Combinatorics and Graph Theory 2.

Recitation 5, October 4.

List coloring

Things to know:

G is a graph and there is a list of colors $L(v)$ assigned to every vertex v of G. G is L colorable if there is a proper coloring of G such that for every vertex v its color $c(v) \in L(v)$.

The list coloring number of G, $ch(G)$ is the smallest k with the property that whenever $|L(v)| = k$ for every v, then G is L colorable.

For every G, $\chi(G) \leq ch(G)$, and for every k there is a graph G with $\chi(G) = 2$ de $ch(G) > k$.

For every G, $ch(G) \leq \Delta(G) + 1$.

List coloring conjecture: If G is a line graph, then $\chi(G) = ch(G)$.
Galvin theorem: If G is the line graph of a bipartite graph, then $\chi(G) = ch(G)$.
Thomassen 94: If G is a planar graph, then $ch(G) \leq 5$.
Voigt 93: There is a G planar graph for which $ch(G) = 5$.

1. Determine $ch(K_{2,4})$. ($ch(K_{2,4})$ is a complete bipartite graph with 2 and 4 vertices in the classes.)
2. Is it true that if $\chi(G) = ch(G)$, then $\chi(\overline{G}) = ch(\overline{G})$?
3. There is a list of colors on each vertex of G. Each list has length at least $ch(G)$. Is it true that for a proper ordering of the vertices it can be colored in a greedy way from the list? That is, we color in the given order and always choose the smallest possible color from the list.
4. Let $K_{2,2,\ldots,2}$ a graph of $2n$ vertices whose complement is n independent edges. Determine $ch\left(K_{2,2,\ldots,2}\right)$.
5. Show a graph which is not a line graph.
6. Show that if G is a line graph, then $ch(G) \leq 2\chi(G) - 1$.
7. Suppose that for a graph G and lists L, $|L(v)| > d(v)$ holds for every vertex v. Prove that G is L-colorable.
8. Show that for any tree T (of at least 2 vertices) $ch(T) = 2$.
9. Prove that if C is an odd cycle, then $ch(C) = 3$.
10. For an arbitrary graph G, let $3G$ be the following graph. We take three disjoint copies of G and connect the corresponding vertices in the three copies. Prove that if G is planar, then $ch(3G) \leq 7$.
11. G is drawn in the plane with just one edge crossing (where exactly two edges cross) Prove that

    a. $ch(G) \leq 6$,

    b. $ch(G) \leq 5$.
12. G is a planar graph with at least 4 vertices. We have 7 colors, $\{1, 2, \ldots, 7\}$, we want to color G with these colors. But somebody already colored 4 vertices that span a K_4 with colors 1, 2, 3, 4 Prove that we can finish the coloring, that is, we can color the rest of the vertices with colors $\{1, 2, \ldots, 7\}$ (so that neighboring vertices get different colors)
13. If we remove any 4 edges from G, we get a planar graph. Prove that $ch(G) \leq 6$.
14. G is a planar graph. There is a list of five colors at each vertex, except one vertex where we have a list of one color. Prove that G can be colored from the lists.

Homework

1. Prove that $ch(K_{n,n^n}) = n + 1$ for every $n > 0$.
2. Prove that if C is an even cycle then $ch(C) = 2$.