Combinatorics and graph theory 2.

Recitation 8, november 6, 2023.

Turán

For any graph H, $ex(n, H)$ is the maximum number of edges of an n-vertex graph that does not contain H as a subgraph.

Let $n, r \geq 1$. The n-vertex, r-class Turán graph $T(n, r)$ is the following. Divide the n vertices into r classes in the possible most uniform way: if $n = ar + b$, $r > b \geq 0$,, then there are $\lceil n/r \rceil$ vertices in b classes, and $\lfloor n/r \rfloor$ vertices in the other $r - b$ classes. Any two vertices from different classes are connected, and two vertices from the same class are never connected.

For any graph G, $|E(G)|$ is its number of edges.

Turán theorem (1941). a. $ex(n, K_{r+1}) = |E(T(n, r))|$.

b. $T(n, r)$ is the only extremal graph. That is, if G has n vertices, does not contain K_{r+1} as a subgraph, and $|E(G)| = ex(n, K_{r+1}) = |E(T(n, r))|$, then G is isomorphic to $T(n, r)$.

Erdős, Stone, Simonovits theorem (1946...).

$$\lim_{n\to\infty} \frac{ex(n, H)}{\binom{n}{2}} = 1 - \frac{1}{\chi(H) - 1}.$$

Erdős, Kővári, Sós, Turán theorem (1954).
For any $r \geq s \geq 2$, $ex(n, K_{r,s}) \leq c_{r,s} n^{2-1/s}$ for some constant $c_{r,s}$.

1. What is the maximum number of edges of an n-vertex graph if it does not contain
   - cycle?
   - odd cycle?
   - even cycle?
   - 2 edge path?
   - triangle and 3 edge path?
   - spanning tree?
2. In a group of 90 people, some pairs are friends. Among any 10 of them, there are two friends. Show that there are at least 405 pairs who are friends.
3. Prove that the Turán graph $T_{n,m}$ does not contain a Hamiltonian cycle if and only if $m = 2$ and n is odd.
4. $v_1, v_2, \ldots, v_n$ are vectors in the plane, $|v_i| \geq 1$. For at least how many pairs should we have $|v_i + v_j| \geq 1$?
5. What is the minimum number of vertices a triangle-free simple graph G if $|E(G)| \geq 2|E(K_k)|$?
6. We have n, not necessarily different points. What is the maximum number of unit-distance pairs?
7. Prove that between n different points and n different lines in the plane there are at most $c \cdot n^{\frac{3}{2}}$ incidences. (Incidence: a (point, line) pair, where the point is on the line.)
8. Prove that n different points in the plane determine at most $c \cdot n^{\frac{3}{2}}$ unit distances.
9. What is the maximum maximum number of edges of an n-vertex graph if its edges can be colored with two colors so that there is no monochromatic triangle.
10. There is a group of n people, they do not know each other. How many introductions are necessary (introduction: between two people, originally they don't know each other, after the introduction they do know each other) can we achieve the following situation: 1. Among any three people there are two who know each other. 2. Any person can send a message to any other person possibly through some others, if the message can passed from anyone to known people only.

11. A graph has 49 vertices and 1030 edges. Prove that its chromatic number is at least 8, and it can be 8.
12. There is a group of n people. Among any k there are two who shaked hands. What is the minimum number of handshakes?
13. H is a graph of 5 vertices, the disjoint union of an edge and a triangle. Determine $ex(n, H)$. (Suppose that $n \geq 100$.)

Házi feladat

1. a. G has n vertices every degree is at least 100. Prove that G contains a path of length 100. (path with 100 vertices)

   b. G has n vertices and e edges, $e > 100n$. Prove that G contains a path of length 100. (path with 100 vertices)
2. For every n, show a graph of n vertices and e edges, $e > 40n - 100000$, which does not contain a path of length 100. (path with 100 vertices)
3. H is a graph of 4 vertices, the disjoint union of two edges. Determine $ex(n, H)$. (Suppose that $n \geq 100$.)