Combinarotics and Graph Theory 2.

Recitation 6, October 18.

Ramsey

Ramsey's theorem (for graphs, with two colors): For any $k, l > 0$ there is a (smallest) number $R = R(k, l)$ with the following property. If we color the edges of the complete graph of R vertices with red and blue, there is either a complete subgraph of k vertices all of whose edges are red or a complete subgraph of l vertices all of whose edges are blue. Erdős–Szekeres: $R(k, l) \leq \binom{k+l-2}{k-1}$.

Ramsey's theorem (for graphs, with r colors): For any $r > 0$, $k_1, k_2, \ldots, k_r > 0$ there is a (smallest) number $R = R(k_1, k_2, \ldots, k_r)$ with the following property. If we color the edges of the complete graph of R vertices with colors $1, 2, \ldots, r$, then for some $1 \leq i \leq r$, there is a complete subgraph of k_i vertices all of whose edges are of color i.

Ramsey's theorem (for k-uniform hypergraphs, with r colors): For any $r > 0$, $k \geq 2$, $k_1, k_2, \ldots, k_r > 0$ there is a (smallest) number $R = R_k(k_1, k_2, \ldots, k_r)$ with the following property. If we color the hyperedges of the complete k-uniform hypergraph of R vertices with colors $1, 2, \ldots, r$, then for some $1 \leq i \leq r$, there is a complete subhypergraph of k_i vertices all of whose hyperedges are of color i.

Schur theorem: For any $t > 0$ there is a number $N = N(t)$ such that for any coloring of the numbers $1, 2, \ldots, N$ with t colors, there are x, y, z of the same color with $x + y = z$.

Van der Waerden theorem: For any $t, k > 0$ there is an $N = N(t, k)$ such that for any coloring of the numbers $1, 2, \ldots, N$ with t colors, there will be a monochromatic arithmetic progression of k terms.

Erdős-Szekeres theorem: For every $n > 0$ there is an $F(n)$ such that $F(n)$ points in general position in the plane always contain n in convex position.

1. Prove that if G has at least 10 vertices then $\omega(G) \geq 4$ or $\alpha(G) \geq 3$.
2. Prove that $R(3, 3, 3) \leq 17$, $R(3, 4) = 9$, $R(3, 3, 3) \leq 17$, and $R(3, 4) = 9$.
3. a. The edges of a complete graph are colored with red and blue. Prove that there is a monochromatic spanning tree.

   b. Is it true with Hamiltonian path instead of spanning tree?
4. Show that $R_3(4, 4) \leq 21$.
5. Prove the following inequalities: $R_3(k, l) \leq R_2(R_3(k-1, l), R_3(k, l-1)) + 1$.

   What upper bound (asymptotically) does it give for $R_3(k, k)$?
6. Prove that if $c \geq 3$ then $R_t(n_1, n_2, \ldots, n_c) \leq R_t(n_1, n_2, \ldots, n_{c-2}, R_t(n_{c-1}, n_c))$ holds.
7. Suppose that we know that the Van der Waerden theorem holds for two colors. Show that is also holds for three and more colors.
8. Let $H(V, E)$ be a k-uniform hypergraph with less than 2^{k-1} hyperedges. Prove that the vertices of H can be colored with red and blue such that no hyperedge is monochromatic.
9. (Infinite Ramsey theorem) We have a complete graph of infinitely many vertices. We color the edges with two colors. Show that there is a monochromatic complete subgraph of infinitely many vertices.
10. Suppose that G is a simple graph oh n vertices. Prove that

    $\max(\alpha(G), \omega(G)) \geq 1 + \log_4 n$.
11. The points of the plane are colored with red, white and blue. Prove that there are two points of the same color at distance one.

12. The points of the plane are colored with red and blue, both colors are used. Prove thet there are two points at distance one, of different colors. Is there always a monochromatic unit regular triangle?
13. The points of the plane are colored with red and blue. There are no two blue points at unit distance. Let T be an arbitrary triangle. Prove that there are three red points that form a triangle congruent to T.
14. Let a_n be an infinite increasing sequence of natural numbers. Prove that there is an arbitrary long subsequence whose elements are pairwise relative prime, or pairwise not relative prime.
15. Prove that for for every $t > 0$ there is an $M = M(t)$ with the following property. For any t-coloring of the numbers $1, 2, \ldots, M$, there are numbers x, y, z of the same color with $x + y = z$ and $x \neq y$.
16. Prove that for any $t, k > 0$ there is an $M = M(t, k)$ such that for any t-coloring of the numbers $1, 2, \ldots, M$ there is a monochromatic geometric series of k elements.
17. Is it possible to color the integers with two colors so that there is no infinite monochromatic arithmetic progression? What about geometric series?
18. Show a graph of $(k - 1)^2$ vertices with no complete or empty spanned subgraph of k vertices.
19. P is a planar point set such that any two points determine an integer distance and not all points of P are collinear. Show that P is finite.
20. Prove that for any n there is a $K(n)$ such that any set of $K(n)$ different points determiner at least n distinct distances.
21. Prove that for any k there is an $N(k)$ such that in any k-coloring of $1, 2, 3, \ldots, N(k)$ there are three different numbers, x, y, z of the same color with $x + y = 2z$.

Homework

1. Decide if the following is true. For every $n \geq 1$ there is an N such that for any two-coloring of the subsets of $[N] = \{1, 2, \ldots, N\}$ there is an n-element subset whose subsets are all of the same color.
2. Prove that for every positive k there is an $N(k)$ such that the following holds. If $n > N(k)$ and we color the subsets of the set $[n] := \{1, 2, \ldots, n\}$ with k colors, then there are two disjoint subsets X_1 and X_2 such that X_1, X_2 and $X_1 \cup X_2$ are of the same color. Is it true with three disjoint subsets (and all unions)?