Combinatorics and Graph Theory 2.

Recitation 3. September 20.

Planar graphs

Things to know

G is a planar graph if it can be drawn on the plane without edge crossings. Suppose that G is drawn on the plane without edge crossings and we have n vertices, e edges, f faces, c connected components.

Euler's formula: $n - e + f = c + 1$.

Corollary: If G is a simple graph of $n \geq 3$ vertices, then $e \leq 3n - 6$. If G is a simple bipartite graph of $n \geq 3$ vertices, then $e \leq 2n - 4$. Moreover, if G is a simple graph of $n \geq 3$ vertices that does not contain triangles, then also $e \leq 2n - 4$.

Corollary of corollary: K_5 and $K_{3,3}$ are not planar.

Four color thoeorem (Appel-Haken 1976) If G is planar, then $\chi(G) \leq 4$. (It cannot be improved, e. g. K_4.)

Topological isomorphism. Define two operations, which are the inverses of each other. Operation 1: u and v are vertices of G and connected. Delete edge uv, add a vertex x and egdes ux and xv. Operation 2: Suppose that x has exactly two neighbors, u and v. Delete x and add the edge uv. Graphs G and H and topologically isomorphic, if we can get H from G by repeated applications of operations 1 and 2.

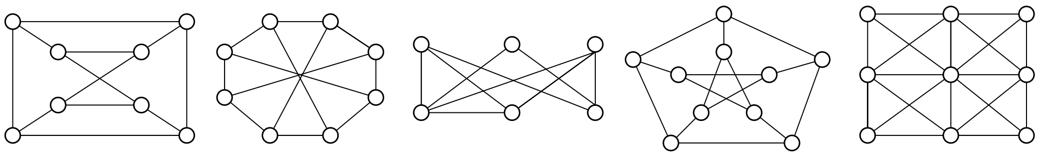
Observation: Suppose that G and H and topologically isomorphic. Then G is planar if and only if H is planar.

Kuratovski's theorem (1930) G is planar if and only if it does not contain a subgraph topologically isomorphic to K_5 or $K_{3,3}$.

Fáry-Wagner theorem (1936, 1948) If G is planar, then it has a crossing-free (planar) drawing such that all edges are straight line segments.

1. (i) A graph, drawn in the plane, has $n \geq 3$ vertices and $3n - 6$ edges. Prove that all faces are triangles.

   (ii) A simple graph, drawn in the plane such that all faces are triangles. Prove that it has $3n - 6$ edges.
2. G has 3 faces with 3 edges, 3 with 4 and 1 with 5. How many vertices does it have?
3. For any planar graph let $n(G)$ be the number of vertices, $e(G)$ the number of edges and $f(G)$ the number of faces. Determine the maximum of $n(G) + e(G) - 2f(G)$ over all graphs G which are planar, simple, connected, and have 100 vertices.
4. Let G be a simple planar graph of n vertices. Prove that
   a) it can also be drawn (crossing-free) on the torus;
   b) if G has less than $3n - 6$ edges, than we can add an edge to G such that it remains simple, planar;
   c) in any planar drawing of G we have the same number of faces;
   d) either G contains a vertex of degree at most 3 or in any planar drawing there is a triangular face.
5. Give a simple planar graph of 8 vertices whose complement is also planar.
6. Show that if $|V(G)| \geq 11$, then one of G and $\overline{G}$ is not planar.
7. A convex polyhedron's faces are all quadrilaterals or octagons. Each vertex belongs to exactly three faces. What is the difference of the numbers of quadrilateral and octagonal faces?
8. We have k houses and k wells in the meadow. From each house there are direct paths to exactly four different wells. Prove that there are two crossing paths.
9. Prove that any plane graph can be made 3-connected by adding further edges, but not creating crossings. Prove that if all faces of G are triangles, then G is 3-connected. (G has at least 4 vertices)
10. Suppose that G is a simple planar graph and g is the length of the shortest cycle. Prove that $|E(G)| \leq \frac{g}{g-2}(n-2)$.
11. A poyhedron of 20 vertices has 12 faces, each face has k sides. Determine k.
12. Determine if the graphs $K_6, K_{4,2}, K_{4,3}, K_5 - e, K_{3,3} - e, \overline{C_7}$ are planar or not. What about the following graphs?



13. Show that in any simple planar graph
    a) the minimum degree is at most 5;
    b) if the min degree is 5, then there are at least 12 vertices of degree 5.
14. In G, all degrees are at most 3 and all cycles have length at most 5. Show that G is planar!
15. Let $cr(G)$ be the minimum number of edge crossing in a drawing of G in the plane. (Three edges cannot cross at the same point.) Determine $cr(K_{4,4})$ and $cr(K_6)$?
16. Prove that $cr(K_{5,5}) \geq 11$.
17. Prove that both K_7 and $K_{4,4}$ can be drawn on the torus (without crossings). Prove that if we add an edge to a planar graph, the resulting graph can be drawn on the torus.
18. Prove that a 4-regular simple bipartite graph is not planar.
19. (Hanani-Tutte theorem) (*) We managed to draw a graph in the plane such that any two edges cross each other an even number of times. Prove that it is a planar graph!
20. a (*). Spiderman (A), Superman (B) and Batman (C) live next to each other. The garages are in a different buildig, also next to each other. (See Figure)

    Unfortunately they hate each other so they want to build paths from the houses to the corresponding garages so that the paths do not cross. (they are old and cannot fly anymore) Is it possible?

    b. Moreover, their weekend houses are in a common garden. They all have a separate gates. Can they build paths from the gates to the corresponding houses without crossings?
21. Suppose that every face of G is bounded by an even number of edges. Prove that in this case G is a bipartite graph.
22. For a graph G, drawn in the plane, let $n(G)$ be the number of vertices, $e(G)$ the number of edges and $f(G)$ the number of faces. Determine the maximum of $n(G) + 2e(G) - f(G)$ if G can be any simple planar graph of 2020 vertices.

Homework.

1. Suppose that $G_1(V, E_1)$, $G_2(V, E_2)$ and $G_3(V, E_3)$ are planar graphs on the same vertex set. Let $G(V, E_1 \cup E_2 \cup E_3)$ be their union.

a. Prove that $\chi(G) \leq 1000$.

b. Prove that $\chi(G) \leq 18$.

2. We have a graph of 10000000 vertices, drawn on a piece of paper with only 2 crossings. Prove that we van remove five edges so that the remaining graph is not connected.

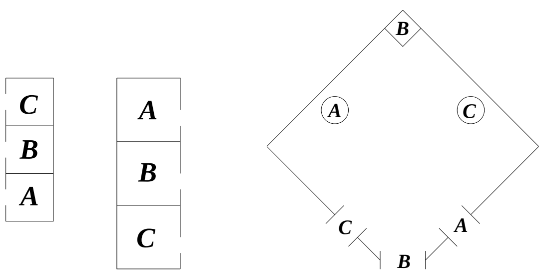


Figure 1: The houses+garages and the weekend houses of Spiderman, Superman and Batman.