$ is a bipartite graph, and $|V| = 163$. Prove that G contains a clique or an independent set of size 10.

Homework

1. Give a poset with a maximal chain and maximal antichain which are disjoint.

2. Let $a_1, \ldots, a_n$ be an arbitrary sequence. The vertices of G are $v_1, \ldots, v_n$, v_i and v_j ($i \neq j$) are connected if and only if $|a_i - a_j| \geq 100$. Prove that G is perfect.

3. Given 32 circular arcs on a circle. Show that there are 6 of them which are either pairwise crossing or pairwise disjoint.