Combinatorics and Graph Theory 2.

Recitation 4, September 27.

Planar graphs, duality, Whitney

Things to know:

G is a graph, drawn in the plane without crossings (here parallel edges and loops are allowed). Its **dual** is G^*: Put a vertex of G^* in every face of G, connect any two vertices of G^* that correspond to neighboring faces of G, through the edge forming the common boundary. If there are more then one such edge, connect the vertices through each common boundary edge.

If G is connected, then G^{**} is isomorphic to G.

H and G are abstract graphs. G and H are *abstract duals* or *Whitney duals* of each other, if there is a bijection between $E(G)$ and $E(H)$, such that cutset $\longleftrightarrow$ cycle and cycle $\longleftrightarrow$ cutset.

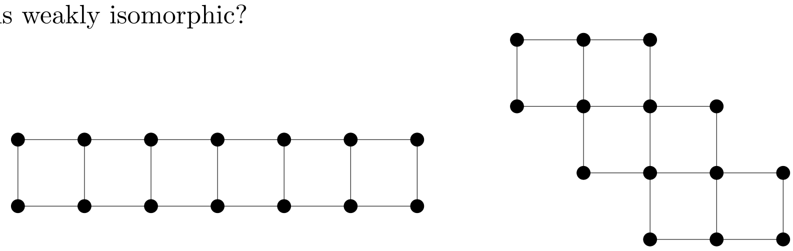
G and H are *weakly isomorphic* if if there is a bijection between $E(G)$ and $E(H)$, such that cutset $\longleftrightarrow$ cutset and cycle $\longleftrightarrow$ cycle.

Whitney 1: G has an abstract dual if and only if G is planar.

Whitney 2: Suppose that G is planar, G and H are weakly isomorphic. Then
1. H is also planar, 2. G^* and H^* are also weakly isomorphic, 3. G and G^{**} are also weakly isomorphic.

Whitney 3: Suppose that G and H are weakly isomorphic. Then one can get H from G with the repeated applications of the following three types of operations. (a) if a graph has a cut vertex, we can cut the graph there (the cut vertex will appear in both parts) (b) two disjoint components are glued together at a vertex. (That is, take a vertex of both components and identify them.) (c) If a graph has a cut of two vertices, then we can separate the two components here (keeping both vertices at both parts) and and glue them together again, with the two points switched in one of the components.

1. Are these graphs weakly isomorphic?



2. Show that two trees are weakly isomorphic if and only if they have the same number of vertices.
3. Show that the edge set of a simple planar graph is always the union of the edge sets of two bipartite graphs.
4. G and G^* are duals of each other. Prove that $\min\{\delta(G), \delta(G^*)\} = 3$. δ is the min degree.
5. Let G be a simple planar graph of $n \geq 3$ vertices and $3n-6$ edges. What is the max degree in G^*, $\Delta(G^*)$?
6. Let $F_n = K_{n,n} - nK_2$, that is, a bipartite graph we get from $K_{n,n}$ by removing a perfect matching. For which n is F_n planar?
7. G is a draph drawn in the plane, all faces of G are triangles, and all faces of G^* are quadrilaterals. Determine the number of vertices and edges of G.
8. G is a connected planar graph, drawn in the plane. It has 200 edges and its dual is a simple bipartite graph. Show that G has at most 100 vertices.
9. G is a simple, connected planar graph, it has $n \geq 3$ vertices and it does not contain cycles of length 3, 4 and 5. Prove that its dual, G^*, is not a simple graph.
10. For an arbitrary connected planar G graph, show a G' self-dual plane graph which contains G as an induced subgraph.

Homework

1. G is a connected plane (drawn in the plane) graph. It has 200 vertices and 300 edges. Its dual, G^* is a simple graph. Prove that $\Delta(G)$, the max degree in G, is 3.

2. For any plane graph G, let $n(G)$, $e(G)$, and $f(G)$ denote the number of vertices, edges and faces, respectively. Determine the maximum of $e(G) - n(G) - 3f(G)$ over all plane graphs G of at least 3 vertices.